Supplement to "Adaptive Inference for Change Points in High-Dimensional Data"

The supplement contains all the technical proofs in Section 6 and some additional simulation results on network change-point detection in Section 7.

6 Technical Appendix

In the following, we will denote $a_n \lesssim b_n$ and $b_n \gtrsim a_n$ if $\limsup_n a_n/b_n < \infty$.

Proof of Theorem 2.2. Recall that under the null, as X_i's have the same mean,

$$D_{n,q}(r;[a,b]) = \sum_{c=0}^{q}(-1)^{q-c}\binom{q}{c}P_{q-c}^{\lfloor nr\rfloor-\lfloor na\rfloor-c}P_{c}^{\lfloor nb\rfloor-\lfloor nr\rfloor-q+c}S_{n,q,c}(r;[a,b]).$$

Therefore, we can calculate the covariance structure of G_q based on that of $Q_{q,c}$ given in Theorem 2.1.

$$\mathrm{var}[G_q(r;[a,b])] = \sum_{c=0}^{q}\binom{q}{c}^2 c!(q-c)!(r-a)^{2q-c}(b-r)^{q+c} = q!(r-a)^q(b-r)^q(b-a)^q.$$

When $r_1 < r_2$,

$$\begin{aligned}
&\mathrm{cov}(G_q(r_1;[a_1,b_1]),G_q(r_2;[a_2,b_2])) \\
=&\sum_{0\le c_1\le c_2\le q}\Big((-1)^{c_1+c_2}\binom{q}{c_1}\binom{q}{c_2}\binom{C}{c}c!(q-C)!\mathbb{1}_{r_1>a_2,r_2<b_1} \\
&\cdot(r_1-a_1)^{q-c_1}(b_1-r_1)^{c_1}(r_2-a_2)^{q-c_2}(b_2-r_2)^{c_2}(r-A)^c(R-r)^{C-c}(b-R)^{q-C}\Big).
\end{aligned}$$

When $r_1 > r_2$,

$$
\begin{aligned}
&\operatorname{cov}(G_q(r_1;[a_1,b_1]), G_q(r_2;[a_2,b_2]))\\
=&\sum_{0\le c_2\le c_1\le q}\Bigg((-1)^{c_1+c_2}\binom{q}{c_1}\binom{q}{c_2}\binom{C}{c}c!(q-C)!\mathbb{1}_{r_2>a_1,r_1<b_2}\\
&\cdot(r_1-a_1)^{q-c_1}(b_1-r_1)^{c_1}(r_2-a_2)^{q-c_2}(b_2-r_2)^{c_2}(r-A)^c(R-r)^{C-c}(b-R)^{q-C}\Bigg).
\end{aligned}
$$

When $r_1 = r_2 = r$,

$$
\begin{aligned}
&\operatorname{cov}(G_q(r;[a_1,b_1]), G_q(r;[a_2,b_2]))\\
=&\sum_{c=0}^{q}\binom{q}{c}^2 c!(q-c)!(r-a_1)^{q-c}(b_1-r)^c(r-a_2)^{q-c}(b_2-r)^c(r-A)^c(b-r)^{q-c}\\
=&q!(r-A)^q(b-r)^q(B-a)^q.
\end{aligned}
$$

For $r_1 \neq r_2$, we have

$$
\operatorname{cov}(G_q(r_1;[a_1,b_1]), G_q(r_2;[a_2,b_2])) = q![(r-A)(b-R)(B-a)-(A-a)(R-r)(B-b)]^q.
$$

For $q_1 \neq q_2$, since covariance of Q_{q_1,c_1} and Q_{q_2,c_2} is 0, we know the covariance of G_{q_1} and G_{q_2} is also 0, since their arbitrary linear combinations are also Gaussian by previous proofs, they are jointly Gaussian and therefore independence is implied by uncorrelation. The rest follows from an application of the continuous mapping theorem. $\diamondsuit$

Note that our Assumption 2.1 is a counterpart to the assumption made by Remark 3.2 in Wang et al. (2019). Their results are derived with some weaker assumption (i.e. Assumption 3.1 therein), whose L_q-norm based counterpart for is given as follows.

Assumption **6.1**. *For any $q \in 2\mathbb{N}$, the following statements hold:*

A.1 $\sum_{l_1,l_2,l_3,l_4=1}^{p}(\Sigma_{l_1l_2}\Sigma_{l_2l_3}\Sigma_{l_3l_4}\Sigma_{l_4l_1})^{q/2} = o\left(\|\Sigma\|_q^{2q}\right)$.

A.2 Z_0 has up to 8−th moments and there exists a constant C independent of n such that

$$\sum_{l_1,\ldots,l_h=1}^{p} |cum(Z_{0,l_1},...,Z_{0,l_h})|^q \leq C\|\Sigma\|_q^{qh/2},$$

for $h = 2,\ldots,8$.

We claim that Assumption 6.1 is implied by Assumption 2.1.

Proof of the claim. Define

$$S_{m,h}(l_1) := \left\{1 \leq l_2,\ldots,l_h \leq p_n : \max_{1\leq i,j\leq h} |l_i - l_j| = m\right\}.$$

By triangular inequality, $|l_1 - l_2| + |l_2 - l_3| + |l_3 - l_4| + |l_4 - l_1| \geq 2\max_{1\leq i,j\leq 4} |l_i - l_j|$, and therefore,

$$\begin{aligned}\sum_{l_1,\cdots,l_4=1}^{p} (\Sigma_{l_1l_2}\Sigma_{l_2l_3}\Sigma_{l_3l_4}\Sigma_{l_4l_1})^{q/2} &= \sum_{l_1=1}^{p_n}\sum_{m=0}^{p_n}\sum_{l_2,\ldots,l_4\in S_{m,4}(l_1)} (\Sigma_{l_1l_2}\Sigma_{l_2l_3}\Sigma_{l_3l_4}\Sigma_{l_4l_1})^{q/2}\\ &\leq \sum_{l_1=1}^{p_n}\sum_{m=0}^{p_n} |S_{m,4}(l_1)|\, C_2^{2q}(1\vee m)^{-qr}\\ &\lesssim p_n \sum_{m=0}^{p_n} (1\vee m)^{4-2-qr}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{l_1,\cdots,l_h=1}^{p} \mathrm{cum}^q(X_{0,l_1,n},\cdots,X_{0,l_h,n}) &= \sum_{l_1=1}^{p_n}\sum_{m=0}^{p_n}\sum_{l_2,\ldots,l_h\in S_{m,h}(l_1)} \mathrm{cum}^q(X_{0,l_1,n},\cdots,X_{0,l_h,n})\\ &\leq \sum_{l_1=1}^{p_n}\sum_{m=0}^{p_n} |S_{m,h}(l_1)|\, C_h^{q}(1\vee m)^{-qr}\\ &\lesssim p_n \sum_{m=0}^{p_n} (1\vee m)^{h-2-qr}.\end{aligned}$$

RHS has order $O\left(p_n^{h-qr}\right)$ if $h - qr - 1 > 0$. Now a simple computation shows that Assumption 6.1 is satisfied if $h - qr < h/2$ for $h = 2,\ldots,8$, and $q = 2,\ldots,$ which is equivalent to $r > 2$. ♢

We are now ready to introduce the following lemma, which is vital in proving the main result.

Lemma **6.1**. *Under Assumption 2.1, for any $i_1^{(h)}, i_2^{(h)}, ..., i_q^{(h)}$ that are all distinct, $h = 1, ..., 8$, and $c = 1, 2, ..., q$,*

$$\left| \sum_{l_1,\ldots,l_8=1}^{p} \delta_{n,l_1}^{q-c} \cdots \delta_{n,l_8}^{q-c} \mathbb{E}[Z_{i_1^{(1)},l_1} \cdots Z_{i_c^{(1)},l_1} \cdots Z_{i_1^{(8)},l_8} \cdots Z_{i_c^{(8)},l_8}] \right| \lesssim \|\Delta_n\|_q^{8(q-c)} \|\Sigma\|_q^{4c} \quad (1)$$

In particular, for $c = q$, we have

$$\left| \sum_{l_1,\ldots,l_8=1}^{p} \mathbb{E}[Z_{i_1^{(1)},l_1} \cdots Z_{i_c^{(1)},l_1} \cdots Z_{i_1^{(8)},l_8} \cdots Z_{i_c^{(8)},l_8}] \right| \lesssim \|\Sigma\|_q^{4q}. \quad (2)$$

In addition, for any $c = 1, 2, ..., q-1$,

$$\left| \sum_{l_1,l_2=1}^{p} \delta_{n,l_1}^{q-c} \delta_{n,l_2}^{q-c} \Sigma_{l_1,l_2}^{c} \right| = o\left(\|\Delta_n\|_q^{2(q-c)} \|\Sigma\|_q^{c} \right). \quad (3)$$

Proof of Lemma 6.1. Applying the generalized Hölder's Inequality, we obtain

$$\left| \sum_{l_1,\ldots,l_8=1}^{p} \delta_{n,l_1}^{q-c} \cdots \delta_{n,l_h}^{q-c} \mathbb{E}[Z_{i_1^{(1)},l_1} \cdots Z_{i_c^{(1)},l_1} \cdots Z_{i_1^{(8)},l_8} \cdots Z_{i_c^{(8)},l_8}] \right| = \left| \mathbb{E}\left(\prod_{u=1}^{8} \left[\sum_{l_u=1}^{p} \delta_{n,l_u}^{q-c} Z_{i_1^{(u)},l_u} \cdots Z_{i_c^{(u)},l_u} \right] \right) \right|$$
$$\leq \prod_{u=1}^{8} \left\{ \mathbb{E}\left(\left[\sum_{l_u=1}^{p} \delta_{n,l_u}^{q-c} Z_{i_1^{(u)},l_u} \cdots Z_{i_c^{(u)},l_u} \right]^8 \right) \right\}^{1/8} = \mathbb{E}\left(\left[\sum_{l_1=1}^{p} \delta_{n,l_1}^{q-c} Z_{i_1^{(1)},l_1} \cdots Z_{i_c^{(1)},l_1} \right]^8 \right)$$
$$= \sum_{l_1,\ldots,l_8=1}^{p} \delta_{n,l_1}^{q-c} ... \delta_{n,l_8}^{q-c} \mathbb{E}\left[Z_{i_1^{(1)},l_1} ... Z_{i_1^{(1)},l_8} \cdots Z_{i_c^{(1)},l_1} ... Z_{i_c^{(1)},l_8} \right] = \sum_{l_1,\ldots,l_8=1}^{p} \delta_{n,l_1}^{q-c} ... \delta_{n,l_8}^{q-c} \left(\mathbb{E}\left[Z_{i_1^{(1)},l_1} ... Z_{i_1^{(1)},l_8} \right] \right)^c,$$

