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Supplemental material - Appendix A

This document describes the *material flow model* (MFM) in details. It presents mathematical basics of the formalism and introduces its properties. A short implementation note is also given.

1. Material flow model (MFM)

The MFM is a combination of three types of concepts: domains, data structures and procedures.

Domains. Each scalar parameter belongs to one of the four domains: $\mathrm{T}_\mathcal{D}$, $\mathrm{F}_\mathcal{D}$, $\mathrm{S}_\mathcal{D}$, $\mathrm{D}_\mathcal{D}$, representing: time, material amount (flow), processing speed, and dimensionless (or percentage) values, respectively. Values from the first three domains have implicitly assigned units which satisfy the obvious relation

$$[\mathrm{F}_\mathcal{D}] = [\mathrm{S}_\mathcal{D}] \cdot [\mathrm{T}_\mathcal{D}].$$

In the MFM time and material flow are considered to be continuous and values from all the defined domains belong to $\mathbb{R}_+ \cup \{0\}$.

Data structures. Data structures are compound data objects based on scalars from the domains. For each data structure two aspects are defined, namely *representation* which defines how to code the data structure by combination of scalars, and *interpretation* which determines meaning of the data structure in the MFM.

A data structure *sp* represents a speed pattern of production resource and $\mathrm{SP}_\mathcal{S}$ is the set of all such data structures. Representation of *sp* is defined by (1) as the sequence of pairs from $\mathrm{T}_\mathcal{D} \times \mathrm{S}_\mathcal{D}$.

$$sp = \left((t_i, s_i)\Big|_{i=1,2,\ldots,n} \right), \quad sp \in \mathrm{SP}_\mathcal{S}, t_i \in \mathrm{T}_\mathcal{D}, s_i \in \mathrm{S}_\mathcal{D}, \quad \underset{i\in\{1,2,\ldots,n-1\}}{\forall} (t_i < t_{i+1} \wedge s_i \neq s_{i+1}). \quad (1)$$

The data structure *sp* is interpreted according to (2a,b) as a piecewise constant time-varying function, obtained by zero order backward linear interpolation based on points defined by *sp* representation.

$$sp\,(\cdot) : \mathrm{T}_\mathcal{D} \longrightarrow \mathrm{S}_\mathcal{D}, \tag{2a}$$

$$s = sp\,(t) = \left((t_i, s_i)\Big|_{i=1,2,\ldots,n} \right)(t) = \begin{cases} s_1 & \text{if} \quad 0 \le t < t_1 \\ s_{i+1} & \text{if} \quad t_i \le t < t_{i+1}\Big|_{i=1,2,\ldots,n-1} \\ 0 & \text{if} \quad t_n \le t \end{cases}. \tag{2b}$$

The speed pattern will also be alternatively represented by multiplication of a *nominal speed* $u \in \mathrm{S}_\mathcal{D}$ and an *efficiency pattern* $ep \in \mathrm{EP}_\mathcal{S}$, such that

$$sp = \left((t_i, s_i)\Big|_{i=1,2,\ldots,n} \right) = \left((t_i, u \cdot e_i)\Big|_{i=1,2,\ldots,n} \right) = u \cdot \left((t_i, e_i)\Big|_{i=1,2,\ldots,n} \right) = u \cdot ep,$$

where the data structure $ep \in \mathrm{EP}_{\mathcal{S}}$ is defined in the same way as $sp \in \mathrm{SP}_{\mathcal{S}}$, except that $e_i \in \mathrm{D}_{\mathcal{D}}$ whereas $s_i \in \mathrm{S}_{\mathcal{D}}$.

A data structure *mf* represents material flow in a production system and $\mathrm{MF}_{\mathcal{S}}$ is the set of all such data structures. Representation of *mf* is defined by (3a,b) as the sequence of pairs from $\mathrm{T}_{\mathcal{D}} \times \mathrm{F}_{\mathcal{D}}$.

