# Supplemental Material On
**"Additive Nonparametric Regression in the Presence of Endogenous Regressors"**

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THIS APPENDIX PROVIDES PROOFS FOR SOME TECHNICAL LEMMAS IN THE ABOVE PAPER.

**Proof of Lemma B.1.** By straightforward moment calculations, we can show that $E||Q_{n,PP} -Q_{PP}||^2 = O\left(\kappa_1^2/n\right)$ under Assumption A1($i$)-($ii$) and A2($vi$). Then ($i$) follows from Markov inequality. By Weyl inequality [e.g., Bernstein (2005, Theorem 8.4.11)] and the fact that $\lambda_{\max}(A) \leq \|A\|$ for any symmetric matrix $A$ (as $|\lambda_{\max}(A)|^2 = \lambda_{\max}(AA) \leq \|A\|^2$), we have

$$\begin{aligned}\lambda_{\min}(Q_{n,PP}) &\leq \lambda_{\min}(Q_{PP}) + \lambda_{\max}(Q_{n,PP} - Q_{PP}) \\ &\leq \lambda_{\min}(Q_{PP}) + \|Q_{n,PP} - Q_{PP}\| = \lambda_{\min}(Q_{PP}) + o_P(1).\end{aligned}$$

Similarly,

$$\begin{aligned}\lambda_{\min}(Q_{n,PP}) &\geq \lambda_{\min}(Q_{PP}) + \lambda_{\min}(Q_{n,PP} - Q_{PP}) \\ &\geq \lambda_{\min}(Q_{PP}) - \|Q_{n,PP} - Q_{PP}\| = \lambda_{\min}(Q_{\kappa_1}) - o_P(1).\end{aligned}$$

Analogously, we can prove the second part of ($ii$). Thus ($ii$) follows. By the submultiplicative property of the spectral norm, ($i$)-($ii$) and Assumption A2($i$),

$$\begin{aligned}\left\|Q_{n,PP}^{-1} - Q_{PP}^{-1}\right\|_{\text{sp}} &= \left\|Q_{n,PP}^{-1}(Q_{PP} - Q_{n,PP})Q_{PP}^{-1}\right\|_{\text{sp}} \leq \left\|Q_{n,PP}^{-1}\right\|_{\text{sp}} \|Q_{PP} - Q_{n,PP}\|_{\text{sp}} \left\|Q_{PP}^{-1}\right\|_{\text{sp}} \\ &= O_P(1)\, O_P\left(\kappa_1/n^{1/2}\right) O_P(1) = O_P\left(\kappa_1/n^{1/2}\right),\end{aligned}$$

where we use the fact that $\left\|Q_{n,PP}^{-1}\right\|_{\text{sp}} = [\lambda_{\min}(Q_{n,PP})]^{-1} = [\lambda_{\min}(Q_{PP}) + o_P(1)]^{-1} = O_P(1)$ by ($ii$) and Assumption A2($i$). Then ($iii$) follows. The proof of ($iv$)-($v$) is analogous to that of ($i$)-($ii$) and thus omitted. ■

**Proof of Lemma B.2.** ($i$) By Assumption A1($i$) and A2($i$), $E\|\xi_{nl}\|^2 = n^{-2}\text{tr}\{\sum_{i=1}^n E(P_i P_i' U_{li}^2)\} \leq n^{-1}(1 + d\kappa_1)\, \lambda_{\max}(Q_{PP,U_l}) = O(\kappa_1/n)$. Then $\|\xi_{nl}\|^2 = O_P(\kappa_1/n)$ by Markov inequality.

