Online Supplement for *Optimizing the Recovery of Disrupted Single Sourced Multi-Echelon Assembly Supply Chain Networks*

Huy Nguyen* Thomas C. Sharkey† John E. Mitchell‡ William A. Wallace§

September 6, 2019

1 Summary of Relevant Notation

Parameters

Notation	Definition
N	set of manufacturers.
A	set of arcs. $(i, j) \in A$ implies that component i is required by manufacturer j.
B	the final manufacturer node.
$\mathcal{E}$	$\mathcal{E} = \{0, 1, 2, \ldots, M\}$: the set of tiers in the network.
$N(E)$	the set of nodes in Tier E, where $E \in \mathcal{E}$.
K	the set of demands for final assembled product.
H	the total number of time periods in the scheduling horizon.
D^k	the due date of demand $k \in K$ at the final assembly node.
$D_r^{i,k}$	the local due date that components i, which will be used for demand k, need to be sent out along path r.
$pred(i)$	the set of sub-component manufacturers for manufacturer i
$R(i)$	the set of paths r from a manufacturer i to B.
$p_{(i,j)}$	the shipping time from manufacturer i to manufacturer j.
p_i	the production time at manufacturer i.
cap^i	the capacity of raw material manufacturer i.

*Department of Industrial and Systems Engineering, Rensselaer Polytechnic Institute, Troy, NY, 12180.

†Corresponding author, e-mail: sharkt@rpi.edu. Associate Professor, Department of Industrial and Systems Engineering, Rensselaer Polytechnic Institute, Troy, NY, 12180.

‡Professor, Department of Mathematical Sciences, Rensselaer Polytechnic Institute, Troy, NY, 12180.

§Yamada Professor and Department Head, Department of Industrial and Systems Engineering, Rensselaer Polytechnic Institute, Troy, NY, 12180.

$cap^{i,1\rightarrow t}$	The cumulative capacity of manufacturer i, i.e., cumulative production up to t is bounded by it.
$Q_{(i,j)}$	the number of components of type i necessary in the production of *one* component of type j .
$Q_r^{i,k}$	the total components i needed to be on path r in a final product k.
p_r^i	the minimum lead time it takes a product from i to be processed and shipped along $r \in R(m)$.
u^i	the time manufacturer i is restored.

Variables

Notation	**Definition**
$C_{r,q}^{i,k}$	the time to ship a product q from manufacturer i to meet final product demand k through path r.
$t_{r,q}^{i,k}$	the tardiness of the product q from manufacturer i for demand k through path r based on the decision $C_{r,q}^{i,k}$.
$t^{i,k}$	the tardiness of the manufacturer i for demand k based on shipping decisions at i alone.
t^k	the tardiness of demand k at the final assembly node.
$T_{in}^{i,k}$	the tardiness of manufacturer i in meeting demand k based on the arrival of needed components.
T_{in}^i	the tardiness of manufacturer i across all demands based on the arrival of needed components.
T_{in}^E	the maximum tardiness of all manufacturers i in Tier $E \in \mathcal{E}$ based on the arrival of needed components.
T_{dis}^i	the tardiness of manufacturer i across all demands based on the disruption at i.
T_{dis}^E	the maximum tardiness of all manufacturers i in Tier $E \in \mathcal{E}$ based on their disruptions.
$T_{out}^{i,k}$	the maximum tardiness of manufacturer i in meeting demand k based on its shipping decisions.
T_{out}^i	the maximum tardiness of manufacturer i across all demands based on its shipping decisions.
T_{out}^E	the maximum tardiness of all manufacturers i in Tier $E \in \mathcal{E}$ based on their shipping decisions.
$\alpha_r^i(\phi)$	the earliest arrival time a product from manufacturer i to be processed and shipped at time ϕ to the final product through path $r \in R(i)$, where $r = \; i - j_1 - j_2..j_n - B$, considering disruptions at the manufacturers on the path.

$\tau^{i,k}$	the tardiness of manufacturer i for demand k based on decisions at i and the disruptions on paths from i to B.
$MS^{i,K}$	$MS^{i,K} = \max_{r,q,k\in K} \alpha_r^i(C_{r,q}^{i,k})$: the makespan of manufacturer i in scheduling the set of demands K.
TTR_r^i	$TTR_r^i = \max(\max_{k,q:t^k>0}(C_{r,q}^{i,k} + p_r^i), 0)$: the time to recover of manufacturer i through path r.
TTR^i	$TTR^i = \max_r TTR_r^i = \max_k TTR^{i,k} = \max_{r,k,q:t^k>0} \alpha_r^i(C_{r,q}^{i,k})$: the time to recover of manufacturer i.
TTR^E	$TTR^E = \max_{i\in N(E)} TTR^i$: the time to recover of Tier E.

2 Polynomial time representation for inputs

Table 2: Polynomial time representation for $C_{r,q}^{i,k}$

Production Completion Time	1	$\dots$	u^i	$u^i + p_i$	$u^i + 2p_i$	$\dots$
Production Capacity	0	0	cap^i	cap^i	cap^i	cap^i
Cumulative Production Capacity	0	0	cap^i	$2cap^i$	$3cap^i$	$\dots$

Table 3: Polynomial time representation for D^k

Time	1	2	$\dots$	H
Local demand quantity	$\sum_{k,r:D_r^{i,k}=1} Q_r^{i,k}$	$\sum_{k,r:D_r^{i,k}=2} Q_r^{i,k}$	$\dots$	$\sum_{k,r:D_r^{i,k}=H} Q_r^{i,k}$
Cumulative demand	$\sum_{k,r:D_r^{i,k}\leq 1} Q_r^{i,k}$	$\sum_{k,r:D_r^{i,k}\leq 2} Q_r^{i,k}$	$\dots$	$\lvert K\rvert \sum_{r\in R(i)} Q_r^{i,k}$

3 Minimizing the Maximum Tardiness: Proof of Optimality of the EDD Rule

These additional decision variables, based on our completion times, are necessary for this proof.

