Online Supplement to "Discerning Solution Concepts for Discrete Games"

Nail Kashaev[*] Bruno Salcedo[†]

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C. Games with Multiple Actions

Our results can be generalized to games with more than two actions. Suppose that firms choose actions from $Y_i = \{1, \ldots, Y_{d_Y}\}$, $d_Y < +\infty$. Player i's payoffs from outcome y are given by

$$\alpha_{i,y}(\mathbf{w}) + [\beta_{i,y_i}(\mathbf{w})\mathbf{z}_{i,y_i} - \mathbf{e}_{i,y_i}]. \tag{1}$$

Note that in contrast to the main that now we have an action-specific covariate and shock for every firm.

The following assumption is a standard location and scale normalizations of the payoffs.

Assumption 1

(i) $\alpha_{i,(0,y_{-i})}(w) = \beta_{i,0}(w) = 0$ for all i, y_{-i}, and w; $\mathbf{e}_{i,0} = 0$ a.s. for all i.

(ii) $\beta_{i,y_i}(w) \neq 0$ for all i, $y_i \neq 0$, and w.

Let $\mathbf{z} = (\mathbf{z}_{i,y_i})_{i\in I, y_i \in Y_i\setminus\{0\}}$ be a $d_Z = \sum_i d_{Y_i}$-dimensional vector of payoff relevant action-specific covariates; $x = (z^\mathsf{T}, w^\mathsf{T})^\mathsf{T}$ be the vector of all observed covariates; and $\mathbf{e} = (\mathbf{e}_{i,y_i})_{i\in I, y_i\in Y_i\setminus\{0\}}$ be a vector of payoff shocks. We allow shocks to be correlated and we impose no restrictions on the sign of $\alpha_{i,y}(\,\cdot\,)$.

We group all payoff parameters and $\Sigma(\,\cdot\,)$ into a single parameter $\theta \in \Theta$.

[*]Department of Economics, Western University, nail.kashaev.ru, nkashaev@uwo.ca.
[†]Department of Economics, Western University, brunosalcedo.com, bsalcedo@uwo.ca.

Proposition C.1 *Under assumptions 1–4, and 11, both θ_0 and h_0 are identified, and any solution concept S nested into $S_R(\theta)$ is discernible relative to the set of parameters that satisfy assumptions 1–3, and 11.*

Proof. Similar to the proof of Proposition 4.1 we can turn a game with many actions to a game with two actions by sending z_{i,y_i} to $+\infty$ or $-\infty$, and then apply Proposition A.2 to identify the payoff parameters. Then similarly to the proof of Proposition 4.2, identification of h_0 follows from completeness of the exponential family of distributions. The latter automatically implies discernibility of Nash solution concept in rationalizability. ■

D. Additional Details for Entry Example

D.1. Proof of Nondiscernibility of PNE and SAA

Let $f_{\mathbf{e}}$ denote the p.d.f. of $\mathbf{e}$. Our assumptions imply that $f_{\mathbf{e}}(e_1, e_2) > 0$ and $f_{\mathbf{e}}(e_1, e_2) = f_{\mathbf{e}}(e_2, e_1)$ almost everywhere on $\mathbb{R}^2$. For each possible outcome $y \in \{0, 1\}^2$, let $p_{\text{PNE}}(y; \eta')$ and $p_{\text{SAA}}(y; \eta)$ denote the probabilities of the outcome according to each of the two solution concepts under consideration.

Fix any parameter value $\eta \geq 0$. We will show that there exists some $\eta' \geq 0$ such that $p_{\text{PNE}}(y; \eta') = p_{\text{SAA}}(y; \eta)$ for every possible outcome y. If $\eta = 0$, then we can simply set $\eta' = 0$. Hence, for the rest of the proof, we assume that $\eta > 0$.

On one hand, if $\eta' = \eta$, then

$$\begin{aligned} p_{\text{PNE}}((0,0); \eta') &= \int_{\eta'}^{\infty} \int_{\eta'}^{\infty} f_{\mathbf{e}}(e_1, e_2)\, de_2\, de_1 \\ &< \int_{\eta}^{\infty} \int_{\eta}^{\infty} f_{\mathbf{e}}(e_1, e_2)\, de_2\, de_1 + \int_{0}^{\eta} \int_{0}^{\eta} f_{\mathbf{e}}(e_1, e_2)\, de_2\, de_1 \\ &= p_{\text{SAA}}((0,0); \eta). \end{aligned}$$

(See Figure 1). On the other hand, if $\eta' = 0$, then

$$p_{\text{PNE}}((0,0); \eta') = \int_{0}^{\infty} \int_{0}^{\infty} f_{\mathbf{e}}(e_1, e_2)\, de_2\, de_1$$

$$
\begin{aligned}
&> \int_{\eta}^{\infty}\int_{\eta}^{\infty} f_{\mathbf{e}}(e_1,e_2)\,de_2\,de_1 + \int_0^{\eta}\int_0^{\eta} f_{\mathbf{e}}(e_1,e_2)\,de_2\,de_1 \\
&= p_{\mathrm{SAA}}((0,0);\eta).
\end{aligned}
$$

Since $p_{\mathrm{PNE}}((0,0);\eta')$ is continuous in η', there exists some $\eta' \in (0,\eta)$ such that $p_{\mathrm{PNE}}((0,0);\eta') = p_{\mathrm{SAA}}((0,0);\eta)$. Fix such η'.