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$(ii)$ By the facts that $\|a\|_{\rm sp}^2 = \|a\|^2$ for any vector $a$, $|a'b| \le \|a\| \, \|b\|$ for any two conformable vectors $a$ and $b$ and that $\varkappa' A \varkappa \le \lambda_{\max}(A) \|\varkappa\|^2$ for any p.s.d. matrix $A$ and conformable vector $\varkappa$, Cauchy-Schwarz inequality, Lemma B.1$(ii)$ and Assumptions A2$(iv)$, we have

$$
\begin{aligned}
\|\zeta_{nl}\|^2 &= \|\zeta_{nl}\|_{\rm sp}^2 = \lambda_{\max}\left(\zeta_{nl}\zeta_{nl}'\right) \\
&= \max_{\|\varkappa\|=1} n^{-2} \sum_{i=1}^n \sum_{j=1}^n \varkappa' P_i P_j' \varkappa \left[m_l(\mathbf{Z}_i) - P_i'\boldsymbol{\alpha}_l\right] \left[m_l(\mathbf{Z}_j) - P_j'\boldsymbol{\alpha}_l\right] \\
&\le \max_{\|\varkappa\|=1} \left\{ n^{-1} \sum_{i=1}^n \left\{ \varkappa' P_i P_i' \varkappa \left[m_l(\mathbf{Z}_i) - P_i'\boldsymbol{\alpha}_l\right]^2 \right\}^{1/2} \right\}^2 \\
&\le O_P(\kappa_1^{-2\gamma}) \max_{\|\varkappa\|=1} \left\{ n^{-1} \sum_{i=1}^n \varkappa' P_i P_i' \varkappa \right\} \le O_P(\kappa_1^{-2\gamma}) \lambda_{\max}(Q_{n,PP}) = O_P(\kappa_1^{-2\gamma}).
\end{aligned}
$$

$(iii)$ Noting that $X_{li} = m_l(\mathbf{Z}_i) + U_{li} = P_i'\boldsymbol{\alpha}_l + U_{li} + \left[m_l(\mathbf{Z}_i) - P_i'\boldsymbol{\alpha}_l\right]$, by Lemma B.1$(ii)$, w.p.a.1 we have

$$
\begin{aligned}
\widetilde{\boldsymbol{\alpha}}_l - \boldsymbol{\alpha}_l &= \left(\sum_{i=1}^n P_i P_i'\right)^{-} \sum_{i=1}^n P_i X_{li} - \boldsymbol{\alpha}_l \\
&= Q_{n,PP}^{-1} n^{-1} \sum_{i=1}^n P_i U_{li} + Q_{n,PP}^{-1} n^{-1} \sum_{i=1}^n P_i \left[m_l(\mathbf{Z}_i) - P_i'\boldsymbol{\alpha}_l\right] \\
&= Q_{n,PP}^{-1} \xi_{nl} + Q_{n,PP}^{-1} \zeta_{nl} \equiv a_{1l} + a_{2l}, \text{ say.}
\end{aligned}
\tag{C.1}
$$

Note that $a_{1l} = Q_{\kappa_1}^{-1} \xi_{nl} + r_{1nl}$ where $r_{1nl} = \left(Q_{n,PP}^{-1} - Q_{PP}^{-1}\right) \xi_{nl}$ satisfies that

$$
\begin{aligned}
\|r_{1l}\| &\le = \left\{ \operatorname{tr}\left[ \left(Q_{n,PP}^{-1} - Q_{PP}^{-1}\right) \xi_{nl} \xi_{nl}' \left(Q_{n,PP}^{-1} - Q_{PP}^{-1}\right) \right] \right\}^{1/2} \\
&\le \|\xi_{nl}\|_{\rm sp} \left\| Q_{n,PP}^{-1} - Q_{PP}^{-1} \right\| = O_P(\kappa_1^{1/2}/n^{1/2}) O_P(\kappa_1^{1/2}/n^{1/2}) = O_P(\kappa_1/n)
\end{aligned}
$$

by Lemmas B.1$(iii)$ and B.2$(i)$. For $a_{2l}$, we have $a_{2l} = Q_{\kappa_1}^{-1} \zeta_{nl} + r_{2nl}$ where $r_{2nl} = \left(Q_{n\kappa_1}^{-1} - Q_{n\kappa_1}^{-1}\right) \zeta_{nl}$ satisfies that

$$
\|r_{2l}\| \le \|\zeta_{nl}\|_{\rm sp} \left\| Q_{n,PP}^{-1} - Q_{PP}^{-1} \right\| = O_P(\kappa_1^{-\gamma}) O_P(\kappa_1^{1/2}/n^{1/2}) = O_P(\kappa_1^{-\gamma+1/2}/n^{1/2})
$$