since $i_1^{(1)}, i_2^{(1)}, ..., i_c^{(1)}$ are all different, and $\{Z_i\}$ are i.i.d. Again by Hölder's Inequality,

$$
\begin{aligned}
&\sum_{l_1,...,l_8=1}^{p} \delta_{n,l_1}^{q-c}...\delta_{n,l_8}^{q-c} \left(\mathbb{E}\left[Z_{i_1^{(1)},l_1}...Z_{i_1^{(1)},l_8}\right]\right)^c \\
\leq &\left\{\sum_{l_1,...,l_8=1}^{p} (\delta_{n,l_1}^{q-c}...\delta_{n,l_8}^{q-c})^{q/(q-c)}\right\}^{(q-c)/q} \left\{\sum_{l_1,...,l_8=1}^{p} \left(\mathbb{E}\left[Z_{i_1^{(1)},l_1}...Z_{i_1^{(1)},l_8}\right]\right)^{cq/c}\right\}^{c/q} \\
\lesssim &\|\Delta_n\|_q^{8(q-c)} \left\{\sum_{l_1,...,l_8=1}^{p} \sum_{\pi} \prod_{B\in\pi} cum(Z_{0,l_i}, i \in B)^q\right\}^{c/q}.
\end{aligned}
$$

The last line in the above inequalities is due to the CR inequality and the definition of joint cumulants, where π runs through the list of all partitions of $\{1, ..., 8\}$, B runs through the list of all blocks of the partition π. As all blocks in a partition are disjoint, we can further bound it as

$$
\begin{aligned}
&\|\Delta_n\|_q^{8(q-c)} \left\{\sum_{l_1,...,l_8=1}^{p} \sum_{\pi} \prod_{B\in\pi} cum(Z_{0,l_i}, i \in B)^q\right\}^{c/q} \\
=&\|\Delta_n\|_q^{8(q-c)} \left\{\sum_{\pi} \prod_{B\in\pi} \sum_{l_i=1, i\in B}^{p} cum(Z_{0,l_i}, i \in B)^q\right\}^{c/q} \lesssim \|\Delta_n\|_q^{8(q-c)} \left\{\sum_{\pi} \|\Sigma\|_q^{q\sum_{B\in\pi}|B|/2}\right\}^{c/q} \\
\lesssim&\|\Delta_n\|_q^{8(q-c)} \|\Sigma\|_q^{4c},
\end{aligned}
$$

where the first inequality in the above is due to Assumption 6.1, A.2, which is a consequence of Assumption 2.1, and the fact that there are only finite number of distinct partitions over $\{1, ..., 8\}$. This completes the proof of the first result.

For the second result, we first define $A^{\circ n}$ as the notation for the element-wise n-th power of any real matrix A, i.e. $A_{i,j}^{\circ n} = A_{i,j}^n$. Then we have

$$
\left|\sum_{l_1,l_2=1}^{p} \delta_{n,l_1}^{q-c}\delta_{n,l_2}^{q-c}\Sigma_{l_1,l_2}^c\right| = \Delta_n^{\circ(q-c)^T}\Sigma^{\circ c}\Delta_n^{\circ(q-c)} \leq \|\Delta_n^{\circ(q-c)}\|_2^2 \sigma_{\max}(\Sigma^{\circ c}),
$$

where $\sigma_{\max}$ is the largest eigenvalue. First observe that $\|\Delta_n^{\circ(q-c)}\|_2^2 = \sum_{l=1}^{p} \delta_{n,l}^{2(q-c)} = \|\Delta_n\|_{2(q-c)}^{2(q-c)}$. By properties of L_q norm, $\|\Delta_n\|_{2(q-c)} \le \|\Delta_n\|_q$, if $q \le 2(q-c)$, and $\|\Delta_n\|_{2(q-c)} \le p^{1/2(q-c)-1/q}\|\Delta_n\|_q$, if $q > 2(q-c)$. This implies $\|\Delta_n\|_{2(q-c)}^{2(q-c)} \le \max(p^{(2c-q)/q}\|\Delta_n\|_q^{2(q-c)}, \|\Delta_n\|_q^{2(q-c)})$.

Next, for any symmetric matrix A, $\sigma_{\max}(A)\| \le \|A\|_\infty = \max_{i=1,...,p}\sum_{j=1}^p |A_{i,j}|$. This, together with Assumption 2.1 (A.2), implies

$$\sigma_{\max}(\Sigma^{\circ c}) \le \max_{i=1,...,p}\sum_{j=1}^p |\Sigma_{i,j}^{\circ c}| \lesssim \max_{i=1,...,p}\sum_{j=1}^p (1 \wedge |i-j|)^{-cr} \le 1 + \sum_{m=1}^p m^{-cr} \le \infty,$$

for some $r > 2$. This is equivalent to $\sigma_{\max}(\Sigma^{\circ c}) = O(1)$. Note that $\|\Sigma\|_q^q \ge tr(\Sigma^{\circ q}) \gtrsim p$, which leads to $p^{c/q} \lesssim \|\Sigma\|_q^c$. So

$$\left|\sum_{l_1,l_2=1}^p \delta_{n,l_1}^{q-c}\delta_{n,l_2}^{q-c}\Sigma_{l_1,l_2}^c\right| \lesssim \max(p^{(2c-q)/q}\|\Delta_n\|_q^{2(q-c)}, \|\Delta_n\|_q^{2(q-c)}) = o(\|\Delta_n\|_q^{2(q-c)}\|\Sigma\|_q^c),$$

since $(2c-q)/q = c/q + (c-q)/q < c/q$, for $c = 1, 2, ..., q-1$. This completes the proof for the second result.

$\diamondsuit$

This lemma is a generalization to its counterpart in Wang et al. (2019), in which we only have $q = 2$. To prove Theorem 2.1, we need the following lemmas to show tightness and finite dimensional convergence.

Lemma **6.2**. *Under Assumption 2.1, for any $c = 0, 1, 2..., q$, and define the 3-dimensional index set $\mathcal{G}_n := \{(i/n, j/n, k/n) : i, j, k = 0, 1, ..., n\}$,*

$$\mathbb{E}[a_n^{-8}(S_{n,q,c}(r_1; [a_1, b_1]) - S_{n,q,c}(r_2; [a_2, b_2]))^8] \le C\|(a_1, r_1, b_1) - (a_2, r_2, b_2)\|^4,$$

for some constant C, any $(a_1, r_1, b_1), (a_2, r_2, b_2) \in \mathcal{G}_n$ such that $\|(a_1, r_1, b_1) - (a_2, r_2, b_2)\| > \delta/n^4$.

Proof of Lemma 6.2. By CR-inequality,

$$\begin{aligned}\mathbb{E}[(S_{n,q,c}(r_1;[a_1,b_1]) - S_{n,q,c}(r_2;[a_2,b_2]))^8] \leq & C\Big\{\mathbb{E}[(S_{n,q,c}(r_1;[a_1,b_1]) - S_{n,q,c}(r_1;[a_1,b_2]))^8] \\ & + \mathbb{E}[(S_{n,q,c}(r_1;[a_1,b_2]) - S_{n,q,c}(r_1;[a_2,b_2]))^8] \\ & + \mathbb{E}[(S_{n,q,c}(r_1;[a_2,b_2]) - S_{n,q,c}(r_2;[a_2,b_2]))^8]\Big\}.\end{aligned}$$

We shall only analyze $\mathbb{E}[(S_{n,q,c}(r;[a,b]) - S_{n,q,c}(r;[a,b']))^8]$, and the analysis of the other 2 terms are similar.

Note that for any $a, r, b, b' \in [0,1]$ and $b < b'$,

$$\begin{aligned}&\mathbb{E}[(S_{n,q,c}(r;[a,b]) - S_{n,q,c}(r;[a,b']))^8] \\ =&\mathbb{E}\left[\left((q-c)\sum_{l=1}^{p}\sum_{\lfloor nb\rfloor+1\leq j\leq\lfloor nb'\rfloor}\sum_{\lfloor na\rfloor+1\leq i_1\neq\cdots\neq i_c\leq\lfloor nr\rfloor}\sum_{\lfloor nr\rfloor+1\leq j_1\neq\cdots\neq j_{q-c-1}\leq j-1}\left(\prod_{t=1}^{c}Z_{i_t,l}\cdot\prod_{s=1}^{q-c-1}Z_{j_s,l}\cdot Z_{j,l}\right)\right)^8\right] \\ =&C\sum_{j^{(\cdot)},i_t^{(\cdot)},j_s^{(\cdot)}}\sum_{l_1,\ldots,l_8=1}^{p}\prod_{h=1}^{8}\left(\mathbb{E}\left[\prod_{t=1}^{c}Z_{i_t^{(h)},l_h}\right]\mathbb{E}\left[\prod_{s=1}^{q-c-1}Z_{j_s^{(h)},l_h}\right]\mathbb{E}\left[Z_{j^{(h)},l_h}\right]\right) \\ \lesssim& n^{4(q-1)}(\lfloor nb'\rfloor - \lfloor nb\rfloor)^4\|\Sigma\|_q^{qh/2} \lesssim n^{4q}\left[(b'-b)^4 + \frac{1}{n^4}\right]\|\Sigma\|_q^{4q},\end{aligned}$$

where we have applied Lemma 6.1-(2) to $i_1^{(h)},\ldots,i_c^{(h)},j_1^{(h)},\ldots,j_{q-c-1}^{(h)},j^{(h)}$, and the summation $\sum_{j^{(\cdot)},i_t^{(\cdot)},j_s^{(\cdot)}}$ is over $\lfloor nb\rfloor+1\leq j^{(h)}\leq\lfloor nb'\rfloor$, $\lfloor na\rfloor+1\leq i_1^{(h)}\neq\cdots\neq i_c^{(h)}\leq\lfloor nr\rfloor$, $\lfloor nr\rfloor+1\leq j_1^{(h)}\neq\cdots\neq j_{q-c-1}^{(h)}\leq j^{(h)}-1$ for $h=1,\ldots,8$. Therefore, we have

$$a_n^{-8}\mathbb{E}[(S_{n,q,c}(r;[a,b]) - S_{n,q,c}(r;[a,b']))^8] \lesssim (b'-b)^4 + \frac{1}{n^4}.$$

