Supplemental Material On

"**Additive Nonparametric Regression in the Presence of Endogenous Regressors**"

Deniz Ozabaci,[1] Daniel J. Henderson,[2] and Liangjun Su[3]

THIS APPENDIX PROVIDES PROOFS FOR SOME TECHNICAL LEMMAS IN THE ABOVE PAPER.

Proof of Lemma B.1. By straightforward moment calculations, we can show that $E||Q_{n,PP} - Q_{PP}||^2 = O\left(\kappa_1^2/n\right)$ under Assumption A1(i)-(ii) and A2(vi). Then (i) follows from Markov inequality. By Weyl inequality [e.g., Bernstein (2005, Theorem 8.4.11)] and the fact that $\lambda_{\max}(A) \leq \|A\|$ for any symmetric matrix A (as $|\lambda_{\max}(A)|^2 = \lambda_{\max}(AA) \leq \|A\|^2$), we have

$$\begin{aligned}\lambda_{\min}(Q_{n,PP}) &\leq \lambda_{\min}(Q_{PP}) + \lambda_{\max}(Q_{n,PP} - Q_{PP}) \\ &\leq \lambda_{\min}(Q_{PP}) + \|Q_{n,PP} - Q_{PP}\| = \lambda_{\min}(Q_{PP}) + o_P(1).\end{aligned}$$

Similarly,

$$\begin{aligned}\lambda_{\min}(Q_{n,PP}) &\geq \lambda_{\min}(Q_{PP}) + \lambda_{\min}(Q_{n,PP} - Q_{PP}) \\ &\geq \lambda_{\min}(Q_{PP}) - \|Q_{n,PP} - Q_{PP}\| = \lambda_{\min}(Q_{\kappa_1}) - o_P(1).\end{aligned}$$

Analogously, we can prove the second part of (ii). Thus (ii) follows. By the submultiplicative property of the spectral norm, (i)-(ii) and Assumption A2(i),

$$\begin{aligned}\left\|Q_{n,PP}^{-1} - Q_{PP}^{-1}\right\|_{\text{sp}} &= \left\|Q_{n,PP}^{-1}\left(Q_{PP} - Q_{n,PP}\right)Q_{PP}^{-1}\right\|_{\text{sp}} \leq \left\|Q_{n,PP}^{-1}\right\|_{\text{sp}} \|Q_{PP} - Q_{n,PP}\|_{\text{sp}} \left\|Q_{PP}^{-1}\right\|_{\text{sp}} \\ &= O_P(1)\, O_P\left(\kappa_1/n^{1/2}\right) O_P(1) = O_P\left(\kappa_1/n^{1/2}\right),\end{aligned}$$

where we use the fact that $\left\|Q_{n,PP}^{-1}\right\|_{\text{sp}} = [\lambda_{\min}(Q_{n,PP})]^{-1} = [\lambda_{\min}(Q_{PP}) + o_P(1)]^{-1} = O_P(1)$ by (ii) and Assumption A2(i). Then (iii) follows. The proof of (iv)-(v) is analogous to that of (i)-(ii) and thus omitted. ■

Proof of Lemma B.2. (i) By Assumption A1(i) and A2(i), $E\|\xi_{nl}\|^2 = n^{-2}\text{tr}\{\sum_{i=1}^n E(P_iP_i'U_{li}^2)\} \leq n^{-1}(1 + d\kappa_1)\lambda_{\max}(Q_{PP,U_l}) = O(\kappa_1/n)$. Then $\|\xi_{nl}\|^2 = O_P(\kappa_1/n)$ by Markov inequality.

[1]Deniz Ozabaci, Dept of Economics, State Univ of New York, Binghamton, NY 13902-6000, (607) 777-2572, Fax: (607) 777-2681, e-mail: dozabac1@binghamton.edu.

[2]Daniel J. Henderson, Dept of Economics, Finance and Legal Studies, Univ of Alabama, Tuscaloosa, AL 35487-0224, (205) 348-8991, Fax: (205) 348-0186, e-mail: djhender@cba.ua.edu.