$$mf = \left((t_i, f_i)\Big|_{i=1,2,\ldots,n} \right), \qquad mf \in \mathrm{MF}_{\mathcal{S}}, \quad t_i \in \mathrm{T}_{\mathcal{D}}, \quad f_i \in \mathrm{F}_{\mathcal{D}}, \tag{3a}$$

$$f_1 = 0, \qquad \underset{i \in \{1,2,\ldots,n-1\}}{\forall} (t_i < t_{i+1} \wedge f_i \leq f_{i+1}), \qquad \neg \underset{i \in \{1,2,\ldots,n-2\}}{\exists} \frac{f_{i+1} - f_i}{t_{i+1} - t_i} = \frac{f_{i+2} - f_{i+1}}{t_{i+2} - t_{i+1}}. \tag{3b}$$

The data structure *mf* is interpreted according to (4a,b) as a function obtained by the first order linear interpolation based on points defined by *mf* representation

$$mf(\cdot) : \mathrm{T}_{\mathcal{D}} \longrightarrow \mathrm{F}_{\mathcal{D}}, \tag{4a}$$

$$f = mf(t) = \left((t_i, f_i)\Big|_{i=1,2,\ldots,n} \right)(t) = \begin{cases} 0 & \text{if} \quad 0 \leq t < t_1 \\ \frac{f_{i+1} - f_i}{t_{i+1} - t_i}(t - t_i) + f_i & \text{if} \quad t_i \leq t < t_{i+1}\Big|_{i=1,2,\ldots,n-1} \\ f_n & \text{if} \quad t_n \leq t \end{cases}. \tag{4b}$$

Procedures. Procedures are formal representations of computational algorithms that return some data related to the MFM.

The procedure $\mathrm{OMF}_{\mathcal{P}}$ (*output material flow*) computes output material flow mf_{out} that is the result of processing performed by a resource with a given speed pattern *sp* characterized by start time t_{s} and total amount of processed material $f_{\max}$ (5). One assumes that at any time sufficient amount of material for processing inflows from a storehouse or a previous process.

$$\mathrm{OMF}_{\mathcal{P}} : \mathrm{SP}_{\mathcal{S}} \times \mathrm{T}_{\mathcal{D}} \times \mathrm{F}_{\mathcal{D}} \longrightarrow \mathrm{MF}_{\mathcal{S}}, \qquad mf_{\mathrm{out}} = \mathrm{OMF}_{\mathcal{P}}(sp, t_{\mathrm{s}}, f_{\max}). \tag{5}$$

The output material flow mf_{out} is defined by the function

$$mf_{\mathrm{out}}(t) = \begin{cases} 0 & \text{if} \quad 0 \leq t < t_{\mathrm{s}} \\ \int_{t_{\mathrm{s}}}^{t} sp(\tau)\, d\tau & \text{if} \quad t_{\mathrm{s}} \leq t < t_{\mathrm{c}} \\ f_{\max} & \text{if} \quad t_c \leq t \end{cases}. \tag{6}$$

The parameter t_{c} in (6) is the upper limit of the integral (7)

$$\int_{t_{\mathrm{s}}}^{t_{\mathrm{c}}} sp(\tau)\, d\tau = f_{\max}. \tag{7}$$

The definitions (2a,b) and (4a,b) implicate that the indefinite integral of any function given by a data structure from $\mathrm{SP}_{\mathcal{S}}$ is represented by a function given by a data structure included in $\mathrm{MF}_{\mathcal{S}}$

$$\int sp(\tau)\, d\tau = mf^{\mathrm{I}}(t), \qquad sp \in \mathrm{SP}_{\mathcal{S}}, \quad mf^{\mathrm{I}} \in \mathrm{MF}_{\mathcal{S}}.$$

To get the value of t_{c} from the relation (7) the inverse function of the indefinite integral $mf^{\mathrm{I}}(t)$ is needed. However, the function $mf^{\mathrm{I}}(t)$ is in general not injective and thus its inverse does not exist in the strict sense. Therefore a special inverse function $\widetilde{mf^{\mathrm{I}}}(f)$ defined by (8a,b,c) is used. This function assigns to each value of the material flow $f \in [0, f_{\max}]$ the minimum value of time t from the function $mf^{\mathrm{I}}(t)$. This is justified, because the material flow retaining some value in the time range $[t_1, t_2]$ reaches this flow at the moment t_1.

