SUPPLEMENTARY MATERIAL TO "SPECTRAL INFERENCE UNDER COMPLEX TEMPORAL DYNAMICS"

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Abstract. This file contains additional lemmas and detailed proofs to the remarks, lemmas, and theorems of the paper.

Remark A.1. Denote $X_{u,i} := G(i/N, \mathcal{F}_{u,i})$ where $\mathcal{F}_{u,i} = (\ldots, \epsilon_{\lfloor uN \rfloor}, \epsilon_{\lfloor uN \rfloor+1}, \ldots, \epsilon_{\lfloor uN \rfloor+i})$. Let ϵ'_k be an i.i.d. copy of ϵ_k and $X'_{u,i} := G(i/N, \mathcal{F}'_{u,i})$ where $\mathcal{F}'_{u,i} = (\ldots, \epsilon'_0, \ldots, \epsilon'_{\lfloor uN \rfloor}, \epsilon_{\lfloor uN \rfloor+1}, \ldots, \epsilon_{\lfloor uN \rfloor+i})$ is a coupled version of $\mathcal{F}_{u,i}$. Then under GMC(p), $p > 0$, there exist $C > 0$ and $0 < \rho = \rho(p) < 1$ that do not depend on u, such that for any u and i, we have

$$\sup_u \mathbb{E}(|X'_{u,i} - X_{u,i}|^p) \leq C\rho^i. \tag{1}$$

This is because, when GMC(p) holds, we have $\sup_u \mathbb{E}(|X'_{u,i} - X_{u,i}|^p) \leq \sum_{k=i}^{\infty} \delta_p(k) \leq \mathcal{O}(\sum_{k=i}^{\infty} \rho^k) = \mathcal{O}(\rho^i)$.

Furthermore, it can be easily shown that if GMC(2) holds, then $\sup_u |r(u,k)| = \mathcal{O}(\rho^k)$ for some $\rho \in (0,1)$. Also, if $\sup_i \|X_i\|_p < \infty$ and GMC(α) holds with any given $\alpha > 0$, then X_i is GMC(α) with any $\alpha \in (0,p)$. In particular, if GMC(α) holds with some $\alpha \geq 2$, then we must have $\sup_u \sum_{k=-\infty}^{\infty} |r(u,k)| < \infty$ since $\sup_u |r(u,k)| = \mathcal{O}(\rho^k) = o(k^{-2})$. Also, if GMC(2) holds as well as $\sup_i \mathbb{E}(|X_i|^{4+\delta}) < \infty$ for some $\delta > 0$, then GMC(4) holds. ◁

Lemma A.2. *(Berry-Esseen) If $\{X_i, i \geq 1\}$ are independent random variables with $\mathbb{E}(X_i) = 0$, $s_n^2 = \sum_{i=1}^n \mathbb{E}(X_i^2) > 0$, $\sum_{i=1}^n \mathbb{E}|X_i|^{2+\delta} < \infty$, for some $\delta \in (0,1]$ and $S_n = \sum_{i=1}^n X_i$, there exists a universal constant C_δ such that*

$$\sup_{-\infty < x < \infty} |\mathbb{P}(S_n < xs_n) - \Phi(x)| \leq C_\delta \left(\frac{\sum_{i=1}^n \mathbb{E}|X_i|^{2+\delta}}{s_n^{2+\delta}} \right). \tag{2}$$

Proof. See [CT88, pp. 304]. □

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A.1. **Proof of Lemma 8.1.** Define $d_{u,n}(h) = \frac{1}{n}\sum_{k=1+h}^{n} \mu_{u,k}\mu_{u,k-h}$ for $0 \le h \le n-1$ and $d_{u,n}(h) = 0$ if $h \ge n$. Since

$$\sum_{k=1}^{n} \cos(k\theta_{j_\ell})\cos((k+h)\theta_{j_{\ell'}}) = \frac{n}{2}\cos(h\theta_{j_\ell})\mathbf{1}_{\{j_\ell = j_{\ell'}\}}, \tag{3}$$

using

$$\begin{aligned} d_{u,n}(h) &= \frac{1}{n}\sum_{k=1+h}^{n+h} \mu_{u,k}\mu_{u,k-h} - \frac{1}{n}\sum_{k=n+1}^{n+h} \mu_{u,k}\mu_{u,k-h} \\ &= \sum_{\ell=1}^{p} c_\ell^2 \frac{\cos(h\theta_{j_\ell})}{2\pi f(u,\theta_{j_\ell})} - \frac{1}{n}\sum_{k=n+1}^{n+h} \mu_{u,k}\mu_{u,k-h}, \end{aligned} \tag{4}$$

we get that uniformly over J, c and u, there exists K_0 such that

$$\left| d_{u,n}(h) - \sum_{\ell=1}^{p} c_\ell^2 \frac{\cos(h\theta_{j_\ell})}{2\pi f(u,\theta_{j_\ell})} \right| \le K_0 \min\left\{\frac{h}{n}, 1\right\}. \tag{5}$$

Next, we can write $\|T_{u,n}\|^2/n$ as

$$\begin{aligned} &\frac{1}{n}\mathbb{E}\left(\sum_{k=1}^{N} \mu_{u,k}\tau\left(\frac{k-\lfloor uN\rfloor}{n}\right)X_k\right)^2 \\ &= d_{u,n}(0)r(u,0)\left[\frac{1}{n}\sum_k \tau\left(\frac{k-\lfloor uN\rfloor}{n}\right)^2\right] \\ &+ 2\sum_{h=1}^{\infty} d_{u,n}(h)r(u,h)\left[\frac{1}{n}\sum_k \tau\left(\frac{k-\lfloor uN\rfloor}{n}\right)\tau\left(\frac{k+h-\lfloor uN\rfloor}{n}\right)\right] + o(1). \end{aligned} \tag{6}$$

Furthermore, defining

$$f_n(u,\theta) := \frac{1}{2\pi}\sum_{h=0}^{\infty} r(u,h)\cos(h\theta)\left[\frac{1}{n}\sum_k \tau\left(\frac{k-\lfloor uN\rfloor}{n}\right)\tau\left(\frac{k+h-\lfloor uN\rfloor}{n}\right)\right], \tag{7}$$

we have that

$$\sum_h \left\{ \left[\frac{1}{n} \sum_k \tau\left(\frac{k - \lfloor uN \rfloor}{n}\right) \tau\left(\frac{k+h-\lfloor uN \rfloor}{n}\right) \right] r(u,h) \sum_{\ell=1}^p c_\ell^2 \frac{\cos(h\theta_{j_\ell})}{2\pi f(u,\theta_{j_\ell})} \right\}$$

$$= \sum_{\ell=1}^p \frac{c_\ell^2}{2\pi f(u,\theta_{j_\ell})} \sum_h \left\{ r(u,h)\cos(h_{\theta_{j_\ell}}) \left[\frac{1}{n} \sum_k \tau\left(\frac{k - \lfloor uN \rfloor}{n}\right) \tau\left(\frac{k+h-\lfloor uN \rfloor}{n}\right) \right] \right\} \tag{8}$$

$$= \sum_{\ell=1}^p c_\ell^2 \frac{f_n(u,\theta_{j_l})}{f(u,\theta_{j_l})}.$$

By the assumptions that $\tau \in \mathcal{C}^1([-1/2,1/2])$, $\int \tau^2(x)\mathrm{d}x = 1$, together with $\sup_u |r(u,h)| = o(h^{-2})$, and $\sum_{h=1}^\infty |r(u,h)| < \infty$, we have $f_n(u,\theta) = f(u,\theta) + o(1)$, uniformly over u and θ. This implies that

$$\sum_{\ell=1}^p c_\ell^2 \frac{f_n(u,\theta_{j_l})}{f(u,\theta_{j_l})} = \sum_{\ell=1}^p c_\ell^2 + o(1) = 1 + o(1). \tag{9}$$

Therefore, uniformly over J and c, we have that

(10)

$$\left| \frac{\|T_{u,n}\|^2}{n} - 1 \right| - o(1)$$

$$\leq 2 \sum_{h=0}^\infty \left| d_{u,n}(h) - \sum_{\ell=1}^p c_\ell^2 \frac{\cos(h\theta_{j_\ell})}{2\pi f_n(u,\theta_{j_\ell})} \right| r(u,h) \left[\frac{1}{n} \sum_k \tau\left(\frac{k - \lfloor uN \rfloor}{n}\right) \tau\left(\frac{k+h-\lfloor uN \rfloor}{n}\right) \right]$$

$$\leq 2 \sum_{h=0}^\infty K_0 \min\left\{\frac{h}{n}, 1\right\} r(u,h) \left[\frac{1}{n} \sum_k \tau\left(\frac{k - \lfloor uN \rfloor}{n}\right) \tau\left(\frac{k+h-\lfloor uN \rfloor}{n}\right) \right].$$

Finally, since $\sup_u \sum_h |r(u,h)| < \infty$, we have $\sup_u \sum_{h>n} |r(u,h)| \to 0$. Also, as $n \to \infty$,

$$\begin{aligned} \sup_u \sum_{h<n} (h/n) r(u,h) &\leq \sup_u \sum_{h<\sqrt{n}} (h/n) r(u,h) + \sup_u \sum_{\sqrt{n}\leq h<n} (h/n) r(u,h) \\ &\leq \sup_u \sum_{h<\sqrt{n}} r(u,h)/\sqrt{n} + \sup_u \sum_{h>\sqrt{n}} r(u,h) \\ &\to 0. \end{aligned} \tag{11}$$

Therefore, $\left| \frac{\|T_{u,n}\|^2}{n} - 1 \right| \to 0$.

A.2. **Proof of Lemma 8.2.** Throughout the proof, we write $\tilde{X}_k^{[\ell]}$ as $\tilde{X}_k$ for short. Note that $\int \tau(x)\tau(x+h)\mathrm{d}x \le \frac{1}{2}\int [\tau(x)^2+\tau(x+h)^2]\,\mathrm{d}x = 1$. For simplicity of the proof, we can assume that there exists some finite τ_* such that

$$\frac{1}{n}\sum_k \tau\left(\frac{k-\lfloor uN\rfloor}{n}\right)\tau\left(\frac{k+h-\lfloor uN\rfloor}{n}\right) \le \tau_*^2. \tag{12}$$

Then we have that

(13)

$$\begin{aligned}
&\frac{\|T_{u,n}-\tilde{T}_{u,n}\|}{\sqrt{n}} = \left[\frac{1}{n}\sum_{j=-\infty}^{\lfloor uN+n/2\rfloor}\|\mathcal{P}_j(T_{u,n}-\tilde{T}_{u,n})\|^2\right]^{1/2} \\
&\le \mu_*\tau_*\left[\frac{1}{n}\sum_{k=1}^{n}\sum_{j=-\infty}^{\lfloor uN+n/2\rfloor}\|\mathcal{P}_j(X_{\lfloor uN\rfloor+k-\lfloor n/2\rfloor}-\tilde{X}_{\lfloor uN\rfloor+k-\lfloor n/2\rfloor})\|^2\right]^{1/2} \\
&\le \mu_*\tau_*\max_{k\in\{1,\dots,n\}}\sum_{j=-\infty}^{\lfloor uN+n/2\rfloor}\|\mathcal{P}_j(X_{\lfloor uN\rfloor+k-\lfloor n/2\rfloor}-\tilde{X}_{\lfloor uN\rfloor+k-\lfloor n/2\rfloor})\| \\
&\le \mu_*\tau_*\max_{k\in\{1,\dots,n\}}\sum_{j=-\infty}^{\lfloor uN+n/2\rfloor}\min\left\{2\|\mathcal{P}_j(X_{\lfloor uN\rfloor+k-\lfloor n/2\rfloor})\|,\|X_{\lfloor uN\rfloor+k-\lfloor n/2\rfloor}-\tilde{X}_{\lfloor uN\rfloor+k-\lfloor n/2\rfloor}\|\right\} \\
&\le \mu_*\tau_*\sup_k\sum_{j=-\infty}^{k+n}\min\{2\|\mathcal{P}_j(X_k)\|,\|X_k-\tilde{X}_k\|\}\to 0, \quad \text{as } \ell\to\infty.
\end{aligned}$$

Since the upper bound does not depend on u, the convergence holds uniformly over u.

A.3. **Proof of Lemma 8.3.** In this proof, we omit subscript u for simplicity and write $\tilde{X}_k^{[\ell]}$ as $\tilde{X}_k$ for short. Since $\sup_k \mathbb{E}(X_k^2)<\infty$, we have that

$$\lim_{t\to\infty}\sup_k \mathbb{E}[X_k^2\mathbf{1}(|X_k|>t)] = 0. \tag{14}$$

By the property of conditional expectation, we have $\mathbb{E}(\tilde{X}_k^2)<\mathbb{E}(X_k^2)$. Therefore, defining

$$g_n(r) = r^2\sup_k \mathbb{E}[\tilde{X}_k^2\mathbf{1}(|\tilde{X}_k|\ge\sqrt{n}/r)], \tag{15}$$

we can get $\lim_{n\to\infty} g_n(r)=0$ for all given $r>0$. Also g_n is non-decreasing with r. Then there exists a sequence $\{r_n\}$ such that $r_n\uparrow\infty$ and $g_n(r_n)\to 0$. Note that r_n does not depend on u.

For simplicity, we use $\tilde{X}_{u,k}$ to denote $\tilde{X}_{\lfloor uN \rfloor + k - \lfloor n/2 \rfloor}$. Let $Y_{u,k} = \tilde{X}_{u,k}\mathbf{1}(|\tilde{X}_{u,k}| \leq \sqrt{n}/r_n)$ and $T_{u,n,Y} = \sum_{k=1}^{n} \mu_{u,k} Y_{u,k}$. Since $\mathbb{E}[\tilde{X}_k^2 \mathbf{1}(|\tilde{X}_k| \geq \sqrt{n}/r)] = o(1/r_n^2)$ by the definition of r_n, we have $\|Y_{u,k} - \tilde{X}_{u,k}\| = o(1/r_n)$. Now since $Y_{u,k} - \tilde{X}_{u,k}$ is ℓ-dependent, we divide each of $\{Y_{u,k}\}$ and $\{\tilde{X}_{u,k}\}$ into ℓ sub-sequences that each sub-sequences has $\lfloor n/\ell \rfloor$ independent elements. Then by the triangle inequality we can get

$$\|T_{u,n,Y} - \tilde{T}_{u,n}\| \leq \sum_{a=1}^{\ell} \left\| \sum_{b=a,a+\ell,\dots}^{n} \mu_{u,b}(Y_{u,b} - \tilde{X}_{u,b}) \right\| = o(\sqrt{n}/r_n). \tag{16}$$

Next, divide the sequence of $\{Y_{u,k}\}$ into pieces of length $p_n + \ell$ where $p_n = \lfloor r_n^{1/4} \rfloor$.

$$U_{u,t} = \sum_{a \in B_t} \mu_{u,a} Y_{u,a} \tag{17}$$

where $B_t = \{a \in \mathbb{N} : 1 + (t-1)(p_n + \ell) \leq a \leq p_n + (t-1)(p_n + \ell)\}$. Note that for given u, $\{U_{u,t}\}$ are independent (but not identically distributed) for different t.