[3]Corresponding author: Liangjun Su, School of Economics, Singapore Management University, 90 Stamford Road, Singapore, 178903; Tel: (65) 6828-0386; e-mail: ljsu@smu.edu.sg.

(ii) By the facts that $\|a\|_{\text{sp}}^2 = \|a\|^2$ for any vector a, $|a'b| \leq \|a\| \, \|b\|$ for any two conformable vectors a and b and that $\varkappa' A \varkappa \leq \lambda_{\max}(A) \|\varkappa\|^2$ for any p.s.d. matrix A and conformable vector $\varkappa$, Cauchy-Schwarz inequality, Lemma B.1(ii) and Assumptions A2(iv), we have

$$\begin{aligned}
\|\zeta_{nl}\|^2 &= \|\zeta_{nl}\|_{\text{sp}}^2 = \lambda_{\max}\left(\zeta_{nl}\zeta_{nl}'\right) \\
&= \max_{\|\varkappa\|=1} n^{-2} \sum_{i=1}^{n}\sum_{j=1}^{n} \varkappa' P_i P_j' \varkappa \left[m_l(\mathbf{Z}_i) - P_i'\boldsymbol{\alpha}_l\right]\left[m_l(\mathbf{Z}_j) - P_j'\boldsymbol{\alpha}_l\right] \\
&\leq \max_{\|\varkappa\|=1} \left\{ n^{-1} \sum_{i=1}^{n} \left\{ \varkappa' P_i P_i' \varkappa \left[m_l(\mathbf{Z}_i) - P_i'\boldsymbol{\alpha}_l\right]^2 \right\}^{1/2} \right\}^2 \\
&\leq O_P(\kappa_1^{-2\gamma}) \max_{\|\varkappa\|=1} \left\{ n^{-1} \sum_{i=1}^{n} \varkappa' P_i P_i' \varkappa \right\} \leq O_P(\kappa_1^{-2\gamma}) \lambda_{\max}(Q_{n,PP}) = O_P(\kappa_1^{-2\gamma}).
\end{aligned}$$

(iii) Noting that $X_{li} = m_l(\mathbf{Z}_i) + U_{li} = P_i'\boldsymbol{\alpha}_l + U_{li} + \left[m_l(\mathbf{Z}_i) - P_i'\boldsymbol{\alpha}_l\right]$, by Lemma B.1$(ii)$, w.p.a.1 we have

$$\begin{aligned}
\widetilde{\boldsymbol{\alpha}}_l - \boldsymbol{\alpha}_l &= \left(\sum_{i=1}^{n} P_i P_i'\right)^{-} \sum_{i=1}^{n} P_i X_{li} - \boldsymbol{\alpha}_l \\
&= Q_{n,PP}^{-1} n^{-1} \sum_{i=1}^{n} P_i U_{li} + Q_{n,PP}^{-1} n^{-1} \sum_{i=1}^{n} P_i \left[m_l(\mathbf{Z}_i) - P_i'\boldsymbol{\alpha}_l\right] \\
&= Q_{n,PP}^{-1}\xi_{nl} + Q_{n,PP}^{-1}\zeta_{nl} \equiv a_{1l} + a_{2l}, \text{ say.} \qquad \text{(C.1)}
\end{aligned}$$

Note that $a_{1l} = Q_{\kappa_1}^{-1}\xi_{nl} + r_{1nl}$ where $r_{1nl} = \left(Q_{n,PP}^{-1} - Q_{PP}^{-1}\right)\xi_{nl}$ satisfies that

$$\begin{aligned}
\|r_{1l}\| &\leq = \left\{ \text{tr}\left[\left(Q_{n,PP}^{-1} - Q_{PP}^{-1}\right)\xi_{nl}\xi_{nl}'\left(Q_{n,PP}^{-1} - Q_{PP}^{-1}\right)\right]\right\}^{1/2} \\
&\leq \|\xi_{nl}\|_{\text{sp}} \left\|Q_{n,PP}^{-1} - Q_{PP}^{-1}\right\| = O_P(\kappa_1^{1/2}/n^{1/2}) O_P(\kappa_1^{1/2}/n^{1/2}) = O_P(\kappa_1/n)
\end{aligned}$$

