Proof of Theorem 1

Consider the information matrix $I(\theta, \xi_t)$ for a single design time t. Then the set $\{1, e^{-2\alpha_2/t}, \frac{1}{t}e^{-2\alpha_2/t}\}$ forms a Tchebycheff system on any fixed interval. Then it follows from the results of Karlin and Studden (1966, p.333) and of Fedorov(1972, pp.85-86) that, for any fixed parameters, the D-optimal design is based on exactly two points of support.

As observed by Mukhopadhyay and Haines (1995), a necessary and sufficient condition for a design ξ to be locally D-optimal is

$$d_{\boldsymbol{\theta}}(\xi, t) = tr[\mathbf{I}(\boldsymbol{\theta}, \xi_t)\mathbf{I}(\boldsymbol{\theta}, \xi)^{-1}] - 2 \leq 0, \tag{1}$$

for all t in the design space.

For any design point t in the design space, it is easy to see that

$$d_{\boldsymbol{\theta}}(\xi, t) = 2e^{-\frac{2\alpha_2}{t}} \frac{e^{-\frac{2\alpha_2}{t_1}}(\frac{1}{t} - \frac{1}{t_1})^2 + e^{-\frac{2\alpha_2}{t_2}}(\frac{1}{t} - \frac{1}{t_2})^2}{e^{-2\alpha_2(\frac{1}{t_1}+\frac{1}{t_2})}(\frac{1}{t_2} - \frac{1}{t_1})^2}, \tag{2}$$

and we want $d_{\boldsymbol{\theta}}(\xi, t) \leq 0$ for all points between $t_{\min}$ and $t_{\max}$. Taking the derivative with respect to t, we have

$$d'_{\boldsymbol{\theta}}(\xi, t) = \frac{4e^{-\frac{\alpha_2}{t}}}{t^2 e^{-2\alpha_2(\frac{1}{t_1}+\frac{1}{t_2})}(\frac{1}{t_2} - \frac{1}{t_1})^2} \times (\alpha - 1)[e^{\frac{2\alpha_2}{t_1}}(\frac{1}{t} - \frac{1}{t_1})^2 + e^{\frac{2\alpha_2}{t_2}}(\frac{1}{t} - \frac{1}{t_2})^2]. \tag{3}$$

Plugging in t_2 in (3), we find $d'_{\boldsymbol{\theta}}(\xi, t_2) \geq 0$, which implies $t_2 = t_{\max}$.

Next, note that ξ will be a locally D-optimal design if $d_{\boldsymbol{\theta}}(\xi, t)$ has a local maximum value at t_1. Setting $d'_{\boldsymbol{\theta}}(\xi, t) = 0$, we find $t_1 = \frac{\alpha_2 t_2}{\alpha_2 + t_2}$. If $\frac{\alpha_2 t_2}{\alpha_2 + t_2} < t_{\min}$, then $d'_\theta(\xi, t) < 0$ for $t \in [t_{\min}, t_{\max})$. It follows that $d_\theta(\xi, t)$ is a decreasing function in $[t_{\min}, t_{\max}]$; thus $d_\theta(\xi, t)$ will have its local maximum at $t_{\min}$. Consequently, $t_1^* = \max(\frac{\alpha_2 t_2}{\alpha_2 + t_2}, t_{\min})$.

Also, by the definition of D-optimal design, the above design identifies an ellipsoid which contains all points on $\mathbf{v}_1(t), t \in [t_{\min}, t_{\max}]$ with minimum volume. There will be two points, $\frac{\alpha_2 t_{\max}}{\alpha_2 + t_{\max}}, t_{\max}$ at the boundary of the above ellipsoid.

References:

Federov, V. V. (1972), *Theory of Optimal Experiments*, New York: Academic Press.

Karlin, S. and Studden, W. J. (1966), "*Tchebycheff Systems, with Applications in Analysis and Statistics*," New York: Interscience Publishers.