Define $V_{u,t} = \sum_{t=1}^{t_n} U_{u,t}$, then the difference between $V_{u,t}$ and $T_{u,n,Y}$ is the sum of those dropped ℓ terms in each piece. Since ℓ is fixed and there are t_n blocks, we have $\|T_{u,n,Y} - V_{u,t}\| = \mathcal{O}(\sqrt{t_n})$.

Furthermore, since

$$(\sqrt{n}/r_n)^2 \mathbb{P}(|\tilde{X}_k| \geq \sqrt{n}/r)) \geq \mathbb{E}[\tilde{X}_k^2 \mathbf{1}(|\tilde{X}_k| \geq \sqrt{n}/r)] = o(1/r_n^2) \tag{18}$$

we have $P(|\tilde{X}_k| \geq \sqrt{n}/r)) = o(1/n)$. Then, using

$$\begin{aligned} [\mathbb{E}(Y_k)]^2 &= [\mathbb{E}(\tilde{X}_k) - \mathbb{E}(Y_k)]^2 = [\mathbb{E}\tilde{X}_k \mathbf{1}(|\tilde{X}_k| \geq \sqrt{n}/r)]^2 \\ &\leq \mathbb{E}(\tilde{X}_k^2 \mathbf{1}(|\tilde{X}_k| \geq \sqrt{n}/r))\mathbb{P}(|\tilde{X}_k| \geq \sqrt{n}/r)) = o(1/r_n^2)o(1/n) \end{aligned} \tag{19}$$

we have $\mathbb{E}(Y_k) = o(\frac{1}{\sqrt{n} r_n})$, which implies $|\mathbb{E}(V_n)| = O(n)|\mathbb{E}(Y_k)| = o(\sqrt{n}/r_n)$.

Next, defining $W = (V_n - \mathbb{E}(V_n))/\sqrt{n}$ and $\Delta = \tilde{T}_n/\sqrt{n} - W$, we get

$$\begin{aligned} \sqrt{n}\|\Delta\| = \|\tilde{T}_n - V_n + \mathbb{E}(V_n)\| &\leq |\mathbb{E}(V_n)| + \|V_n - \tilde{T}_n\| \\ &\leq |\mathbb{E}(V_n)| + \|V_n - T_{n,Y}\| + \|T_{n,Y} - \tilde{T}_n\| \\ &= o(\sqrt{n}/r_n) + \mathcal{O}(\sqrt{t_n} + \sqrt{n}/r_n) = \mathcal{O}(\sqrt{t_n}). \end{aligned} \tag{20}$$

Next, we apply Lemma A.2 to $\{U_t - \mathbb{E}(U_t), t = 1, \ldots, t_n\}$. Recall that $V_n = \sum_{t=1}^{t_n} U_t$ and $W = (V_n - \mathbb{E}(V_n))/\sqrt{n}$, then

$$
\begin{aligned}
&\sup_x |\mathbb{P}(V_n - \mathbb{E}(V_n) < x\|V_n - \mathbb{E}(V_n)\|) - \Phi(x)| \\
&= \sup_x |\mathbb{P}(W < x\|W\|) - \Phi(x)| \\
&\leq C\sum_{t=1}^{t_n} \mathbb{E}|U_t - \mathbb{E}(U_t)|^3 \|V_n - \mathbb{E}(V_n)\|^{-3} \\
&\leq C\sum_{t=1}^{t_n} \mathbb{E}|U_t|^3 \|V_n - \mathbb{E}(V_n)\|^{-3}.
\end{aligned} \tag{21}
$$

Next, we get upper bounds of $\mathbb{E}|U_t|^3$ and $\|V_n - \mathbb{E}(V_n)\|^{-3}$. First, by Hölder's inequality $\sum_{a\in B_t} |Y_a| \leq (\sum_{a\in B_t} |Y_a|^3)^{1/3}(\sum_{a\in B_t} 1)^{2/3}$, we have that

$$
\mathbb{E}|U_t|^3 \leq \mu_*^3\mathbb{E}\left|\sum_{a\in B_t} Y_a\right|^3 \leq \mu_*^3 p_n^2 \sum_{a\in B_t} \mathbb{E}|Y_a|^3 \leq \mu_*^3 p_n^2 \sum_{a\in B_t} \mathbb{E}(\frac{\sqrt{n}}{r_n}|Y_a|^2) = \mathcal{O}\left(\mu_*^3 p_n^3 \frac{\sqrt{n}}{r_n}\right). \tag{22}
$$

For sequences a_n and b_n, we define $a_n = \Theta(b_n)$ if both $a_n = \mathcal{O}(b_n)$ and $b_n = \mathcal{O}(a_n)$. Then, using the definition of $\Theta(\cdot)$, the variance of $\sum_{a\in B_t} \mu_a Y_a$ has the order of $\Theta(p_n)$ because Y_a is ℓ-dependent. Then the variance of V_n has the order of $\Theta(t_n p_n) = \Theta(n)$. Thus, $\|V_n - \mathbb{E}(V_n)\|^{-3}$ has an order of $\Theta(n^{-3/2})$. Overall, we have that

$$
\sup_x |\mathbb{P}(W < x\|W\|) - \Phi(x)| \leq \mathcal{O}(\mu_*^3 p_n^3(\sqrt{n}/r_n))\Theta(n) = \mathcal{O}(p_n^{-2}). \tag{23}
$$

To complete the proof, we first replace $V_n = \sum_t \sum_{a\in B_t} \mu_a Y_a$ by $\tilde{T}_n = \sum_k \mu_k \tilde{X}_k$ then by $T_n = \sum_k X_k$. Since

$$
\{W \leq x - \delta, |\Delta| < \delta\} \subseteq \{W + \Delta \leq x\} \subseteq \{W \leq x + \delta\} \cup \{|\Delta| \geq \delta\}, \tag{24}
$$

we have that

$$
\mathbb{P}(W \leq x - \delta) - \mathbb{P}(|\Delta| \geq \delta) \leq \mathbb{P}(W + \Delta \leq x) \leq \mathbb{P}(W \leq x + \delta) + \mathbb{P}(|\Delta| \geq \delta). \tag{25}
$$

Furthermore, one can get

$$
\begin{aligned}
&\sup_x |\mathbb{P}(W < x\|W\|) - \Phi(x)| \\
&= \sup_x |\mathbb{P}(W < x) - \Phi(x/\|W\|)| \\
&= \sup_x \left|\mathbb{P}(\tilde{T}_n/\sqrt{n} - \Delta < x) - \Phi(x/\|W\|)\right|.
\end{aligned} \tag{26}
$$

Using

$$
\mathbb{P}(W < x - \delta) - \mathbb{P}(|\Delta| \geq \delta) \leq \mathbb{P}(\tilde{T}_n/\sqrt{n} < x) \leq \mathbb{P}(W < x + \delta) + \mathbb{P}(|\Delta| \geq \delta), \tag{27}
$$

we get

$$
\sup_x \left|\mathbb{P}(\tilde{T}_n/\sqrt{n} < x) - \mathbb{P}(W < x)\right| \leq \mathbb{P}(|\Delta| \geq \delta) = \mathcal{O}(\|\Delta\|^2/\delta^2) = \mathcal{O}(p_n^{-1}/\delta^2). \tag{28}
$$

Also

$$
\begin{aligned}
&\sup_x |\Phi(x/\|W\|) - \phi(x/\|W + \Delta\|)| \\
&= \mathcal{O}(\|W + \Delta\|/\|W\| - 1) = \mathcal{O}(\|\Delta\|) = \mathcal{O}(\sqrt{t_n/n}) = \mathcal{O}(p_n^{-1/2}).
\end{aligned} \tag{29}
$$

Letting $\delta = p_n^{-1/4}$ we have that

$$
\sup_x \left|\mathbb{P}(\tilde{T}_n/\sqrt{n} < x) - \Phi(x/\|W + \Delta\|)\right| = \mathcal{O}(p_n^{-2}) + \mathcal{O}(p_n^{-1/2}) + \mathcal{O}(p_n^{-1/2}). \tag{30}
$$

Finally, use the above technique again with $\Delta_1 = (T_n - \tilde{T}_n)/\sqrt{n}$ and $\delta = \|\Delta_1\|^{1/2}$, we get

$$
\sup_x \left|\mathbb{P}(T_n/\sqrt{n} < x) - \Phi(\sqrt{n}x/\|T_n\|)\right| = \mathcal{O}(\mathbb{P}(|\Delta_1| \geq \|\Delta_1\|^{1/2}) + p_n^{-1/2} + \|\Delta_1\|). \tag{31}
$$

Lemma A.3. *(Bernstein's inequality) Let $X_1, \ldots, X_n$ be independent zero-mean random variables. Suppose $|X_i| \leq M$ a.s., for all i. Then for all positive t,*

$$
\mathbb{P}\left(\sum_i X_i > t\right) \leq \exp\left(\frac{-\frac{1}{2}t^2}{\sum \mathbb{E}(X_i^2) + \frac{1}{3}Mt}\right). \tag{32}
$$

Definition A.4. Let $(U_1, \ldots, U_k)$ be a random vector. Then the joint cumulant is defined as

$$
\operatorname{cum}(U_1, \ldots, U_k) = \sum (-1)^p (p-1)! \mathbb{E}\left(\prod_{j \in V_1} U_j\right) \ldots \mathbb{E}\left(\prod_{j \in V_p} U_j\right), \tag{33}
$$

where $V_1, \dots, V_p$ is a partition of the set $\{1, 2, \dots, k\}$ and the sum is taken over all such partitions.

Lemma A.5. *Assume GMC(α) with $\alpha = k$ for some $k \in \mathbb{N}$, and* $\sup_t \mathbb{E}(|X_t|^k) < \infty$ *Then there exists a constant $C > 0$ such that for all u and $0 \le m_1 \le \cdots \le m_{k-1}$,*

$$|cum(X_{u,0}, X_{u,m_1}, \dots, X_{u,m_{k-1}})| \le C\rho^{m_{k-1}/[k(k-1)]}, \tag{34}$$

where $X_{u,i} := \tau\left(\frac{i-\lfloor n\rfloor/2\rfloor}{n}\right) X_{\lfloor uN\rfloor+i-\lfloor n/2\rfloor}$.

Proof. Since $\tau(\cdot)$ is bounded, we have $\sup_u \sup_i \mathbb{E}(|X_{u,i}|^k) < \infty$. We extend [WS04, Proposition 2] to the cases of locally stationary time series.

Given $1 \le l \le k-1$, by multi-linearity of joint cumulants, we replace X_{u,m_i} by independent X'_{u,m_i} for all $i \ge l$ as follows

$$\begin{aligned} J &:= \mathrm{cum}(X_{u,0}, X_{u,m_1}, \dots, X_{u,m_{k-1}}) \\ &= \mathrm{cum}(X_{u,0}, X_{u,m_1}, \dots, X_{u,m_{l-1}}, X'_{u,m_l}, \dots, X'_{u,m_{k-1}}) \\ &\quad + \mathrm{cum}(X_{u,0}, X_{u,m_1}, \dots, X_{u,m_{l-1}}, X_{u,m_l} - X'_{u,m_l}, \dots, X_{u,m_{k-1}}) \\ &\quad \dots \\ &\quad + \mathrm{cum}(X_{u,0}, X_{u,m_1}, \dots, X_{u,m_{l-1}}, X'_{u,m_l}, \dots, X_{u,m_{k-1}} - X'_{u,m_{k-1}}) \\ &=: B + \sum_{i=l}^{k-1} A_i. \end{aligned} \tag{35}$$

Note that $(X_{u,0}, X_{u,m_1}, \dots, X_{u,m_{l-1}})$ is independent with $(X'_{u,m_l}, \dots, X'_{u,m_{k-1}})$. By [Ros85, pp.35], we have $B = 0$. Suppose we have that

$$|A_i| \le \frac{C}{k}\rho^{(m_i-m_{l-1})/k} \le \frac{C}{k}\rho^{(m_l-m_{l-1})/k} \tag{36}$$

for $l \le i \le k-1$ and some constant C that does not depend on l. Then $|J| \le C\rho^{(m_l-m_{l-1})/k}$ for any $1 \le l \le k-1$. Then we get

$$|J| \le C \min_l \rho^{(m_l-m_{l-1})/k} = C\rho^{\max_l \frac{m_l-m_{l-1}}{k}} \le C\rho^{m_{k-1}/k(k-1)}. \tag{37}$$

Next, we show Eq. (36). In particular, we show the case $i = l$ and the other cases can be proven similarly. Note that $\mathbb{E}(|X_{u,i}|^k)$ is uniformly bounded, by the definition of joint cumulants in Definition A.4, we only need to show that for $V \subset \{0, \dots, k-1\}$ such that

$l \notin V$, we have that

$$\mathbb{E}\left((X_{u,m_l} - X'_{u,m_l}) \prod_{j\in V} X_{u,m_j}\right) \le C\rho^{(m_l - m_{l-1})/k}. \tag{38}$$

Letting $|V|$ be the cardinality of the set V, then $|V| \le k-1$, and we have

$$\begin{aligned} \left|\mathbb{E}\left[\left(\prod_{j\in V} X_{u,m_j}\right)^{\frac{1+|V|}{|V|}}\right]\right| &\le \mathbb{E}\left[\left(\frac{1}{|V|}\sum_{j\in V} |X_{u,m_j}|^{|V|}\right)^{\frac{1+|V|}{|V|}}\right] \\ &\le \mathbb{E}\left(\frac{1}{|V|}\sum_{j\in V} |X_{u,m_j}|^{1+|V|}\right) \\ &\le \max_{j\in V} \mathbb{E}\left(|X_{u,m_j}|^{1+|V|}\right) \le M. \end{aligned} \tag{39}$$

By Hölder's inequality and Jensen's inequality

$$\begin{aligned} &\left|\mathbb{E}\left((X_{u,m_l} - X'_{u,m_l}) \prod_{j\in V} X_{u,m_j}\right)\right| \\ &\le \left\|X_{u,m_l} - X'_{u,m_l}\right\|_{1+|V|} \left\|\prod_{j\in V} X_{u,m_j}\right\|_{\frac{1+|V|}{|V|}} \\ &\le \left\|X_{u,m_l} - X'_{u,m_l}\right\|_k M^{\frac{|V|}{1+|V|}} \le (C'\rho^{m_l - m_{l-1}})^{1/k} M' \le C\rho^{(m_l-m_{l-1})/k}. \end{aligned} \tag{40}$$

□

A.4. **Proof of Lemma 8.4.** Throughout this proof, we write $\tilde{X}_k^{[\ell]}$ as $\tilde{X}_k$ and $\tilde{X}_{u,i}^{[\ell]}$ as $\tilde{X}_{u,i}$ for short. First, letting $\alpha_k = a(k/B_n)\cos(k\theta)$, we have that

$$h_n(u,\theta) = \frac{1}{2\pi\sqrt{nB_n}}\left(\sum_{k=0}^{B_n}\sum_{j=n-k+1}^{n} X_{u,j}X_{u,j+k}\alpha_k + \sum_{k=-B_n}^{-1}\sum_{j=n+k+1}^{n} X_{u,j}X_{u,j+k}\alpha_k\right). \tag{41}$$

By the summability of cumulants of orders 2 and 4 [Ros85, page 185], one can get

$$\sup_\theta \sup_u \operatorname{var}\left(\sum_{k=0}^{B_n}\sum_{j=n-k+1}^{n} X_{u,j}X_{u,j+k}\alpha_k\right) = \mathcal{O}(B_n^2). \tag{42}$$

Therefore, we have $\sup_\theta \sup_u \|h_n(u,\theta)\| = (nB_n)^{-1/2}\mathcal{O}(B_n)$.