$$\underset{mf^{\mathrm{I}} \in \mathrm{MF}_{\mathcal{S}}}{\forall} \ \underset{\widetilde{mf^{\mathrm{I}}}}{\exists} \ mf^{\mathrm{I}} = \left((t_i, f_i)\Big|_{i=1,2,\ldots,n} \right) \Longrightarrow \widetilde{mf^{\mathrm{I}}} = \left((f_i, t_i)\Big|_{i=1,2,\ldots,n} \right), \tag{8a}$$

$$\widetilde{mf^{\mathrm{I}}}(\cdot) : \mathrm{F}_{\mathcal{D}}^{[0, f_{\max}]} \longrightarrow \mathrm{T}_{\mathcal{D}}, \tag{8b}$$

$$\widetilde{mf^{\mathrm{I}}}(f) = \left((f_i, t_i)\Big|_{i=1,2,\ldots,n} \right)(f) = \begin{cases} 0 & \text{if} \quad f = 0 \\ \frac{t_{i+1}-t_i}{f_{i+1}-f_i}(f - f_i) + t_i & \text{if} \quad f_i < f \le f_{i+1}\Big|_{i=1,2,\ldots,n-1} \end{cases}. \tag{8c}$$

Now, the value t_{c} can be calculated as follows

$$\int_{t_{\mathrm{s}}}^{t_{\mathrm{c}}} sp(\tau)\, d\tau = f_{\max} \quad \Longrightarrow \quad mf^{\mathrm{I}}(\tau)\Big|_{t_s}^{t_c} = f_{\max} \quad \Longrightarrow \quad t_{\mathrm{c}} = \widetilde{mf^{\mathrm{I}}}\left(mf^{\mathrm{I}}(t_s) + f_{\max} \right). \tag{9}$$

It is important to notice that the function $t_{\mathrm{c}}(t_{\mathrm{s}})$ characterized by (9) is non-decreasing, because $\widetilde{mf^{\mathrm{I}}}(\cdot)$ is strictly increasing and its argument $mf^{\mathrm{I}}(t_s) + f_{\max}$ is non-decreasing. The parameters t_{s} and t_{c} will be called *start time* and *completion time* of an operation represented by *mf*.

The procedure $\mathrm{CMT}_{\mathcal{P}}$ (*completion time*) returning the value of t_{c} on the basis of the parameters *sp*, t_{s} and $f_{\max}$, according to (9), is defined as

$$t_{\mathrm{c}} = \mathrm{CMT}_{\mathcal{P}}(sp, t_{\mathrm{s}}, f_{\max}), \qquad \mathrm{CMT}_{\mathcal{P}} : \mathrm{SP}_{\mathcal{S}} \times \mathrm{T}_{\mathcal{D}} \times \mathrm{F}_{\mathcal{D}} \longrightarrow \mathrm{T}_{\mathcal{D}}. \tag{10}$$

The material flow returned by the procedure $\mathrm{OMF}_{\mathcal{P}}$ is reliable only if there is enough amount of material inflowing to the input of the process. If this inflowing material is represented by the data structure $mf_{\mathrm{in}} \in \mathrm{MF}_{\mathcal{S}}$ and the processing machine has the speed pattern $sp_{\mathrm{out}} \in \mathrm{SP}_{\mathcal{S}}$, the *earliest start time* t_{es} can be defined that ensures reliability of the procedure $\mathrm{OMF}_{\mathcal{P}}$

$$t_{\mathrm{es}} = \min \left\{ t \in [t_{\mathrm{s,in}}, t_{\mathrm{c,in}}] \,\Big|\, \underset{\tau \in [t, t_{\mathrm{c,in}}]}{\forall} mf_{\mathrm{in}}(\tau) + mf^{\mathrm{I}}_{\mathrm{out}}(t) - mf^{\mathrm{I}}_{\mathrm{out}}(\tau) \ge bf \cdot f_{\max} \right\}, \tag{11}$$

where $t_{\mathrm{s,in}}$ and $t_{\mathrm{c,in}}$ are the start and completion times of mf_{in}, $f_{\max}$ is the maximum flow of mf_{in} and $mf^{\mathrm{I}}_{\mathrm{out}}$ is the indefinite integral of sp_{out}. The parameter *bf* (*buffer*) forces some additional margin between material flows mf_{in} and mf_{out} such that the relation is more general and more adequate for industrial practice. This margin is expressed as a fraction of the maximum flow and thus it has the values restricted to the range $[0, 1]$. It is assumed that the value of maximum flow ($f_{\max}$) is the same for both material flows being in the in-out relation.