Since $p_{\mathrm{PNE}}((1,1);\eta') = p_{\mathrm{SAA}}((1,1);\eta)$ and there are only four possible outcomes it follows that

$$
p_{\mathrm{PNE}}((1,0);\eta') + p_{\mathrm{PNE}}((0,1);\eta') = p_{\mathrm{SAA}}((1,0);\eta) + p_{\mathrm{SAA}}((0,1);\eta).
$$

Now, we will show that $p_{\mathrm{PNE}}((1,0);\eta') = p_{\mathrm{PNE}}((0,1);\eta')$ and $p_{\mathrm{SAA}}((1,0);\eta) = p_{\mathrm{SAA}}((0,1);\eta)$. This implies that the probabilities of all outcomes are the same under both solution concepts. For PNE, we have that

$$
\begin{aligned}
p_{\mathrm{PNE}}((1,0);\eta') &= \int_{-\infty}^{0}\int_{0}^{\infty} f_{\mathbf{e}}(e_1,e_2)\,de_2\,de_1 + \int_0^{\eta'}\int_{e_1}^{\eta'} f_{\mathbf{e}}(e_1,e_2)\,de_2\,de_1 \\
&= \int_{-\infty}^{0}\int_{0}^{\infty} f_{\mathbf{e}}(e_2,e_1)\,de_1\,de_2 + \int_0^{\eta'}\int_{e_2}^{\eta'} f_{\mathbf{e}}(e_2,e_1)\,de_1\,de_2 \\
&= \int_{-\infty}^{0}\int_{0}^{\infty} f_{\mathbf{e}}(e_1,e_2)\,de_1\,de_2 + \int_0^{\eta'}\int_{e_2}^{\eta'} f_{\mathbf{e}}(e_1,e_2)\,de_1\,de_2 \\
&= p_{\mathrm{PNE}}((0,1);\eta'),
\end{aligned}
$$

where the second equality follows from using the change of variables $(e_1,e_2) \to (e_2,e_1)$, and the third one from the symmetry fo $f_{\mathbf{e}}$. The argument for SAA is completely analogous. ■

D.2. Discernibility of PNE and SAA with an Excluded Covariate

Let us consider a modified version of the entry example. Suppose that everything is as in Section 2, except that firm i's profit is given by

$$
y_i \cdot \Big[\eta_0(1-y_{-i}) + \mathbf{z} - \mathbf{e}_i\Big],
$$

where $\mathbf{z}$ is a covariate supported on the whole real line, independent of $\mathbf{e}$, and such that Assumption 3 holds. The researcher observes the joint distribution of outcomes $(\mathbf{y}_1,\mathbf{y}_2,\mathbf{z})$. We claim that, with this added covariate, PNE and SAA are no longer observationally equivalent. Now, if the firm behavior corresponds to SAA, then it cannot be explained by PNE, and vice versa.

Proposition D.1 *In the entry example with a covariate, for every $\eta, \eta' > 0$ there exists $z \in \mathbb{R}$ such that SSA with $\eta_0 = \eta$ and PNE with $\eta_0 = \eta'$ imply different outcome distributions conditional on $\mathbf{z} = z$.*

Proof. It suffices to consider the probability of no entry, i.e., $\mathbf{y} = (0, 0)$. Fix any $\eta, \eta' > 0$. If $\eta' \geq \eta$, then PNE implies a higher probability of no entry than SAA regardless of the realization of $\mathbf{z}$ (see Figure 1). Hence, we assume for the rest of the proof that $\eta > \eta'$.

The probability of no entry conditional on $\mathbf{z} = z$ is $[1 - \Phi(z + \eta)]^2$ under PNE and $[\Phi(z + \eta') - \Phi(z)]^2 + [1 - \Phi(z + \eta')]^2$ under SAA. Let $\chi : \mathbb{R} \to \mathbb{R}$ give the difference between these probabilities as a function of z, i.e.,

$$\chi(z) = [\Phi(z + \eta') - \Phi(z)]^2 + [1 - \Phi(z + \eta')]^2 - [1 - \Phi(z + \eta)]^2.$$

We will show that there exist numbers z such that $\chi(z) \neq 0$.