by Lemmas B.1$(iii)$ and B.2$(ii)$. The result follows. ■

**Proof of Lemma B.3.** $(i)$ We only prove the $r = 1$ case as the proof of the other case is almost identical. By the definition of $\widetilde{U}_{li}$ and (C.1), we can decompose $\widetilde{U}_{li} - U_{li} = \left[X_{li} - \widetilde{m}_l(\mathbf{Z}_i)\right] - U_{li}$

as follows

$$\begin{aligned}
\widetilde{U}_{li}-U_{li} &= (\mu_l-\widetilde{\mu}_l)+\sum_{k=1}^{d_1}\left[m_{l,k}\left(Z_{1k,i}\right)-\widetilde{m}_{l,k}\left(Z_{1k,i}\right)\right]+\sum_{k=1}^{d_2}\left[m_{l,d_1+k}\left(Z_{2k,i}\right)-\widetilde{m}_{l,d_1+k}\left(Z_{2k,i}\right)\right] \\
&= -\left(\widetilde{\mu}_l-\mu_l\right)-\sum_{k=1}^{d_1}p^{\kappa_1}\left(Z_{1k,i}\right)'\mathbb{S}_{1k}a_{1l}-\sum_{k=1}^{d_2}p^{\kappa_1}\left(Z_{2k,i}\right)'\mathbb{S}_{1,d_1+k}a_{1l} \\
&\quad -\sum_{k=1}^{d_1}p^{\kappa_1}\left(Z_{1k,i}\right)'\mathbb{S}_{1k}a_{2l}-\sum_{k=1}^{d_2}p^{\kappa_1}\left(Z_{2k,i}\right)'\mathbb{S}_{1,d_1+k}a_{2l} \\
&\equiv -u_{1l,i}-u_{2l,i}-u_{3l,i}-u_{4l,i}-u_{5l,i},\ \text{say.}
\end{aligned} \tag{C.2}$$

Then by Cauchy-Schwarz inequality, $n^{-1}\sum_{i=1}^n(\widetilde{U}_{li}-U_{li})^2\sigma_i^2\leq 5\sum_{s=1}^5 n^{-1}\sum_{i=1}^n u_{sl,i}^2\sigma_i^2\equiv 5\sum_{s=1}^5 V_{nl,s}$, say. Apparently, $V_{nl,1}=O_P\left(n^{-1}\right)$ as $\widetilde{\mu}_l-\mu_l=O_P\left(n^{-1/2}\right)$.

$$\begin{aligned}
V_{nl,2} &= n^{-1}\sum_{i=1}^n\left(\sum_{k=1}^{d_1}p^{\kappa_1}\left(Z_{1k,i}\right)'\mathbb{S}_{1k}a_{1l}\right)^2\sigma_i^2 \\
&\leq d_1\sum_{k=1}^{d_1}n^{-1}\sum_{i=1}^n\left(p^{\kappa_1}\left(Z_{1k,i}\right)'\mathbb{S}_{1k}a_{1l}\right)^2\sigma_i^2=d_1\sum_{k=1}^{d_1}\operatorname{tr}\left(\mathbb{S}_{1k}a_{1l}a_{1l}'\mathbb{S}_{1k}'Q_{n1k,pp}\right) \\
&\leq d_1\sum_{k=1}^{d_1}\lambda_{\max}\left(Q_{n1k,pp}\right)\operatorname{tr}\left(a_{1l}a_{1l}'\mathbb{S}_{1k}'\mathbb{S}_{1k}\right)\leq d_1\sum_{k=1}^{d_1}\lambda_{\max}\left(Q_{n1k,pp}\right)\left\|\mathbb{S}_{1k}\right\|_{\text{sp}}^2\left\|a_{1l}\right\|^2.
\end{aligned}$$