$\diamondsuit$

LEMMA **6.3**. *Fix q, c, for any $0\leq a_1<r_1<b_1\leq 1, 0\leq a_2<r_2<b_2$, any $\alpha_1,\alpha_2\in\mathbb{R}$, we have*

$$\frac{\alpha_1}{a_n}S_{n,q,c}(r_1;[a_1,b_1]) + \frac{\alpha_2}{a_n}S_{n,q,c}(r_2,[a_2,b_2]) \xrightarrow{\mathcal{D}} \alpha_1 Q_{q,c}(r_1;[a_1,b_1]) + \alpha_2 Q_{q,c}(r_2;[a_2,b_2]),$$

where

$$\operatorname{cov}\left(Q_{q,c}(r_1;[a_1,b_1]),Q_{q,c}(r_2;[a_2,b_2])\right)=c!(q-c)!(r-A)^c(b-R)^{q-c},$$

Proof of Lemma 6.3. WLOG, we can assume $a_1 < a_2 < r_1 < r_2 < b_1 < b_2$. The other terms are similar. Define

$$\begin{aligned}
\xi_{1,i} =& \frac{q-c}{a_n}\sum_{l=1}^{p}\sum^{*}_{\lfloor na_1\rfloor+1\le i_1,\cdots,i_c\le\lfloor nr_1\rfloor}\sum^{*}_{\lfloor nr_1\rfloor+1\le j_1,\cdots,j_{q-c-1}\le i-1}\left(\prod_{t=1}^{c}Z_{i_t,l}\cdot\prod_{s=1}^{q-c-1}Z_{j_s,l}\cdot Z_{i,l}\right)\\
\xi_{2,i} =& \frac{q-c}{a_n}\sum_{l=1}^{p}\sum^{*}_{\lfloor na_2\rfloor+1\le i_1,\cdots,i_c\le\lfloor nr_2\rfloor}\sum^{*}_{\lfloor nr_2\rfloor+1\le j_1,\cdots,j_{q-c-1}\le i-1}\left(\prod_{t=1}^{c}Z_{i_t,l}\cdot\prod_{s=1}^{q-c-1}Z_{j_s,l}\cdot Z_{i,l}\right),
\end{aligned}$$

and

$$\widetilde{\xi}_{n,i}=\begin{cases}\alpha_1\xi_{1,i} & \text{if } \lfloor nr_1\rfloor+q-c\le i\le\lfloor nr_2\rfloor+q-c-1\\ \alpha_1\xi_{1,i}+\alpha_2\xi_{2,i} & \text{if } \lfloor nr_2\rfloor+q-c\le i\le\lfloor nb_1\rfloor\\ \alpha_2\xi_{2,i} & \text{if } \lfloor nb_1\rfloor+1\le i\le\lfloor nb_2\rfloor\end{cases}.$$

Define $\mathcal{F}_i=\sigma\left(Z_i,Z_{i-1},\cdots\right)$, we can see that under the null $\mathbb{E}[Z_1]=0$, $\widetilde{\xi}_{n,i}$ is a martingale difference sequence w.r.t. $\mathcal{F}_i$, and

$$\frac{\alpha_1}{a_n}S_{n,q,c}\left(r_1;[a_1,b_1]\right)+\frac{\alpha_2}{a_n}S_{n,q,c}\left(r_2,[a_2,b_2]\right)=\sum_{i=\lfloor nr_1\rfloor+q-c}^{\lfloor nb_2\rfloor}\widetilde{\xi}_{n,i}.$$

To apply the martingale CLT (Theorem 35.12 in Billingsley (2008)), we need to verify the following two conditions

(1) $\quad\forall\epsilon>0,\ \displaystyle\sum_{i=\lfloor nr_1\rfloor+q-c}^{\lfloor nb_2\rfloor}\mathbb{E}\left[\widetilde{\xi}^2_{n,i}\mathbf{1}\left\{\left|\widetilde{\xi}_{n,i}\right|>\epsilon\right\}\middle|\mathcal{F}_{i-1}\right]\xrightarrow{p}0.$

(2) $\quad V_n = \sum_{i=\lfloor nr_1 \rfloor + q - c}^{\lfloor nb_2 \rfloor} \mathbb{E}\left[\widetilde{\xi}_{n,i}^2 | \mathbb{F}_{i-1}\right] \xrightarrow{p} \sigma^2$. To prove (1), it suffices to show that

$$\sum_{i=\lfloor nr_1 \rfloor + q - c}^{\lfloor nb_2 \rfloor} \mathbb{E}\left[\widetilde{\xi}_{n,i}^4\right] \to 0.$$

Observe that

$$\begin{aligned}
&\sum_{i=\lfloor nr_1 \rfloor + q - c}^{\lfloor nb_2 \rfloor} \mathbb{E}\left[\widetilde{\xi}_{n,i}^4\right] \\
=&\alpha_1^4 \sum_{i=\lfloor nr_1 \rfloor + q - c}^{\lfloor nr_2 \rfloor + q - c - 1} \mathbb{E}\left[\xi_{1,i}^4\right] + \sum_{i=\lfloor nr_2 \rfloor + q - c}^{\lfloor nb_1 \rfloor} \mathbb{E}\left[(\alpha_1 \xi_{1,i} + \alpha_2 \xi_{2,i})^4\right] + \alpha_2^4 \sum_{i=\lfloor nb_1 \rfloor + 1}^{\lfloor nb_2 \rfloor} \mathbb{E}\left[\xi_{2,i}^4\right] \\
\leq& 8\alpha_1^4 \sum_{i=\lfloor nr_1 \rfloor + q - c}^{\lfloor nb_1 \rfloor} \mathbb{E}\left[\xi_{1,i}^4\right] + 8\alpha_2^4 \sum_{i=\lfloor nr_2 \rfloor + q - c}^{\lfloor nb_2 \rfloor} \mathbb{E}\left[\xi_{2,i}^4\right].
\end{aligned}$$

Straightforward calculations show that

$$\begin{aligned}
&\mathbb{E}\left[\xi_{1,i}^4\right] \\
&= \frac{C}{n^{2q} \|\Sigma\|_q^{2q}} \sum_{i_t^{(h)}, j_s^{(h)}} \sum_{l_1, l_2, l_3, l_4 = 1}^{p} \prod_{h=1}^{4} \left(\mathbb{E}\left[\prod_{t=1}^{c} Z_{i_t^{(h)}, l_h}\right] \mathbb{E}\left[\prod_{s=1}^{q-c-1} Z_{j_s^{(h)}, l_h}\right] \mathbb{E}\left[Z_{i, l_h}\right] \right) \\
&\lesssim \frac{1}{n^{2q} \|\Sigma\|_q^{2q}} n^{2(q-1)} \|\Sigma\|_q^{2q} = O(\frac{1}{n^2}).
\end{aligned}$$

The same result holds for $\xi_{2,i}$. Therefore,

$$\sum_{i=\lfloor nr_1 \rfloor + q - c}^{\lfloor nb_2 \rfloor} \mathbb{E}\left[\widetilde{\xi}_{n,i}^4\right] \lesssim \sum_{i=\lfloor nr_1 \rfloor + q - c}^{\lfloor nb_1 \rfloor} \mathbb{E}\left[\xi_{1,i}^4\right] + \sum_{i=\lfloor nr_2 \rfloor + q - c}^{\lfloor nb_2 \rfloor} \mathbb{E}\left[\xi_{2,i}^4\right] = O(\frac{1}{n}) \to 0.$$

As regards (2), we decompose V_n as follows,

$$
\begin{aligned}
&\sum_{i=\lfloor nr_1\rfloor+q-c}^{\lfloor nb_2\rfloor} \mathbb{E}\left[\widetilde{\xi}_{n,i}^2|\mathbb{F}_{i-1}\right]\\
=&\alpha_1^2\sum_{i=\lfloor nr_1\rfloor+q-c}^{\lfloor nr_2\rfloor+q-c-1} \mathbb{E}\left[\xi_{1,i}^2|\mathbb{F}_{i-1}\right]+\sum_{i=\lfloor nr_2\rfloor+q-c}^{\lfloor nb_1\rfloor} \mathbb{E}\left[\left(\alpha_1\xi_{1,i}+\alpha_2\xi_{2,i}\right)^2|\mathbb{F}_{i-1}\right]+\alpha_2^2\sum_{i=\lfloor nb_1\rfloor+1}^{\lfloor nb_2\rfloor} \mathbb{E}\left[\xi_{2,i}^2|\mathbb{F}_{i-1}\right]\\
=&\alpha_1^2\sum_{i=\lfloor nr_1\rfloor+q-c}^{\lfloor nb_1\rfloor} \mathbb{E}\left[\xi_{1,i}^2|\mathbb{F}_{i-1}\right]+\alpha_2^2\sum_{i=\lfloor nr_2\rfloor+q-c}^{\lfloor nb_2\rfloor} \mathbb{E}\left[\xi_{2,i}^2|\mathbb{F}_{i-1}\right]+2\alpha_1\alpha_2\sum_{i=\lfloor nr_2\rfloor+q-c}^{\lfloor nb_1\rfloor} \mathbb{E}\left[\xi_{1,i}\xi_{2,i}|\mathbb{F}_{i-1}\right]\\
=:&\ \alpha_1^2 V_{1,n}+\alpha_2^2 V_{2,n}+2\alpha_1\alpha_2 V_{3,n}.
\end{aligned}
$$