Next, note that by the assumption of GMC(4) defined in Eq. (1), we have that

$$\sup_u \sup_i \mathbb{E}(|X_{u,i} - \tilde{X}_{u,i}|^4) \leq C\rho^{\ell_n}. \tag{43}$$

Then we have

$$\begin{aligned}
\sup_\theta \sup_u \sup_i \|Y_{u,i} - \tilde{Y}_{u,i}\| &\leq \sup_u \sup_i \frac{1}{2\pi} \sum_{k=-B_n}^{B_n} \|X_{u,i}X_{u,i+k} - \tilde{X}_{u,i}\tilde{X}_{u,i+k}\|(\sup_\theta |\alpha_k|) \\
&\leq C \sup_u \sup_i \sum_{k=-B_n}^{B_n} \|(X_{u,i} - \tilde{X}_{u,i})X_{u,i+k} + \tilde{X}_{u,i}(X_{u,i+k} - \tilde{X}_{u,i+k})\| \\
&= \mathcal{O}(B_n) \sup_u \sup_i \|X_{u,i} - \tilde{X}_{u,i}\| \\
&= \mathcal{O}(B_n) \sup_u \sup_i (\mathbb{E}(|X_{u,i} - \tilde{X}_{u,i}|)^4)^{1/4} \\
&= \mathcal{O}(B_n\rho^{\ell_n/4}).
\end{aligned} \tag{44}$$

Finally

$$\sup_\theta \sup_u \|g_n(u,\theta) - \tilde{g}_n(u,\theta)\| = \mathcal{O}\left(\sup_\theta \sup_u \sum_{i=1}^n \|Y_{u,i} - \tilde{Y}_{u,i}\|\right) = \mathcal{O}(nB_n\rho^{\ell_n/4}) = o(1). \tag{45}$$

A.5. **Proof of Lemma 8.5.** We write $\tilde{X}_k^{[\ell]}$ as $\tilde{X}_k$ and $\tilde{X}_{u,i}^{[\ell]}$ as $\tilde{X}_{u,i}$ for short. To show Eq. (72), since α_k is bounded, letting $z_n = k_n(p_n + q_n) + 1 - q_n$, we have that

$$\sup_u \mathbb{E}(\max_\theta |V_{u,k_n}(\theta)|) \leq C \sum_{j=-B_n}^{B_n} \sup_u \mathbb{E}|\sum_{i=z_n}^n \tilde{X}_{u,i}\tilde{X}_{u,i+j}|. \tag{46}$$

Since $\tilde{X}_{u,i}\tilde{X}_{u,i+j}$ is $2\ell_n$-dependent, if $|j| < \ell_n$, we have

$$\sup_u \left\|\sum_{i=z_n}^n \tilde{X}_{u,i}\tilde{X}_{u,i+j}\right\| = \mathcal{O}(2\ell_n\sqrt{(n-z_n)/2\ell_n}) = \mathcal{O}(\sqrt{q_n\ell_n}) = \mathcal{O}(\sqrt{p_n\ell_n}). \tag{47}$$

If $|j| \le \ell_n$, since $\mathbb{E}(\tilde{X}_{u,i}\tilde{X}_{u,i+j}\tilde{X}_{u,i'}\tilde{X}_{u,i'+j}) = 0$ if $|i-i'| > \ell_n$, we have that

$$\begin{aligned}\sup_u \left\| \sum_{i=z_n}^{n} \tilde{X}_{u,i}\tilde{X}_{u,i+j} \right\|^2 &= \sup_u \sum_{i,i'=z_n}^{n} \mathbb{E}(\tilde{X}_{u,i}\tilde{X}_{u,i+j}\tilde{X}_{u,i'}\tilde{X}_{u,i'+j}) \\ &= \sup_u \sum_{i'=i-\ell_n}^{i+\ell_n} \sum_{i=z_n}^{n} \mathbb{E}(\tilde{X}_{u,i}\tilde{X}_{u,i+j}\tilde{X}_{u,i'}\tilde{X}_{u,i'+j}) \\ &= \mathcal{O}(q_n\ell_n) = \mathcal{O}(p_n\ell_n),\end{aligned} \tag{48}$$

where we have used the assumption $\sup_i \mathbb{E}(|X_i|^{4+\delta}) < M$. Therefore, we get Eq. (72).

To show Eq. (73), we first define $\tilde{h}_n(u,\theta)$ by replacing X_i by $\tilde{X}_i$. Then we can prove similarly to Eq. (44) that

$$\sup_u \mathbb{E}(\max_\theta |h_n(u,\theta) - \tilde{h}_n(u,\theta)|) = o(1). \tag{49}$$

Therefore, it suffices to show $\sup_u \mathbb{E}(\max_\theta |\tilde{h}_n(u,\theta)|) = o(1)$. Using similar technique to Eq. (46) we can show that

$$\sup_u \mathbb{E}(\max_\theta |\tilde{h}_n(u,\theta)|) = \frac{1}{\sqrt{nB_n}}\mathcal{O}(\sqrt{B_n\ell_n}B_n) = \mathcal{O}(\sqrt{\ell_n}B_n/\sqrt{n}) = o(1), \tag{50}$$

where we have used $\eta < \frac{1}{2}$ and $\sqrt{\ell_n}B_n/\sqrt{n} = \mathcal{O}((\log n)^{1/2}n^{\eta-1/2}) = o(1)$.

To show Eq. (74), we note that GMC(4) implies the absolute summability of cumulants up to the fourth order. Also, for zero-mean random variables X, Y, Z, W, the joint cumulants

$$\operatorname{cum}(X,Y,Z,W) = \mathbb{E}(XYZW) - \mathbb{E}(XY)\mathbb{E}(ZW) - \mathbb{E}(XZ)\mathbb{E}(YW) - \mathbb{E}(XW)\mathbb{E}(YZ). \tag{51}$$

Therefore, letting $\mathcal{L}_r$ be the set of the indices i's such that $Y_{u,i}$ belongs to the block corresponding to $U_{u,r}$, we have that

$$\begin{aligned}
\mathrm{var}(U_{u,r}(\theta)) &= \left\| \sum_{i\in\mathcal{L}_r} \sum_{k=-B_n}^{B_n} [X_{u,i}X_{u,i+k} - \mathbb{E}(X_{u,i}X_{u,i+k})]\alpha_k \right\|^2 \\
&= \sum_{i,j\in\mathcal{L}_r} \sum_{k,l=-B_n}^{B_n} \mathbb{E}\{[X_{u,i}X_{u,i+k} - \mathbb{E}(X_{u,i}X_{u,i+k})][X_{u,j}X_{u,j+l} - \mathbb{E}(X_{u,j}X_{u,j+l})]\alpha_k\alpha_l\} \\
&= \sum_{i,j\in\mathcal{L}_r} \sum_{k,l=-B_n}^{B_n} \mathrm{cum}(X_{u,i}, X_{u,i+k}, X_{u,j}, X_{u,j+l})\alpha_k\alpha_l \\
&\quad + \sum_{i,j\in\mathcal{L}_r} \sum_{k,l=-B_n}^{B_n} \mathbb{E}(X_{u,i}X_{u,j})\mathbb{E}(X_{u,i+k}X_{u,j+l})\alpha_k\alpha_l \\
&\quad + \sum_{i,j\in\mathcal{L}_r} \sum_{k,l=-B_n}^{B_n} \mathbb{E}(X_{u,i}X_{u,j+l})\mathbb{E}(X_{u,i+k}X_{u,j})\alpha_k\alpha_l,
\end{aligned} \tag{52}$$

where the first term is finite since the fourth cumulants are summable. For the second term (the last term can also be shown similarly), we use the condition Eq. (11), so that

$$\mathbb{E}(X_{u,i}X_{u,j})\mathbb{E}(X_{u,i+k}X_{u,j+l}) = [r(u, i-j) + o(1/n)][r(u, i-j+k-j) + o(1/n)]. \tag{53}$$

Then using $p_n = o(n)$, $B_n = o(p_n)$ and $\sup_u \sum_{k=-\infty}^{\infty} |r(u,k)| < \infty$, one can get

$$\begin{aligned}
&\sup_u \max_r \max_\theta \sum_{i,j\in\mathcal{L}_r} \sum_{k,l=-B_n}^{B_n} [r(u, i-j) + o(1/n)][r(u, i-j+k-j) + o(1/n)] \\
&= \sup_u \max_r \max_\theta \sum_{i,j\in\mathcal{L}_r} r(u, i-j) \left[\sum_{k,l=-B_n}^{B_n} r(u, i-j+k-j) + o(B_n/n) \right] \\
&\leq (2p_n+1)(2B_n+1)(\sup_u \sum_{k=-\infty}^{\infty} |r(u,k)|^2) + o(p_nB_n/n) = \mathcal{O}(p_nB_n).
\end{aligned} \tag{54}$$

To show Eq. (75), we note that

$$\mathrm{var}(U'_{u,r}) = \mathrm{var}(U_{u,r}) \left[1 + \frac{2\mathbb{E}(U'_{u,r})\mathbb{E}(U_{u,r} - U'_{u,r}) - 2\mathrm{var}(U_{u,r} - U'_{u,r})}{\mathrm{var}(U_{u,r})} \right]. \tag{55}$$

From Lemma 8.8, we know that $\operatorname{var}(U_{u,r}(\theta)) \sim p_n B_n \sigma_u^2(\theta)$ and $\sigma_u^2(\theta) = [1+\eta(2\theta)] f^2(u,\theta) \int_{-1}^1 a^2(t)\mathrm{d}t \geq f_*^2 \int_{-1}^1 a^2(t)\mathrm{d}t > 0$. Thus, it suffices to show that

(56)
$$\sup_u \sup_r \sup_\theta \mathbb{E}(U'_{u,r})\mathbb{E}(U_{u,r} - U'_{u,r}) = o(p_n B_n), \quad \sup_u \sup_r \sup_\theta \operatorname{var}(U_{u,r} - U'_{u,r}) = o(p_n B_n).$$

By Lemma 8.6, applying similar inequalities as Eq. (82), we have that

$$\begin{aligned}
&\sup_u \sup_i \sup_\theta \operatorname{var}(U_{u,r} - U'_{u,r}) \\
&\leq \sup_u \sup_i \sup_\theta \frac{\|U_{u,r}\|_{2+\delta/2}^{2+\delta/2}}{d_n^{\delta/2}} \\
&= \mathcal{O}((\ell_n \sqrt{p_n B_n})^{2+\delta/2} (\sqrt{n B_n} (\log n)^{-1/2})^{-\delta/2}) \\
&= \mathcal{O}(p_n B_n) \mathcal{O}((\log n)^{2+3\delta/4} (\sqrt{p_n B_n})^{\delta/2} (\sqrt{n B_n})^{-\delta/2}) \\
&= o(p_n B_n).
\end{aligned} \tag{57}$$

Finally, since $\mathbb{E}(U'_{u,r}) \leq \mathbb{E}(|U_{u,r}|) \leq [\mathbb{E}(|U_{u,r}|^{2+\delta/2})]^{\frac{1}{2+\delta/2}}$, using again similar inequalities as Eq. (82), we have that

$$\begin{aligned}
&\sup_u \sup_r \sup_\theta \mathbb{E}(U'_{u,r})\mathbb{E}(U_{u,r} - U'_{u,r}) \\
&\leq \sup_u \sup_r \sup_\theta \|U_{u,r}\|_{2+\delta/2} \frac{\|U_{u,r}\|_{2+\delta/2}^{2+\delta/2}}{d_n^{1+\delta/2}} \\
&= \mathcal{O}(\|U_{u,r}\|_{2+\delta/2}/d_n) o(p_n B_n) \\
&= \mathcal{O}(\sqrt{p_n/n} (\log n)^{3/2}) o(p_n B_n) = o(p_n B_n).
\end{aligned} \tag{58}$$

A.6. **Proof of Lemma 8.6.** In this proof, we write $\tilde{X}_k^{[\ell]}$ as $\tilde{X}_k$ and $\tilde{X}_{u,i}^{[\ell]}$ as $\tilde{X}_{u,i}$ for short. For simplicity, we first consider that u and i are fixed. Without loss of generality, we consider the first block sum ($i = 1$) so

$$U_{u,1}(\theta) = \sum_{j=1}^{p_n} \tilde{Y}_{u,j}(\theta). \tag{59}$$

We will first show that

$$\left\| \sum_{j=1}^{p_n} \sum_{k=-B_n}^{B_n} \tilde{X}_{u,}\tilde{X}_{u,j+k}\alpha_k \right\|_{2+\delta/2} = \mathcal{O}(\ell_n \sqrt{p_n B_n}), \tag{60}$$

where $\alpha_k = a(k/B_n)\cos(k\theta)$. Then we conclude that $\mathcal{O}(\ell_n\sqrt{p_n B_n})$ is also uniformly over u and i since the assumption $\sup_u \sup_i \mathbb{E}(|X_{u,i}|^{4+\delta}) < M$. We first write by the triangle inequality

$$\begin{aligned} &\left\| \sum_{j=1}^{p_n} \sum_{k=-B_n}^{B_n} \tilde{X}_{u,j}\tilde{X}_{u,j+k}\alpha_k \right\|_{2+\delta/2} \\ &\leq \left\| \sum_{j=1}^{p_n} \sum_{k=-B_n}^{0} \tilde{X}_{u,j}\tilde{X}_{u,j+k}\alpha_k \right\|_{2+\delta/2} + \left\| \sum_{j=1}^{p_n} \sum_{k=0}^{B_n} \tilde{X}_{u,j}\tilde{X}_{u,j+k}\alpha_k \right\|_{2+\delta/2}. \end{aligned} \tag{61}$$

Now consider two cases (i) $\ell_n = o(B_n)$, then

$$\sum_{j=1}^{p_n} \sum_{k=-B_n}^{0} \tilde{X}_{u,j}\tilde{X}_{u,j+k}\alpha_k = \sum_{j=1}^{p_n} \left(\tilde{X}_{u,j} \sum_{k=-B_n}^{-\ell_n} \tilde{X}_{u,j+k}\alpha_k \right) + \sum_{j=1}^{p_n} \sum_{k=1-\ell_n}^{0} \tilde{X}_{u,j}\tilde{X}_{u,j+k}\alpha_k, \tag{62}$$

where the first term of the right hand side of Eq. (62) satisfies

$$\begin{aligned} &\left\| \sum_{j=1}^{p_n} \left(\tilde{X}_{u,j} \sum_{k=-B_n}^{-\ell_n} \tilde{X}_{u,j+k}\alpha_k \right) \right\|_{2+\delta/2} \\ &\leq \sum_{h=1}^{\ell_n} \left\| \sum_{j=1}^{\lfloor (p_n-h)/\ell_n \rfloor} \tilde{X}_{u,h+(j-1)\ell_n} \sum_{k=-B_n}^{-\ell_n} \tilde{X}_{u,h+(j-1)\ell_n+k}\alpha_k \right\|_{2+\delta/2}. \end{aligned} \tag{63}$$