by Lemmas B.1(iii) and B.2(i). For a_{2l}, we have $a_{2l} = Q_{\kappa_1}^{-1}\zeta_{nl} + r_{2nl}$ where $r_{2nl} = \left(Q_{n\kappa_1}^{-1} - Q_{n\kappa_1}^{-1}\right)\zeta_{nl}$ satisfies that

$$\|r_{2l}\| \leq \|\zeta_{nl}\|_{\text{sp}} \left\|Q_{n,PP}^{-1} - Q_{PP}^{-1}\right\| = O_P(\kappa_1^{-\gamma}) O_P(\kappa_1^{1/2}/n^{1/2}) = O_P(\kappa_1^{-\gamma+1/2}/n^{1/2})$$

by Lemmas B.1(iii) and B.2(ii). The result follows. ■

Proof of Lemma B.3. (i) We only prove the $r = 1$ case as the proof of the other case is almost identical. By the definition of $\widetilde{U}_{li}$ and (C.1), we can decompose $\widetilde{U}_{li} - U_{li} = \left[X_{li} - \widetilde{m}_l(\mathbf{Z}_i)\right] - U_{li}$

as follows

$$\begin{aligned}
\widetilde{U}_{li}-U_{li} &= (\mu_l-\widetilde{\mu}_l)+\sum_{k=1}^{d_1}\left[m_{l,k}\left(Z_{1k,i}\right)-\widetilde{m}_{l,k}\left(Z_{1k,i}\right)\right]+\sum_{k=1}^{d_2}\left[m_{l,d_1+k}\left(Z_{2k,i}\right)-\widetilde{m}_{l,d_1+k}\left(Z_{2k,i}\right)\right] \\
&= -\left(\widetilde{\mu}_l-\mu_l\right)-\sum_{k=1}^{d_1}p^{\kappa_1}\left(Z_{1k,i}\right)'\mathbb{S}_{1k}a_{1l}-\sum_{k=1}^{d_2}p^{\kappa_1}\left(Z_{2k,i}\right)'\mathbb{S}_{1,d_1+k}a_{1l} \\
&\quad -\sum_{k=1}^{d_1}p^{\kappa_1}\left(Z_{1k,i}\right)'\mathbb{S}_{1k}a_{2l}-\sum_{k=1}^{d_2}p^{\kappa_1}\left(Z_{2k,i}\right)'\mathbb{S}_{1,d_1+k}a_{2l} \\
&\equiv -u_{1l,i}-u_{2l,i}-u_{3l,i}-u_{4l,i}-u_{5l,i},\ \text{say.} \qquad \text{(C.2)}
\end{aligned}$$

Then by Cauchy-Schwarz inequality, $n^{-1}\sum_{i=1}^{n}(\widetilde{U}_{li}-U_{li})^2\sigma_i^2 \leq 5\sum_{s=1}^{5}n^{-1}\sum_{i=1}^{n}u_{sl,i}^2\sigma_i^2 \equiv 5\sum_{s=1}^{5}$ $V_{nl,s}$, say. Apparently, $V_{nl,1}=O_P\left(n^{-1}\right)$ as $\widetilde{\mu}_l-\mu_l=O_P\left(n^{-1/2}\right)$.

$$\begin{aligned}
V_{nl,2} &= n^{-1}\sum_{i=1}^{n}\left(\sum_{k=1}^{d_1}p^{\kappa_1}\left(Z_{1k,i}\right)'\mathbb{S}_{1k}a_{1l}\right)^2\sigma_i^2 \\
&\leq d_1\sum_{k=1}^{d_1}n^{-1}\sum_{i=1}^{n}\left(p^{\kappa_1}\left(Z_{1k,i}\right)'\mathbb{S}_{1k}a_{1l}\right)^2\sigma_i^2=d_1\sum_{k=1}^{d_1}\operatorname{tr}\left(\mathbb{S}_{1k}a_{1l}a_{1l}'\mathbb{S}_{1k}'Q_{n1k,pp}\right) \\
&\leq d_1\sum_{k=1}^{d_1}\lambda_{\max}\left(Q_{n1k,pp}\right)\operatorname{tr}\left(a_{1l}a_{1l}'\mathbb{S}_{1k}'\mathbb{S}_{1k}\right)\leq d_1\sum_{k=1}^{d_1}\lambda_{\max}\left(Q_{n1k,pp}\right)\left\|\mathbb{S}_{1k}\right\|_{\mathrm{sp}}^2\left\|a_{1l}\right\|^2.
\end{aligned}$$