where $Q_{n1k,pp}=n^{-1}\sum_{i=1}^n p^{\kappa_1}\left(Z_{1k,i}\right)p^{\kappa_1}\left(Z_{1k,i}\right)'\sigma_i^2$ such that $\lambda_{\max}(Q_{n1k,pp})=O_P(1)$ by Assumption A3($ii$) and arguments analogous to those used in the proof of Lemma B.1($ii$). In addition, $\left\|\mathbb{S}_{1k}\right\|_{\text{sp}}^2=\lambda_{\max}\left(\mathbb{S}_{1k}\mathbb{S}_{1k}'\right)=1$ and $\left\|a_{1l}\right\|^2\leq\left\|Q_{n,PP}^{-1}\right\|_{\text{sp}}^2\left\|\xi_{nl}\right\|^2=O_P(1)\,O_P(\kappa_1/n)=O_P(\kappa_1/n)$ by Lemma B.1($iii$) and B.2($i$) and Assumption A2($i$). It follows that $V_{nl,2}=O_P(1)\times 1\times O_P\left(\kappa_1/n\right)=O_P\left(\kappa_1/n\right)$. Similarly, using the fact that $\left\|a_{2l}\right\|^2\leq\left\|Q_{n,PP}^{-1}\right\|_{\text{sp}}^2\left\|\zeta_{nl}\right\|^2=O_P(1)$ $O_P(\kappa_1^{-2\gamma})$, we have

$$\begin{aligned}
V_{nl,4} &= n^{-1}\sum_{i=1}^n\left(\sum_{k=1}^{d_1}p^{\kappa_1}\left(Z_{1k,i}\right)'\mathbb{S}_{1k}a_{2l}\right)^2\sigma_i^2\leq d_1\sum_{k=1}^{d_1}\lambda_{\max}\left(Q_{n1k,pp}\right)\left\|\mathbb{S}_{1k}\right\|_{\text{sp}}^2\operatorname{tr}\left(a_{2l}a_{2l}'\right) \\
&= O_P(1)\times 1\times O_P(\kappa_1^{-2\gamma})=O_P(\kappa_1^{-2\gamma}).
\end{aligned}$$

By the same token, $V_{nl,3}=O_P(\kappa_1 n^{-1})$ and $V_{nl,5}=O_P(\kappa_1^{-2\gamma})$.

($ii$) The result follows from ($i$) and the fact that $\max_{1\leq i\leq n}\left\|\Phi_i\right\|=O_P\left(\varsigma_{0\kappa}\right)$ under Assumption A2($vi$).

($iii$) By Assumption A2($vi$), Taylor expansion and ($i$),

$$\begin{aligned}
n^{-1}\sum_{i=1}^n\left\|p^{\kappa}\left(\widetilde{U}_{li}\right)-p^{\kappa}\left(U_{li}\right)\right\|^2 &= n^{-1}\sum_{i=1}^n\left\|\dot{p}^{\kappa}\left(U_{li}^{\dagger}\right)\left(\widetilde{U}_{li}-U_{li}\right)\right\|^2 \\
&\leq O\left(\varsigma_{1\kappa}^2\right)n^{-1}\sum_{i=1}^n\left(\widetilde{U}_{li}-U_{li}\right)^2=O_P\left(\varsigma_{1\kappa}^2\nu_{1n}^2\right),
\end{aligned}$$

where $U_{li}^{\dagger}$ lies between $\widetilde{U}_{li}$ and $U_{li}$.

$(iv)$ By Assumption A2$(vi)$, Taylor expansion and triangle inequality,

$$\left\|n^{-1}\sum_{i=1}^{n}\left[p^{\kappa}\left(\widetilde{U}_{li}\right)-p^{\kappa}\left(U_{li}\right)\right]\Phi_{i}'\right\|_{\mathrm{sp}}$$

is bounded by $\left\|n^{-1}\sum_{i=1}^{n}\dot{p}^{\kappa}\left(U_{li}\right)\Phi_{i}'\left(\widetilde{U}_{li}-U_{li}\right)\right\|_{\mathrm{sp}}+\frac{1}{2}\left\|n^{-1}\sum_{i=1}^{n}\ddot{p}^{\kappa}\left(U_{li}^{\ddagger}\right)\Phi_{i}'\left(\widetilde{U}_{li}-U_{li}\right)^{2}\right\|_{\mathrm{sp}}\equiv$ $T_{nl,1}+T_{nl,2}$, where $U_{li}^{\ddagger}$ lies between $\widetilde{U}_{li}$ and $U_{li}$. By triangle and Cauchy-Schwarz inequalities and $(ii)$,