We still focus on the case $a_1<a_2<r_1<r_2<b_1<b_2$. Note that

$$
\begin{aligned}
&\sum_{i=\lfloor nr_1\rfloor+q-c}^{\lfloor nb_1\rfloor} \mathbb{E}\left[\xi_{1,i}^2|\mathbb{F}_{i-1}\right]\\
=&\frac{(q-c)^2}{n^q\|\Sigma\|_q^q}c!(q-c-1)!\sum_{i=\lfloor nr_1\rfloor+q-c}^{\lfloor nb_1\rfloor}\sum_{i_t^{(h)},j_s^{(h)}}\sum_{l_1,l_2=1}^{p}\Sigma_{l_1l_2}\prod_{h=1}^{2}\left(\prod_{t=1}^{c}Z_{i_t^{(h)},l_h}\cdot\prod_{s=1}^{q-c-1}Z_{j_s^{(h)},l_h}\right)\\
=&\frac{(q-c)^2}{n^q\|\Sigma\|_q^q}c!(q-c-1)!\sum_{i=\lfloor nr_1\rfloor+q-c}^{\lfloor nb_1\rfloor}\sum_{i_t^{(h)},j_s^{(h)}}^{(1)}\sum_{l_1,l_2=1}^{p}\Sigma_{l_1l_2}\prod_{h=1}^{2}\left(\prod_{t=1}^{c}Z_{i_t^{(h)},l_h}\cdot\prod_{s=1}^{q-c-1}Z_{j_s^{(h)},l_h}\right)\\
&+\frac{(q-c)^2}{n^q\|\Sigma\|_q^q}c!(q-c-1)!\sum_{i=\lfloor nr_1\rfloor+q-c}^{\lfloor nb_1\rfloor}\sum_{i_t^{(h)},j_s^{(h)}}^{(2)}\sum_{l_1,l_2=1}^{p}\Sigma_{l_1l_2}\prod_{h=1}^{2}\left(\prod_{t=1}^{c}Z_{i_t^{(h)},l_h}\cdot\prod_{s=1}^{q-c-1}Z_{j_s^{(h)},l_h}\right)\\
=:&\ V_{1,n}^{(1)}+V_{1,n}^{(2)},
\end{aligned}
$$

where $\sum_{i_t^{(h)},j_s^{(h)}}^{(1)}$ denotes the summation over terms s.t. $i_t^{(1)}=i_t^{(2)}, j_s^{(1)}=j_s^{(2)}, \forall t,s$, and $\sum_{i_t^{(h)},j_s^{(h)}}^{(2)}$ is over the other terms.

It is straightforward to see that $\mathbb{E}[V_{1,n}^{(2)}]=0$ as Z_i's are independent, and

$$
\begin{aligned}
\mathbb{E}[V_{1,n}^{(1)}] &= \frac{(q-c)^2}{n^q\|\Sigma\|_q^q} c!(q-c-1)! n^c (r_1-a_1)^c \sum_{k=1}^{\lfloor nb_1\rfloor - \lfloor nr_1\rfloor} k^{q-c-1} \sum_{l_1,l_2=1}^{p} \Sigma_{l_1 l_2}^p + o(1) \\
&= c!(q-c)!(r_1-a_1)^c (b_1-r_1)^{q-c} + o(1).
\end{aligned}
$$

Note that

$$
\begin{aligned}
\mathbb{E}[(V_{1,n}^{(1)})^2] =& \frac{(q-c)^4}{n^{2q}\|\Sigma\|_q^{2q}} [c!(q-c-1)!]^2 \sum_{l_1,l_2,l_3,l_4=1}^{p} \sum_{i=\lfloor nr_1\rfloor+q-c}^{\lfloor nb_1\rfloor} \sum_{j=\lfloor nr_1\rfloor+q-c}^{\lfloor nb_1\rfloor} \sum_{i_t^{(h)},j_s^{(h)}}^{*} \Bigg[\Sigma_{l_1 l_2}\Sigma_{l_3 l_4} \\
& \prod_{h=1}^{4} \left(\prod_{t=1}^{c} Z_{i_t^{(h)},l_h} \cdot \prod_{s=1}^{q-c-1} Z_{j_s^{(h)},l_h} \right) \Bigg] + o(1),
\end{aligned}
$$

where the summation $\sum^{*}_{i_t^{(h)},j_s^{(h)}}$ is over the range of $i_t^{(h)}, j_s^{(h)}, h = 1,2,3,4$ s.t. $i_t^{(1)} = i_t^{(2)}, j_s^{(1)} = j_s^{(2)}, i_t^{(3)} = i_t^{(4)}, j_s^{(3)} = j_s^{(4)}, \forall t, s$. Note that RHS can be further decomposed into 2 parts. The first part corresponds to the summation of the terms s.t. $\{i_t^{(h)}, j^{(s)}\}$ for $h = 1$ and has no intersection with that for $h = 3$, which has order

$$
\begin{aligned}
& \frac{(q-c)^4}{n^{2q}\|\Sigma\|_q^{2q}} [c!(q-c-1)!]^2 n^{2c} (r_1-a_1)^{2c} \sum_{i=1}^{\lfloor nb_1\rfloor - \lfloor nr_1\rfloor} i^{q-c-1} \sum_{j=1}^{\lfloor nb_1\rfloor - \lfloor nr_1\rfloor} j^{q-c-1} \sum_{l_1,l_2,l_3,l_4}^{p} \Sigma_{l_1 l_2}^q \Sigma_{l_3 l_4}^q \\
=& [c!(q-c)!(r_1-a_1)^c (b_1-r_1)^{q-c}]^2 + o(1) = \mathbb{E}^2[V_{1,n}^{(1)}] + o(1).
\end{aligned}
$$

For the second part, it corresponds to the summation of the terms s.t. $\{i_t^{(h)}, j^{(s)}\}$ for $h = 1$ and has at least one intersection with that for $h = 3$. Since at least one "degree of freedom" for n is lost, the summation still has the form $\sum_{l_1,l_2,l_3,l_4=1}^{p} \mathbb{E}\left[Z_{i_1^{(1)},l_1} \cdots Z_{i_q^{(1)},l_1} \cdots Z_{i_1^{(h)},l_h} \cdots Z_{i_q^{(h)},l_h} \right]$ as in Lemma 6.1-(2), which has order $O(\|\Sigma\|_q^{2q})$. We can conclude that the second part has order $O(\frac{1}{n})$, and hence goes to 0.

Therefore, $\limsup \left(\mathbb{E}[(V_{1,n}^{(1)})^2] - \mathbb{E}^2[V_{1,n}^{(1)}] \right) \leq 0$, which implies $\lim \mathrm{var}(V_{1,n}^{(1)}) = 0$. Therefore, we can conclude that $V_{1,n}^{(1)} \xrightarrow{p} \lim \mathbb{E}[V_{1,n}^{(1)}] = c!(q-c)!(r_1-a_1)^c (b_1-r_1)^{(q-c)}$. It remains to show that $V_{1,n}^{(2)} \xrightarrow{p} 0$.

It suffices to show that $\mathbb{E}\left[(V_{1,n}^{(2)})^2\right] \to 0$. Based on the same argument as before, by applying Lemma 6.1-(2) we know that every kind of summation has the same order $O(\frac{1}{n})$ no matter how $i_t^{(h)}, j_s^{(h)}, i, j$ intersects with each other. Therefore, the terms in the expansion of $\mathbb{E}\left[(V_{1,n}^{(2)})^2\right]$ for which n has highest degree of freedom should dominate. For these terms, each index in $i_t^{(h)}, j_s^{(h)}, i, j$ should have exactly one pair. The number of these terms is of order $O(n^{2q})$. The summation has forms $\sum_{l_1,l_2,l_3,l_4=1}^{p}(\Sigma_{l_1l_2}^d\Sigma_{l_3l_4}^d\Sigma_{l_1l_4}^d\Sigma_{l_2l_3}^e\Sigma_{l_1l_3}^f\Sigma_{l_2l_4}^f)$, s.t. $d>0, e+f>0$ and $d+e+f=q$. We need to show that it is of order $o(\|\Sigma\|_q^{2q})$ to complete the proof. By symmetry, we can assume $e>0$, and therefore $d, e \leq 1$. Note that for $q>2$,

$$
\begin{aligned}
&\sum_{l_1,l_2,l_3,l_4=1}^{p}(\Sigma_{l_1l_2}^d\Sigma_{l_3l_4}^d\Sigma_{l_1l_4}^e\Sigma_{l_2l_3}^e\Sigma_{l_1l_3}^f\Sigma_{l_2l_4}^f)\\
=&\sum_{l_1,l_2,l_3,l_4=1}^{p}(\Sigma_{l_1l_2}\Sigma_{l_2l_3}\Sigma_{l_3l_4}\Sigma_{l_4l_1})(\Sigma_{l_1l_2}^{d-1}\Sigma_{l_3l_4}^{d-1}\Sigma_{l_1l_4}^{e-1}\Sigma_{l_2l_3}^{e-1}\Sigma_{l_1l_3}^f\Sigma_{l_2l_4}^f)\\
\leq&\left[\sum_{l_1,l_2,l_3,l_4=1}^{p}|\Sigma_{l_1l_2}\Sigma_{l_2l_3}\Sigma_{l_3l_4}\Sigma_{l_4l_1}|^{q/2}\right]^{2/q}\left[\sum_{l_1,l_2,l_3,l_4=1}^{p}|\Sigma_{l_1l_2}^{d-1}\Sigma_{l_3l_4}^{d-1}\Sigma_{l_1l_4}^{e-1}\Sigma_{l_2l_3}^{e-1}\Sigma_{l_1l_3}^f\Sigma_{l_2l_4}^f|^{q/(q-2)}\right]^{1-2/q}\\
\lesssim& o(\|\Sigma\|_q^4)\cdot\|\Sigma\|_q^{2q-4}=o(\|\Sigma\|_q^{2q}),
\end{aligned}
$$

where we have used Hölder's inequality, along with A.1 and the fact that

$$
\begin{aligned}
&\sum_{l_1,l_2,l_3,l_4=1}^{p}|\Sigma_{l_1l_2}^{d-1}\Sigma_{l_3l_4}^{d-1}\Sigma_{l_1l_4}^{e-1}\Sigma_{l_2l_3}^{e-1}\Sigma_{l_1l_3}^f\Sigma_{l_2l_4}^f|^{q/(q-2)}\\
\lesssim&\sum_{l_1,l_2,l_3,l_4=1}^{p}(\Sigma_{l_1l_2}^q\Sigma_{l_3l_4}^q+\Sigma_{l_1l_3}^q\Sigma_{l_2l_4}^q+\Sigma_{l_1l_4}^q\Sigma_{l_2l_3}^q)=3\|\Sigma\|_q^{2q}.
\end{aligned}
$$

When $q=2$, it must be the case that $d=e=1$, the term becomes $\sum_{l_1,l_2,l_3,l_4=1}^{p}|\Sigma_{l_1l_2}\Sigma_{l_2l_3}\Sigma_{l_3l_4}\Sigma_{l_4l_1}|$, and directly applying A.1 can yield the desired order.