In this work, the material flow used as mf_{in} in calculations of t_{es} will always be equal to the output material flow of the preceding operation. Hence, mf_{in} can be considered as the result from some machine working with the speed pattern sp_{in} that has the indefinite integral $mf^{\mathrm{I}}_{\mathrm{in}}$, and the

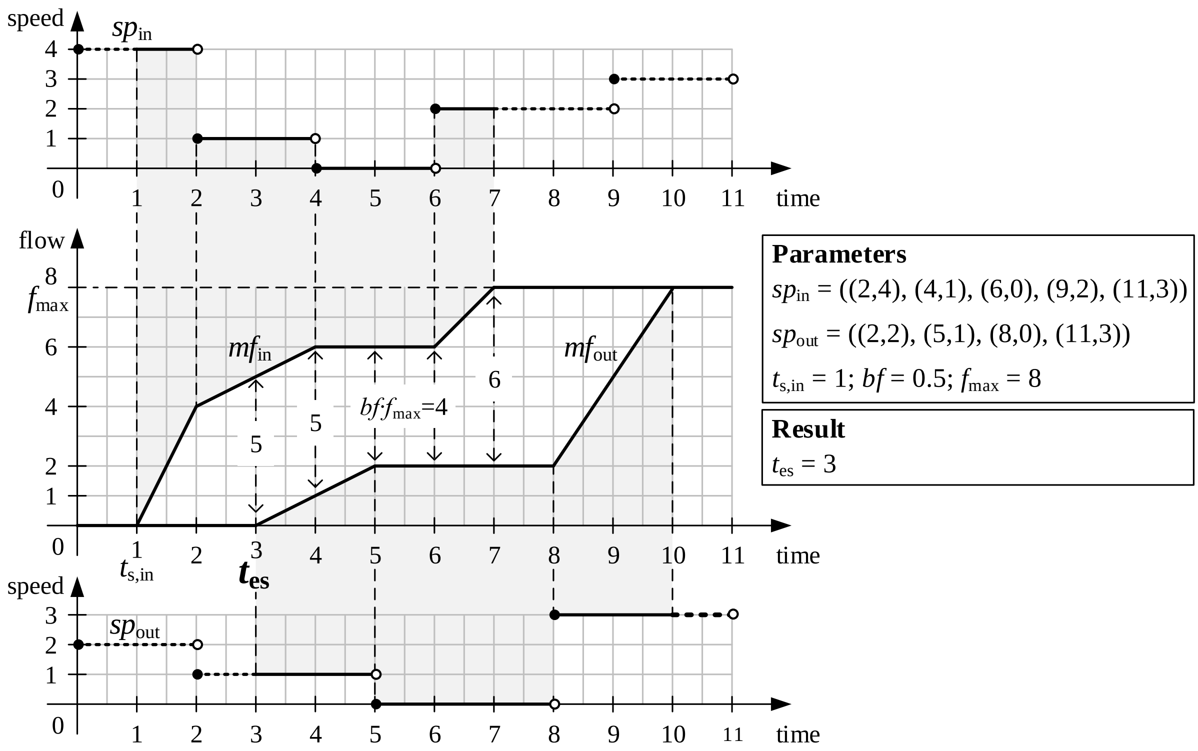

Figure 1. Interpretation of the earliest start time (t_{es}).

substitution is possible

$$mf_{\text{in}}(\tau) \longmapsto mf_{\text{in}}^{\text{I}}(\tau) - mf_{\text{in}}^{\text{I}}(t_{\text{s,in}})\,. \tag{12}$$

After this substitution, the relation (11) takes the form

$$t_{\text{es}} = \min\left\{ t \in [t_{\text{s,in}}, t_{\text{c,in}}] \;\middle|\; \underset{\tau\in[t,t_{\text{c,in}}]}{\forall} mf_{\text{in}}^{\text{I}}(\tau) - mf_{\text{in}}^{\text{I}}(t_{\text{s,in}}) + mf_{\text{out}}^{\text{I}}(t) - mf_{\text{out}}^{\text{I}}(\tau) \geq bf \cdot f_{\max} \right\}, \tag{13}$$