$T_{in}^{i,k} = \max_{s\in pred(i), r\in R(s):\ i\in r, q\in Q_{(s,i)}} t_{r,q}^{s,k}$: the tardiness of manufacturer i in meeting demand k based on the arrival of needed components, before i makes its production/shipping decisions.

$T_{in}^{i} = \max_k \max_{s\in pred(i), r\in R(s):\ i\in r, q\in Q_{(s,i)}} t_{r,q}^{s,k}$: the tardiness of manufacturer i across all demands based on the arrival of needed components, before i makes its production/shipping decisions.

$T_{in}^{E} = \max_{i\in N(E)} T_{in}^{i}$: the maximum tardiness of all manufacturers i in Tier $E \in \mathcal{E}$ based on the arrival of needed components, before the manufacturers make their production/shipping decisions.

$T_{dis}^{i} = \max_{k,r}(u^i + p_i - D_r^{i,k}, 0)$: the tardiness of manufacturer i across all demands based on the disruption at i.

$T_{dis}^{E} = \max_{i\in N(E)} T_{dis}^{i}$: the maximum tardiness of all manufacturers s in Tier $E \in \mathcal{E}$ based on their disruptions.

$T_{out}^{i,k} = \max_{r,q} t_{r,q}^{i,k}$: the maximum tardiness of manufacturer i in meeting demand k based on its shipping decisions.

$T_{out}^{i} = \max_{k} T_{out}^{i,k}$: the maximum tardiness of manufacturer i across all demands based on its decisions. Hence, T_{out}^{B} is the maximum tardiness of the whole system.

$T_{out}^{E} = \max_{i\in N(E)} T_{out}^{i}$: the maximum tardiness of all manufacturers s in Tier $E \in \mathcal{E}$ based on their decisions.

By definition, the tardiness at Tier E does not account for any further disruptions up the MEASC network, i.e., from Tier $E-1,\dots,1,0$. Therefore, we have $C_{r,q}^{i,k} \geq \max_{s\in pred(i)} \left(C_{r,q}^{s,k} + p_{(s,i)}, u^i\right) + p_i$, i.e., the completion time of a task k at i is greater than or equal to the arrival of any required sub-component for task k or the disruption time of i plus the lead time of i. In addition, we have $T_{out}^{E} = \max_{i\in N(E)} T_{out}^{i,k} = \max_{i\in N(E)} \max_{k\in K} \max_{r\in R(i),q} t_{r,q}^{i,k} = \max_{i\in N(E),k\in K,r\in R(i),q} \left(C_{r,q}^{i,k} - D_r^{i,k}, 0\right)$. We then have that

$$
\begin{aligned}
& T_{out}^{E} \geq \max_{s\in N(E+1),r\in R(s),q} \left(\max\left(C_{r,q}^{s,k} + p_{(s,i)}, u^i\right) + p_i - D_r^{i,k}, 0\right) \\
\rightarrow\ & T_{out}^{E} \geq \max_{s\in N(E+1),r\in R(s),q} \left(\max\left(C_{r,q}^{s,k} + p_{(s,i)} + p_i - D_r^{i,k}, u^i + p_i - D_r^{i,k}\right), 0\right) \\
\rightarrow\ & T_{out}^{E} \geq \max_{s\in N(E+1),r\in R(s),q} \left(\max\left(C_{r,q}^{s,k} - D_r^{s,k}, u^i + p_i - D_r^{i,k}\right), 0\right) \\
\rightarrow\ & T_{out}^{E} \geq \max\left(T_{out}^{E+1}, T_{dis}^{E}\right)
\end{aligned}
$$

This means that for any Tier E, the tardiness is bounded from below by both the schedule-caused tardiness of the previous tier, T_{out}^{E+1}, and its own disruption-caused tardiness, T_{dis}^{E}.

The system then has the following property for any schedule:

$$T_{out}^{M-1} \geq \max\left(T_{out}^{M}, T_{dis}^{M-1}\right)$$

$$T_{out}^{M-2} \geq \max\left(T_{out}^{M-1}, T_{dis}^{M-2}\right) \geq \max\left(T_{out}^{M}, \max_{E=M-2,M-1} T_{dis}^{E}\right)$$

$$\dots \qquad \dots \qquad \dots \qquad \dots$$

$$T_{out}^{1} \geq \max\left(T_{out}^{M}, \max_{E=1\dots M-1} T_{dis}^{E}\right)$$

$$T_{out}^{B} \geq \max\left(T_{out}^{M}, \max_{E=0\dots M-1} T_{dis}^{E}\right)$$

Let T^{M*} be the optimal maximum tardiness at tier M. If we can show that the EDD rule can achieve $T_{out}^{B} = \max\left(T_{out}^{M*}, \max_{E=0\dots M-1} T_{dis}^{E}\right)$, then $T_{out}^{B} = T_{out}^{B*}$, i.e, the EDD rule is the optimal solution.