We can then conclude that $\mathbb{E}[V_{1,n}^{(2)}] \to 0$ and hence $V_{1,n}^{(2)} \xrightarrow{p} 0$. Combining what we have proved so far, we obtain $V_{1,n} \xrightarrow{p} c!(q-c)!(r_1-a_1)^c(b_1-r_1)^{q-c}$.

Similar argument shows that

$$V_{2,n} \xrightarrow{p} c!(q-c)!(r_2-a_2)^c(b_2-r_2)^{q-c}, \quad V_{3,n} \xrightarrow{p} c!(q-c)!(r_1-a_2)^c(b_1-r_2)^{q-c}.$$

Therefore, we conclude that

$$\begin{aligned} V_n \xrightarrow{p} & \alpha_1^2 c!(q-c)!(r_1-a_1)^c(b_1-r_1)^{q-c} + \alpha_2^2 c!(q-c)!(r_2-a_2)^c(b_2-r_2)^{q-c} \\ & + 2\alpha_1\alpha_2 c!(q-c)!(r_1-a_2)^c(b_1-r_2)^{q-c}, \end{aligned}$$

which completes the proof. ♢

We can generalize the above lemma to the case when c_i, q_i are not identical.

LEMMA **6.4**. *Fix q_1, c_1, q_2, c_2 for any $0 \leq a_1 < r_1 < b_1 \leq 1, 0 \leq a_2 < r_2 < b_2$, any $\alpha_1, \alpha_2 \in \mathbb{R}$, we have*

$$\frac{\alpha_1}{a_n} S_{n,q_1,c_1}(r_1;[a_1,b_1]) + \frac{\alpha_2}{a_n} S_{n,q_2,c_2}(r_2,[a_2,b_2]) \xrightarrow{\mathcal{D}} \alpha_1 Q_{q_1,c_1}(r_1;[a_1,b_1]) + \alpha_2 Q_{q_2,c_2}(r_2;[a_2,b_2]),$$

where Q_{q_1,r_1} and Q_{q_2,r_2} are independent Gaussian processes if $q_1 \neq q_2$, or $(c_1-c_2)(r_1-r_2) < 0$ or $r_1 = r_2, c_1 \neq c_2$. And when $q_1 = q_2 = q, (c_1-c_2)(r_1-r_2) >= 0$, we have

$$\operatorname{cov}(Q_{q,c_1}(r_1;[a_1,b_1]), Q_{q,c_2}(r_2;[a_2,b_2])) = \binom{C}{c} c!(q-C)!(r-A)^c(R-r)^{C-c}(b-R)^{q-C},$$

Proof of Lemma 6.4. We use the same notations in proving last lemma, as the proof is similar to the previous one and involves applying martingale CLT, where we have decomposed V_n into 2 parts. Since the argument there can be directly applied, the only additional work is about calculating the mean.

To prove the second statement, we take $c_1 < c_2, a_1 < a_2 < r_1 < r_2 < b_1 < b_2$, as the example case, since the proof for other cases are similar. With the same technique we have used, it can be shown that

$$
\begin{aligned}
E[V_n] \to & \alpha_1^2 c_1!(q-c_1)!(r_1-a_1)^{c_1}(b_1-r_1)^{q-c_1} + \alpha_2^2 c_2!(q-c_2)!(r_2-a_2)^{c_2}(b_2-r_2)^{q-c_2} \\
& + 2\alpha_1\alpha_2 \binom{c_2}{c_1} c_1!(q-c_1)!(r_1-a_2)^{c_1}(r_2-r_1)^{c_2-c_1}(b_1-r_2)^{q-c_2}.
\end{aligned}
$$

To derive the convergence in the statement, we can follow the same argument as before to show the variance goes to 0, and therefore, we have the convergence in distribution, with desired covariance structure.

As for the first statement, it is straightforward to see that the expectation for the crossing term (corresponding to $\alpha_1\alpha_2$) is 0 for each of the cases in the first statement, which implies that the Gaussian processes have to be independent due to asymptotic normality. $\diamondsuit$

Now we are ready to complete the proof of Theorem 2.1.

Proof of Theorem 2.1. The tightness is guaranteed by Lemma 6.2 and applying Lemma 7.1 in Kley et al. (2016) with $\Phi(x) = x^4, T = T_n, d(u, u') = \|u - u'\|^{3/4}, \bar{\eta} = n^{-3/4}/2$. We omit the detailed proof as the argument is similar to the tightness proof in Wang et al. (2019). Lemma 6.4 has provided finite dimensional convergence of $S_{n,q,c}$, which has asymptotic covariance structure as $Q_{q,c}$ after normalization. Therefore, we have derived desired process convergence. $\diamondsuit$

Proof Theorem 2.3. Let $(s, k, m) = (\lfloor an \rfloor + 1, \lfloor rn \rfloor, \lfloor bn \rfloor)$ and define

$$
D_{n,q}^Z(r; a, b) = \sum_{l=1}^{p} \sum_{s \le i_1, \ldots, i_q \le k}^{*} \sum_{k+1 \le j_1, \ldots, j_q \le m}^{*} (Z_{i_1,l} - Z_{j_1,l}) \cdots \left(Z_{i_q,l} - Z_{j_q,l}\right).
$$

Recall that Theorem 2.2 holds for $D_{n,q}^Z$ since under the null $D_{n,q}^Z = D_{n,q}$.

Now we are under the alternative, with the location point $k_1 = \lfloor n\tau_1 \rfloor$ and the change of mean equal to Δ_n. Suppose WLOG $s < k_1 < k < m$.

$$\begin{aligned}
D_{n,q}(r;a,b) =& \sum_{l=1}^{p} \sum_{s\le i_1,\ldots,i_q\le k}^{*} \sum_{k+1\le j_1,\ldots,j_q\le m}^{*} (X_{i_1,l}-X_{j_1,l})\cdots\left(X_{i_q,l}-X_{j_q,l}\right)\\
=& q! \sum_{l=1}^{p} \sum_{s\le i_1<\ldots<i_q\le k} \sum_{k+1\le j_1,\ldots,j_q\le m}^{*} (X_{i_1,l}-X_{j_1,l})\cdots\left(X_{i_q,l}-X_{j_q,l}\right)\\
=& q! \sum_{l=1}^{p} \sum_{s\le i_1<\ldots<i_q\le k_1} \sum_{k+1\le j_1,\ldots,j_q\le m}^{*} (Z_{i_1,l}+\delta_{n,l}-Z_{j_1,l})\cdots\left(Z_{i_q,l}+\delta_{n,l}-Z_{j_q,l}\right)\\
&+ q! \sum_{l=1}^{p}\sum_{c=1}^{q-1}\Big[\sum_{s\le i_1<\ldots<i_c\le k_1<i_{c+1}<\ldots<i_q\le k} \sum_{k+1\le j_1,\ldots,j_q\le m}^{*}\\
&(Z_{i_1,l}+\delta_{n,l}-Z_{j_1,l})\cdots(Z_{i_c,l}+\delta_{n,l}-Z_{j_c,l})(Z_{i_{c+1},l}-Z_{j_{c+1},l})\cdots(Z_{i_q,l}-Z_{j_q,l})\Big]\\
&+ q! \sum_{l=1}^{p} \sum_{k_1+1\le i_1<\ldots<i_q\le k} \sum_{k+1\le j_1,\ldots,j_q\le m}^{*} (Z_{i_1,l}-Z_{j_1,l})\cdots\left(Z_{i_q,l}-Z_{j_q,l}\right)\\
=& D_{n,q}^{Z} + P_q^{k_1-s+1}P_q^{m-k}\|\Delta_n\|_q^q + R_{n,q}. \qquad (*)
\end{aligned}$$

First suppose $\gamma_{n,q}\to\gamma\in[0,\infty)$, which is equivalent to $n^{q/2}\left\|\Delta_n\right\|_q^q \lesssim \|\Sigma\|_q^{q/2}$. It suffices to show that in this case,

$$\left\{n^{-q}a_{n,q}^{-1}D_{n,q}(\cdot;[\cdot,\cdot])\right\} \rightsquigarrow \left\{G_q(\cdot;[\cdot,\cdot])+\gamma J_q(\cdot;[\cdot,\cdot])\right\} \text{ in } \ell_\infty\left([0,1]^3\right).$$

Since $n^{-q}a_{n,q}^{-1}D_{n,q}^{Z}(r;[a,b])$ converges to some non-degenerate process, and

$$n^{-q}a_{n,q}^{-1}P_q^{k^*-s+1}P_q^{m-k}\|\Delta_n\|_q^q = \gamma(r^*-a)^q(b-r)^q + o(1),$$

it remains to show that $n^{-q}a_{n,q}^{-1}R_{n,q}\rightsquigarrow 0$.

Note that $R_{n,q}$ consists of terms that are each ratio consistent to

$$Cn^{2(q-c)}\sum_{l=1}^{p}\delta_{n,l}^{q-c}D_{n,c,l}(r;a,b),$$

for some constant C depending on q, a, b, r and $c = 1, ..., q-1$, where

$$D_{n,c,l}(r;a,b) = \sum_{s\leq i_1<\ldots<i_c\leq k} \sum_{k+1\leq j_1,\ldots,j_c\leq m}^{*} (Z_{i_1,l} - Z_{j_1,l}) \cdots (Z_{i_c,l} - Z_{j_c,l}),$$

which can be further decomposed as

$$D_{n,c,l}(r;a,b) \asymp \sum_{d=0}^{c} C_d n^c \sum_{s\leq i_1\ldots,i_d\leq k}^{*} \sum_{k+1\leq j_1,\ldots,j_{c-d}\leq m}^{*} \left(\prod_{t=1}^{d} Z_{i_t,l} \prod_{s=1}^{c-d} Z_{j_s,l}\right),$$

for some constants depending on d, c, q. Therefore, it suffices to show

$$\begin{aligned}
&n^{q-c} a_{n,q}^{-1} \sum_{l=1}^{p} \delta_{n,l}^{q-c} \sum_{s\leq i_1,\ldots,i_d\leq k}^{*} \sum_{k+1\leq j_1,\ldots,j_{c-d}\leq m}^{*} \left(\prod_{t=1}^{d} Z_{i_t,l} \prod_{s=1}^{c-d} Z_{j_s,l}\right) \\
=&n^{q/2-c} \|\Sigma\|_q^{-q/2} \sum_{l=1}^{p} \delta_{n,l}^{q-c} \sum_{s\leq i_1,\ldots,i_d\leq k}^{*} \sum_{k+1\leq j_1,\ldots,j_{c-d}\leq m}^{*} \left(\prod_{t=1}^{d} Z_{i_t,l} \prod_{s=1}^{c-d} Z_{j_s,l}\right) \rightsquigarrow 0.
\end{aligned}$$