$$\begin{aligned}
T_{nl,1} &\leq n^{-1}\sum_{i=1}^{n}\left\|\dot{p}^{\kappa}\left(U_{li}\right)\right\|_{\mathrm{sp}}\left\|\Phi_{i}'\left(\widetilde{U}_{li}-U_{li}\right)\right\|_{\mathrm{sp}}\\
&\leq \left\{n^{-1}\sum_{i=1}^{n}\left\|\dot{p}^{\kappa}\left(U_{li}\right)\right\|^{2}\right\}^{1/2}\left\{n^{-1}\sum_{i=1}^{n}\left\|\Phi_{i}\right\|^{2}\left|\widetilde{U}_{li}-U_{li}\right|^{2}\right\}^{1/2}\\
&= O_{P}\left(\kappa^{1/2}\right)O_{P}\left(\varsigma_{0\kappa}\nu_{1n}\right)=O_{P}\left(\kappa^{1/2}\varsigma_{0\kappa}\nu_{1n}\right).
\end{aligned}$$

By triangle inequality and $(i)$, $T_{nl,2}\leq O\left(\varsigma_{0\kappa}\varsigma_{2\kappa}\right)n^{-1}\sum_{i=1}^{n}(\widetilde{U}_{li}-U_{li})^{2}=O_{P}(\varsigma_{0\kappa}\varsigma_{2\kappa}\nu_{1n}^{2})$. Then $(iv)$ follows.

$(v)$ Let $\Gamma_{nl}\equiv[p^{\kappa}(\widetilde{U}_{l1})-p^{\kappa}\left(U_{l1}\right),...,[p^{\kappa}(\widetilde{U}_{ln})-p^{\kappa}\left(U_{ln}\right)]]'$ and $\mathbf{e}=(e_{1},...,e_{n})'$. Then we can write $n^{-1}\sum_{i=1}^{n}[p^{\kappa}(\widetilde{U}_{li})-p^{\kappa}\left(U_{li}\right)]e_{i}$ as $n^{-1}\Gamma_{nl}'\mathbf{e}$. Let $\mathbb{D}_{n}\equiv\{(\mathbf{X}_{i},\mathbf{Z}_{i},\mathbf{U}_{i})\}_{i=1}^{n}$. By the law of iterated expectations, Taylor expansion, Assumptions A1$(i)$, A3$(ii)$ and A2$(vi)$ and $(i)$

$$\begin{aligned}
E\left\{\left\|n^{-1}\Gamma_{nl}'\mathbf{e}\right\|^{2}|\mathbb{D}_{n}\right\} &= n^{-2}E\left[\mathrm{tr}\left(\Gamma_{n}'\mathbf{e}\mathbf{e}'\Gamma_{n}\right)\right]=n^{-2}E\left[\mathrm{tr}\left(\Gamma_{n}'E\left(\mathbf{e}\mathbf{e}'|\mathbb{D}_{n}\right)\Gamma_{n}\right)\right]\\
&= n^{-2}\sum_{i=1}^{n}[p^{\kappa}(\widetilde{U}_{li})-p^{\kappa}\left(U_{li}\right)]^{2}\sigma_{i}^{2}\\
&\leq O_{P}\left(\varsigma_{1\kappa}\right)n^{-2}\sum_{i=1}^{n}\left(\widetilde{U}_{li}-U_{li}\right)^{2}\sigma_{i}^{2}=O_{P}\left(n^{-1}\varsigma_{1\kappa}^{2}\nu_{1n}^{2}\right).
\end{aligned}$$

It follows that $\left\|n^{-1}\Gamma_{nl}'\mathbf{e}\right\|=O_{P}\left(n^{-1/2}\varsigma_{1\kappa}\nu_{1n}\right)$ by the conditional Chebyshev inequality. ■

**Proof of Lemma B.4.** $(i)$ Noting that $n^{-1}\sum_{i=1}^{n}\left\|\widetilde{\Phi}_{i}-\Phi_{i}\right\|^{2}=\sum_{l=1}^{d_{x}}n^{-1}\sum_{i=1}^{n}\left\|p^{\kappa}\left(\widetilde{U}_{li}\right)-p^{\kappa}\left(U_{li}\right)\right\|^{2}$, the result follows from Lemma B.3$(iii)$.