Lemma 3.1. *Applying the EDD rule at all raw material manufacturers (Tier M) minimizes the maximum tardiness for tier M, $T_{out}^{M} = T_{out}^{M*}$.*

Proof. At Tier M, consider a schedule O, which violates the EDD, that is optimal. We can apply the a similar interchange argument to this problem as for the proof of the one machine problem $1||T_{max}$ in the traditional scheduling literature (Pinedo 2012) : In this schedule there must be at least two adjacent shipments by a manufacturer $s \in N(M)$ such that $C_{r_1,q_1}^{s,k_1} > C_{r_2,q_2}^{s,k_2}$ and $D_{r_1}^{s,k_1} < D_{r_2}^{s,k_2}$ (either $r_1 \neq r_2$ *or* $k_1 \neq k_2$). This implies that

$$\rightarrow \begin{cases} C_{r_1,q_1}^{s,k_1} - D_{r_1}^{s,k_1} > C_{r_2,q_2}^{s,k_2} - D_{r_1}^{s,k_1} \\ \\ C_{r_1,q_1}^{s,k_1} - D_{r_1}^{s,k_1} > C_{r_1,q_1}^{s,k_1} - D_{r_2}^{s,k_2} \end{cases}$$

The maximum tardiness across in Tier M based on the current schedule is

$$\begin{aligned} T_{out}^{M} = \max_{i \in N(M)} T_{out}^{i,k} &= \max_{i \in N(M)} \max_{\text{k}} \max_{r \in R(i),q} t_{r,q}^{i,k} \\ &= \max\left(0, \max_{i \in N(M),(i,k,r,q) \neq (s,k_1,r_1,q_1),(s,k_2,r_2,q_2)} t_{r}^{i,k}, C_{r_1,q_1}^{s,k_1} - D_{r_1}^{s,k_1}, C_{r_2,q_2}^{s,k_2} - D_{r_2}^{s,k_2}\right) \end{aligned}$$

If we exchange shipment C_{r_1,q_1}^{s,k_1} and C_{r_2,q_2}^{s,k_2} in our schedule O, then the maximum tardiness is then

$$\hat{T}_{out}^{M} = \max\left(0, \max_{i \in N(2),(i,k,r,q) \neq (s,k_1,r_1,q_1),(s,k_2,r_2,q_2)} t_{r,q}^{i,k}, C_{r_2,q_2}^{s,k_2} - D_{r_1}^{s,k_1}, C_{r_1,q_1}^{s,k_1} - D_{r_2}^{s,k_2}\right) \leq T_{out}^{M}$$

Therefore, schedule O can be modified to follow the EDD rule without increasing the objective function value, i.e., applying the EDD rule gives the optimal value at tier M, i.e., $T_{out}^{M} = T_{out}^{M*}$. $\square$

Lemma 3.2. *The maximum tardiness does not increase between shipping from Tier $E+1$ and receiving at Tier E, $T_{in}^{E*} = T_{out}^{E+1*}$*

Proof. The maximum tardiness of all manufacturers s in Tier $E \geq 1$ based on the arrival of needed components is:

$$T_{in}^{E} = \max_{s \in N(E)} T_{in}^{s} = \max_{s \in N(E)} \max_{k} \max_{i \in pred(s), r \in R(i):\ s \in r} \max_{q=1..Q_r} t_{r,q}^{i,k}$$

Based on our assumptions, each path entering Tier E must be from a manufacturer in Tier $E+1$, we have

$$T_{in}^{E} = \max_{s} \max_{k} \max_{i \in pred(s), r \in R(i):\ s \in r} \max_{q=1..Q_r} t_{r,q}^{i,k} = \max_{i \in N(E+1)} \max_{\text{k}} \max_{r \in R(i), q} t_{r,q}^{i,k} = T_{out}^{E+1}$$

$$\rightarrow T_{in}^{E*} = T_{out}^{E+1*}$$

$\square$

Lemma 3.3. *It is optimal to apply the EDD at any Tier $E,\ 1 \leq E \leq M-1$ to minimize the maximum tardiness T_{out}^{E}. The optimal maximum tardiness is equal to either the tardiness of previous tier T_{out}^{E+1*}, or the the disruption-caused tardiness T_{dis}^{E}. $T_{out}^{E*} = \max\left(T_{out}^{E+1*}, T_{dis}^{E}\right)$.*

Proof. At Tier $E,\ 1 \leq E \leq M-1$, we have

$$T_{out}^{E} = \max_{i \in N(E)} T_{out}^{i,k} = \max_{i \in N(E)} \max_{\text{k}} \max_{r \in R(i), q} t_{r,q}^{i,k} = \max_{i \in N(E)} \max_{\text{k}} \max_{r \in R(i), q} \left(C_{r,q}^{i,k} - D_{r}^{i,k}, 0\right)$$

Using the EDD and applying the same proof as we did for the raw material manufacturers in Tier M, $T_{out}^{E,EDD} = T_{out}^{E*}$. For any schedule, we also have

$$T_{out}^{E} \geq \max\left(T_{out}^{E+1}, T_{dis}^{E}\right) \rightarrow T_{out}^{E} \geq \max\left(T_{in}^{E}, T_{dis}^{E}\right)$$

We will show $T_{out}^{E*} = \max\left(T_{in}^{E*}, T_{dis}^{E}\right)$, by showing $T_{out}^{i*} = \max\left(T_{in}^{i*}, T_{dis}^{i}\right) \forall i \in N(E),\ 1 \leq E \leq M-1$.

We have

$$T_{in}^{i} = \max_{k} \max_{s\in in(i), r\in R(s):\ i\in r, q\in 1..Q_r} \left(C_{r,q}^{s,k} - D_{r}^{s,k}, 0\right)$$

$$T_{out}^{i} = \max_{k} \max_{r\in R(i), q\in 1..Q_r} \left(C_{r,q}^{i,k} - D_{r}^{i,k}, 0\right)$$

$$T_{dis}^{i} = \max_{k,r}(u^{i} + p_i - D_{r}^{i,k}, 0)$$

By way of contradiction, suppose $\exists i,\ T_{out}^{i*} > \max\left(T_{in}^{i*}, T_{dis}^{i}\right)$.