where $Q_{n1k,pp}=n^{-1}\sum_{i=1}^{n}p^{\kappa_1}\left(Z_{1k,i}\right)p^{\kappa_1}\left(Z_{1k,i}\right)'\sigma_i^2$ such that $\lambda_{\max}(Q_{n1k,pp})=O_P\left(1\right)$ by Assumption A3(ii) and arguments analogous to those used in the proof of Lemma B.1(ii). In addition, $\left\|\mathbb{S}_{1k}\right\|_{\mathrm{sp}}^2=\lambda_{\max}\left(\mathbb{S}_{1k}\mathbb{S}_{1k}'\right)=1$ and $\left\|a_{1l}\right\|^2\leq\left\|Q_{n,PP}^{-1}\right\|_{\mathrm{sp}}^2\left\|\xi_{nl}\right\|^2=O_P\left(1\right)O_P(\kappa_1/n)=$ $O_P(\kappa_1/n)$ by Lemma B.1(iii) and B.2(i) and Assumption A2(i). It follows that $V_{nl,2}=O_P\left(1\right)\times$ $1\times O_P\left(\kappa_1/n\right)=O_P\left(\kappa_1/n\right)$. Similarly, using the fact that $\left\|a_{2l}\right\|^2\leq\left\|Q_{n,PP}^{-1}\right\|_{\mathrm{sp}}^2\left\|\zeta_{nl}\right\|^2=O_P\left(1\right)$ $O_P(\kappa_1^{-2\gamma})$, we have

$$\begin{aligned}
V_{nl,4} &= n^{-1}\sum_{i=1}^{n}\left(\sum_{k=1}^{d_1}p^{\kappa_1}\left(Z_{1k,i}\right)'\mathbb{S}_{1k}a_{2l}\right)^2\sigma_i^2\leq d_1\sum_{k=1}^{d_1}\lambda_{\max}\left(Q_{n1k,pp}\right)\left\|\mathbb{S}_{1k}\right\|_{\mathrm{sp}}^2\operatorname{tr}\left(a_{2l}a_{2l}'\right) \\
&= O_P\left(1\right)\times 1\times O_P(\kappa_1^{-2\gamma})=O_P(\kappa_1^{-2\gamma}).
\end{aligned}$$

By the same token, $V_{nl,3}=O_P(\kappa_1 n^{-1})$ and $V_{nl,5}=O_P(\kappa_1^{-2\gamma})$.

(ii) The result follows from (i) and the fact that $\max_{1\leq i\leq n}\left\|\Phi_i\right\|=O_P\left(\varsigma_{0\kappa}\right)$ under Assumption A2(vi).

(iii) By Assumption A2(vi), Taylor expansion and (i),

$$\begin{aligned}
n^{-1}\sum_{i=1}^{n}\left\|p^{\kappa}\left(\widetilde{U}_{li}\right)-p^{\kappa}\left(U_{li}\right)\right\|^2 &= n^{-1}\sum_{i=1}^{n}\left\|\dot{p}^{\kappa}\left(U_{li}^{\dagger}\right)\left(\widetilde{U}_{li}-U_{li}\right)\right\|^2 \\
&\leq O\left(\varsigma_{1\kappa}^2\right)n^{-1}\sum_{i=1}^{n}\left(\widetilde{U}_{li}-U_{li}\right)^2=O_P\left(\varsigma_{1\kappa}^2\nu_{1n}^2\right),
\end{aligned}$$

where $U_{li}^{\dagger}$ lies between $\widetilde{U}_{li}$ and U_{li}.