which is helpful in analysing the dependency between the start time $t_{\text{s,in}}$ of some operation and earliest possible start time t_{es} of a next operation in the processing sequence. An example is presented on Figure 1 which additionally explains relation (13). This relation is the base of the procedure $\text{EST}_{\mathcal{P}}$ (*earliest start time*) defined as

$$\text{EST}_{\mathcal{P}} : \text{T}_{\mathcal{D}} \times \text{SP}_{\mathcal{S}} \times \text{SP}_{\mathcal{S}} \times \text{D}_{\mathcal{D}}^{[0,1]} \times \text{F}_{\mathcal{D}} \longrightarrow \text{T}_{\mathcal{D}} \qquad t_{\text{es}} = \text{EST}_{\mathcal{P}}(t_{\text{s,in}}, sp_{\text{in}}, sp_{\text{out}}, bf, f_{\max})\,. \tag{14}$$

Finally, the *material flow model* is the 10-tuple of the previously introduced concepts

$$\text{MFM} = (\text{T}_{\mathcal{D}}, \text{F}_{\mathcal{D}}, \text{S}_{\mathcal{D}}, \text{D}_{\mathcal{D}}, \text{MF}_{\mathcal{S}}, \text{SP}_{\mathcal{S}}, \text{EP}_{\mathcal{S}}, \text{OMF}_{\mathcal{P}}, \text{CMT}_{\mathcal{P}}, \text{EST}_{\mathcal{P}})\,.$$

2. MFM properties

The following two properties of the MFM have a significant impact on features of scheduling problems defined on the basis of it.

MFM Property 1. The completion time of an operation is a non-decreasing function of its start time, i.e.

$$\underset{\substack{sp \in \mathrm{SP}_{\mathcal{S}} \\ t_{\mathrm{s}_1}, t_{\mathrm{s}_2} \in \mathrm{T}_{\mathcal{D}} \\ f_{\max} \in \mathrm{F}_{\mathcal{D}} \wedge f_{\max} > 0}}{\forall} \quad t_{\mathrm{s}_2} > t_{\mathrm{s}_1} \implies \mathrm{CMT}_{\mathcal{P}}\left(sp, t_{\mathrm{s}_2}, f_{\max}\right) \geq \mathrm{CMT}_{\mathcal{P}}\left(sp, t_{\mathrm{s}_1}, f_{\max}\right).$$

MFM Property 2. The earliest start time of an operation is a non-decreasing function of the start time of the previous one, i.e.

$$\underset{\substack{sp_{\mathrm{in}}, sp_{\mathrm{out}} \in \mathrm{SP}_{\mathcal{S}} \\ t_{\mathrm{s}_1}, t_{\mathrm{s}_2} \in \mathrm{T}_{\mathcal{D}},\, bf \in \mathrm{D}_{\mathcal{D}}^{[0,1]} \\ f_{\max} \in \mathrm{F}_{\mathcal{D}} \wedge f_{\max} > 0}}{\forall} \quad t_{\mathrm{s}_2} > t_{\mathrm{s}_1} \implies \mathrm{EST}_{\mathcal{P}}\left(t_{\mathrm{s}_2}, sp_{\mathrm{in}}, sp_{\mathrm{out}}, bf, f_{\max}\right) \geq \mathrm{EST}_{\mathcal{P}}\left(t_{\mathrm{s}_1}, sp_{\mathrm{in}}, sp_{\mathrm{out}}, bf, f_{\max}\right).$$

Property 1 reflects the fact that the function (9) is non-decreasing, as it has been noticed earlier. In classic scheduling problems processing times are usually constant, i.e. for any operation

$$\left[\underset{t_{\mathrm{s}}, t_{\mathrm{c}} \in \mathbb{T}}{\forall} p = t_{\mathrm{c}} - t_{\mathrm{s}} = \mathrm{const}\right] \implies \left[\underset{t_{\mathrm{s}_1}, t_{\mathrm{s}_2}, t_{\mathrm{c}_1}, t_{\mathrm{c}_2} \in \mathbb{T}}{\forall} t_{\mathrm{s}_2} > t_{\mathrm{s}_1} \implies t_{\mathrm{c}_2} > t_{\mathrm{c}_1}\right],$$

where p is processing time of the operation and $\mathbb{T}$ is the time domain of the relevant scheduling problem. In the problems formulated using the MFM only the weaker relation, given by Property 1, takes place.