Similar argument for showing tightness and finite dimensional convergence in proving Theorem 2.1 can be applied. More precisely, we can get a similar moment bound as in Lemma 6.2 and follow the argument there to show the tightness, since we have

$$\begin{aligned}
&n^{4q-8c} \|\Sigma\|_q^{-4q} n^{4c} \sum_{l_1,\cdots,l_8=1}^{p} \mathbb{E}\left[\delta_{n,l_1}^{q-c} \cdots \delta_{n,l_8}^{q-c} Z_{i_1^{(1)},l_1} \cdots Z_{i_c^{(1)},l_1} \cdots Z_{i_1^{(8)},l_h} \cdots Z_{i_c^{(8)},l_8}\right] \\
=&n^{4(q-c)} \|\Sigma\|_q^{-4q} \sum_{l_1,\cdots,l_8=1}^{p} \mathbb{E}\left[\delta_{n,l_1}^{q-c} \cdots \delta_{n,l_8}^{q-c} Z_{i_1^{(1)},l_1} \cdots Z_{i_c^{(1)},l_1} \cdots Z_{i_1^{(8)},l_8} \cdots Z_{i_c^{(8)},l_8}\right] \\
\lesssim&\|\Delta_n\|_q^{-8(q-c)} \|\Sigma\|_q^{-4c} \sum_{l_1,\cdots,l_8=1}^{p} \mathbb{E}\left[\delta_{n,l_1}^{q-c} \cdots \delta_{n,l_8}^{q-c} Z_{i_1^{(1)},l_1} \cdots Z_{i_c^{(1)},l_1} \cdots Z_{i_1^{(8)},l_8} \cdots Z_{i_c^{(8)},l_8}\right] \lesssim 1,
\end{aligned}$$

by Lemma 6.1-(1).

Furthermore, following the proof of Lemma 6.3, Lemma 6.1-(3) implies finite dimensional convergence to 0, as

$$
\begin{aligned}
& n^{q-2c}\|\Sigma\|_q^{-q} n^c \sum_{l_1,l_2=1}^{p} \delta_{n,l_1}^{q-c}\delta_{n,l_2}^{q-c}\Sigma_{l_1l_2}^c \\
=& n^{q-c}\|\Sigma\|_q^{-q} \sum_{l_1,l_2=1}^{p} \delta_{n,l_1}^{q-c}\delta_{n,l_2}^{q-c}\Sigma_{l_1l_2}^c \\
\lesssim& \|\Delta_n\|_q^{-2(q-c)}\|\Sigma\|_q^{-c} \sum_{l_1,l_2=1}^{p} \delta_{n,l_1}^{q-c}\delta_{n,l_2}^{q-c}\Sigma_{l_1l_2}^c \to 0.
\end{aligned}
$$

We have the desired process convergence for $\gamma_{n,q} \to \gamma < \infty$., which along with the continuous mapping theorem further implies the convergence of the statistic.

When $\gamma = +\infty$, note that $\tilde{T}_{n,q} \geq \frac{U_{n,q}(k_1;1,n)^2}{W_{n,q}(k_1;1,n)}$. Since k_1 is the location of the change point, the denominator has the same value as the null. On the contrary, it is immediate to see that the numerator diverges to infinity after normalizing (with $n^{-q}a_{n,q}^{-1}$). Therefore, we have $\tilde{T}_{n,q} \to +\infty$. $\diamondsuit$

Before we prove the convergence rate for SN-based estimator, we state the following useful propositions.

Proposition **6.1**. *For any $1 \leq l < k < m \leq n$, $k \geq l+1$ and $m \geq k+2$, we have:*

1. *if $k^* < l$ or $k^* \geq m$, $U_{n,2}(k;l,m) = U_{n,2}^Z(k;l,m)$;*

2. *if $l \leq k \leq k^* < m$,*

$$
\begin{aligned}
U_{n,2}(k;l,m) =& U_{n,2}^Z(k;l,m) + (k-l+1)(k-l)(m-k^*)(m-k^*-1)\|\Delta_n\|_2^2 \\
& - 2(k-l+1)(m-k^*)(m-k)\sum_{i=l}^{k}\Delta_n^T Z_i + 2(k-l)(k-l+1)(m-k^*)\sum_{i=k+1}^{m}\Delta_n^T Z_i \\
& + 2(k-l)(m-k^*)\sum_{i=l}^{k}\Delta_n^T Z_i - 2(k-l+1)(k-l+1)\sum_{i=k^*+1}^{m}\Delta_n^T Z_i;
\end{aligned}
$$

3. if $l \le k^* \le k < m$,

$$
\begin{aligned}
U_{n,2}(k;l,m) =& U_{n,2}^Z(k;l,m) + (k^*-l+1)(k^*-l)(m-k)(m-k-1)\|\Delta_n\|_2^2 \\
&- 2(k^*-l+1)(m-k)(m-k-1)\sum_{i=l}^{k}\Delta_n^T Z_i + 2(m-k-1)(k^*-l+1)(k-l+1)\sum_{i=k+1}^{m}\Delta_n^T Z_i \\
&+ 2(m-k-1)(m-k)\sum_{i=l}^{k^*}\Delta_n^T Z_i - 2(m-k-1)(k^*-l+1)\sum_{i=k+1}^{m}\Delta_n^T Z_i.
\end{aligned}
$$

Let $\epsilon_n = n\gamma_{n,2}^{-1/4+\kappa}$. We have the following result.

PROPOSITION **6.2**. *Under Assumption 3.1,*

1. $P\left(\sup_{k\in\Omega_n} U_{n,2}(k;1,n)^2 - U_{n,2}(k^*;1,n)^2 \ge 0\right) \to 0$*;*

2. $P\left(W_{n,2}(k^*;1,n) - \inf_{k\in\Omega_n} W_{n,2}(k;1,n) \ge 0\right) \to 0,$

where $\Omega_n = \{k : |k-k^*| > \epsilon_n\}$.

Now we are ready to prove the convergence rate for SN-based statistic $\hat{\tau}$.

Proof of Theorem 3.1. Due to the fact that $\hat{k}$ is the global maximizer, we have

$$
\begin{aligned}
0 \le& \frac{U_{n,2}(\hat{k};1,n)^2}{W_{n,2}(\hat{k};1,n)} - \frac{U_{n,2}(k^*;1,n)^2}{W_{n,2}(k^*;1,n)} \\
=& \frac{U_{n,2}(\hat{k};1,n)^2}{W_{n,2}(\hat{k};1,n)} - \frac{U_{n,2}(k^*;1,n)^2}{W_{n,2}(\hat{k};1,n)} + \frac{U_{n,2}(k^*;1,n)^2}{W_{n,2}(\hat{k};1,n)} - \frac{U_{n,2}(k^*;1,n)^2}{W_{n,2}(k^*;1,n)} \\
=& \frac{1}{W_{n,2}(\hat{k};1,n)}(U_{n,2}(\hat{k};1,n)^2 - U_{n,2}(k^*;1,n)^2) + \frac{U_{n,2}(k^*;1,n)^2}{W_{n,2}(\hat{k};1,n)W_{n,2}(k^*;1,n)}(W_{n,2}(k^*;1,n) - W_{n,2}(\hat{k};1,n)).
\end{aligned}
$$

Since $U_{n,2}(k^*;1,n)^2$, $W_{n,2}(\hat{k};1,n)$ and $W_{n,2}(k^*;1,n)$ are all strictly positive almost surely, we can then conclude that $U_{n,2}(\hat{k};1,n)^2 - U_{n,2}(k^*;1,n)^2 \ge 0$ or $W_{n,2}(k^*;1,n) - W_{n,2}(\hat{k};1,n) \ge 0$.

Define $\Omega_n = \{k : |k - k^*| > \epsilon_n\}$. If $\hat{k} \in \Omega_n$, then there exists at least one $k \in \Omega_n$ such that $U_{n,2}(k;1,n)^2 - U_{n,2}(k^*;1,n)^2 \geq 0$ or $W_{n,2}(k^*;1,n) - W_{n,2}(k;1,n) \geq 0$. This implies

$$P(\hat{k} \in \Omega_n) \leq P\left(\sup_{k\in\Omega_n} U_{n,2}(k;1,n)^2 - U_{n,2}(k^*;1,n)^2 \geq 0\right) + P\left(W_{n,2}(k^*;1,n) - \inf_{k\in\Omega_n} W_{n,2}(k;1,n) \geq 0\right).$$

By Proposition 6.2, it is straightforward to see that $P(\hat{k} \in \Omega_n) \to 0$, and this completes the proof. $\diamondsuit$

Proof of Proposition 6.1. If $k^* < l$ or $k^* \geq m$, then $\mathbb{E}[X_i]$ are all identical, for $i = l, ..., m$. This implies that $U_{n,2}(k;l,m) = \sum_{l\leq i_1\neq i_2\leq k} \sum_{k+1\leq j_1\neq j_2\leq m} (X_{i_1} - X_{j_1})^T (X_{i_1} - X_{j_2}) = \sum_{l\leq i_1\neq i_2\leq k} \sum_{k+1\leq j_1\neq j_2\leq m} (Z_{i_1} - Z_{j_1})^T (Z_{i_1} - Z_{j_2}) = U^Z_{n,2}(k;l,m)$.