(iv) By Assumption A2(vi), Taylor expansion and triangle inequality,

$$\left\| n^{-1}\sum_{i=1}^{n}\left[p^{\kappa}\left(\widetilde{U}_{li}\right)-p^{\kappa}\left(U_{li}\right)\right]\Phi_i' \right\|_{\text{sp}}$$

is bounded by $\left\| n^{-1}\sum_{i=1}^{n}\dot{p}^{\kappa}\left(U_{li}\right)\Phi_i'\left(\widetilde{U}_{li}-U_{li}\right)\right\|_{\text{sp}}+\frac{1}{2}\left\| n^{-1}\sum_{i=1}^{n}\ddot{p}^{\kappa}\left(U_{li}^{\ddagger}\right)\Phi_i'\left(\widetilde{U}_{li}-U_{li}\right)^2\right\|_{\text{sp}}\equiv$ $T_{nl,1}+T_{nl,2}$, where $U_{li}^{\ddagger}$ lies between $\widetilde{U}_{li}$ and U_{li}. By triangle and Cauchy-Schwarz inequalities and (ii),

$$\begin{aligned} T_{nl,1} &\leq n^{-1}\sum_{i=1}^{n}\left\|\dot{p}^{\kappa}\left(U_{li}\right)\right\|_{\text{sp}}\left\|\Phi_i'\left(\widetilde{U}_{li}-U_{li}\right)\right\|_{\text{sp}} \\ &\leq \left\{n^{-1}\sum_{i=1}^{n}\left\|\dot{p}^{\kappa}\left(U_{li}\right)\right\|^2\right\}^{1/2}\left\{n^{-1}\sum_{i=1}^{n}\left\|\Phi_i\right\|^2\left|\widetilde{U}_{li}-U_{li}\right|^2\right\}^{1/2} \\ &= O_P\left(\kappa^{1/2}\right)O_P\left(\varsigma_{0\kappa}\nu_{1n}\right)=O_P\left(\kappa^{1/2}\varsigma_{0\kappa}\nu_{1n}\right). \end{aligned}$$

By triangle inequality and (i), $T_{nl,2}\leq O\left(\varsigma_{0\kappa}\varsigma_{2\kappa}\right)n^{-1}\sum_{i=1}^{n}(\widetilde{U}_{li}-U_{li})^2=O_P(\varsigma_{0\kappa}\varsigma_{2\kappa}\nu_{1n}^2)$. Then (iv) follows.

(v) Let $\Gamma_{nl}\equiv[p^{\kappa}(\widetilde{U}_{l1})-p^{\kappa}\left(U_{l1}\right),...,[p^{\kappa}(\widetilde{U}_{ln})-p^{\kappa}\left(U_{ln}\right)]]'$ and $\mathbf{e}=(e_1,...,e_n)'$. Then we can write $n^{-1}\sum_{i=1}^{n}[p^{\kappa}(\widetilde{U}_{li})-p^{\kappa}\left(U_{li}\right)]e_i$ as $n^{-1}\Gamma_{nl}'\mathbf{e}$. Let $\mathbb{D}_n\equiv\{(\mathbf{X}_i,\mathbf{Z}_i,\mathbf{U}_i)\}_{i=1}^{n}$. By the law of iterated expectations, Taylor expansion, Assumptions A1(i), A3(ii) and A2(vi) and (i)

$$\begin{aligned} E\left\{\left\|n^{-1}\Gamma_{nl}'\mathbf{e}\right\|^2|\mathbb{D}_n\right\} &= n^{-2}E\left[\text{tr}\left(\Gamma_n'\mathbf{e}\mathbf{e}'\Gamma_n\right)\right]=n^{-2}E\left[\text{tr}\left(\Gamma_n'E\left(\mathbf{e}\mathbf{e}'|\mathbb{D}_n\right)\Gamma_n\right)\right] \\ &= n^{-2}\sum_{i=1}^{n}[p^{\kappa}(\widetilde{U}_{li})-p^{\kappa}\left(U_{li}\right)]^2\sigma_i^2 \\ &\leq O_P\left(\varsigma_{1\kappa}\right)n^{-2}\sum_{i=1}^{n}\left(\widetilde{U}_{li}-U_{li}\right)^2\sigma_i^2=O_P\left(n^{-1}\varsigma_{1\kappa}^2\nu_{1n}^2\right). \end{aligned}$$