Let $\left\{\hat{s}, \hat{k}, \hat{r}, \hat{q}\right\} = \text{argmax}_{k, s\in pred(i), r\in R(s):\ i\in r, q\in 1..Q_r} \left(C_{r,q}^{s,k*} - D_{r}^{s,k}, 0\right)$, i.e., this quadruple has the largest tardiness into manufacturer i. There must exist $\left\{\tilde{k}, \tilde{r}, \tilde{q}\right\}$ such that $C_{\tilde{r},\tilde{q}}^{i,\tilde{k}*} - D_{\tilde{r}}^{i,\tilde{k}} > C_{\hat{r},\hat{q}}^{\hat{s},\hat{k}*} - D_{\hat{r}}^{\hat{s},\hat{k}}$ and $C_{\tilde{r},\tilde{q}}^{i,\tilde{k}*} - D_{\tilde{r}}^{i,\tilde{k}} > u^{i} + p_i - D_{\tilde{r}}^{i,\tilde{k}}$.

Using the fact that we start production as soon as i is restored and all components are available,

$$C_{\tilde{r},\tilde{q}}^{i,\tilde{k}*} = \max\left(\max_{s\in pred(i)} \left(C_{\tilde{r},\tilde{q}}^{s,\tilde{k}*} + p_{(s,i)}\right), u^{i}\right) + p_i$$

If $C_{\tilde{r},\tilde{q}}^{i,\tilde{k}*} = u^{i} + p_i$, then $C_{\tilde{r},\tilde{q}}^{i,\tilde{k}*} - D_{\tilde{r}}^{i,\tilde{k}} = u^{i} + p_i - D_{\tilde{r}}^{i,\tilde{k}}$ contradicting $T_{out}^{i*} > T_{dis}^{i}$

If $C_{\tilde{r},\tilde{q}}^{i,\tilde{k}*} > u^{i} + p_i$, let $\tilde{s} \in pred\,(i)$ be the manufacturer that has $C_{\tilde{r},\tilde{q}}^{i,\tilde{k}*} = C_{\tilde{r},\tilde{q}}^{\tilde{s},\tilde{k}*} + p_{(\tilde{s},i)} + p_i$. We then have

$$C_{\tilde{r},\tilde{q}}^{i,\tilde{k}*} - D_{\tilde{r}}^{i,\tilde{k}} = C_{\tilde{r},\tilde{q}}^{\tilde{s},\tilde{k}*} + p_{(\tilde{s},i)} + p_i - D_{\tilde{r}}^{i,\tilde{k}} = C_{\tilde{r},\tilde{q}}^{\tilde{s},\tilde{k}*} - D_{\tilde{r}}^{\tilde{s},\tilde{k}}$$
$$\rightarrow C_{\tilde{r},\tilde{q}}^{\tilde{s},\tilde{k}*} - D_{\tilde{r}}^{\tilde{s},\tilde{k}} > C_{\hat{r},\hat{q}}^{\hat{s}\hat{k}*} - D_{\hat{r}}^{\hat{s},\hat{k}},$$

which contradicts that $\left\{\hat{s}, \hat{k}, \hat{r}, \hat{q}\right\} = \text{argmax}_{k, s\in pred(i), r\in R(s):\ i\in r, q\in 1..Q_r} \left(C_{r,q}^{s,k*} - D_{r}^{s,k}, 0\right)$. Therefore, $T_{out}^{E*} = \max\left(T_{in}^{E*}, T_{dis}^{E}\right)$, which means $T_{out}^{E*} = \max\left(T_{out}^{E+1*}, T_{dis}^{E}\right)$. □

Theorem 3.4. *Applying the EDD rule for each individual manufacturer minimizes the maximum tardiness of the whole MEASC network, i.e.,* T_{out}^{B}.

Proof. Lemma 3.1 shows that if the EDD rule is applied for all raw material manufacturers (Tier M) then the maximum tardiness for Tier M is optimal. Lemma 3.2 shows that the maximum tardiness does not increase between shipping from Tier $E+1$ and receiving at Tier E. Lemma 3.3 shows that as we move up the tiers and continue to apply the EDD rule, the maximum tardiness T^E_{out} at each Tier E is optimal, since it is either the optimal maximum tardiness of the previous Tier T^{E+1*}, or the disruption-caused tardiness T^E_{dis} of that tier. Consequently, the maximum tardiness out of the final assembly T^B_{out} is optimal as applying the EDD rule for each manufacturer in the whole system gives the best possible scheduling based on the disruptions at all the tiers, i.e., we have that

$$T^B_{out} = \max\left(T^{B+1*}_{out}, T^B_{dis}\right) = \max\left(T^{B+2*}_{out}, T^{B+1}_{dis}, T^B_{dis}\right) = \cdots = \max\left(T^{M*}_{out}, \max_{E=0\ldots M-1} T^E_{dis}\right) = T^{B*}_{out}.$$

$\square$

4 Minimizing the Time to Recover: Proof of Optimality of the Decision Rule

We have the following additional decision variables that are necessary for this proof.

$\tau^{i,k} = \max_{r,q}(\alpha^i_r(C^{i,k}_{r,q}) - D^k, 0)$: the tardiness of manufacturer i for demand k based on the production/shipping decisions at i and the disruptions at upper tier manufacturers on paths from i.

$TTR^i_r = \max(\max_{k,q:t^k>0}(C^{i,k}_{r,q} + p^i_r), 0)$: the time to recover of manufacturer i through path r.

$TTR^{i,K} = \max_{r,q,k\in K:t^k>0} \alpha^i_r(C^{i,k}_{r,q})$: the time to recover of manufacturer i for a particular set of final demands K.