$(ii)$ Noting that $\left\|n^{-1}\sum_{i=1}^{n}\left(\widetilde{\Phi}_{i}-\Phi_{i}\right)\Phi_{i}'\right\|^{2}=\sum_{l=1}^{d_{x}}\left\|n^{-1}\sum_{i=1}^{n}\left[p^{\kappa}\left(\widetilde{U}_{li}\right)-p^{\kappa}\left(U_{li}\right)\right]\Phi_{i}'\right\|^{2}$, the result follows from Lemma B.3$(iv)$.

$(iii)$ Noting that $\widetilde{Q}_{n,\Phi\Phi}-Q_{n,\Phi\Phi}=n^{-1}\sum_{i=1}^{n}(\widetilde{\Phi}_{i}\widetilde{\Phi}_{i}'-\Phi_{i}\Phi_{i}')=n^{-1}\sum_{i=1}^{n}(\widetilde{\Phi}_{i}-\Phi_{i})(\widetilde{\Phi}_{i}-\Phi_{i})'$ $+n^{-1}\sum_{i=1}^{n}(\widetilde{\Phi}_{i}-\Phi_{i})\Phi_{i}'+n^{-1}\sum_{i=1}^{n}\Phi_{i}(\widetilde{\Phi}_{i}-\Phi_{i})'$, the result follows from $(i)$-$(ii)$ and the triangle inequality.

$(iv)$ By the triangle inequality $\left\|\widetilde{Q}_{n,\Phi\Phi}^{-1}-Q_{\Phi\Phi}^{-1}\right\|_{\mathrm{sp}}\leq\left\|\widetilde{Q}_{n,\Phi\Phi}^{-1}-Q_{n,\Phi\Phi}^{-1}\right\|_{\mathrm{sp}}+\left\|Q_{n,\Phi\Phi}^{-1}-Q_{\Phi\Phi}^{-1}\right\|_{\mathrm{sp}}.$ Arguments like those used in the proof of Lemma B.1(ii) show that $\left\|\widetilde{Q}_{n,\Phi\Phi}^{-1}\right\|_{\mathrm{sp}}=\left[\lambda_{\min}\left(\widetilde{Q}_{n,\Phi\Phi}\right)\right]^{-1}$ $=\left[\lambda_{\min}\left(Q_{\Phi\Phi}\right)+o_P\left(1\right)\right]^{-1}=O_P\left(1\right)$ where the second equality follows from $(iii)$ and Lemma B.1$(ii)$. By the submultiplicative property of the spectral norm and $(iii)$,

$$\begin{aligned}\left\|\widetilde{Q}_{n,\Phi\Phi}^{-1}-Q_{n,\Phi\Phi}^{-1}\right\|_{\mathrm{sp}} &= \left\|\widetilde{Q}_{n,\Phi\Phi}^{-1}\left(\widetilde{Q}_{n,\Phi\Phi}-\widetilde{Q}_{n,\Phi\Phi}\right)Q_{n,\Phi\Phi}^{-1}\right\|_{\mathrm{sp}}\\ &\leq \left\|\widetilde{Q}_{n,\Phi\Phi}^{-1}\right\|_{\mathrm{sp}}\left\|\widetilde{Q}_{n,\Phi\Phi}-\widetilde{Q}_{n,\Phi\Phi}\right\|_{\mathrm{sp}}\left\|Q_{n,\Phi\Phi}^{-1}\right\|_{\mathrm{sp}}\\ &= O_P\left(\kappa^{1/2}\varsigma_{0\kappa}\nu_{1n}+\varsigma_{0\kappa}\varsigma_{2\kappa}\nu_{1n}^2\right).\end{aligned}$$