It follows that $\left\|n^{-1}\Gamma_{nl}'\mathbf{e}\right\|=O_P\left(n^{-1/2}\varsigma_{1\kappa}\nu_{1n}\right)$ by the conditional Chebyshev inequality. ■

Proof of Lemma B.4. (i) Noting that $n^{-1}\sum_{i=1}^{n}\left\|\widetilde{\Phi}_i-\Phi_i\right\|^2=\sum_{l=1}^{d_x}n^{-1}\sum_{i=1}^{n}\left\|p^{\kappa}\left(\widetilde{U}_{li}\right)-p^{\kappa}\left(U_{li}\right)\right\|^2$, the result follows from Lemma B.3(iii).

(ii) Noting that $\left\|n^{-1}\sum_{i=1}^{n}\left(\widetilde{\Phi}_i-\Phi_i\right)\Phi_i'\right\|^2=\sum_{l=1}^{d_x}\left\|n^{-1}\sum_{i=1}^{n}\left[p^{\kappa}\left(\widetilde{U}_{li}\right)-p^{\kappa}\left(U_{li}\right)\right]\Phi_i'\right\|^2$, the result follows from Lemma B.3(iv).

(iii) Noting that $\widetilde{Q}_{n,\Phi\Phi}-Q_{n,\Phi\Phi}=n^{-1}\sum_{i=1}^{n}(\widetilde{\Phi}_i\widetilde{\Phi}_i'-\Phi_i\Phi_i')=n^{-1}\sum_{i=1}^{n}(\widetilde{\Phi}_i-\Phi_i)(\widetilde{\Phi}_i-\Phi_i)'$ $+n^{-1}\sum_{i=1}^{n}(\widetilde{\Phi}_i-\Phi_i)\Phi_i'+n^{-1}\sum_{i=1}^{n}\Phi_i(\widetilde{\Phi}_i-\Phi_i)'$, the result follows from (i)-(ii) and the triangle inequality.

(iv) By the triangle inequality $\left\|\widetilde{Q}_{n,\Phi\Phi}^{-1}-Q_{\Phi\Phi}^{-1}\right\|_{\text{sp}}\leq\left\|\widetilde{Q}_{n,\Phi\Phi}^{-1}-Q_{n,\Phi\Phi}^{-1}\right\|_{\text{sp}}+\left\|Q_{n,\Phi\Phi}^{-1}-Q_{\Phi\Phi}^{-1}\right\|_{\text{sp}}$. Arguments like those used in the proof of Lemma B.1(ii) show that $\left\|\widetilde{Q}_{n,\Phi\Phi}^{-1}\right\|_{\text{sp}}=\left[\lambda_{\min}\left(\widetilde{Q}_{n,\Phi\Phi}\right)\right]^{-1}$ $=\left[\lambda_{\min}\left(Q_{\Phi\Phi}\right)+o_P\left(1\right)\right]^{-1}=O_P\left(1\right)$ where the second equality follows from (iii) and Lemma B.1(ii). By the submultiplicative property of the spectral norm and (iii),

$$\begin{aligned}\left\|\widetilde{Q}_{n,\Phi\Phi}^{-1}-Q_{n,\Phi\Phi}^{-1}\right\|_{\text{sp}} &= \left\|\widetilde{Q}_{n,\Phi\Phi}^{-1}\left(\widetilde{Q}_{n,\Phi\Phi}-\widetilde{Q}_{n,\Phi\Phi}\right)Q_{n,\Phi\Phi}^{-1}\right\|_{\text{sp}}\\ &\leq \left\|\widetilde{Q}_{n,\Phi\Phi}^{-1}\right\|_{\text{sp}}\left\|\widetilde{Q}_{n,\Phi\Phi}-\widetilde{Q}_{n,\Phi\Phi}\right\|_{\text{sp}}\left\|Q_{n,\Phi\Phi}^{-1}\right\|_{\text{sp}}\\ &= O_P\left(\kappa^{1/2}\varsigma_{0\kappa}\nu_{1n}+\varsigma_{0\kappa}\varsigma_{2\kappa}\nu_{1n}^2\right).\end{aligned}$$