$TTR^i = \max_r TTR^i_r = \max_{r,k,q:t^k>0} \alpha^i_r(C^{i,k}_{r,q})$: the time to recover of manufacturer i.

$TTR^E = \max_{i\in N(E)} TTR^i$ the time to recover of Tier E.

<u>Properties of any schedules:</u> For any schedule, we have the following properties for the TTR^E:

$$TTR^E = \max_{i\in N(E)} TTR^i$$
$$TTR^E = \max_{i\in N(E),r,k,q:t^k>0} \alpha^r_i(C^{i,k}_{r,q})$$
$$TTR^E = \max_{i\in N(E),r,k,q:t^k>0} \left(C^{i,k}_{r,q} + p^i_r, \max_{m\in r}(u^m + p^m_r)\right)$$

It should be noted that the term $\max_{m\in r}(u^m + p^m_r)$ ensures TTR^E accounted for the disruptions between i and B. This differs from how the maximum tardiness is defined. We also have

$$C^{i,k}_{r,q} \geq \max\left(\max_{s\in pred(i)}\left(C^{s,k}_{r,q} + p_{(s,i)}, u^i\right) + p_i\right)$$

$$\rightarrow TTR^E \geq \max_{s\in N(E+1), r\in R(s), q:t^k>0} \max\left(C^{s,k}_{r,q} + p_{(s,i)} + p_i + p^i_r, \max_{m\in r}(u^m + p^m_r)\right)$$

$$\rightarrow TTR^E \geq \max_{s\in N(E+1), r\in R(s), q:t^k>0} \max\left(C^{s,k}_{r,q} + p^s_r, \max_{m\in r}(u^m + p^m_r)\right)$$

$$\rightarrow TTR^E \geq TTR^{E+1}$$

This means that for any Tier $E \leq M-1$, the time to recover is bounded from below by that of the time for recover of Tier $E+1$. The system then has the following property for any schedule:

$$TTR^M \leq TTR^{M-1} \leq \cdots \leq TTR^2 \leq TTR^1 \leq TTR^B$$

We will show $TTR^B = TTR^M = TTR^{M*}$, thus implying $TTR^B = TTR^{B*}$, i.e., the decision rule is optimal.

4.1 On the Optimality of the Rule for a Single Raw Material Manufacturer

Theorem 4.1. *For a set of demands $\tilde{K}$, let $MS^{i,\tilde{K}} = \max_{r,k\in\tilde{K},q}\left(C^{i,k}_{r,q} + p^i_r, \max_{m\in r}(u^m + p^m_r)\right) = \max_{k\in\tilde{K}}(D^k + \tau^{i,k})$ be the makespan of manufacturer i if we assume that the makespan of the system only depends on the scheduling of i. Applying the LLT rule at a manufacturer i minimizes the makespan for i for the set of demands $\tilde{K}$.*

Proof. We have: $MS^i = \max_{r,k,q}\left(C^{i,k}_{r,q} + p^i_r, \max_{m\in r}(u^m + p^m_r)\right)$. At i, consider a schedule O, which violates the LLT rule. In this schedule there must be at least two adjacent shipments such that $C^{i,k_1}_{r_1,q_1} > C^{i,k_2}_{r_2,q_2}$ and $p^i_{r_1} > p^i_{r_2}$

$$\rightarrow \begin{cases} C^{i,k_1}_{r_1,q_1} + p^i_{r_1} > C^{i,k_2}_{r_2,q_2} + p^i_{r_1} \\ \\ C^{i,k_1}_{r_1,q_1} + p^i_{r_1} > C^{i,k_1}_{r_1,q_1} + p^i_{r_2} \end{cases}$$

The makespan of manufacturers in Tier M based on schedule O is

$$MS^{i,\tilde{K}} = \max_{r,k,q}\left(C^{i,k}_{r,q} + p^i_r, \max_{m\in r}(u^m + p^m_r)\right)$$

$$MS^{i,\tilde{K}} = \max_{r,k,q}\left(C^{i,k_1}_{r_1,q_1} + p^i_{r_1}, C^{i,k_2}_{r_2,q_2} + p^i_{r_2}, \max_{(s,k,r,q)\neq(i,k_1,r_1,q_1),(i,k_2,r_2,q_2)} C^{s,k}_{r,q} + p^s_r, \max_{m\in r}(u^m + p^m_r)\right)$$

If we exchange shipment $C^{i,k_1}_{r_1,q_1}$ and $C^{i,k_2}_{r_2,q_2}$ in our schedule O, the makespan is then

$$\widehat{MS}^{i,\tilde{K}} = \max_{r,k,q}\left(C^{i,k_2}_{r_2,q_2} + p^i_{r_1}, C^{i,k_1}_{r_1,q_1} + p^i_{r_2}, \max_{(s,k,r,q)\neq(i,k_1,r_1,q_1),(i,k_2,r_2,q_2)} C^{s,k}_{r,q} + p^s_r, \max_{m\in r}(u^m + p^m_r)\right)$$

which is less than or equal to $MS^{i,\tilde{K}}$, since $C^{i,k_1}_{r_1,q_1} + p^i_{r_1} > C^{i,k_2}_{r_2,q_2} + p^i_{r_1}$ and $C^{i,k_1}_{r_1,q_1} + p^i_{r_1} > C^{i,k_1}_{r_1,q_1} + p^i_{r_2}$. Therefore, schedule O can be modified to follow the LLT rule without increasing the objective function value, i.e., the LLT rule gives the optimal value $MS^{i,\tilde{K}} = MS^{i,\tilde{K}*}$. □

Lemma 4.2. *For a subset of demands $\bar{K}$ that must be late, applying the LLT rule to a manufacturer i minimizes the time to recover $TTR^{i(\bar{K})}$ at i for group $\bar{K}$.*