Continuing to divide the sum of $\sum_{k=-B_n}^{-\ell_n} \tilde{X}_{u,h+(j-1)\ell_n+k}\alpha_k$ into ℓ_n parts, then by $\sup_{u,i} \mathbb{E}(|X_{u,i}|^{4+\delta}) < M$, we have that

$$\begin{aligned} \left\| \sum_{j=1}^{p_n} \left(\tilde{X}_{u,j} \sum_{k=-B_n}^{-\ell_n} \tilde{X}_{u,j+k}\alpha_k \right) \right\|_{2+\delta/2} &= \mathcal{O}(\ell_n)\mathcal{O}(\sqrt{p_n/\ell_n})\mathcal{O}(\ell_n)\mathcal{O}(\sqrt{B_n/\ell_n}) \\ &= \mathcal{O}(\ell_n\sqrt{p_n B_n}), \end{aligned} \tag{64}$$

which holds uniformly over u and i. Similarly, for the second term of the right hand side of Eq. (62)

$$\begin{aligned}\left\|\sum_{j=1}^{p_n}\sum_{k=1-\ell_n}^{0}\tilde{X}_{u,j}\tilde{X}_{u,j+k}\alpha_k\right\|_{2+\delta/2} &\leq \sum_{k=1-\ell_n}^{0}\left\|\sum_{j=1}^{p_n}\tilde{X}_{u,j}\tilde{X}_{u,j+k}\alpha_k\right\|_{2+\delta/2} \\ &= \sum_{k=1-\ell_n}^{0}\sum_{h=1}^{3\ell_n}\left\|\sum_{j=1}^{\lfloor(p_n-h)/3\ell_n\rfloor}\tilde{X}_{u,h+3j\ell_n}\tilde{X}_{u,h+3j\ell_n+k}\alpha_k\right\|_{2+\delta/2} = \mathcal{O}(\ell_n^2\sqrt{p_n/\ell_n}).\end{aligned} \tag{65}$$

Note that the order $\mathcal{O}(\ell_n^2\sqrt{p_n/\ell_n})$ also holds uniformly over u and i. This is because $\|\tilde{X}_{u,h+3j\ell_n}\tilde{X}_{u,h+3j\ell_n+k}\|_{2+\delta/2}$ is uniformly bounded, which can be shown using Cauchy–Schwarz's inequality and $\sup_u\sup_i\mathbb{E}(|X_{u,i}|^{4+\delta}) < M$. Therefore, we have proven that, for case (i), we have $\sup_u\sup_i\sup_\theta\|U_{u,i}(\theta)\|_{2+\delta/2} = \mathcal{O}(\ell_n\sqrt{p_nB_n})$.

For the second case (ii) $B_n = \mathcal{O}(\ell_n)$, we have that

$$\begin{aligned}\left\|\sum_{j=1}^{p_n}\sum_{k=-B_n}^{0}\tilde{X}_{u,j}\tilde{X}_{u,j+k}\alpha_k\right\|_{2+\delta/2} &\leq \sum_{k=-B_n}^{0}\left\|\sum_{j=1}^{p_n}\tilde{X}_{u,j}\tilde{X}_{u,j+k}\alpha_k\right\|_{2+\delta/2} \\ &= \sum_{k=-B_n}^{0}\sum_{h=1}^{3\ell_n}\left\|\sum_{j=1}^{\lfloor(p_n-h)/3\ell_n\rfloor}\tilde{X}_{u,h+3j\ell_n}\tilde{X}_{u,h+3j\ell_n+k}\alpha_k\right\|_{2+\delta/2} \\ &= \mathcal{O}(B_n\ell_n\sqrt{p_n/\ell_n}) = \mathcal{O}(\ell_n\sqrt{p_nB_n}),\end{aligned} \tag{66}$$

which is also uniform over u and i.

A.7. **Proof of Lemma 8.8.** Using the property of cumulants in Eq. (51), similarly to Eqs. (52) and (53), one can get that

$$
\begin{aligned}
&\left\| \sum_{i=-s_n/2}^{s_n/2} \{Y_{u,i}(\theta) - \mathbb{E}(Y_{u,i}(\theta))\} \right\|^2 \\
&= \sum_{i,j=-s_n}^{s_n} \sum_{k,l=-B_n}^{B_n} \mathrm{cum}(X_{u,i}, X_{u,i+k}, X_{u,j}, X_{u,j+l})\alpha_k\alpha_l \\
&\quad + \sum_{i,j=-s_n}^{s_n} \sum_{k,l=-B_n}^{B_n} r(u, i-j)r(u, i+k-j-l)\alpha_k\alpha_l + o(s_nB_n/n) \\
&\quad + \sum_{i,j=-s_n}^{s_n} \sum_{k,l=-B_n}^{B_n} r(u, i-j-l)r(u, i+k-j)\alpha_k\alpha_l + o(s_nB_n/n).
\end{aligned} \tag{67}
$$

By Lemma A.5, we have that

$$
\sum_{m_1,m_2,m_3\in\mathbb{Z}} \mathrm{cum}(X_{u,0}, X_{u,m_1}, X_{u,m_2}, X_{u,m_3}) < C\sum_{s=0}^{\infty} \rho^{s/[4(4-1)]} < \infty, \tag{68}
$$

which implies that the first term of the right hand side of Eq. (67) is finite.

Finally, according to [Ros84, Theorem 2, Eqs. (3.9)–(3.12)], one can show that

$$
\begin{aligned}
&\sum_{i,j=-s_n}^{s_n} \sum_{k,l=-B_n}^{B_n} r(u, i-j)r(u, i+k-j-l)\alpha_k\alpha_l \\
&+ \sum_{i,j=-s_n}^{s_n} \sum_{k,l=-B_n}^{B_n} r(u, i-j-l)r(u, i+k-j)\alpha_k\alpha_l \sim s_nB_n\sigma_u^2(\theta).
\end{aligned} \tag{69}
$$

Lemma A.6. *Let $\{X_k\}$ be ℓ-dependent with $\mathbb{E}X_k = 0$ and $X_k \in \mathcal{L}^p$ with $p \geq 2$. Let $W_n = \sum_{k=1}^n X_k$. Then for any $Q > 0$, there exists $C_1, C_2 > 0$ only depending on Q such that*

$$
\mathbb{P}(|W_n| \geq x) \leq C_1 \left(\frac{\ell}{x^2}\mathbb{E}W_n^2\right)^Q + C_1 \min\left[\frac{\ell^{p-1}}{x^p}\sum_{k=1}^n \|X_k\|_p^p, \sum_{k=1}^n \mathbb{P}\left(|X_k| \geq C_2\frac{x}{\ell}\right)\right]. \tag{70}
$$

Proof. See [LW10, Lemma 2]. □

Lemma A.7. *Let $\{X_t\}$ be ℓ-dependent with $\mathbb{E}X_t = 0$, $|X_t| \leq M$ a.s., $\ell \leq n$, and $M \geq 1$. Let $S_{k,l} = \sum_{t=l+1}^{l+k} X_t \sum_{s=1}^{t-1} \alpha_{n,t-s}X_s$, where $l \geq 0$, $l+k \leq n$ and assume that $\max_{1\leq t\leq n} |\alpha_{n,t}| \leq K_0$,*

$\max_{1\le t\le n}\mathbb{E}X_t^2 \le K_0$, $\max_{1\le t\le n}\mathbb{E}X_t^4 \le K_0$ *for some* $K_0 > 0$. *Then for any* $x \ge 1$, $y \ge 1$, *and* $Q > 0$,

$$\mathbb{P}(|S_{k,l} - \mathbb{E}S_{k,l}| \ge x) \le 2e^{-y/4} + C_1 n^3 M^2 \left(x^{-2}y^2\ell^3(M^2+k)\sum_{s=1}^{n}\alpha_{n,s}^2\right)^Q + C_1 n^3 M^2 \sum_{i=1}^{n}\mathbb{P}\left(|X_i| \ge \frac{C_2 x}{y\ell^2(M+k^{1/2})}\right), \tag{71}$$

where $C_1, C_2 > 0$ *are constants depending only on* Q *and* K_0.

Proof. See [LW10, Proposition 3]. □

Lemma A.8. *Assume that* $X_k \in \mathcal{L}^p$, *with* $p > 1$, *and* $\mathbb{E}X_k = 0$. *Let* $C_p = 18p^{3/2}(p-1)^{-1/2}$ *and* $p' = \min(2,p)$. *Let* $\alpha_1, \ldots, \in \mathbb{C}$. *Then under GMC, we have that*

$$\left\|\sum_{k=1}^{n}\alpha_k(X_k - \tilde{X}_k^{[\ell]})\right\|_p \le C_p\left(\sum_{k=1}^{n}|\alpha_k|^{p'}\right)^{1/p'} o(\rho^\ell), \tag{72}$$

and

$$\|\sum_{k=1}^{n}\alpha_k X_k\|_p \le C\left(\sum_{k=1}^{n}|\alpha_k|^{p'}\right)^{1/p'}, \quad \|\sum_{k=1}^{n}\alpha_k \tilde{X}_k^{[\ell]}\|_p \le C\left(\sum_{k=1}^{n}|\alpha_k|^{p'}\right)^{1/p'}, \tag{73}$$

for some constant C.

Proof. This lemma follows from [LW10, Lemma 1] with $\Theta_{\ell+1,p} = o(\sum_{j=\ell+1}^{\infty}\rho^j) = o(\rho^\ell)$. □

Lemma A.9. *Assume* $\mathbb{E}X_{u,k} = 0$, $\sup_u \mathbb{E}|X_{u,k}|^{2p} < \infty$, $p \ge 2$. *Let*

$$L_{n,u} = \sum_{1\le j\le j'\le n}\alpha_{j'-j}X_{u,j}X_{u,j'}, \quad \tilde{L}_{n,u} = \sum_{1\le j\le j'\le n}\alpha_{j'-j}\tilde{X}_{u,j}^{[\ell]}\tilde{X}_{u,j'}^{[\ell]}, \tag{74}$$

where $\alpha_1, \ldots, \in \mathbb{C}$. *Then under GMC, we have that*

$$\frac{\sup_u \|L_{n,u} - \mathbb{E}L_{n,u} - (\tilde{L}_{n,u} - \mathbb{E}\tilde{L}_{n,u})\|_p}{n^{1/2}(\sum_{s=1}^{n-1}|\alpha_s|^2)^{1/2}} = o(\ell\rho^\ell). \tag{75}$$

Proof. For fixed u, if $\mathbb{E}|X_{u,k}|^{2p} < \infty$, the result follow from [LW10, Proposition 1] with $\Theta_{0,2p} = o(1)$ and $d_{\ell,2p} = \sum_{t=0}^{\infty}\min\{o(\rho^t), o(\rho^\ell)\} = o(\ell\rho^\ell)$. Since we have $\sup_u \mathbb{E}|X_{u,k}|^{2p} < \infty$ the proof of [LW10, Proposition 1] also holds uniformly over u. □

Lemma A.10. *Assume that* $\mathbb{E}X_{u,k} = 0$, $\sup_u \mathbb{E}X_{u,k}^4 < \infty$ *and* GMC(2)*. Let* $\alpha_j = \beta_j \exp(ij\theta)$, *where* $i = \sqrt{-1}$, $\theta \in \mathbb{R}$, $\beta_j \in \mathbb{R}$, $1-n \le j \le -1$, $m \in \mathbb{N}$ *and* $\tilde{L}_{n,u} = \sum_{1\le j<t\le n} \alpha_{j-t}\tilde{X}_{u,j}^{[\ell]}\tilde{X}_{u,t}^{[\ell]}$. *Define*

$$D_k(u,\theta) = A_{u,k} - \mathbb{E}(A_{u,k} \,|\, \mathcal{F}_{u,k-1}), \quad M_n(u,\theta) = \sum_{t=1}^{n} D_t(u,\theta)^* \sum_{j=1}^{t-1} \alpha_{j-t} D_j(u,\theta), \tag{76}$$

where $(\cdot)^*$ *denotes the complex conjugate,* $A_{u,k} = \sum_{t=0}^{\infty} \mathbb{E}(\tilde{X}_{u,t+k}^{[\ell]} \,|\, \mathcal{F}_{u,k}) \exp(ij\theta)$ *where* $\mathcal{F}_{u,k-1} := \mathcal{F}_{\lfloor uN-n/2\rfloor+k-1}$. *Then*

$$\sup_u \frac{\|\tilde{L}_{n,u} - \mathbb{E}\tilde{L}_{n,u} - M_n(u,\theta)\|}{m^{3/2}n^{1/2}\sup_k \|X_{u,k}\|_4^2} \le CV_m^{1/2}(\beta), \tag{77}$$

where

$$V_m(\beta) = \max_{1-n\le i\le -1} \beta_i^2 + m\sum_{j=-1}^{-n-1} |\beta_j - \beta_{j-1}|^2. \tag{78}$$

Proof. For fixed u, the result comes from [LW10, Proposition 2]. Since here we have assumed $\sup_u \mathbb{E}X_{u,k}^4 < \infty$, following the proof of [LW10, Proposition 2], the upper bound also holds uniformly over u. □

Lemma A.11. *Suppose that* $\mathbb{E}X_k = 0$, $\sup_u \mathbb{E}X_k^4 < \infty$, *and* GMC(2) *holds, then*

(1) *We have that*

$$\left|\frac{\mathbb{E}[(g_n(u_1,\theta_1) - \mathbb{E}g_n(u_1,\theta_1))(g_n(u_2,\theta_2) - \mathbb{E}g_n(u_2,\theta_2))]}{nB_n}\right| = \mathcal{O}(1/(\log B_n)^2), \tag{79}$$

uniformly on $(u_1,u_2,\theta_1,\theta_2)$ *such that either* $(u_1,u_2) \in \mathcal{U}^2$ *or* $(\theta_1,\theta_2) \in \Theta^2$ *where* $\mathcal{U}^2 = \{(u_1,u_2) : \frac{n}{2N} \le u_1 \le u_2 \le 1 - \frac{n}{2N}, |u_1 - u_2| \ge \frac{n}{N}(1 - 1/(\log B_n)^2)\}$ *and* $\Theta^2 = \{(\theta_1,\theta_2) : 0 \le \theta_1 < \theta_2 \le \pi - B_n^{-1}(\log B_n)^2, |\theta_1 - \theta_2| \ge B_n^{-1}(\log B_n)^2\}$.

(2) *For* $\alpha_n > 0$ *with* $\limsup \alpha_n < 1$, *we have that*

$$\left|\frac{\mathbb{E}[(g_n(u_1,\theta_1) - \mathbb{E}g_n(u_1,\theta_1))(g_n(u_2,\theta_2) - \mathbb{E}g_n(u_2,\theta_2))]}{4\pi^2 nB_n f(u_1,\theta_1)f(u_2,\theta_2)\int_{t=-1}^{1} a^2(t)\mathrm{d}t}\right| \le \alpha_n, \tag{80}$$

uniformly on $(u_1,u_2,\theta_1,\theta_2)$ *such that either* $(u_1,u_2) \in \mathcal{U}^2$ *or* $(\theta_1,\theta_2) \in \bar{\Theta}^2$ *where* $\mathcal{U}^2 = \{(u_1,u_2) : \frac{n}{2N} \le u_1 \le u_2 \le 1 - \frac{n}{2N}, |u_1 - u_2| \ge \frac{n}{N}(1 - 1/(\log B_n))\}$ *and* $\bar{\Theta}^2 = \{(\theta_1,\theta_2) : B_n^{-1}(\log B_n)^2 \le \theta_1 < \theta_2 \le \pi - B_n^{-1}(\log B_n)^2, |\theta_1 - \theta_2| \ge B_n^{-1}\}$.