Similarly, $\left\|Q_{n,\Phi\Phi}^{-1}-Q_{\Phi\Phi}^{-1}\right\|_{\text{sp}}=O_P\left(\kappa/n^{1/2}\right)$ by Lemma B.1(iii). It follows that $\left\|\widetilde{Q}_{n,\Phi\Phi}^{-1}-Q_{\Phi\Phi}^{-1}\right\|_{\text{sp}}$ $=O_P\left(\kappa^{1/2}\varsigma_{0\kappa}\nu_{1n}+\varsigma_{0\kappa}\varsigma_{2\kappa}\nu_{1n}^2\right)$.

(v) Noting that $\left\|n^{-1}\sum_{i=1}^n\left(\widetilde{\Phi}_i-\Phi_i\right)e_i\right\|^2=\sum_{l=1}^{d_x}\left\|n^{-1}\sum_{i=1}^n\left[p^\kappa\left(\widetilde{U}_{li}\right)-p^\kappa\left(U_{li}\right)\right]e_i\right\|^2$, the result follows from Lemma B.3(v).

(vi) Let $\delta_i\equiv\bar{g}\left(X_i,Z_{1i},U_i\right)-\Phi_i'\boldsymbol{\beta}$. By triangle inequality, Assumption A2(v), Jensen inequality and (i), we have $\left\|n^{-1}\sum_{i=1}^n\left(\widetilde{\Phi}_i-\Phi_i\right)\delta_i\right\|\leq O_P\left(\kappa^{-\gamma}\right)n^{-1}\sum_{i=1}^n\left\|\widetilde{\Phi}_i-\Phi_i\right\|=O_P\left(\kappa^{-\gamma}\right)$ $O_P\left(\varsigma_{1\kappa}\nu_{1n}\right)=O_P\left(\kappa^{-\gamma}\varsigma_{1\kappa}\nu_{1n}\right)$. ■

Proof of Lemma B.5. The proof of (i)-(ii) is analogous to that of Lemma B.2(i)-(ii), respectively. Noting that $\left\|Q_{\Phi\Phi}^{-1}\right\|_{\text{sp}}=O\left(1\right)$ by Assumption A2(ii), we can prove (iii) by showing that $\|T_{nl}\|=O_P(\nu_{1n})$, where $T_{nl}=n^{-1}\sum_{i=1}^n\Phi_i\delta_{li}(\widetilde{U}_{li}-U_{li})$ where $\delta_{li}=\dot{p}^\kappa\left(U_{li}\right)'\boldsymbol{\beta}_{d_x+d_1+l}$. By triangle inequality and Assumptions A1(ii) and A2(iii) and (v)

$$\begin{aligned}c_{\delta_l} &\equiv \max_{1\leq i\leq n}\|\delta_{li}\|\leq\sup_{u_l\in\mathcal{U}_l}\left\|\dot{g}_{d_x+d_1+l}\left(u_l\right)-\dot{p}^\kappa\left(u_l\right)'\boldsymbol{\beta}_{d_x+d_1+l}\right\|+\sup_{u_l\in\mathcal{U}_l}\|\dot{g}_{d_x+d_1+l}\left(u_l\right)\|\\ &= O\left(\kappa^{-\gamma}\right)+O\left(1\right)=O\left(1\right).\end{aligned}$$

By (C.2), $T_{nl}=n^{-1}\sum_{i=1}^n\Phi_i\delta_{li}(\widetilde{U}_{li}-U_{li})=\sum_{s=1}^5n^{-1}\sum_{i=1}^n\Phi_i\delta_{li}u_{sl,i}=\sum_{s=1}^5T_{nl,s}$, say.