When $l \leq k^* < m$, there are two scenarios depending on the value of k. If $k \leq k^*$, note that $\mathbb{E}[X_i] = \Delta_n$ for any $i > k^*$ and zero otherwise, then by straightforward calculation we have

$$\begin{aligned}
&U_{n,2}(k;l,m) = \sum_{l\leq i_1\neq i_2\leq k} \sum_{k+1\leq j_1\neq j_2\leq m} (X_{i_1} - X_{j_1})^T (X_{i_1} - X_{j_2}) \\
=& \sum_{l\leq i_1\neq i_2\leq k} \sum_{k+1\leq j_1\neq j_2\leq m} (Z_{i_1} - Z_{j_1} - \mathbb{E}[X_{j_1}])^T (Z_{i_1} - Z_{j_2} - \mathbb{E}[X_{j_2}]) \\
=& U_{n,2}(k;l,m) + (k-l+1)(k-l)(m-k^*)(m-k^*-1)\|\Delta_n\|_2^2 - 2(k-l)(m-k^*) \sum_{i=l}^{k} \sum_{j=k+1}^{k^*} \Delta_n^T (Z_i - Z_j) \\
&- 2(k-l)(m-k^*-1) \sum_{i=l}^{k} \sum_{j=k^*+1}^{m} \Delta_n^T (Z_i - Z_j) \\
=& U^Z_{n,2}(k;l,m) + (k-l+1)(k-l)(m-k^*)(m-k^*-1)\|\Delta_n\|_2^2 - 2(k-l)(m-k^*)(m-k) \sum_{i=l}^{k} \Delta_n^T Z_i \\
&+ 2(k-l)(m-k^*)(k-l+1) \sum_{i=k+1}^{m} \Delta_n^T Z_i + 2(k-l)(m-k^*) \sum_{i=l}^{k} \Delta_n^T Z_i \\
&- 2(k-l)(k-l+1) \sum_{i=k^*+1}^{m} \Delta_n^T Z_i.
\end{aligned}$$

Similarly if $k \geq k^*$ we have

$$
\begin{aligned}
&U_{n,2}(k;l,m) = \sum_{l \leq i_1 \neq i_2 \leq k} \sum_{k+1 \leq j_1 \neq j_2 \leq m} (X_{i_1} - X_{j_1})^T (X_{i_1} - X_{j_2}) \\
&= \sum_{l \leq i_1 \neq i_2 \leq k} \sum_{k+1 \leq j_1 \neq j_2 \leq m} (Z_{i_1} - Z_{j_1} + \mathbb{E}[X_{i_1}] - \Delta_n)^T (Z_{i_1} - Z_{j_2} + \mathbb{E}[X_{i_2}] - \Delta_n) \\
&= U_{n,2}(k;l,m) + (k^* - l + 1)(k^* - l)(m-k)(m-k-1)\|\Delta_n\|_2^2 - 2(m-k-1)(k^*-l) \sum_{i=l}^{k^*} \sum_{j=k+1}^{m} \Delta_n^T (Z_i - Z_j) \\
&\quad - 2(m-k-1)(k^*-l+1) \sum_{i=k^*+1}^{k} \sum_{j=k+1}^{m} \Delta_n^T (Z_i - Z_j) \\
&= U_{n,2}^Z(k;l,m) + (k^* - l + 1)(k^* - l)(m-k)(m-k-1)\|\Delta_n\|_2^2 - 2(k^*-l+1)(m-k)(m-k-1) \sum_{i=l}^{k} \Delta_n^T Z_i \\
&\quad + 2(m-k-1)(k^*-l+1)(k-l+1) \sum_{i=k+1}^{m} \Delta_n^T Z_i + 2(m-k-1)(m-k) \sum_{i=l}^{k^*} \Delta_n^T Z_i \\
&\quad - 2(m-k-1)(k^*-l+1) \sum_{i=k+1}^{m} \Delta_n^T Z_i.
\end{aligned}
$$

$\diamondsuit$

Proof of Proposition 6.2. To show the first result, we first assume $k < k^* - \epsilon_n$. Then according to Proposition 6.1,

$$
\begin{aligned}
U_{n,2}(k;1,n) = &\, U_{n,2}^Z(k;1,n) + k(k-1)(n-k^*)(n-k^*-1)\|\Delta_n\|_2^2 - 2(k-1)(n-k^*)(n-k) \sum_{i=1}^{k} \Delta_n^T Z_i \\
&+ 2k(k-1)(n-k^*) \sum_{i=k+1}^{n} \Delta_n^T Z_i + 2(k-1)(n-k^*) \sum_{i=1}^{k} \Delta_n^T Z_i - 2k(k-1) \sum_{i=k^*+1}^{n} \Delta_n^T Z_i.
\end{aligned}
$$

Similarly we have

$$U_{n,2}(k^*;1,n) = U_{n,2}^Z(k^*;1,n) + k^*(k^*-1)(n-k^*)(n-k^*-1)\|\Delta_n\|_2^2 - 2(k^*-1)(n-k^*)(n-k^*-1)\sum_{i=1}^{k^*}\Delta_n^T Z_i$$
$$+ 2k^*(k^*-1)(n-k^*-1)\sum_{i=k^*+1}^{n}\Delta_n^T Z_i.$$

It is easy to verify that $\mathbb{E}[U_{n,2}(k;1,n)] = k(k-1)(n-k^*)(n-k^*-1)\|\Delta_n\|_2^2$, for $k \le k^*$. Furthermore, by Theorem 2.1 in Wang et al. (2019) and the argument therein, we have

$$\sup_{k=2,\ldots,n-2} |U_{n,2}^Z(k;1,n)| = O(n^3\|\Sigma\|_F) = o_p(n^{3.5}\sqrt{\|\Sigma\|_F}\|\Delta_n\|_2),$$

since $\sqrt{\|\Sigma\|_F} = o(\sqrt{n}\|\Delta_n\|_2)$ by Assumption 3.1 (3), and

$$\sup_{1\le a\le b\le n}\left|\sum_{i=a}^{b}\Delta_n^T Z_i\right| = O_p(\sqrt{n}\sqrt{\Delta_n^T\Sigma\Delta_n}) \le O_p(\sqrt{n\|\Sigma\|_2}\|\Delta_n\|_2) \le O_p(\sqrt{n\|\Sigma\|_F}\|\Delta_n\|_2).$$

These imply that

$$\begin{aligned} U_{n,2}(k^*;1,n) =& k^*(k^*-1)(n-k^*)(n-k^*-1)\|\Delta_n\|_2^2 + O_p(n^{3.5}\|\Delta_n\|_2\sqrt{\|\Sigma\|_F}) \\ =& k^*(k^*-1)(n-k^*)(n-k^*-1)\|\Delta_n\|_2^2 + o_p(n^4\|\Delta_n\|_2^2), \end{aligned}$$

since $\sqrt{\|\Sigma\|_F} = o(\sqrt{n}\|\Delta_n\|_2)$ by Assumption 3.1 (3). Therefore, we have

$$P(\sup_{k<k^*-\epsilon_n} |U_{n,2}(k;1,n)| + U_{n,2}(k^*;1,n) > 0) \to 1.$$

In addition,

$$
\begin{aligned}
&\sup_{k<k^*-\epsilon_n} |U_{n,2}(k;1,n)| - U_{n,2}(k^*;1,n) \\
\leq &\sup_{k<k^*-\epsilon_n} k(k-1)(n-k^*)(n-k^*-1)\|\Delta_n\|_2^2 + O_p(n^{3.5}\|\Delta_n\|_2\sqrt{\|\Sigma\|_F}) \\
&- k^*(k^*-1)(n-k^*)(n-k^*-1)\|\Delta_n\|_2^2 - O_p(n^{3.5}\|\Delta_n\|_2\sqrt{\|\Sigma\|_F}) \\
= &-\epsilon_n(2k^*-\epsilon_n-1)(n-k^*)(n-k^*-1)\|\Delta_n\|_2^2 + O_p(n^{3.5}\|\Delta_n\|_2\sqrt{\|\Sigma\|_F}) \\
= &-\epsilon_n(2k^*-\epsilon_n-1)(n-k^*)(n-k^*-1)\|\Delta_n\|_2^2 + O_p(n^4\|\Delta_n\|_2^2/\sqrt{\gamma_{n,2}}).
\end{aligned}
$$

Since $n/\sqrt{\gamma_{n,2}} = o(n\gamma_{n,2}^{-1/4+\kappa}) = o(\epsilon_n)$, we have

$$
\begin{aligned}
&P(\sup_{k<k^*-\epsilon_n} |U_{n,2}(k;1,n)| - U_{n,2}(k^*;1,n) < 0) \\
\geq &P\Big(-\epsilon_n(2k^*-\epsilon_n-1)(n-k^*)(n-k^*-1)\|\Delta_n\|_2^2 + O_p(n^4\|\Delta_n\|_2^2/\sqrt{\gamma_{n,2}}) < 0\Big) \to 1.
\end{aligned}
$$

Finally, it is straightforward to see that

$$
\begin{aligned}
&\sup_{k\leq k^*-\epsilon_n} U_{n,2}(k;1,n)^2 - U_{n,2}(k^*;1,n)^2 \leq \left(\sup_{k\leq k^*-\epsilon_n} |U_{n,2}(k;1,n)|\right)^2 - U_{n,2}(k^*;1,n)^2 \\
= &\left(\sup_{k\leq k^*-\epsilon_n} |U_{n,2}(k;1,n)| - U_{n,2}(k^*;1,n)\right)\left(\sup_{k\leq k^*-\epsilon_n} |U_{n,2}(k;1,n)| + U_{n,2}(k^*;1,n)\right).
\end{aligned}
$$

And

$$
\begin{aligned}
&P\left(\left(\sup_{k\leq k^*-\epsilon_n} |U_{n,2}(k;1,n)| - U_{n,2}(k^*;1,n)\right)\left(\sup_{k\leq k^*-\epsilon_n} |U_{n,2}(k;1,n)| + U_{n,2}(k^*;1,n)\right) < 0\right) \\
\geq &P\left(\left\{\sup_{k\leq k^*-\epsilon_n} |U_{n,2}(k;1,n)| - U_{n,2}(k^*;1,n) < 0\right\} \bigcap \left\{\sup_{k\leq k^*-\epsilon_n} |U_{n,2}(k;1,n)| + U_{n,2}(k^*;1,n) > 0\right\}\right) \to 1,
\end{aligned}
$$

since both $P(\sup_{k<k^*-\epsilon_n} |U_{n,2}(k;1,n)| + U_{n,2}(k^*;1,n) > 0)$ and $P(\sup_{k<k^*-\epsilon_n} |U_{n,2}(k;1,n)| - U_{n,2}(k^*;1,n) <$

0) converge to 1. This is equivalent to

$$P\left(\left(\sup_{k\le k^*-\epsilon_n}|U_{n,2}(k;1,n)|-U_{n,2}(k^*;1,n)\right)\left(\sup_{k\le k^*-\epsilon_n}|U_{n,2}(k;1,n)|+U_{n,2}(k^*;1,n)\right)\ge 0\right)\to 0,$$

and it implies that $P\left(\sup_{k<k^*-\epsilon_n}U_{n,2}(k;1,n)^2-U_{n,2}(k^*;1,n)^2\ge 0\right)\to 0$. Similar tactics can be applied to the case $k>k^*+\epsilon_n$ and by combining the two parts we have $P\left(\sup_{k\in\Omega_n}U_{n,2}(k;1,n)^2-U_{n,2}(k^*;1,n)^2\ge 0\right)\to 0$. Therefore this completes the proof for the first result.