(3) *We have that*

(81) $$\left|\frac{\mathbb{E}[g_n(u,\theta)-\mathbb{E}g_n(u,\theta)]^2}{4\pi^2 nB_n f^2(u,\theta)\int_{t=-1}^{1}a^2(t)\mathrm{d}t}-1\right| = \mathcal{O}(1/(\log B_n)^2),$$

uniformly on $\{(u,\theta): B_n^{-1}(\log B_n)^2 \le \theta \le \pi - B_n^2(\log B_n)^2, \frac{n}{2N} < u < 1-\frac{n}{2N}\}$.

Proof. Throughout this proof, we write $\tilde{X}_k^{[\ell]}$ as $\tilde{X}_k$ and $\tilde{X}_{u,i}^{[\ell]}$ as $\tilde{X}_{u,i}$ for simplicity.

(1) By Lemma A.9 we approximate $g_n-\mathbb{E}g_n$ first by $\tilde{g}_n-\mathbb{E}\tilde{g}_n$. Then by Lemma A.10, we approximate $\tilde{g}_n-\mathbb{E}\tilde{g}_n$ by $M_n(u,\theta)$, where $M_n(u,\theta)=\sum_{t=1}^{n}D_t(u,\theta)^*\sum_{j=1}^{t-1}\alpha_{n,j-t}D_j(u,\theta)$. Then it is suffices to show that $|\mathbb{E}[M_n(u_1,\theta_1)-M_n^*(u_1,\theta_1)][M_n(u_2,\theta_2)-M_n^*(u_2,\theta_2)]| \le C\frac{nB_n}{(\log B_n)^2}$ and $|\mathbb{E}[M_n(u_1,\theta_1)+M_n^*(u_1,\theta_1)][M_n(u_2,\theta_2)+M_n^*(u_2,\theta_2)]| \le C\frac{nB_n}{(\log B_n)^2}$. We only prove the first inequality here, since the other inequality can be proved similarly. Define

(82) $$r_n(u_1,\theta_1,u_2,\theta_2) := |\mathbb{E}[M_n(u_1,\theta_1)+M_n^*(u_1,\theta_1)][M_n(u_2,\theta_2)+M_n^*(u_2,\theta_2)]|.$$

Since the martingale differences $\{D_t(u,\theta)\}$ are uncorrelated but not independent, we further define $N_n(u,\theta)=\sum_{t=1}^{n}D_t(u,\theta)^*\sum_{j=1}^{t-\ell-1}\alpha_{n,j-t}D_j(u,\theta)$, then $\|M_n(u,\theta)-N_n(u,\theta)\| = \mathcal{O}(\sqrt{n\ell})$ and $|r_n(u_1,\theta_1,u_2,\theta_2)| \le |\tilde{r}_n(u_1,\theta_1,u_2,\theta_2)| + \mathcal{O}(\sqrt{n\ell(nB_n)}+\sqrt{n\ell(B_n^2)})$, where

(83) $$\tilde{r}_n(u_1,\theta_1,u_2,\theta_2) := |\mathbb{E}[N_n(u_1,\theta_1)+N_n^*(u_1,\theta_1)][N_n(u_2,\theta_2)+N_n^*(u_2,\theta_2)]|.$$

Since $\ell=\lfloor n^\gamma\rfloor$ where γ is small enough, it suffices to show that $\tilde{r}_n(u_1,\theta_1,u_2,\theta_2) = \mathcal{O}(nB_n/(\log B_n)^2)$. Now we substitute $N_n(u,\theta)=\sum_{t=1}^{n}D_t(u,\theta)^*\sum_{j=1}^{t-\ell-1}\alpha_{n,j-t}D_j(u,\theta)$ to $\tilde{r}_n(u_1,\theta_1,u_2,\theta_2)$.

If $\theta_1\ne\theta_2$ and $u_1=u_2$, we have that

(84)
$$\sum_{t=1}^{n}\sum_{j=1}^{t-\ell-1}2\mathbb{E}|D_t(u,\theta)D_j(u,\theta)|^2 a^2\left(\frac{t-j}{B_n}\right)[\cos((t-j)(\theta_1+\theta_2))+\cos((t-j)(\theta_1-\theta_2))].$$

Now it suffices to show that

$$\sum_{t=1}^{n}\sum_{j=1}^{t-\ell-1}a^2\left(\frac{t-j}{B_n}\right)\cos((t-j)(\theta_1\pm\theta_2)) = \mathcal{O}(nB_n/(\log B_n)^2).$$

Since $|\theta_1 - \theta_2| \geq B_n^{-1}(\log B_n)^2$, using $1 + 2\sum_{k=1}^n \cos(k\theta) = \sin((n+1)\theta/2)/\sin(\theta/2) \leq 1/\sin(\theta/2)$, $\sin(x) = \Theta(x)$ when $x \to 0$, and denoting $j = t - s$, we have that

$$\sum_{t=1}^{n} \left| \sum_{j=1}^{B_n} a^2(j/B_n) \cos[j(\theta_1 \pm \theta_2)] \right| \leq Cn/(B_n^{-1}(\log B_n)^2) = \mathcal{O}(nB_n/(\log B_n)^2). \tag{85}$$

If $\theta_1 = \theta_2$ but $u_1 \neq u_2$, using Eq. (88) and $n - N|u_1 - u_2| \leq n/(\log B_n)^2$, we have that

$$\tilde{r}_n(u_1, \theta, u_2, \theta) \leq C\tilde{r}_{n-N|u_1-u_2|}(u, \theta, u, \theta) = \mathcal{O}((n - N|u_1 - u_2|)B_n) = \mathcal{O}(nB_n/(\log B_n)^2).$$

(2) When $\theta_1 \neq \theta_2$, using [WN67, Lemma 3.2(ii)] with the assumption on the continuity of $a(\cdot)$ in Theorem 5.3, we have that

$$\limsup_n 2(nB_n)^{-1} \sum_{t=1}^{n} \sum_{j=1}^{t-\ell-1} a^2\left(\frac{t-j}{B_n}\right) \cos((t-j)(\theta_1 - \theta_2)) < \int a^2(t)\mathrm{d}t. \tag{86}$$

If $\theta_1 = \theta_2$ and $u_1 \neq u_2$ then

$$\begin{aligned}
&\limsup_n 2(nB_n)^{-1} \sum_{t=1}^{n-N|u_1-u_2|} \sum_{j=1}^{t-\ell-1} a^2\left(\frac{t-j}{B_n}\right) \\
&\leq \limsup_n 2(nB_n)^{-1}(n - N|u_1 - u_2|) \sum_{j=-B_n}^{B_n} a^2\left(\frac{t-j}{B_n}\right) \\
&\leq \limsup_n 2(nB_n)^{-1}[nB_n/(\log B_n)^2] \int a^2(t)\mathrm{d}t < \int a^2(t)\mathrm{d}t.
\end{aligned} \tag{87}$$

(3) Since $\|D_t(u,\theta)\|^2 = \sum_{j=\ell}^{\ell} \mathbb{E}(\tilde{X}_{u,t}\tilde{X}_{u,t+j}) \exp(ij\theta)$, we have that

$$\begin{aligned}
\tilde{r}_n(u, \theta, u, \theta) &= \mathcal{O}(nB_n/(\log B_n)^2) + \sum_{t=1}^{n} \|D_t(u,\theta)\|^2 \sum_{s=-B_n}^{B_n} a^2(s/B_n) \\
&= \mathcal{O}(nB_n/(\log B_n)^2) + 4\pi^2 f^2(u,\theta) nB_n \int a^2(t)\mathrm{d}t.
\end{aligned} \tag{88}$$

□

Lemma A.12. *Let $X_1, \ldots, X_m$ be independent mean zero d-dimensional random vectors such that $|X_i| \leq M$. If the underlying probability space is rich enough, one can define independent normally distributed mean zero random vectors $V_1, \ldots, V_m$ such that the covariance matrices* *of V_i and X_i are equal, for all $1 \le i \le m$; furthermore*

$$\mathbb{P}\left(\left|\sum_{i=1}^{m}(X_i - V_i)\right| \ge \delta\right) \le c_1 \exp(-c_2\delta/M). \tag{89}$$

Proof. See [EM97, Fact 2.2]. ☐

Lemma A.13. *If X and Y have a bi-variate normally distributed distribution with expectations 0, unit variances, and correlation coefficient r, then*

$$\lim_{c\to\infty} \frac{\mathbb{P}(\{X > c\} \cap \{Y > c\})}{[2\pi(1-r)^{\frac{1}{2}}c^2]^{-1}\exp\left(-\frac{c^2}{1+r}\right)(1+r)^{\frac{3}{2}}} = 1, \tag{90}$$

uniformly for all r such that $|r| \le \delta$, for all $0 < \delta < 1$.

Proof. See [Ber62, Lemma 2]. ☐

A.8. **Proof of Lemma 8.9.** By Markov's inequality, we have that

$$\begin{aligned}
&\mathbb{P}\left(\max_{u\in\mathcal{U}}\max_{0\le i\le B_n}\frac{|g_n(u,\theta_i)-\tilde{g}_n(u,\theta_i)|}{\sqrt{nB_n}} \ge 1/\log D_n\right)\\
&\le \sum_{u\in\mathcal{U}}\sum_{0\le i\le B_n}\mathbb{P}\left(\frac{|g_n(u,\theta_i)-\tilde{g}_n(u,\theta_i)|}{\sqrt{nB_n}} \ge 1/\log D_n\right)\\
&\le CB_nC_n\frac{\mathbb{E}[\frac{|g_n(u,\theta_i)-\tilde{g}_n(u,\theta_i)|}{\sqrt{nB_n}}]^{p/2}}{(1/\log D_n)^{p/2}}.
\end{aligned} \tag{91}$$

By Lemma A.9, $\mathbb{E}|g_n(u,\theta_i)-\tilde{g}_n(u,\theta_i)| = o(n^{1+\gamma}\rho^{\lfloor n^\gamma\rfloor})$ uniformly on u and θ_i. Since $D_n = B_nC_n$ is polynomial of n, the GMC assumption guarantees

$$\mathbb{P}\left(\max_{u\in\mathcal{U}}\max_{0\le i\le B_n}\frac{|g_n(u,\theta_i)-\tilde{g}_n(u,\theta_i)|}{\sqrt{nB_n}} \ge 1/\log D_n\right) = o(1). \tag{92}$$

A.9. **Proof of Lemma 8.10.**

Lemma A.14. *Let $X_i, i = 1, \ldots, n$ be an arbitrary sequence of real-valued random variables with finite mean and variance. Then*

$$\mathbb{E}(\max_{1\le i\le n} X_i) \le \max_{1\le i\le n}\mathbb{E}X_i + \sqrt{\frac{n-1}{n}\sum_{i=1}^{n}\operatorname{var}(X_i)}. \tag{93}$$

Proof. See [Ave85, Theorem 2.1]. ☐

In this proof, we write $\tilde{X}_k^{[\ell]}$ and $\tilde{X}_{u,i}^{[\ell]}$ as $\tilde{X}_k$ and $\tilde{X}_{u,i}$ for simplicity. First of all, since $a(\cdot)$ has bounded support $[-1,1]$, we only need to consider the case that $|s-k| \le B_n$. Furthermore, let α_* be an upper bound of $\alpha_{n,i}$ uniformly over u. Then, we have that

(94)

$$\begin{aligned}
&\mathbb{E}\left(\max_{u\in\mathcal{U}}\max_{\theta}|\tilde{g}_n(u,\theta)-\bar{g}_n(u,\theta)|\right)\\
&\le \alpha_*\,\mathbb{E}\left[\max_{u\in\mathcal{U}}\sum_{2\le k\le n,\max(1,k-B_n)\le s\le k-1}\left|\tilde{X}_{k,u}\tilde{X}_{s,u}-\mathbb{E}(\tilde{X}_{k,u}\tilde{X}_{s,u})-\bar{X}_{k,u}\bar{X}_{s,u}+\mathbb{E}(\bar{X}_{k,u}\bar{X}_{s,u})\right|\right]\\
&\le 2\alpha_*\,\mathbb{E}\left[\max_{u\in\mathcal{U}}\sum_{2\le k\le n,\max(1,k-B_n)\le s\le k-1}\left|\tilde{X}_{k,u}\tilde{X}_{s,u}-\bar{X}_{k,u}\bar{X}_{s,u}\right|\right]\\
&= 2\alpha_*\,\mathbb{E}\left[\max_{u\in\mathcal{U}}\sum_{2\le k\le n,\max(1,k-B_n)\le s\le k-1}\left|\tilde{X}_{k,u}\tilde{X}_{s,u}-\bar{X}_{k,u}\bar{X}_{s,u}-\tilde{X}_{k,u}\bar{X}_{s,u}+\tilde{X}_{k,u}\bar{X}_{s,u}\right|\right]\\
&\le 2\alpha_*\,\mathbb{E}\left[\max_{u\in\mathcal{U}}\left(\sum_{k=2}^{n}|\tilde{X}_{k,u}|\sum_{s=\max\{1,k-B_n\}}^{k-1}|\tilde{X}_{s,u}-\bar{X}_{s,u}|\right)\right]\\
&\quad + 2\alpha_*\,\mathbb{E}\left[\max_{u\in\mathcal{U}}\left(\sum_{k=2}^{n}|\tilde{X}_{k,u}-\bar{X}_{k,u}|\sum_{s=\max\{1,k-B_n\}}^{k-1}|\bar{X}_{s,u}|\right)\right].
\end{aligned}$$

Next, we show that the first term of the right hand side of Eq. (94) satisfies

$$\mathbb{E}\left(\max_{u\in\mathcal{U}}\frac{\sum_{k=2}^{n}|\tilde{X}_{k,u}|\sum_{s=\max\{1,k-B_n\}}^{k-1}|\tilde{X}_{s,u}-\bar{X}_{s,u}|}{\sqrt{nB_n}}\right)=o(1). \tag{95}$$

Similar arguments yield the same result for the second term of the right hand side of Eq. (94). Note that

$$\begin{aligned}
&\mathbb{E}\left(\max_{u\in\mathcal{U}} \frac{\sum_{k=2}^{n}|\tilde{X}_{u,k}|\sum_{s=\max\{1,k-B_n\}}^{k-1}|\tilde{X}_{u,s}-\bar{X}_{u,s}|}{\sqrt{nB_n}}\right)\\
&\le \mathbb{E}\left(\max_{u\in\mathcal{U}} \frac{\sum_{k=2}^{n}|\tilde{X}_{u,k}|\sum_{s=\max\{1,k-B_n\}}^{k-\ell}|\tilde{X}_{u,s}-\bar{X}_{u,s}|}{\sqrt{nB_n}}\right)\\
&\quad + \mathbb{E}\left(\max_{u\in\mathcal{U}} \frac{\sum_{k=2}^{n}|\tilde{X}_{u,k}|\sum_{s=\max\{1,k-\ell+1\}}^{k-1}|\tilde{X}_{u,s}-\bar{X}_{u,s}|}{\sqrt{nB_n}}\right).
\end{aligned} \tag{96}$$