Note that χ is differentiable and

$$\begin{aligned}\chi'(z) = {} & 2[\Phi(z + \eta') - \Phi(z)][\phi(z + \eta') - \phi(z)] \\ & - 2[1 - \Phi(z + \eta')]\phi(z + \eta') + 2[1 - \Phi(z + \eta)]\phi(z + \eta).\end{aligned}$$

Let $z^* := -\eta'/2 < 0$, so that $z^* = -(z^* + \eta')$ and $z^* + \eta' > 0$. Since $\phi(\,\cdot\,)$ is symmetric around 0, this implies that $\phi(z^*) = \phi(z^* + \eta')$. Therefore the first term of $\chi'(z^*)$ is equal to zero and we have

$$\begin{aligned}\chi'(z^*) &= -2[1 - \Phi(z^* + \eta')]\phi(z^* + \eta') + 2[1 - \Phi(z^* + \eta)]\phi(z^* + \eta) \\ &= 2[\phi(z^* + \eta) - \phi(z^* + \eta')] + 2\Phi(z^* + \eta')\phi(z^* + \eta') - 2\Phi(z^* + \eta)\phi(z^* + \eta) \\ &< 2\Phi(z^* + \eta')\phi(z^* + \eta) - 2\Phi(z^* + \eta)\phi(z^* + \eta) \\ &= -2\phi(z^* + \eta)[\Phi(z^* + \eta) - \Phi(z^* + \eta')] < 0,\end{aligned}$$

where the first inequality follows because ϕ is decreasing on the positive real line, and thus $\phi(z^* + \eta') > \phi(z^* + \eta)$. Since $\chi'(z^*) \neq 0$, there exists an open set Z' such that $\chi(z) \neq 0$ for all $z \in Z'$. ∎

Note that the proof of Proposition D.1 considers only the probability of $\mathbf{y} =$

$(0,0)$. The probability of this outcome in the multiplicity region is zero under any PNE. Hence, the conclusion of the proposition does not depend on any assumptions about equilibrium selection.

D.3. Alternative Entry Subsidy

A form of subsidy that is more common in practice consists of giving a lump sum subsidy $\hat{\tau} > 0$ to any firm that enters a market with some observable characteristics (see, e.g., Goolsbee (2002)). Under the PNE assumption, every market that would be served without the policy would also be served with the policy. Hence, the policy has an unambiguously positive effect (abstracting from the cost). However, this need not be the case under SAA.

Proposition D.2 *Suppose that firms profits are given by*

$$\boldsymbol{\pi}_i(y) = y_i \cdot \Big[\alpha + \eta(1 - y_{-i}) - \mathbf{e}_i\Big],$$

firms make entry decisions in accordance with the SAA model, and $\mathbf{e}$ *is normally distributed with zero mean and the identity matrix as a covariance matrix. There exists an open set* $\Xi \subseteq \mathbb{R}^2$ *and a threshold* $\bar{\tau}$ *such that if* $(\alpha, \eta) \in \Xi$ *and* $\hat{\tau} < \bar{\tau}$*, then the probability that a market is not served is increasing in the size of the subsidy.*

Proof. Under strategic ambiguity there is no entry if either $\mathbf{e}_i > \alpha + \eta + \hat{\tau}$ for $i = 1, 2$, or $\alpha + \tau < \mathbf{e}_i < \alpha + \eta + \hat{\tau}$ for $i = 1, 2$ (See Figure 1). Hence, the probability that a market is not served as a function of $\hat{\tau}$ is given by

$$P(\hat{\tau}) = \Big[1 - \Phi(\alpha + \eta + \hat{\tau})\Big]^2 + \Big[\Phi(\alpha + \eta + \hat{\tau}) - \Phi(\alpha + \hat{\tau})\Big]^2. \tag{2}$$

Taking derivatives

$$\begin{aligned} P'(\hat{\tau}) = &-2\phi(\alpha + \eta + \hat{\tau})\Big[1 - \Phi(\alpha + \eta + \hat{\tau})\Big] \ldots \\ &\ldots + 2\Big[\phi(\alpha + \eta + \hat{\tau}) - \phi(\alpha + \hat{\tau})\Big] \cdot \Big[\Phi(\alpha + \eta + \hat{\tau}) - \Phi(\alpha + \hat{\tau})\Big]. \end{aligned} \tag{3}$$

Evaluating when $\hat{\tau} = 0$, $\alpha < 0$, and $\eta = -\alpha + \sqrt{-\alpha}$ yields

$$\frac{P'(0)}{2\phi(-\alpha)} = \left[\Phi\left(\sqrt{-\alpha}\right) - 1\right] + \left[1 - \frac{\phi(\alpha)}{\phi\left(\sqrt{-\alpha}\right)}\right] \cdot \left[\Phi\left(\sqrt{-\alpha}\right) - \Phi(\alpha)\right] \qquad (4)$$

When $\alpha \longrightarrow -\infty$, the right-hand side converges to 1. Hence, we must have $P'(0) > 0$ when $-\alpha$ is sufficiently large. Since P is continuous, this must also be true in an open set. ■

References

Goolsbee, A. (2002). Subsidies, the value of broadband, and the importance of fixed costs. In R. Crandall & J. H. Alleman (Eds.), *Broadband: Should we regulate high-speed internet access?* (pp. 278–294). Brookings Institution Press.