Let $\eta_{nlk}\equiv n^{-1}\sum_{i=1}^n\delta_{li}\Phi_ip^{\kappa_1}\left(Z_{1k,i}\right)'$ and $\bar{\eta}_{lk}=E\left(\eta_{nlk}\right)$. Then $\|\eta_{nlk}-\bar{\eta}_{lk}\|=O_P((\kappa\kappa_1/n)^{1/2})$ by Chebyshev inequality and

$$\|\bar{\eta}_k\|_{\text{sp}}^2=\left\|E\left[\delta_{li}\Phi_ip^{\kappa_1}\left(Z_{1k,i}\right)'\right]\right\|_{\text{sp}}^2\leq c_\delta^2\lambda_{\max}\left(M\right)=O\left(1\right),$$

where $M\equiv E\left[\Phi_ip^{\kappa_1}\left(Z_{1k,i}\right)'\right]E\left[p^{\kappa_1}\left(Z_{1k,i}\right)\Phi_i'\right]$ and we use the fact that M has bounded largest eigenvalue. To see the last point, first note that for $\kappa_1\leq\kappa$, $E\left[\Phi_ip^{\kappa_1}\left(Z_{1k,i}\right)'\right]$ is a submatrix of $A\equiv E\left(\Phi_i\Phi_i'\right)$ which has bounded largest eigenvalue. Partition A as follows

$$A=\begin{bmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{bmatrix}$$

where $A_{ij} = A'_{ji}$ for i, $j = 1, 2, 3$ and $E\left[\Phi_i p^{\kappa_1}(Z_{1k,i})'\right] = \begin{bmatrix} A'_{12} & A_{22} & A'_{32} \end{bmatrix}'$. Then

$$M = \begin{bmatrix} A_{12}A'_{12} & A_{12}A_{22} & A_{12}A'_{32} \\ A_{22}A'_{12} & A_{22}A_{22} & A_{22}A'_{32} \\ A_{32}A'_{12} & A_{32}A_{22} & A_{32}A'_{32} \end{bmatrix}.$$

By Thompson and Freede (1970, Theorem 2), $\lambda_{\max}(M) \leq \lambda_{\max}(A_{12}A'_{12}) + \lambda_{\max}(A_{22}A'_{22}) + \lambda_{\max}(A_{32}A'_{32})$. By Fact 8.9.3 in Bernstein (2005), the positive definiteness of A ensures that both $A_{12}A'_{12}$ and $A_{32}A'_{32}$ have finite maximum eigenvalues as both $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and $\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}$ are also positive definite. In addition, $\lambda_{\max}(A_{22}A_{22}) = [\lambda_{\max}(A_{22})]^2$ is finite as A has bounded maximum eigenvalue. It follows that $\lambda_{\max}(M) = O(1)$. Consequently, $\|\eta_{nlk}\| = O_P(1 + (\kappa\kappa_1/n)^{1/2}) = O_P(1)$.

Analogously, noting that 1 is the first element of Φ_i, we can show that $\left\|n^{-1}\sum_{i=1}^n \Phi_i\delta_i\right\|_{\text{sp}} = O_P(1 + (\kappa/n)^{1/2}) = O_P(1)$. It follows that

$$\begin{aligned} \|T_{nl,1}\| &= \left\|n^{-1}\sum_{i=1}^n \Phi_i\delta_{li}\right\|_{\text{sp}} |\widetilde{\mu}_l - \mu_l| = O_P(1)\, O_P\left(n^{-1/2}\right) = O_P\left(n^{-1/2}\right), \\ \|T_{nl,2} + T_{nl,4}\| &\leq \sum_{k=1}^{d_1} \|\eta_{nlk}\| \, \|\mathbb{S}_{1k}\|_{\text{sp}} \left(\|a_{1l}\| + \|a_{2l}\|\right) = O_P(1)\, O(1)\, O(\nu_{1n}) = O(\nu_{1n}), \end{aligned}$$

and $\|T_{nl,3} + T_{nl,5}\| = O(\nu_{1n})$ by the same token. Thus we have shown that $\|T_{nl}\| = O_P(\nu_{1n})$. ■

References

Bernstein, D. S. (2005), *Matrix Mathematics: Theory, Facts and Formulas with Application to Linear Systems Theory,* Princeton: Princeton University Press.

Thompson, R., and Freede, L. J. (1974), "Eigenvalues of Partitioned Hermitian Matrices," *Bulletin Australian Mathematical Society,* 3, 23-37.