Applying Lemma A.14 and using ℓ-independence and Hölder's inequality, we have that uniformly on u

$$\begin{aligned}
&\mathbb{E}\left(\sum_{k=2}^{n}|\tilde{X}_{u,k}|\sum_{s=\max\{1,k-B_n\}}^{k-\ell}|\tilde{X}_{u,s}-\bar{X}_{u,s}|\right)\\
&= \mathbb{E}\left(\sum_{k=2}^{n}|\tilde{X}_{u,k}|\right)\mathbb{E}\left(\sum_{s=\max\{1,k-B_n\}}^{k-\ell}|\tilde{X}_{u,s}-\bar{X}_{u,s}|\right)\\
&= \mathcal{O}(n)\mathcal{O}(B_n)\mathbb{E}\left|\tilde{X}_{u,k}\mathbf{1}_{|\tilde{X}_{u,k}|>(nB_n)^{\alpha}} - \mathbb{E}\tilde{X}_{u,k}\mathbf{1}_{|\tilde{X}_{u,k}|>(nB_n)^{\alpha}}\right|\\
&\le \mathcal{O}(nB_n)(\mathbb{E}\tilde{X}_{u,k}^{p})^{1/p}(\mathbb{P}(|\tilde{X}_{u,k}|^{p} > (nB_n)^{\alpha p}))^{1-1/p}\\
&= \mathcal{O}(nB_n)\mathcal{O}((nB_n)^{-\alpha p})^{1-1/p} = \mathcal{O}((nB_n)^{1-\alpha(p-1)}).
\end{aligned} \tag{97}$$

Furthermore, uniformly on u, we also have that

$$\begin{aligned}
&\sqrt{\mathbb{E}\left(\sum_{k=2}^{n}|\tilde{X}_{u,k}|\sum_{s=\max\{1,k-B_n\}}^{k-\ell}|\tilde{X}_{u,s}-\bar{X}_{u,s}|\right)^2}\\
&=\sqrt{\mathbb{E}\left(\sum_{k=2}^{n}|\tilde{X}_{u,k}|\right)^2\mathbb{E}\left(\sum_{s=\max\{1,k-B_n\}}^{k-\ell}|\tilde{X}_{u,s}-\bar{X}_{u,s}|\right)^2}\\
&=\sqrt{\mathcal{O}(n^2)}\sqrt{\mathcal{O}(B_n^2)\mathbb{E}\left|\tilde{X}_{u,k}\mathbf{1}_{|\tilde{X}_{u,k}|>(nB_n)^\alpha}-\mathbb{E}\tilde{X}_{u,k}\mathbf{1}_{|\tilde{X}_{u,k}|>(nB_n)^\alpha}\right|^2}\\
&\leq \mathcal{O}(nB_n)(\mathbb{E}\tilde{X}_{u,k}^p)^{1/p}(\mathbb{P}(|\tilde{X}_{u,k}|^p>(nB_n)^{\alpha p}))^{1-1/p}\\
&=\mathcal{O}(nB_n)\mathcal{O}((nB_n)^{-\alpha p})^{1-1/p}=\mathcal{O}((nB_n)^{1-\alpha(p-1)}).
\end{aligned}\tag{98}$$

By the assumptions $p>4$ and $(p-1)\alpha>3/4$, we have that

$$\mathbb{E}\left(\max_{u\in\mathcal{U}}\frac{\sum_{k=2}^{n}|\tilde{X}_{u,k}|\sum_{s=\max\{1,k-B_n\}}^{k-\ell}|\tilde{X}_{u,s}-\bar{X}_{u,s}|}{\sqrt{nB_n}}\right)=\mathcal{O}\left(\frac{C_n^{1/2}(nB_n)^{1-\alpha(p-1)}}{(nB_n)^{1/2}}\right)=o(1),\tag{99}$$

since we have assumed $C_n^{1/2}=o[(nB_n)^{\alpha(p-1)-\frac{1}{2}}]$. Next, uniformly on u, the second term of the right hand side of Eq. (94) satisfies that

$$\begin{aligned}
&\mathbb{E}\left(\sum_{k=2}^{n}|\tilde{X}_{u,k}|\sum_{s=\max\{1,k-\ell+1\}}^{k-1}|\tilde{X}_{u,s}-\bar{X}_{u,s}|\right)\\
&=\mathcal{O}(n\ell)\mathbb{E}\left|\tilde{X}_{u,k}^2\mathbf{1}_{\tilde{X}_{u,k}^2>(nB_n)^{2\alpha}}-\mathbb{E}\tilde{X}_{u,k}^2\mathbf{1}_{\tilde{X}_{u,k}^2>(nB_n)^{2\alpha}}\right|\\
&\leq\mathcal{O}(n\ell)\left(\mathbb{E}|\tilde{X}_{u,k}|^p\right)^{2/p}\left(\mathbb{P}(\tilde{X}_{u,k}^p<(nB_n)^{p\alpha})\right)^{1-2/p}\\
&=\mathcal{O}(n\ell)\mathcal{O}((nB_n)^{-\alpha p})^{1-2/p}\\
&=\mathcal{O}(n\ell)\mathcal{O}(nB_n)^{-\alpha(p-2)}.
\end{aligned}\tag{100}$$

Furthermore, uniformly on u, we have that

$$\begin{aligned}
&\sqrt{\mathbb{E}\left(\sum_{k=2}^{n}|\tilde{X}_{u,k}|\sum_{s=\max\{1,k-\ell+1\}}^{k-1}|\tilde{X}_{u,s}-\bar{X}_{u,s}|\right)^2}\\
&=\sqrt{\mathcal{O}(n^2\ell^2)\mathbb{E}\left|\tilde{X}_{u,k}^2\mathbf{1}_{\tilde{X}_{u,k}^2>(nB_n)^{2\alpha}}-\mathbb{E}\tilde{X}_{u,k}^2\mathbf{1}_{\tilde{X}_{u,k}^2>(nB_n)^{2\alpha}}\right|^2}\\
&\leq \mathcal{O}(n\ell)\left(\mathbb{E}|\tilde{X}_{u,k}|^p\right)^{2/p}\left(\mathbb{P}(\tilde{X}_{u,k}^p<(nB_n)^{p\alpha})\right)^{1-2/p}\\
&=\mathcal{O}(n\ell)\mathcal{O}((nB_n)^{-\alpha p})^{1-2/p}\\
&=\mathcal{O}(n\ell)\mathcal{O}(nB_n)^{-\alpha(p-2)}.
\end{aligned} \tag{101}$$

Overall, we have that

$$\mathbb{E}\left(\max_{u\in\mathcal{U}}\frac{\sum_{k=2}^{n}|\tilde{X}_{u,k}|\sum_{s=\max\{1,k-\ell+1\}}^{k-1}|\tilde{X}_{u,s}-\bar{X}_{u,s}|}{\sqrt{nB_n}}\right)=\mathcal{O}\left(\frac{C_n^{1/2}n\ell(nB_n)^{-\alpha(p-2)}}{\sqrt{nB_n}}\right)=o(1), \tag{102}$$

since we have assumed $C_n=o(B_n^{1+2\alpha(p-2)}n^{-2-2\gamma})$.

A.10. **Proof of Lemma 8.11.** We prove this lemma by first showing that

$$\max_{u\in\mathcal{U}}\max_{0\leq i\leq B_n}\left|\frac{\sum_{j=1}^{k_n+1}V_j(u,\theta_i)}{\sqrt{nB_n}}\right|=o_{\mathbb{P}}(1), \tag{103}$$

and then showing

$$\max_{u\in\mathcal{U}}\max_{0\leq i\leq B_n}\left|\frac{\sum_{j=1}^{k_n}U_j(u,\theta_i)-\sum_{j=1}^{k_n}\bar{U}_j(u,\theta_i)}{\sqrt{nB_n}}\right|=o_{\mathbb{P}}(1). \tag{104}$$

To show Eq. (103), we note that $\{V_j\}$ are independent. Applying Lemma A.6, we have that

$$\mathbb{P}\left(\left|\frac{\sum_{j=1}^{k_n+1}V_j}{\sqrt{nB_n}}\right|\geq\frac{1}{\log B_n}\right)\leq C_1\left(\frac{\sum_{j=1}^{k_n+1}\mathbb{E}V_j^2}{nB_n(\log B_n)^{-2}}\right)^Q+C_1\sum_{j=1}^{k_n+1}\mathbb{P}\left(\frac{|V_j|}{\sqrt{nB_n}}\geq\frac{C_2}{\log B_n}\right). \tag{105}$$

Similar to the proof of Lemma 8.12, one can show $\sum_{j=1}^{k_n+1}\mathbb{E}V_j^2 = \mathcal{O}(n^{1+\gamma}B_n^{1-\beta})$. Therefore, by choosing γ close to zero and Q large enough, we have that

$$\left(\frac{\sum_{j=1}^{k_n+1}\mathbb{E}V_j^2}{nB_n(\log B_n)^{-2}}\right)^Q = \mathcal{O}(n^{-c}), \tag{106}$$

for any $c > 0$. For the other term

$$\sum_{j=1}^{k_n+1}\mathbb{P}\left(\frac{|V_j|}{\sqrt{nB_n}} \geq \frac{C_2}{\log B_n}\right), \tag{107}$$

we apply Lemma A.7 with $M = (nB_n)^\alpha$, $k = B_n + \ell$, $\ell = \lfloor n^\gamma \rfloor$ and $y = (\log B_n)^2$, which yields

$$\begin{aligned} &\mathbb{P}\left(\frac{|V_j|}{\sqrt{nB_n}} \geq \frac{C_2}{\log B_n}\right) \\ &\leq 2\exp\left(-\frac{(\log B_n)^2}{4}\right) + \mathcal{O}\left(n^3(nB_n)^{2\alpha}\left(\frac{(\log B_n)^2}{nB_n}(\log B_n)^4\lfloor n^{3\gamma}\rfloor((nB_n)^{2\alpha} + B_n)\right)^Q\right) \\ &\quad + \mathcal{O}\left(n^3(nB_n)^{2\alpha}\sum_{i=1}^n \mathbb{P}\left(|\bar{X}_{i,\ell}| \geq \frac{C_2\frac{\sqrt{nB_n}}{\log B_n}}{(\log B_n)^2\lfloor n^{2\gamma}\rfloor((nB_n)^\alpha + (B_n + \lfloor n^\gamma\rfloor)^{1/2})}\right)\right), \end{aligned} \tag{108}$$

where the second term of the right hand side is $\mathcal{O}(n^{-c})$ by choosing Q large enough. Since $\alpha < 1/4$ and $|\bar{X}_{i,\ell}| < (nB_n)^\alpha$ almost surely, the last term of the right hand side converges to zero almost surely if

$$(nB_n)^\alpha = o\left(\frac{\frac{\sqrt{nB_n}}{\log B_n}}{(\log B_n)^2\lfloor n^{2\gamma}\rfloor((nB_n)^\alpha + (B_n + \lfloor n^\gamma\rfloor)^{1/2})}\right), \tag{109}$$

which can be satisfied by choosing γ close enough to zero. Therefore, by choosing Q large enough so that $\mathcal{O}(C_nB_nn^{-c}) = o(1)$ (Note that this only requires $C_n = o(n^cB_n^{-1})$ for some c, which is always satisfied when C_n is polynomial of n), we have that

$$\mathbb{P}\left(\max_{u\in\mathcal{U}}\max_{0\leq i\leq B_n}\left|\frac{\sum_{j=1}^{k_n+1}V_j}{\sqrt{nB_n}}\right| \geq \frac{1}{\log B_n}\right) \leq \mathcal{O}(C_nB_n)\mathbb{P}\left(\left|\frac{\sum_{j=1}^{k_n+1}V_j}{\sqrt{nB_n}}\right| \geq \frac{1}{\log B_n}\right) = o(1), \tag{110}$$

which implies Eq. (103).

To prove Eq. (104), note that

(111)
$$U_j - \bar{U}_j(u,\theta) = U_j(u,\theta)\mathbf{1}\left(\frac{|U_j(u,\theta)|}{\sqrt{nB_n}} > \frac{1}{(\log B_n)^4}\right) - \mathbb{E}U_j(u,\theta)\mathbf{1}\left(\frac{|U_j(u,\theta)|}{\sqrt{nB_n}} > \frac{1}{(\log B_n)^4}\right).$$

Therefore, other than using $p_n = B_n^{1+\beta}$ instead of $q_n = B_n + \ell$, the proof of Eq. (104) is essentially the same as the proof of Eq. (103).

A.11. **Proof of Lemma 8.12.** By Lemma A.3, we have that

(112)
$$\begin{aligned}
&\mathbb{P}\left(\max_{u\in\mathcal{U}} \max_{i\notin[(\log B_n)^2, B_n-(\log B_n)^2]} \left|\frac{\sum_{j=1}^{k_n}\bar{U}_j}{\sqrt{nB_n}}\right| \geq x\sqrt{\log(B_nC_n)}\right) \\
&= \mathcal{O}(C_n) \sum_{i\notin[(\log B_n)^2, B_n-(\log B_n)^2]} \mathbb{P}\left(\left|\frac{\sum_{j=1}^{k_n}\bar{U}_j}{\sqrt{nB_n}}\right| \geq x\sqrt{\log(B_nC_n)}\right) \\
&= \mathcal{O}(C_nB_n + C_n(\log B_n)^2)\mathbb{P}\left(\frac{\sum_{j=1}^{k_n}|\bar{U}_j|}{\sqrt{nB_n}} \geq x\sqrt{\log(B_nC_n)}\right) \\
&= \mathcal{O}(C_nB_n)\exp\left(\frac{-\frac{1}{2}x^2 nB_n(\log B_n + \log C_n)}{\sum_{j=1}^{k_n}\mathbb{E}\bar{U}_j^2 + \frac{1}{3}\frac{\sqrt{nB_n}}{(\log B_n)^4}x\sqrt{nB_n(\log B_n + \log C_n)}}\right).
\end{aligned}$$

Note that $U_j = \sum_{k\in H_j}(\bar{Y}_{k,\ell} - \mathbb{E}\bar{Y}_{k,\ell})$, we first divide $\sum_{k\in H_j}(\bar{Y}_{k,\ell} - \mathbb{E}\bar{Y}_{k,\ell})$ into ℓ sums of subsequences. Note that $\bar{Y}_{k,\ell} = \bar{X}_{k,\ell}\sum_{s=1}^{k-1}\alpha_{n,k-s}\bar{X}_{s,\ell} = \bar{X}_{k,\ell}\sum_{s=\max(1,k-B_n)}^{k-1}\alpha_{n,k-s}\bar{X}_{s,\ell}$. Thus, one can get $\|\bar{U}_j\|^2 = \mathcal{O}(\ell B_n^2)$. Then using $\ell = \mathcal{O}(n^\gamma)$ and $k_n = \lfloor n/(p_n+q_n)\rfloor = \mathcal{O}(n/B_n^{1+\beta})$, one can get $\sum_{j=1}^{k_n}\mathbb{E}\bar{U}_j^2 = \mathcal{O}(n^{1+\gamma}B_n^{1-\beta}) = o(nB_n)$ by choosing γ and β such that $n^\gamma B_n^{-\beta} = o(1)$.