Property 2 can be proved as follows

Proof. (MFM Property 2)
Let us assume that material inflowing to some process B as a result of a preceding operation A performed by a machine with speed pattern $sp_{\mathrm{in}} \in \mathrm{SP}_{\mathcal{S}}$, starts at the time $t_{\mathrm{s}_1} \in \mathrm{T}_{\mathcal{D}}$ and has the maximum flow $f_{\max} \in \mathrm{F}_{\mathcal{D}}$. The process B is to be performed by a machine with speed pattern $sp_{\mathrm{out}} \in \mathrm{SP}_{\mathcal{S}}$, and the minimum material buffer between the operations A and B is defined as $bf \in \mathrm{D}_{\mathcal{D}}^{[0,1]}$. Under these conditions, the earliest start time for the operation B equals

$$t_{\mathrm{es}_1} = \mathrm{EST}_{\mathcal{P}}\left(t_{\mathrm{s}_1}, sp_{\mathrm{in}}, sp_{\mathrm{out}}, bf, f_{\max}\right).$$

Let us select an optional $t_{\mathrm{s}_2} \in \mathrm{T}_{\mathcal{D}}$ such that

$$t_{\mathrm{s}_2} > t_{\mathrm{s}_1} \wedge t_{\mathrm{es}_2} = \mathrm{EST}_{\mathcal{P}}\left(t_{\mathrm{s}_2}, sp_{\mathrm{in}}, sp_{\mathrm{out}}, bf, f_{\max}\right). \tag{15}$$

From (13) we obtain

$$\underset{\tau \in \left[t_{\mathrm{es}_2}, t_{\mathrm{c}_2}\right]}{\forall} mf_{\mathrm{in}}^{\mathrm{I}}(\tau) - mf_{\mathrm{in}}^{\mathrm{I}}\left(t_{\mathrm{s}_2}\right) + mf_{\mathrm{out}}^{\mathrm{I}}\left(t_{\mathrm{es}_2}\right) - mf_{\mathrm{out}}^{\mathrm{I}}(\tau) \geq bf \cdot f_{\max}, \tag{16}$$

where $t_{c_2} = \mathrm{CMT}_{\mathcal{P}}\left(sp_{\mathrm{in}}, t_{s_2}, f_{\max}\right)$. On the basis of (16) the following also holds

$$\underset{\tau \in \left[t_{\mathrm{es}_2}, t_{c_1}\right]}{\forall} \; mf^{\mathrm{I}}_{\mathrm{in}}\left(\tau\right) - mf^{\mathrm{I}}_{\mathrm{in}}\left(t_{s_1}\right) + mf^{\mathrm{I}}_{\mathrm{out}}\left(t_{\mathrm{es}_2}\right) - mf^{\mathrm{I}}_{\mathrm{out}}\left(\tau\right) \geq bf \cdot f_{\max}, \tag{17}$$

because $t_{c_2} \geq t_{c_1}$ that results from (15) and MFM Property 1, as well as $mf^{\mathrm{I}}_{\mathrm{in}}\left(t_{s_2}\right) \geq mf^{\mathrm{I}}_{\mathrm{in}}\left(t_{s_1}\right)$, since $mf^{\mathrm{I}}_{\mathrm{in}}\left(\cdot\right)$ is non-decreasing. From the initial assumption and sentences (17), (13) we can conclude

$$\left[t_{\mathrm{es}_2}, t_{c_1}\right] \subseteq \left[t_{\mathrm{es}_1}, t_{c_1}\right] \implies t_{\mathrm{es}_2} \geq t_{\mathrm{es}_1}.$$

□

3. MFM implementation note

The algorithmic layer of the MFM is based on the procedures $\mathrm{OMF}_{\mathcal{P}}$, $\mathrm{CMT}_{\mathcal{P}}$ and $\mathrm{EST}_{\mathcal{P}}$. The procedures $\mathrm{OMF}_{\mathcal{P}}$ (*output material flow*) and $\mathrm{CMT}_{\mathcal{P}}$ (*completion time*) have identical list of arguments (sp, t_s, $f_{\max}$). In fact, the two procedures perform equivalent computations with the difference that $\mathrm{OMF}_{\mathcal{P}}$ returns a complete material flow function, whereas $\mathrm{CMT}_{\mathcal{P}}$ returns only a specially extracted parameter of this function. In both cases, simple computations are executed for integrating a piecewise constant time-varying function and finding the argument for which the resulting function reaches the value $f_{\max}$.