Proof. We know that $\bar{K}$ must be late. Therefore, $TTR^{i,(\bar{K})} > \max_{k\in\bar{K}} D^k$ meaning that there exists $k \in \bar{K}, t^{i,k} > 0$ for any schedule. We have that $TTR^{i,\bar{K}} = \max_{k\in\bar{K}}(D^k + t^{i,k}) = MS^{i,\bar{K}}$ From Theorem 4.1, we know that applying the LLT minimizes the makespan. Hence, the LLT rule minimizes the time to recover at manufacturer i, $TTR^{i,\bar{K}*} = MS^{i,\bar{K}*}$ for a subset of demands $\bar{K}$ that must be late. □

Preliminary Results for the Reverse EDD Rule

Theorem 4.3. *For any set of demands K, if there exists a schedule with maximum tardiness of 0, then the reverse EDD schedule has the maximum tardiness of 0.*

Proof. We know that applying the EDD rule minimizes maximum tardiness (with value 0), as shown in Theorem 3.4. From the EDD schedule, if we exchange production completion time $C^{i,\tilde{k}}_{\tilde{r},\tilde{q}}$ assigned to the last demand $\tilde{k}$ (i.e., the demand with largest deadline) with largest production completion time $C^{i,\bar{k}}_{\bar{r},\bar{q}}$ that is less than or equal to $D^{i,\tilde{k}}_{\tilde{r}}$. Then the tardiness for $\tilde{k}$ is still 0. Since $C^{i,\bar{k}}_{\bar{r},\bar{q}} \leq C^{i,\tilde{k}}_{\tilde{r},\tilde{q}}$, the maximum tardiness for the remaining demands in K is still 0. Hence, the EDD schedule can be transformed into the reverse EDD schedule while maintaining the maximum tardiness of 0. □

Determining the Time to Recover of the Raw Materials manufacturers

Theorem 4.4. *For each raw material manufacturer i, decomposing the subproblems at $k(i)$ generated by the algorithm in Subsection 5.1 gives the optimal local time to recover $TTR^{i*} = \mathcal{C}(i)$, where TTR^{i*} is the optimal time to recover considering just manufacturer i and $\mathcal{C}(i)$ is the time to recover given by the algorithm.*

Proof. We have $K2 = \{k(i)+1, k(i)+2, \dots |K|\}$ is the on-time subset of demand since there is a reverse EDD schedule with maximum tardiness of 0 (Theorem 4.3). Let $K1 = \{1, 2, \dots, k(i)\}$ be the subset of remaining demands. We have two situations:

Case 1: $K1 = \emptyset$. Then, $TTR^{i*} = TTR^{i,K1*} = 0 = \mathcal{C}(i)$ since all jobs can be completed on time.

Case 2: $K1 \neq \emptyset$ We want to show that $K1$ must be late, i.e., $MS^{i,K1*} > D^{k(i)}$. Suppose $MS^{i,K1*} \leq D^{k(i)}$. This implies that $MS^{i,\{1,2,\dots k(i-1)\}*} \leq D^{k(i)}$. Because the iteration did not break at the previous iteration, that means it is not possible to find a schedule with maximum tardiness of 0 for the subset of demands $\{k(i), k(i)+1, \dots |K|\}$.Since the maximum tardiness of $K2 = \{k(i)+1, k(i)+2, \dots |K|\}$ is 0, this implies that the tardiness of $k(i)$ must be positive. Hence, for the subset $K1 = \{1, 2, \dots, k(i)\}$, $MS^{i,K1*} > D^{k(i)}$, or $K1$ must be late. Since $\mathcal{C}(i)$ is generated by the LLT rule and $K1$ must be late, $TTR^{i,K1*} = MS^{i,K1*} = \mathcal{C}(i)$ from Theorem 4.1 and Lemma 4.2. We have $TTR^{i*} = TTR^{i,K1*} = \mathcal{C}(i)$. □

Theorem 4.5. *Let $\bar{\mathcal{C}} = \max_{i \in N(M)} \mathcal{C}(i)$ be the maximum optimal individual TTR across raw material manufacturers i calculated by the algorithm in Subsection 5.1. Decomposing the main problem into 2 subproblems based on $\bar{k}$ and applying the decision rules at a raw material manufacturer i gives an updated time to recover, $TTR^i \leq \bar{\mathcal{C}}$*

Proof. Let $K1 = \bar{K} = \{k \in K | k \leq \bar{k}\}$ and $K2 = \{k \in K | k \geq \bar{k} + 1\}$. If $k(i) = \bar{k}$, from Theorem 4.4, we have that $TTR^{i*} = \mathcal{C}(i) \leq \bar{\mathcal{C}}$ since the sets $K1$ and $K2$ will not change.

Now, consider the case where $k(i) < \bar{k}$. Applying the LLT rule for $K1$ gives $TTR^{i,K1} = MS^{i,\{k \in K1 | t_k > 0*\}}$ $\leq MS^{i,K1*} \leq D^{\bar{k}} < \bar{\mathcal{C}}$ where the second to last inequality comes directly from the definition of $k(i)$ (i.e., $k(i) < \bar{k}$ implying that $MS^{i,K1*} \leq D^{\bar{k}}$). In other words, the schedule that was created for manufacturer i initially for K1 had all tasks being completed by $D^{\bar{k}}$ and applying the LLT rule can only decrease the makespan for this set of jobs at i. □

4.2 The decision rules minimize TTR for the whole system: Proofs

Subproblem 1: For subset of final demands $k \leq \bar{k}$

We need to show that the LLT rule minimizes the time to recover of the whole MEASC network, i.e., TTR^B for the subset of final demands with deadline before cutoff $\bar{k}$ determined by the algorithm in Subsection 5.1, i.e. the subset of final demands that must be late. We will show that applying the LLT rule for all raw material manufacturers gives the optimal value TTR^{M*} (Theorem 4.6). Then, Theorem 4.7 will show that as we move up the tiers and continue to apply the LLT rule, the time to recover at a Tier E does not increase and equals the optimal value of the previous tier time to recover, $TTR^{E*} = TTR^{E+1*}$. Combining these two theorems, we can show that the LLT rule gives optimal value of the time to recover at the final assembly node $TTR^{B*} = TTR^{M*} = \bar{\mathcal{C}}$.