Finally, we have that

(113)
$$\begin{aligned}
&\mathcal{O}(C_nB_n)\exp\left(\frac{-\frac{1}{2}x^2 nB_n(\log B_n + \log C_n)}{\sum_{j=1}^{k_n}\mathbb{E}\bar{U}_j^2 + \frac{1}{3}\frac{\sqrt{nB_n}}{(\log B_n)^4}x\sqrt{nB_n(\log B_n + \log C_n)}}\right) \\
&= o\left[C_nB_n\exp\left(\frac{-\frac{1}{2}x^2\log(B_nC_n)}{o(nB_n)/(nB_n) + \frac{1}{3}x\frac{\log(B_nC_n)}{(\log B_n)^4}}\right)\right] \\
&\to o\left[C_nB_n\exp\left(-\frac{3}{2}x(\log B_n)^4\right)\right] = o(1),
\end{aligned}$$

since $\log C_n + \log B_n = o(\log B_n)^4$ when C_n and B_n are polynomials of n.

A.12. **Proof of Lemma 8.13.**

(1) We first show that for $|i_1 - i_2| \geq (\log B_n)^2/B_n$ or $|u_1 - u_2| \geq \frac{n}{N}(1 - 1/(\log B_n)^2)$, we have that

$$\left| \frac{\mathbb{E} \sum_{j=1}^{k_n} \bar{U}_j(u_1, \theta_{i_1}) \sum_{j=1}^{k_n} \bar{U}_j(u_2, \theta_{i_2})}{nB_n} \right| = \mathcal{O}(1/(\log B_n)^2). \tag{114}$$

Note that $\sum_j \bar{U}_j$ can be approximated by $\bar{g}_n$. This is because according to the proof of Lemma 8.11, we have that

$$\max_u \max_i \frac{\mathbb{E}|\sum_{j=1}^{k_n} \bar{U}_j(u, \theta_i) - \bar{g}_n(u, \theta_i)|^2}{nB_n} = \mathcal{O}(B_n^{-\epsilon/2}). \tag{115}$$

Next, we can approximate $\bar{g}_n$ by $\tilde{g}_n$. This is because by Lemma 8.10 we have that

$$\max_u \max_\theta \frac{\mathbb{E}|\tilde{g}_n(u, \theta) - \bar{g}_n(u, \theta)|^2}{nB_n} = \mathcal{O}(1/(\log B_n)^2). \tag{116}$$

Finally, we only need to show that

$$\frac{|\mathrm{Cov}(\tilde{g}_n(u_1, \theta_{i_1}), \tilde{g}_n(u_2, \theta_{i_2}))|}{nB_n} = \mathcal{O}(1/(\log B_n)^2), \tag{117}$$

which has been proved in Lemma A.11(i).

(2) For convenience, we assume $\int a^2(t)\mathrm{d}t = 1$. Select d distinct tuples $(\theta_{i_1}, u_i), i = 1, \ldots, d$ that $(\log B_n)^2 \leq i_1 \leq \cdots \leq i_d \leq B_n - (\log B_n)^2$ and $u_i \in \mathcal{U}, i = 1, \ldots, d$. Let $\boldsymbol{W}_n = \sum_{j=1}^{k_n} W_j$ where

$$W_j = \left(\frac{\bar{U}_j(u_1, \theta_{i_1})}{f(u_1, \theta_{i_1})}, \ldots, \frac{\bar{U}_j(u_d, \theta_{i_d})}{f(u_d, \theta_{i_d})} \right), \quad 1 \leq j \leq k_n. \tag{118}$$

Note that by Lemma A.11(iii), we have that

$$\left| \mathbb{E} \frac{\left(\sum_{j=1}^{k_n} \bar{U}_j(u, \theta) \right)^2}{nB_n} - 4\pi^2 f^2(u, \theta) \right| = \mathcal{O}(1/(\log B_n)^2). \tag{119}$$

Together with Eq. (114), we have that

$$\left| \frac{\mathrm{Cov}(\boldsymbol{W}_n)}{nB_n} - 4\pi^2 \boldsymbol{I}_d \right| = \mathcal{O}(1/(\log B_n)^2). \tag{120}$$

Then we approximate $\boldsymbol{W}_n$ by $\boldsymbol{W}'_n = \sum_{j=1}^{k_n} W'_j$ using Lemma A.12, where $\{W'_j\}$ are independent centered normally distributed random vectors. Then by Lemma A.12, we have $\mathrm{Cov}(W_j) = \mathrm{Cov}(W'_j)$, for $1 \le j \le k_n$, and

$$\mathbb{P}\left(\frac{|\boldsymbol{W}_n - \boldsymbol{W}'_n|}{\sqrt{nB_n}} \ge 1/\log B_n\right) = \mathcal{O}(e^{-(\log B_n)^3}). \tag{121}$$

Therefore, we have that

$$\left|\frac{\mathrm{Cov}(\boldsymbol{W}'_n)}{nB_n} - 4\pi^2 \boldsymbol{I}_d\right| = \mathcal{O}(1/(\log B_n)^2). \tag{122}$$

(3) Next, for $z = (z_1, \ldots, z_d)$, we define the minimum of $\{z_i\}$ by $|z|_d := \min_{1\le i\le d}\{z_i\}$. Then we show that

$$\mathbb{P}\left(\frac{|\boldsymbol{W}_n|_d}{\sqrt{nB_n}} \ge y_n\right) = (1+o(1))\left(\sqrt{8\pi}y_n^{-1}\exp\left(-\frac{y_n^2}{8\pi^2}\right)\right)^d, \tag{123}$$

uniformly on distinct tuples of $\{(u_j, \theta_{i_j}), j = 1, \ldots, d : (\log B_n)^2 \le j_1 \le \cdots \le j_d \le B_n - (\log B_n)^2, \frac{n}{2N} < u_j < 1 - \frac{n}{2N}\}$ such that for any two tuples $(u_{j_1}, \theta_{i_{j_1}})$ and $(u_{j_2}, \theta_{i_{j_2}})$, if $u_{j_1} = u_{j_2}$ then $|\theta_{i_{j_1}} - \theta_{i_{j_2}}| \ge (\log B_n)^2/B_n$; if $\theta_{i_{j_1}} = \theta_{i_{j_2}}$ then $|u_{j_1} - u_{j_2}| \ge \frac{n}{N}(1 - 1/(\log B_n)^2)$.

According to Eq. (121), we have that

$$\begin{aligned} &\mathbb{P}\left(\frac{|\boldsymbol{W}'_n|_d}{\sqrt{nB_n}} \ge y_n - \frac{1}{\log B_n}\right) - \mathcal{O}(e^{-(\log B_n)^3}) \\ &\le \mathbb{P}\left(\frac{|\boldsymbol{W}_n|_d}{\sqrt{nB_n}} \ge y_n\right) \le \mathbb{P}\left(\frac{|\boldsymbol{W}'_n|_d}{\sqrt{nB_n}} \ge y_n - \frac{1}{\log B_n}\right) + \mathcal{O}(e^{-(\log B_n)^3}). \end{aligned} \tag{124}$$

From Eq. (122), we have that

$$\left|\frac{\mathrm{Cov}^{1/2}(\boldsymbol{W}'_n)}{\sqrt{nB_n}} - 2\pi \boldsymbol{I}_d\right| = \mathcal{O}(1/(\log B_n)^2), \tag{125}$$

so that for a standard normally distributed R^d-valued random vector, $\tilde{W}$, the tail probability of $\frac{\mathrm{Cov}^{1/2}(\boldsymbol{W}'_n)}{\sqrt{nB_n}}\tilde{W} - 2\pi\boldsymbol{I}_d\tilde{W}$ satisfies that

$$\begin{aligned} &\mathbb{P}\left(\left|\left(\frac{\mathrm{Cov}^{1/2}(\boldsymbol{W}'_n)}{\sqrt{nB_n}} - 2\pi\boldsymbol{I}_d\right)\tilde{W}\right| \ge 1/\log B_n\right) \\ &= \mathcal{O}(e^{-(\log B_n)^2/4}). \end{aligned} \tag{126}$$

Putting together the above results we can use $2\pi|\tilde{W}|_d$ (recall that we defined the minimum of $\{z_i\}$ by $|z|_d := \min_{1\le i\le d}\{z_i\}$) instead of $\frac{|\boldsymbol{W}'_n|_d}{\sqrt{nB_n}}$ to bound the tail probability of $\frac{|\boldsymbol{W}_n|_d}{\sqrt{nB_n}}$:

$$\begin{aligned}
&\mathbb{P}(2\pi|\tilde{W}|_d \geq y_n - 2/\log B_n) - \mathcal{O}(e^{-(\log B_n)^2/4}) \\
&\leq \mathbb{P}\left(\frac{|\boldsymbol{W}_n|_d}{\sqrt{nB_n}} \geq y_n\right) \\
&\leq \mathbb{P}(2\pi|\tilde{W}|_d \geq y_n - 2/\log B_n) + \mathcal{O}(e^{-(\log B_n)^2/4}).
\end{aligned} \tag{127}$$

Using the following approximation of tail probability of a standard normally distributed random variable Z,

$$\mathbb{P}(Z > z) = 1 - \Phi(z) \leq \frac{1}{z\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right), \tag{128}$$

we can get that

$$\mathbb{P}\left(|Z| > \frac{y_n}{2\pi}\right) = 2\,\mathbb{P}\left(Z > \frac{y_n}{2\pi}\right) \leq \sqrt{8\pi} y_n^{-1} \exp\left(-\frac{y_n^2}{8\pi^2}\right). \tag{129}$$

Then we have shown that

$$\mathbb{P}\left(\frac{|\boldsymbol{W}_n|_d}{\sqrt{nB_n}} \geq y_n\right) = (1 + o(1))\left(\sqrt{8\pi} y_n^{-1} \exp\left(-\frac{y_n^2}{8\pi^2}\right)\right)^d. \tag{130}$$

Similarly, using Lemma A.13 and Lemma A.11(ii), we can also have that

$$\begin{aligned}
&\mathbb{P}\left(\left|\frac{\sum_{j=1}^{k_n} \bar{U}_j(u_k, \theta_{i_k})}{\sqrt{nB_n} f(u_k, \theta_{i_k})}\right| \geq y_n, k = 1, \ldots, d\right) \\
&\leq C\left(\sqrt{8\pi} y_n^{-1} \exp\left(-\frac{y_n^2}{8\pi^2}\right)\right)^{d-2} y_n^{-2} \exp\left(-\frac{y_n^2}{8\pi^2}(1+\delta)\right),
\end{aligned} \tag{131}$$

for some $\delta > 0$, uniformly on distinct tuples of $\{(u_j, \theta_{i_j}), j = 1, \ldots, d : (\log B_n)^2 \leq j_1 \leq \cdots \leq j_d \leq B_n - (\log B_n)^2, \frac{n}{2N} < u_j < 1 - \frac{n}{2N}\}$ such that for any two tuples $(u_{j_1}, \theta_{i_{j_1}})$ and $(u_{j_2}, \theta_{i_{j_2}})$with $j_1 \leq j_2$, if $u_{j_1} = u_{j_2}$ then if $\theta_{i_{j_1}} = \min_j \theta_{i_j}$ then $|\theta_{i_{j_1}} - \theta_{i_{j_2}}| \geq B_n^{-1}$; otherwise $|\theta_{i_{j_1}} - \theta_{i_{j_2}}| \geq (\log B_n)^2/B_n$; if $\theta_{i_{j_1}} = \theta_{i_{j_2}}$ then $|u_{j_1} - u_{j_2}| \geq \frac{n}{N}(1 - 1/(\log B_n)^2)$.

(4) Finally, we define

$$A_{u,i} = \left\{\frac{|\sum_{j=1}^{k_n} \bar{U}_j(u, \theta_i)|^2}{4\pi^2 nB_n f^2(u, \theta_i)} \geq 2\log B_n + 2\log C_n - \log(\pi \log B_n + \pi \log C_n) + x\right\} \tag{132}$$

and we show

$$\mathbb{P}\left(\bigcup_{(\log B_n)^2 \le i \le B_n - (\log B_n)^2, u \in \mathcal{U}} A_{u,i}\right) \to 1 - e^{-e^{-x/2}}. \tag{133}$$

To this end, we define

$$\tilde{A}_u = \bigcup_{(\log B_n)^2 \le i \le B_n - (\log B_n)^2} A_{u,i} \tag{134}$$

and

$$P_{t,u} := \sum_{(\log B_n)^2 \le i_1 < \cdots < i_t \le B_n - (\log B_n)^2} \mathbb{P}(A_{u,i_1} \cap \cdots \cap A_{u,i_t}). \tag{135}$$

Then by Bonferroni's inequality, we have for every fixed k and u that

$$\sum_{t=1}^{2k} (-1)^{t-1} P_{t,u} \le \mathbb{P}(\tilde{A}_u) \le \sum_{t=1}^{2k-1} (-1)^{t-1} P_{t,u}. \tag{136}$$

Next following the proof of [Wat54, Theorem] and [WN67, Theorem 3.3] based on Eq. (123) and Eq. (131), we can show that

$$P_{t,u} \to [B_n \mathbb{P}(A_{u,i})]^t / t! \tag{137}$$

as $n \to \infty$. As shown in [Wat54, pp.799], with Eq. (123) and Eq. (131), when $n \to \infty$, we have that

$$P_{t,u} \to [(B_n - 2(\log B_n)^2)^t / t! + \mathcal{O}(B_n - 2(\log B_n)^2)^{t-1}] \mathbb{P}(A_{u,i})^t. \tag{138}$$

Therefore, we have shown that

$$\mathbb{P}(\tilde{A}_u) \to 1 - e^{-[B_n \mathbb{P}(A_{u,i})]}. \tag{139}$$

Finally, we use the above techniques again to show

$$\mathbb{P}\left(\bigcup_{u \in \mathcal{U}} \tilde{A}_u\right) \to 1 - e^{-e^{-x/2}}, \tag{140}$$

which means we only need to show

$$C_n \mathbb{P}(\tilde{A}_u) \to \exp(-x/2). \tag{141}$$

Letting $y_n^2/4\pi^2 = 2\log B_n + 2\log C_n - \log(\pi \log B_n + \pi \log C_n) + x$, as in Eq. (123), we have that

$$
\begin{aligned}
&C_n \mathbb{P}(\tilde{A}_u) \to C_n B_n \mathbb{P}(A_{u,i}) \to C_n B_n \mathbb{P}\left(|N| > \frac{y_n}{2\pi}\right) \\
&\to \frac{C_n B_n}{y_n}\sqrt{8\pi}\exp\left(-\frac{y_n^2}{8\pi^2}\right) \\
&\to C_n B_n \frac{\sqrt{8\pi}}{\sqrt{8\pi^2}\sqrt{\log B_n + \log C_n}} \exp\left(-\frac{x}{2}\right) \frac{\sqrt{\pi \log B_n + \pi \log C_n}}{B_n C_n} \\
&\to \exp\left(-\frac{x}{2}\right).
\end{aligned} \tag{142}
$$

