12 POLYNOMIALS AND RATES OF CHANGE

Motivating Questions:

- What are the important characteristics of polynomials and how do these characteristics determine the shape of the graph?
- What is end behavior? What information do you need about a polynomial to determine its end behavior?
- What is instantaneous rate of change and how does it relate to average rate of change?

Reading Apprenticeship Assignment:

- Based on the example in 12.3.4, write down a method for how to give a sketch of a polynomial.

12.1 POLYNOMIALS

Some of the most essential functions to the study of calculus are polynomials. A **polynomial**, *in standard form*, is a function that looks like

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots\cdots + a_1 x + a_0,$$

where $n \geq 0$ is an integer and the **coefficients**, $a_n \neq 0$, a_{n-1}, $\cdots\cdots$, a_0 are all real numbers. The term $a_n x^n$ with $a_n \neq 0$, is called the **leading term** of the polynomial. The power of x in the leading term is called the **degree of the polynomial**. In other words, the degree of the polynomial is the highest power of x that appears in the polynomial.

For example, the polynomial $f(x) = 2x + 5x^4 + 3$ is written in standard form as:

$$f(x) = 5x^4 + 0x^3 + 0x^2 + 2x + 3$$

This polynomial has leading term $5x^4$ and degree 4.

Lastly, every polynomial has a domain of all real numbers, $(-\infty, \infty)$.

12.1.1 Check your understanding

1. Determine which of the following functions are polynomials. Also, for the functions that are polynomials, identify the leading term and the degree of the polynomial. (Can you do with *without* simplifying?)

a. $y = e^x$

b. $y = 2x + 1$

c. $y = 5x^5 - 3x^2 + 4 - 1 - 6x^7$

e. $g(x) = |x|$

d. $h(x) = (2x + 3)^2 + 1$

f. $f(x) = 1$

2. Why is it the case that polynomial functions have a domain of all real numbers?

12.1.2 A Quick Note on the Different Types of Polynomials

We have seen examples of many types of polynomials in earlier sections. A degree 0 polynomial is called a **constant** function. A degree 1 polynomial is called a **linear** function and looks like $f(x) = a_1x + a_0$; a degree 2 polynomial is a **quadratic** function and looks like $f(x) = a_2x^2 + a_1x + a_0$; a degree 3 polynomial is a **cubic** function and looks like $f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, and so on.

12.2 The End Behavior of a Polynomial

We already know how to graph a polynomial of degree 0, 1, or 2 since polynomials of these degrees are either constant functions, linear functions, or quadratic functions. Graphing polynomials of degree greater than two is somewhat more complicated. However, because of the nature of calculus, it is more important to have a rough idea of what a function looks like rather (the general *shape*) than being able to graph it perfectly. One characteristic of a polynomial that will help determine its shape is its end behavior.

12.2.1 End Behavior of a polynomial

The **end behavior** of a polynomial is what happens to the output of the polynomial when we let the input get very, very big and very, very small. More precisely, we see what happens to the function when x approaches ∞ and when x approaches $-\infty$. This is very cumbersome to write, so we will use the following notation. Let $f(x)$ be a function then we will write the *process* of "letting x go to ∞ and observing the output of $f(x)$" as

$$\lim_{x \to \infty} f(x)$$