Theorem 4.6. *For $SP1$ where the set of demands that must be late are those such that $k \leq \bar{k}$, applying the LLT rule for all raw material manufacturers gives the optimal value $TTR^{M} = TTR^{M*} = \bar{\mathcal{C}}$.*

Proof. From Theorem 4.5, we know that applying the decision rule gives $TTR^{i} \leq \bar{\mathcal{C}}\ \forall\, i \in N(M)$ with $\bar{\mathcal{C}} = \max_i TTR^i$. Thus, $TTR^{M*} = \max_{i\in N(M)} TTR^i = \bar{\mathcal{C}}$. Therefore, $TTR^{M*} = \bar{\mathcal{C}}$ is the lowest time to recover that can be obtained at Tier M since TTR^i is a lower bound on the overall time to recover. □

Theorem 4.7. *For SP1, as we moves up the tiers, applying the LLT rule does not increase the time to recover, i.e., $TTR^{E*} = TTR^{E+1*}$*

Proof. At Tier $E,\ 1 \leq E \leq M-1$, we have

$$TTR^{E} = \max\nolimits_{i\in N(E),r,k,q}\left(C^{i,k}_{r,q} + p^{i}_{r}, \max_{m\in r}\left(u^{m} + p^{m}_{r}\right)\right)$$

Using the LLT rule and applying the same proof as in Theorem 4.4, $TTR^{E} = TTR^{E*}$. We want to show that $TTR^{E*} = TTR^{E+1*}$ by showing that $TTR^{i*} = \max_{s\in pred(i)} TTR^{s*} \forall\, i \in N(E),\ 1 \leq E \leq M-1$. We have $TTR^{i} = \max_{r} TTR^{i}_{r} = \max_{r,k,q:t^k>0}\alpha^{i}_{r}(C^{i,k}_{r,q})$. By way of contradiction, suppose $\exists i$, where $TTR^{i} > \max_{s\in pred(i)} TTR^{s}$. This implies that

$$\max\nolimits_{r,k,q}\left(C^{i,k}_{r,q} + p^{i}_{r}, \max_{m\in r}\left(u^{m} + p^{m}_{r}\right)\right) > \max\nolimits_{s\in pred(i),r\in R(s):i\in r,k,q}\left(C^{s,k}_{r,q} + p^{s}_{r}, \max_{m\in r}\left(u^{m} + p^{m}_{r}\right)\right)$$

$$\rightarrow \begin{cases} \max_{r,k,q}\left(C^{i,k}_{r,q} + p^{i}_{r}\right) > \max_{s\in pred(i),r\in R(s):i\in r,k,q}\left(C^{s,k}_{r,q} + p^{s}_{r}\right) \\ TTR^{i} = \max_{r,k,q}\left(C^{i,k}_{r,q} + p^{i}_{r}\right) \\ \max_{r,k,q}\left(C^{i,k}_{r,q} + p^{i}_{r}\right) > \max_{m\in r}\left(u^{m} + p^{m}_{r}\right) \end{cases}$$

Let $\left\{\hat{k}, \hat{r}, \hat{q}\right\} = \text{argmax}_{r,k,q} \left(C_{r,q}^{i,k} + p_r^i\right) = TTR^i$

$$\rightarrow \begin{cases} C_{\hat{r},\hat{q}}^{i,\hat{k}} + p_{\hat{r}}^i > \max_{s \in pred(i), r \in R(s): i \in r, k, q} \left(C_{r,q}^{s,k} + p_r^s\right) \\ C_{\hat{r},\hat{q}}^{i,\hat{k}} + p_{\hat{r}}^i > \max_{m \in r} \left(u^m + p_r^m\right) \end{cases}$$

Because we start production as soon as all components are available or when the manufacturer is restored, $C_{\hat{r},\hat{q}}^{i,\hat{k}} = \max\left(\max_{s \in pred(i)} \left(C_{r,q}^{s,k} + p_{(s,i)}\right), u^i\right) + p_i$ if the ordering $C_{r,q}^{s,k}$ and $C_{r,q}^{i,k}$ are identical. Note that since $\forall s \in pred(i), r \in R(s), p_r^s = p_r^i + p_{(s,i)}$, the order of the schedule in the LLT rule of s and i based on r should be identical because if $p_{r_1}^s > p_{r_2}^s, p_{r_1}^s + p_{(s,i)} > p_{r_2}^s + p_{(s,i)} \rightarrow p_{r_1}^i > p_{r_2}^i$. Therefore, we have a contradiction. Hence, $TTR^{E*} = TTR^{E+1*}$. □

Theorem 4.8. *The LLT rule minimizes the time to recover of the whole system for demands before or at cutoff* $\bar{k}$.

Proof. Combining Theorems 4.6 and 4.7 results in the LLT rule achieving $TTR^{B*} = TTR^{M*} = \bar{\mathcal{C}}$. □

SP2: For the demands after $\bar{k}$

Applying the EDD rule achieves optimal the maximum tardiness of 0 as shown in Theorem 3.4 since the process of finding $\bar{k}$ includes verifying that there exists a schedule with tardiness of 0 for every demand after $\bar{k}$. Therefore, $TTR^B = \bar{\mathcal{C}}$ remains.