A.13. **Proof of Remark 4.2.** First of all, by the assumption GMC(2)

$$
\begin{aligned}
\mathbb{E}\hat{f}_n(u,\theta) - f(u,\theta) &= \frac{1}{2\pi}\left[\sum_{k=-B_n}^{B_n} \mathbb{E}\hat{r}(u,k)a(k/B_n) - \sum_{k\in\mathbb{Z}} r(u,k)\right]\exp(\sqrt{-1}k\theta) \\
&= \frac{1}{2\pi}\sum_{k=-B_n}^{B_n}\left[\mathbb{E}\hat{r}(u,k)a(k/B_n) - r(u,k)\right]\exp(\sqrt{-1}k\theta) + \mathcal{O}(\rho^{B_n}).
\end{aligned} \tag{143}
$$

Next, by the SLC condition, we know $r(u,k)$ is Lipschitz. Together with the Lipschitz condition of $\tau(\cdot)$, we have that

$$
\mathbb{E}\hat{r}(u,k) = \frac{1}{n}\sum_{i=1}^{N} \tau\left(\frac{i-\lfloor uN\rfloor}{n}\right)\tau\left(\frac{i+k-\lfloor uN\rfloor}{n}\right)\mathbb{E}(X_i X_{i+k}) \tag{144}
$$

$$
= \frac{1}{n}\sum_{i=\lfloor uN\rfloor - \frac{n}{2}}^{\lfloor uN\rfloor + \frac{n}{2}} \tau\left(\frac{i-\lfloor uN\rfloor}{n}\right)\tau\left(\frac{i+k-\lfloor uN\rfloor}{n}\right)\left[r(i/N,k) + \mathcal{O}(k/N)\right] \tag{145}
$$

$$
= \frac{1}{n}\sum_{i=\lfloor uN\rfloor - \frac{n}{2}}^{\lfloor uN\rfloor + \frac{n}{2}} \left[\tau\left(\frac{i-\lfloor uN\rfloor}{n}\right)^2 + o(k/n)\right] r(i/N,k) + \mathcal{O}(k/N). \tag{146}
$$

Since $r(u,k)$ is twice continuously differentiable with respect to u, we have that

$$
\mathbb{E}\hat{r}(u,k) = \frac{1}{n}\sum_{i=\lfloor uN\rfloor - \frac{n}{2}}^{\lfloor uN\rfloor + \frac{n}{2}} \tau\left(\frac{i-\lfloor uN\rfloor}{n}\right)^2\left[r(u,k) + \left(\frac{i-\lfloor uN\rfloor}{N}\right)\frac{\partial r(u,k)}{\partial u} + \mathcal{O}(n^2/N^2)\right] \tag{147}
$$

$$
+ o(k/n) r(i/N,k) + \mathcal{O}(k/N). \tag{148}
$$

Furthermore, since $\tau(\cdot)$ is an even function

$$\frac{1}{n}\sum_{i=\lfloor uN\rfloor-\frac{n}{2}}^{\lfloor uN\rfloor+\frac{n}{2}}\tau\left(\frac{i-\lfloor uN\rfloor}{n}\right)^2\left(\frac{i-\lfloor uN\rfloor}{N}\right)\frac{\partial r(u,k)}{\partial u}=0. \tag{149}$$

Therefore, we have that

$$\mathbb{E}\hat{r}(u,k)=\left[\int\tau^2(x)\mathrm{d}x+o(1/n)\right]r(u,k)+\mathcal{O}(n^2/N^2)+o(k/n)r(u,k)+\mathcal{O}(k/N) \tag{150}$$

$$=r(u,k)+o(k/n+1/n)r(u,k)+\mathcal{O}(k/N+n^2/N^2). \tag{151}$$

Therefore, by the locally quadratic property of $a(\cdot)$ at 0, we have that

$$\begin{aligned}&\mathbb{E}\hat{r}(u,k)a(k/B_n)-r(u,k)\\&=\mathbb{E}\hat{r}(u,k)\left[a(0)+a'(0)k/B_n+\frac{1}{2}a''(0)k^2/B_n^2+o(k^2/B_n^2)\right]-r(u,k)\\&=-C\left(\frac{k^2}{B_n^2}+o(k/n)\right)r(u,k)+\mathcal{O}(k/N+n^2/N^2).\end{aligned} \tag{152}$$

Then, using the fact that if $\theta\notin\{0,\pi\}$, we know that

$$\sum_{k=0}^{B_n}\cos(k\theta)=\frac{1}{2}+\frac{\sin(\frac{2B_n+1}{2}\theta)}{2\sin(\theta/2)},\quad\sum_{k=1}^{B_n}\sin(k\theta)=\frac{\sin\frac{B_n\theta}{2}\sin\frac{(B_n+1)\theta}{2}}{\sin(\theta/2)}. \tag{153}$$

Then, for fixed $\theta\notin\{0,\pi\}$, we have that

$$\sum_{k=0}^{B_n}\cos(k\theta)=\mathcal{O}(1),\quad\sum_{k=0}^{B_n}k\cos(k\theta)=\mathcal{O}(B_n). \tag{154}$$

If $\sup_u\sum_{k\in\mathbb{Z}}|r(u,k)|k^2<\infty$ and $B_n=o(n)$, then

$$\mathbb{E}\hat{f}_n(u,\theta)-f(u,\theta)+\frac{C}{2\pi}\sum_{k\in\mathbb{Z}}\frac{k^2r(u,k)\exp(\sqrt{-1}k\theta)}{B_n^2}=\mathcal{O}(B_n/N+n^2/N^2). \tag{155}$$

Finally, $B_n=o(N^{1/3})$ implies $\mathcal{O}(B_n/N)=o(1/B_n^2)$. Also, $n=o(N^{2/3})$ and $B_n=o(N^{1/3})$ implies $\mathcal{O}(B_n^2n^2/N^2)=o(1)$.

A.14. **Proof of Theorem 5.5.** For simplicity, we denote $\delta_{u,n}$ as δ_u and $\delta_{\theta,n}$ as δ_θ. First, we write

$$\hat{f}_n(u,\theta) - \hat{f}_n(u_i,\theta_j) = \hat{f}_n(u,\theta) - \hat{f}_n(u_i,\theta_j) - \mathbb{E}[\hat{f}_n(u,\theta) - \hat{f}_n(u_i,\theta_j)] + \mathbb{E}[\hat{f}_n(u,\theta) - \hat{f}_n(u_i,\theta_j)]. \tag{156}$$

Then by continuity we have that

$$\max_{\{u_i,\theta_j\}} \sup_{\{u:|u-u_i|\le\delta_u,\theta:|\theta-\theta_j|\le\delta_\theta\}} |\mathbb{E}\hat{f}_n(u,\theta) - \mathbb{E}\hat{f}_n(u_i,\theta_j)| = o_{\mathbb{P}}(\sqrt{\log n}). \tag{157}$$

Letting $\hat{g}_n(u,u_i,\theta,\theta_j) := \hat{f}_n(u,\theta) - \hat{f}_n(u_i,\theta_j)$, it suffices to show that

$$\max_{\{u_i,\theta_j\}} \sup_{\{u:|u-u_i|\le\delta_u,\theta:|\theta-\theta_j|\le\delta_\theta\}} |\hat{g}_n(u,u_i,\theta,\theta_j) - \mathbb{E}\hat{g}_n(u,u_i,\theta,\theta_j)| = o_{\mathbb{P}}(\sqrt{\log n}). \tag{158}$$

Note that

$$\hat{g}_n(u,u_i,\theta,\theta_j) = [\hat{f}_n(u,\theta) - \mathbb{E}\hat{f}_n(u,\theta)]\left[1 - \frac{\hat{f}_n(u_i,\theta_j)}{\hat{f}_n(u,\theta)}\right] + \mathbb{E}\hat{f}_n(u,\theta)\left[1 - \frac{\hat{f}_n(u_i,\theta_j)}{\hat{f}_n(u,\theta)}\right]. \tag{159}$$

Then we can write

$$\begin{aligned} &\sup_{\{u,\theta\}} \hat{g}_n(u,u_i,\theta,\theta_j) \\ &\le \sup_{\{u,\theta\}} \left[\hat{f}_n(u,\theta) - \mathbb{E}\hat{f}_n(u,\theta)\right] \sup_{\{u,\theta\}} \left|\frac{\hat{f}_n(u_i,\theta_j)}{\hat{f}_n(u,\theta)} - 1\right| \\ &+ \sup_{\{u,\theta\}} \mathbb{E}\hat{f}_n(u,\theta) \sup_{\{u,\theta\}} \left|\frac{\hat{f}_n(u_i,\theta_j)}{\hat{f}_n(u,\theta)} - 1\right|. \end{aligned} \tag{160}$$

Since by Theorem 4.1, we have that

$$\sup_{\{u,\theta\}} [\hat{f}_n(u,\theta) - \mathbb{E}\hat{f}_n(u,\theta)] = \mathcal{O}_{\mathbb{P}}(\sqrt{\log n}). \tag{161}$$

Therefore, the following result completes the proof.

Lemma A.15. *If $\delta_u = \mathcal{O}(\frac{n}{N(\log n)^\alpha})$ and $\delta_\theta = \mathcal{O}(\frac{1}{B_n(\log n)^\alpha})$ for some $\alpha > 0$, then*

$$\max_{\{u_i,\theta_j\}} \sup_{\{u:|u-u_i|\le\delta_u,\theta:|\theta-\theta_j|\le\delta_\theta\}} \left| \frac{\hat{f}_n(u_i,\theta_j)}{\hat{f}_n(u,\theta)} - 1 \right| = o_{\mathbb{P}}(1). \tag{162}$$

Proof. See Appendix A.15. □

A.15. **Proof of Lemma A.15.** First, we pick any (u_0, θ_0) such that $|u_0 - u| \le \delta_u$ and $|\theta_0 - \theta| \le \delta_\theta$. Then

$$\hat{f}_n(u_0,\theta_0) - \hat{f}_n(u,\theta) = \frac{1}{2\pi}\sum_{k=-B_n}^{B_n} a(k/B_n)[\hat{r}(u_0,k)\exp(\sqrt{-1}k\theta_0) - \hat{r}(u,k)\exp(\sqrt{-1}k\theta)]. \tag{163}$$

Using $\tau\left(\frac{i-\lfloor u_0N\rfloor}{n}\right) = \tau\left(\frac{i-\lfloor uN\rfloor}{n}\right) + \mathcal{O}\left(\frac{\delta_u N}{n}\right)$, we have that

$$\hat{r}(u_0,k)\exp(\sqrt{-1}k\theta_0) = \frac{1}{n}\sum_{i=1}^{N}\tau\left(\frac{i-\lfloor u_0N\rfloor}{n}\right)\tau\left(\frac{i+k-\lfloor u_0N\rfloor}{n}\right)(X_iX_{i+k})\exp(\sqrt{-1}k\theta_0) \tag{164}$$

$$= \frac{1}{n}\sum_{i=\lfloor uN\rfloor-\frac{n}{2}}^{\lfloor uN\rfloor+\frac{n}{2}}\left[\tau\left(\frac{i-\lfloor uN\rfloor}{n}\right)\tau\left(\frac{i+k-\lfloor uN\rfloor}{n}\right) + \mathcal{O}\left(\frac{\delta_u N}{n}\right)\right](X_iX_{i+k})\exp(\sqrt{-1}k\theta_0). \tag{165}$$

Note that $\exp(\sqrt{-1}k\theta_0) = \exp(\sqrt{-1}k\theta)[\exp(\sqrt{-1}k(\theta_0-\theta))]$ and $\cos(k\theta_0) = \cos(k\theta)\cos(k(\theta_0-\theta)) - \sin(k\theta)\sin(k(\theta_0-\theta))$. Therefore, we have that

$$\hat{f}_n(u_0,\theta_0) = \frac{1}{2\pi}\sum_{k=-B_n}^{B_n} a(k/B_n)\hat{r}(u_0,k)\exp(\sqrt{-1}k\theta)\exp(\sqrt{-1}k(\theta_0-\theta)) \tag{166}$$

$$= \frac{1}{2\pi}\sum_{k=-B_n}^{B_n} a(k/B_n)\hat{r}(u,k)\cos(k\theta)\left[1+\mathcal{O}\left(\frac{\delta_u N}{n}\right)\right][1+\mathcal{O}(k\delta_\theta)] \tag{167}$$

$$-\frac{1}{2\pi}\sum_{k=-B_n}^{B_n} a(k/B_n)\hat{r}(u_0,k)\sin(k\theta)\mathcal{O}(k\delta_\theta) \tag{168}$$

$$= \hat{f}_n(u,\theta)\left[1+\mathcal{O}\left(\frac{\delta_u N}{n}\right)\right][1+\mathcal{O}(B_n\delta_\theta)] + \mathcal{O}_{\mathbb{P}}(B_n\delta_\theta), \tag{169}$$

where we have used the fact that the GMC condition implies $\sum_{k=0}^{B_n} kr(u,k) = \mathcal{O}(\sum_{k=0}^{B_n} k\rho^k) = \mathcal{O}(B_n)$. Note that we have assumed that $f(u,\theta) > f_* > 0$ uniformly over u and θ, so we can write $\mathcal{O}_{\mathbb{P}}(B_n\delta_\theta) = (B_n\delta_\theta)\mathcal{O}_{\mathbb{P}}(\hat{f}_n(u,\theta))$. Therefore, we have that

$$\hat{f}_n(u_0,\theta_0) - \hat{f}_n(u,\theta) = \mathcal{O}(\delta_u N/n + B_n\delta_\theta)\mathcal{O}_{\mathbb{P}}(\hat{f}_n(u,\theta)), \tag{170}$$

which implies that

$$\left|\frac{\hat{f}_n(u_0,\theta_0)}{\hat{f}_n(u,\theta)} - 1\right| = \mathcal{O}_{\mathbb{P}}(\delta_u N/n + B_n\delta_\theta). \tag{171}$$

In order to make it equal to $o_{\mathbb{P}}(1)$, we only need $\delta_u = o(n/N)$ and $\delta_\theta = o(1/B_n)$. Therefore, choosing $\alpha > 0$, $\delta_u = \mathcal{O}\left(\frac{n}{N(\log n)^\alpha}\right)$ and $\delta_\theta = \mathcal{O}\left(\frac{1}{B_n(\log n)^\alpha}\right)$ is sufficient.

A.16. **Proof of Remark 2.9.** By the triangle inequality and Hölder's inequality, we have that

$$|r(u,k) - r(s,k)| \tag{172}$$

$$= |\mathbb{E}\left[G(u,\mathcal{F}_i)G(u,\mathcal{F}_{i+k}) - G(s,\mathcal{F}_i)G(s,\mathcal{F}_{i+k})\right]| \tag{173}$$

$$\leq \|\left[G(u,\mathcal{F}_i) - G(s,\mathcal{F}_i)\right]G(u,\mathcal{F}_{i+k})\|_1 + \|\left[G(u,\mathcal{F}_{i+k}) - G(s,\mathcal{F}_{i+k})\right]G(s,\mathcal{F}_i)\|_1 \tag{174}$$

$$\leq \|G(u,\mathcal{F}_i) - G(s,\mathcal{F}_i)\|_q\|G(u,\mathcal{F}_{i+k})\|_p + \|G(u,\mathcal{F}_{i+k}) - G(s,\mathcal{F}_{i+k})\|_q\|G(s,\mathcal{F}_i)\|_p \tag{175}$$

$$\leq C|u-s|. \tag{176}$$

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