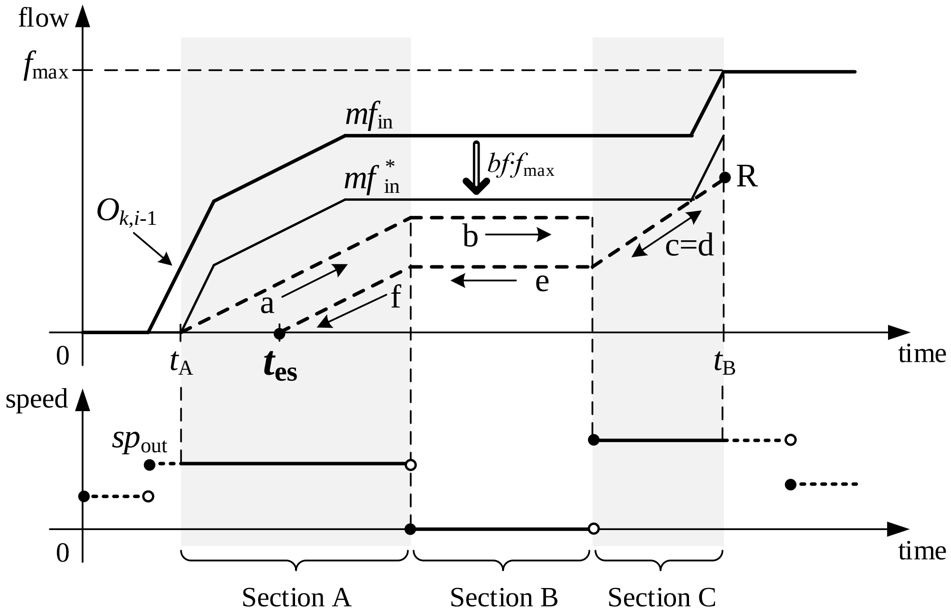

Figure 2. Graphical interpretation of the procedure $\mathrm{EST}_{\mathcal{P}}$ implementation.

The algorithm related to the procedure $\mathrm{EST}_{\mathcal{P}}$ is more complex. An example is presented on Figure 2 which shows graphical interpretation of the procedure implementation. The input data consists of the parameters: mf_{in}, sp_{out}, bf, $f_{\max}$. The input material flow mf_{in} is given arbitrary for simplicity, however, it can also be represented by input speed pattern sp_{in} and start time $t_{\mathrm{s,in}}$ of the operation $O_{k,i-1}$, as it is defined in (12) and (13). Hence, the input data is actually equal to that defined for the procedure $\mathrm{EST}_{\mathcal{P}}$ (14). The resulting value of the parameter t_{es} is obtained by means of the following operations

- Shift mf_{in} vertically down by the absolute value of minimal buffer, i.e. $bf \cdot f_{\max}$. Denote shifted plot by mf^{*}_{in}.
- Select the two time values: t_{A}, i.e. the maximum time for which mf^{*}_{in} has the value 0, and t_{B}, i.e the minimum time for which mf_{in} has the value $f_{\max}$.
- Split the interval $[t_{\mathrm{A}}, t_{\mathrm{B}}]$ into sections where the value of sp_{out} remains constant (sections A, B, C on Figure 2).
- For each section, from left to right, consecutively construct a line segment which has slope defined by sp_{out} and is located as high as possible, but respecting the following constraints
  - each point of the line segment has to be not higher than a point of mf^{*}_{in} for the same time argument (the constraint active for the line segments a and c on Figure 2), and also
  - the location of a line segment that leads to connection with preceding one is the highest possible for a line segment in each section, except for the first (the constraint active for the line segment b on Figure 2).
- Denote the last point in the last section as R.
- Construct a polyline starting from the point R by adding in each section, from right to left, a line segment with slope defined by sp_{out} (the line segments d, e and f on Figure 2). This polyline crosses the time axis at t_{es}.

The described algorithm can be easily implemented in a chosen programming language. In the implementation the geometric operations have to be substituted by analytic geometry transformations.