Similarly, $\left\|Q_{n,\Phi\Phi}^{-1}-Q_{\Phi\Phi}^{-1}\right\|_{\mathrm{sp}}=O_P\left(\kappa/n^{1/2}\right)$ by Lemma B.1$(iii)$. It follows that $\left\|\widetilde{Q}_{n,\Phi\Phi}^{-1}-Q_{\Phi\Phi}^{-1}\right\|_{\mathrm{sp}}$ $=O_P\left(\kappa^{1/2}\varsigma_{0\kappa}\nu_{1n}+\varsigma_{0\kappa}\varsigma_{2\kappa}\nu_{1n}^2\right).$

$(v)$ Noting that $\left\|n^{-1}\sum_{i=1}^n\left(\widetilde{\Phi}_i-\Phi_i\right)e_i\right\|^2=\sum_{l=1}^{d_x}\left\|n^{-1}\sum_{i=1}^n\left[p^\kappa\left(\widetilde{U}_{li}\right)-p^\kappa\left(U_{li}\right)\right]e_i\right\|^2$, the result follows from Lemma B.3$(v)$.

$(vi)$ Let $\delta_i\equiv\bar{g}\left(X_i,Z_{1i},U_i\right)-\Phi_i'\boldsymbol{\beta}$. By triangle inequality, Assumption A2$(v)$, Jensen inequality and $(i)$, we have $\left\|n^{-1}\sum_{i=1}^n\left(\widetilde{\Phi}_i-\Phi_i\right)\delta_i\right\|\leq O_P\left(\kappa^{-\gamma}\right)n^{-1}\sum_{i=1}^n\left\|\widetilde{\Phi}_i-\Phi_i\right\|=O_P\left(\kappa^{-\gamma}\right)$ $O_P\left(\varsigma_{1\kappa}\nu_{1n}\right)=O_P\left(\kappa^{-\gamma}\varsigma_{1\kappa}\nu_{1n}\right).$ ■

**Proof of Lemma B.5.** The proof of $(i)$-$(ii)$ is analogous to that of Lemma B.2$(i)$-$(ii)$, respectively. Noting that $\left\|Q_{\Phi\Phi}^{-1}\right\|_{\mathrm{sp}}=O\left(1\right)$ by Assumption A2$(ii)$, we can prove $(iii)$ by showing that $\|T_{nl}\|=O_P(\nu_{1n})$, where $T_{nl}=n^{-1}\sum_{i=1}^n\Phi_i\delta_{li}(\widetilde{U}_{li}-U_{li})$ where $\delta_{li}=\dot{p}^\kappa\left(U_{li}\right)'\boldsymbol{\beta}_{d_x+d_1+l}$. By triangle inequality and Assumptions A1$(ii)$ and A2$(iii)$ and $(v)$

$$\begin{aligned}c_{\delta_l} &\equiv \max_{1\leq i\leq n}\|\delta_{li}\|\leq\sup_{u_l\in\mathcal{U}_l}\left\|\dot{g}_{d_x+d_1+l}\left(u_l\right)-\dot{p}^\kappa\left(u_l\right)'\boldsymbol{\beta}_{d_x+d_1+l}\right\|+\sup_{u_l\in\mathcal{U}_l}\left\|\dot{g}_{d_x+d_1+l}\left(u_l\right)\right\|\\ &= O\left(\kappa^{-\gamma}\right)+O\left(1\right)=O\left(1\right).\end{aligned}$$

By (C.2), $T_{nl}=n^{-1}\sum_{i=1}^n\Phi_i\delta_{li}(\widetilde{U}_{li}-U_{li})=\sum_{s=1}^5 n^{-1}\sum_{i=1}^n\Phi_i\delta_{li}u_{sl,i}=\sum_{s=1}^5 T_{nl,s}$, say.