5 Integer Programming Models

This section shows the formulations of the IP models that we use as a comparison with the decision rules. We begin with the required additional notation:

N': the set of nodes that need to be restored.

$BigM$: a sufficiently large number

qd^t: the quantity of final products with demand deadline of t

$ship_{(i,j),t}, (i,j) \in A$: the variable presents amount part i is shipped on arc (i,j) at time t.

$pro_{i,t}, i \in N$: the variable presents amount of part i produced at node i at time t where t is the time production ends.

$inv_{i,z,t}, i \in N$: the variable presents amount of part z stored at node i at time t.

5.1 Minimizing Maximum Tardiness

Model specific notations:

$fd^{d,t}$: the variable presents the quantity of the final products received at B at time t, allocated to meeting the final demands with deadline d

$ud^{d,t}$: the variable presents the quantity of the final demands with deadline d remaining unmet at time t

$tbo^{d,t}$: the variable presents total amount of back-ordered final demands with deadline d at time t.

$ed^{d,t}$:the variable presents total amount of excess final demands with deadline d at time t.

$pbo^{d,\delta}$: the binary variable indicates whether there is a positive amount of back-ordered final demands with deadline d at time $t+\delta$. i.e., δ is the tardiness.

$$\text{Minimize } T^B_{out} \tag{1a}$$

$$\text{s.t.} \quad pro_{i,t} + inv_{i,i,t-1} = \sum_{(i,j)\in A} ship_{(i,j),t} + inv_{i,i,t}, \qquad \forall i \in N, t = 1 \dots H \tag{1b}$$

$$ship_{(s,i),t-p_{(s,i)}} + inv_{i,s,t-1} = Q_{(s,i)} pro_{i,t+p_i} + inv_{i,s,t} \quad \forall i \in N, s \in pred(i), t = 1 \dots H \tag{1c}$$

$$\sum_{(i,B)\in A} ship_{(i,B),t-p_{(i,B)}} = \sum_{d=1}^{H} fd^{d,t} \qquad \forall t = 1 \dots H \tag{1d}$$

$$fd^{t,t} + ud^{t,t} = qd^t + ed^{t,t} \qquad \forall t = 1 \dots H \tag{1e}$$

$$fd^{d,t} + ud^{d,t} = ed^{d,t} \qquad d = 1 \dots H, \forall t = 1 \dots H, t \neq d \tag{1f}$$

$$tbo^{d,t} = tbo^{d,t-1} + ud^{d,t} - ed^{d,t} \qquad d = 1 \dots H, t = 2 \dots H, \tag{1g}$$

$$pbo^{d,\delta} BigM \geq tbo^{d,d+\delta} \qquad \forall d = 1 \dots H, \delta = 0 \dots H-d \tag{1h}$$

$$T^B_{out} \geq \sum_{\delta=1}^{H-\delta} pbo^{d,\delta} \qquad \forall d = 1 \dots H \tag{1i}$$

$$pro_{i,t} \leq cap^i \qquad \forall i \in N \setminus N(M), t = 1 \dots H \tag{1j}$$

$$pro_{i,t} = 0 \qquad \forall i \in N', t = 1 \dots u^i \tag{1k}$$

$$ship_{(i,j),t}, pro_{i,t}, inv_{i,z,t}, fd^{t,t}, ud^{d,t}, tbo^{d,t}, ed^{d,t} \geq 0 \tag{1l}$$

$$pbo^{d,\delta} binary \tag{1m}$$

5.2 Minimizing the time to recover

Model specific notations:

ud^t: the variable presents unmet final demand at time t.

ed^t:the variable presents excess final demand at time t.

tbo^t: the variable presents total amount of back-ordered product at time t.

pbo^t: the binary variable indicates whether there is a positive backorders(1 if yes) at time t, i.e. there is a late demand

dis^t: the binary variable indicates (with value 1) if the supply chain has not recovered at time t

$$\text{Minimize } TTR^B \tag{2a}$$

$$\text{s.t.} \quad pro_{i,t} + inv_{i,i,t-1} = \sum_{(i,j)\in A} ship_{(i,j),t} + inv_{i,i,t}, \qquad \forall i \in N, t = 1 \dots H \tag{2b}$$

$$ship_{(s,i),t-p_{(s,i)}} + inv_{i,s,t-1} = Q_{(s,i)} pro_{i,t+p_i} + inv_{i,s,t} \quad \forall i \in N, s \in pred(i), t = 1 \dots H \tag{2c}$$

$$\sum_{(i,B)\in A} ship_{(i,B),t-p_{(i,B)}} + ud^t = qd^t + ed^t \qquad t = 1 \dots H \tag{2d}$$

$$tbo^t = tbo^{t-1} + ud^t - ed^t \qquad t = 2 \ldots H \tag{2e}$$

$$pbo^t BigM \geq tbo^t \qquad t = 1 \ldots H \tag{2f}$$

$$pbo^t \leq dis^t \qquad t = 1 \ldots H \tag{2g}$$

$$TTR^B \geq t\, dis^t \qquad t = 1 \ldots H \tag{2h}$$

$$pro_{i,t} \leq cap^i \qquad \forall i \in N \setminus N(M), t = 1 \ldots H \tag{2i}$$

$$pro_{i,t} = 0 \qquad \forall i \in N', t = 1 \ldots u^i \tag{2j}$$

$$ship_{(i,j),t}, pro_{i,t}, inv_{i,z,t}, ud^t, tbo^t, ed^t \geq 0 \tag{2k}$$

$$pbo^t, dis^t \quad \text{binary} \tag{2l}$$