Let $\eta_{nlk}\equiv n^{-1}\sum_{i=1}^n\delta_{li}\Phi_ip^{\kappa_1}\left(Z_{1k,i}\right)'$ and $\bar{\eta}_{lk}=E\left(\eta_{nlk}\right)$. Then $\|\eta_{nlk}-\bar{\eta}_{lk}\|=O_P((\kappa\kappa_1/n)^{1/2})$ by Chebyshev inequality and

$$\|\bar{\eta}_k\|_{\mathrm{sp}}^2=\left\|E\left[\delta_{li}\Phi_ip^{\kappa_1}\left(Z_{1k,i}\right)'\right]\right\|_{\mathrm{sp}}^2\leq c_\delta^2\lambda_{\max}\left(M\right)=O\left(1\right),$$

where $M\equiv E\left[\Phi_ip^{\kappa_1}\left(Z_{1k,i}\right)'\right]E\left[p^{\kappa_1}\left(Z_{1k,i}\right)\Phi_i'\right]$ and we use the fact that $M$ has bounded largest eigenvalue. To see the last point, first note that for $\kappa_1\leq\kappa$, $E\left[\Phi_ip^{\kappa_1}\left(Z_{1k,i}\right)'\right]$ is a submatrix of $A\equiv E\left(\Phi_i\Phi_i'\right)$ which has bounded largest eigenvalue. Partition $A$ as follows

$$A=\begin{bmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{bmatrix}$$

where $A_{ij} = A'_{ji}$ for $i,\, j = 1,2,3$ and $E\left[\Phi_i p^{\kappa_1}(Z_{1k,i})'\right] = \begin{bmatrix} A'_{12} & A_{22} & A'_{32} \end{bmatrix}'$. Then

$$M = \begin{bmatrix} A_{12}A'_{12} & A_{12}A_{22} & A_{12}A'_{32} \\ A_{22}A'_{12} & A_{22}A_{22} & A_{22}A'_{32} \\ A_{32}A'_{12} & A_{32}A_{22} & A_{32}A'_{32} \end{bmatrix}.$$

By Thompson and Freede (1970, Theorem 2), $\lambda_{\max}(M) \leq \lambda_{\max}(A_{12}A'_{12}) + \lambda_{\max}(A_{22}A'_{22}) + \lambda_{\max}(A_{32}A'_{32})$. By Fact 8.9.3 in Bernstein (2005), the positive definiteness of $A$ ensures that both $A_{12}A'_{12}$ and $A_{32}A'_{32}$ have finite maximum eigenvalues as both $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and $\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}$ are also positive definite. In addition, $\lambda_{\max}(A_{22}A_{22}) = [\lambda_{\max}(A_{22})]^2$ is finite as $A$ has bounded maximum eigenvalue. It follows that $\lambda_{\max}(M) = O(1)$. Consequently, $\|\eta_{nlk}\| = O_P(1 + (\kappa\kappa_1/n)^{1/2}) = O_P(1)$.

Analogously, noting that 1 is the first element of $\Phi_i$, we can show that $\left\|n^{-1}\sum_{i=1}^n \Phi_i\delta_i\right\|_{\text{sp}} = O_P(1 + (\kappa/n)^{1/2}) = O_P(1)$. It follows that

$$\begin{aligned} \|T_{nl,1}\| &= \left\|n^{-1}\sum_{i=1}^{n}\Phi_i\delta_{li}\right\|_{\text{sp}} |\widetilde{\mu}_l - \mu_l| = O_P(1)\, O_P\left(n^{-1/2}\right) = O_P\left(n^{-1/2}\right), \\ \|T_{nl,2} + T_{nl,4}\| &\leq \sum_{k=1}^{d_1} \|\eta_{nlk}\|\, \|\mathbb{S}_{1k}\|_{\text{sp}} (\|a_{1l}\| + \|a_{2l}\|) = O_P(1)\, O(1)\, O(\nu_{1n}) = O(\nu_{1n}), \end{aligned}$$

and $\|T_{nl,3} + T_{nl,5}\| = O(\nu_{1n})$ by the same token. Thus we have shown that $\|T_{nl}\| = O_P(\nu_{1n})$. ■

## References


Bernstein, D. S. (2005), *Matrix Mathematics: Theory, Facts and Formulas with Application to Linear Systems Theory,* Princeton: Princeton University Press.

Thompson, R., and Freede, L. J. (1974), "Eigenvalues of Partitioned Hermitian Matrices," *Bulletin Australian Mathematical Society,* 3, 23-37.