It remains to show the second part. Let us again assume $k<k^*-\epsilon_n$ first. By Proposition 6.1 we have

$$\begin{aligned}
W_{n,2}(k^*;1,n) &= \frac{1}{n}\sum_{t=2}^{k^*-2}U_{n,2}(t;1,k^*)^2+\frac{1}{n}\sum_{t=k^*+2}^{n-2}U_{n,2}(t;k^*+1,n)^2\\
&= \frac{1}{n}\sum_{t=2}^{k^*-2}U^Z_{n,2}(t;1,k^*)^2+\frac{1}{n}\sum_{t=k^*+2}^{n-2}U^Z_{n,2}(t;k^*+1,n)^2,
\end{aligned}$$

and

$$\begin{aligned}
W_{n,2}(k;1,n) &= \frac{1}{n}\sum_{t=2}^{k-2}U_{n,2}(t;1,k)^2+\frac{1}{n}\sum_{t=k+2}^{n-2}U_{n,2}(t;k+1,n)^2\\
&= \frac{1}{n}\sum_{t=2}^{k-2}U^Z_{n,2}(t;1,k)^2+\frac{1}{n}\sum_{t=k+2}^{n-2}U_{n,2}(t;k+1,n)^2.
\end{aligned}$$

When t is between $k+2$ and k^*, by Proposition 6.1 we have

$$\begin{aligned}
U_{n,2}(t;k+1,n) =& U^Z_{n,2}(t;k+1,n)+(t-k)(t-k-1)(n-k^*)(n-k^*-1)\|\Delta_n\|_2^2\\
&-2(t-k-1)(n-k^*)(n-t)\sum_{i=k+1}^{t}\Delta_n^TZ_i+2(t-k-1)(n-k^*)(t-k)\sum_{i=t+1}^{n}\Delta_n^TZ_i\\
&+2(t-k-1)(n-k^*)\sum_{i=k+1}^{t}\Delta_n^TZ_i-2(t-k-1)(t-k)\sum_{i=k^*+1}^{n}\Delta_n^TZ_i,
\end{aligned}$$

and from the above decomposition we observe that $\mathbb{E}[U_{n,2}(t;k+1,n)]=(t-k)(t-k-1)(n-k^*)(n-$

$k^* - 1)\|\Delta_n\|_2^2$, which is the second term in the above equality. Then

$$
\begin{aligned}
U_{n,2}(t;k+1,n)^2 &= (U_{n,2}(t;k+1,n) - \mathbb{E}[U_{n,2}(t;k+1,n)] + \mathbb{E}[U_{n,2}(t;k+1,n)])^2 \\
&\geq \mathbb{E}[U_{n,2}(t;k+1,n)]^2 + 2\mathbb{E}[U_{n,2}(t;k+1,n)](U_{n,2}(t;k+1,n) - \mathbb{E}[U_{n,2}(t;k+1,n)]) \\
&\geq \mathbb{E}[U_{n,2}(t;k+1,n)]^2 - 2\mathbb{E}[U_{n,2}(t;k+1,n)] \sup_{t=k+2,\ldots,n-2} |U_{n,2}(t;k+1,n) - \mathbb{E}[U_{n,2}(t;k+1,n)]|,
\end{aligned}
$$

since $\mathbb{E}[U_{n,2}(t;k+1,n)] > 0$. Furthermore,

$$
\begin{aligned}
&\sup_{t=k+2,\ldots,n-2} |U_{n,2}(t;k+1,n) - \mathbb{E}[U_{n,2}(t;k+1,n)]| \\
\leq &\sup_{t=k+2,\ldots,n-2} |U^Z_{n,2}(t;k+1,n)| + 8n^3 \sup_{a<b,a,b=1,\ldots,n} \left|\sum_{i=a}^{b} \Delta_n^T Z_i\right| \\
= &O_p(n^3\|\Sigma\|_F) + O_p(n^{3.5}\sqrt{\Delta_n^T \Sigma \Delta_n}) = o_p(n^4\|\Delta_n\|^2/\sqrt{a_n}),
\end{aligned}
$$

due to Assumption 3.1, Theorem 2.1 and the argument in Wang et al. (2019).

Similarly when t is between k^* and $n-2$, we have

$$
U_{n,2}(t;k+1,n)^2 \geq \mathbb{E}[U_{n,2}(t;k+1,n)]^2 - 2\mathbb{E}[U_{n,2}(t;k+1,n)] \sup_{t=k+2,\ldots,n-2} |U_{n,2}(t;k+1,n) - \mathbb{E}[U_{n,2}(t;k+1,n)]|,
$$

where $\mathbb{E}[U_{n,2}(t;k+1,n)] = (k^*-k)(k^*-k-1)(n-t)(n-t-1)\|\Delta_n\|_2^2 > 0$, and

$$
\begin{aligned}
&\sup_{t=k+2,\ldots,n-2} |U_{n,2}(t;k+1,n) - \mathbb{E}[U_{n,2}(t;k+1,n)]| \\
\leq &O_p(n^3\|\Sigma\|_F) + O_p(n^{3.5}\sqrt{\Delta_n^T \Sigma \Delta_n}) = O_p(n^4\|\Delta_n\|^2/\sqrt{a_n})
\end{aligned}
$$

Therefore by combining the above results we obtain that

$$
\begin{aligned}
&W_{n,2}(k;1,n)\\
\geq&\frac{1}{n}\sum_{t=2}^{k-2}U_{n,2}^Z(t;1,k)^2+\frac{1}{n}\sum_{t=k+2}^{k^*}\mathbb{E}[U_{n,2}(t;k+1,n)]^2+\frac{1}{n}\sum_{t=k^*+1}^{n-2}\mathbb{E}[U_{n,2}(t;k+1,n)]^2\\
&-\frac{2}{n}\sup_{t=k+2,\dots,n-2}|U_{n,2}(t;k+1,n)-\mathbb{E}[U_{n,2}(t;k+1,n)]|\sum_{t=k+2}^{k^*}\mathbb{E}[U_{n,2}(t;k+1,n)]\\
&-\frac{2}{n}\sup_{t=k+2,\dots,n-2}|U_{n,2}(t;k+1,n)-\mathbb{E}[U_{n,2}(t;k+1,n)]|\sum_{t=k^*+1}^{n-2}\mathbb{E}[U_{n,2}(t;k+1,n)]\\
\gtrsim&(k^*-k)^5n^3\|\Delta_n\|_2^4-(k^*-k)^3n\|\Delta_n\|_2^2\sup_{t=k+2,\dots,n-2}|U_{n,2}(t;k+1,n)-\mathbb{E}[U_{n,2}(t;k+1,n)]|\\
&+(k^*-k)^4n^4\|\Delta_n\|_2^4-(k^*-k)^2n^2\|\Delta_n\|_2^2\sup_{t=k+2,\dots,n-2}|U_{n,2}(t;k+1,n)-\mathbb{E}[U_{n,2}(t;k+1,n)]|\\
&-\left(\sup_k\sup_{t=2,\dots,k-2}|U_{n,2}^Z(t;1,k)|\right)^2\\
=&(k^*-k)^3n^3\|\Delta_n\|_2^4[(k^*-k)^2-o_p(n^2/\sqrt{\gamma_{n,2}})]+(k^*-k)^2n^4\|\Delta_n\|_2^4[(k^*-k)^2-o_p(n^2/\sqrt{\gamma_{n,2}})]-O_p(n^6\|\Sigma\|_F^2)\\
\geq&(k^*-k)^3n^3\|\Delta_n\|_2^4[\epsilon_n^2-o_p(n^2/\sqrt{\gamma_{n,2}})]+(k^*-k)^2n^4\|\Delta_n\|_2^4[\epsilon_n^2-o_p(n^2/\sqrt{\gamma_{n,2}})]-O_p(n^6\|\Sigma\|_F^2)\\
=&((k^*-k)^3n^3+(k^*-k)^2n^4)\|\Delta_n\|_2^4\epsilon_n^2(1-o_p(1))-O_p(n^6\|\Sigma\|_F^2),
\end{aligned}
$$

since $\epsilon_n=na_n^{-1/4+\kappa}$. And

$$\inf_{k<k^*-\epsilon}W_{n,2}(k;1,n)\gtrsim(\epsilon_n^3n^3+\epsilon_n^2n^4)\|\Delta_n\|_2^4\epsilon_n^2(1-o_p(1))-O_p(n^6\|\Sigma\|_F^2)=\epsilon_n^4n^4\|\Delta_n\|^4(1-o_p(1)),$$

since $\epsilon_n=o(n)$ and $\epsilon_n^4n^4\|\Delta_n\|^4/(n^6\|\Sigma\|_F^2)=\gamma_{n,2}^{1+4\kappa}\to\infty$. By very similar arguments, we can obtain the same bound for $\inf_{k>k^*+\epsilon}W_{n,2}(k;1,n)$, and hence $\inf_{k\in\Omega_n}W_{n,2}(k;1,n)\gtrsim\epsilon_n^4n^4\|\Delta_n\|^4(1-o_p(1))$. On the other hand, Theorem 2.1 implies that $W_{n,2}(k^*;1,n)=\frac{1}{n}\sum_{t=2}^{k^*-2}U_{n,2}^Z(t;1,k^*)^2+\frac{1}{n}\sum_{t=k^*+2}^{n-2}U_{n,2}^Z(t;k^*+1,n)^2=O_p(n^6\|\Sigma\|_F^2)$. This indicates that $W_{n,2}(k^*;1,n)=\epsilon_n^4n^4\|\Delta_n\|^4o_p(1)$, and consequently,

$$P\left(W_{n,2}(k^*;1,n) - \inf_{k\in\Omega_n} W_{n,2}(k;1,n) \geq 0\right) \leq P\Big(\epsilon_n^4 n^4 \|\Delta_n\|^4 o_p(1) - \epsilon_n^4 n^4 \|\Delta_n\|^4 (1-o_p(1)) \geq 0\Big) \to 0.$$

This completes the whole proof. $\diamondsuit$

7 Application to network change-point detection

Our change-point testing and estimation methods are applicable to network change-point detection in the following sense. Suppose we observe n independent networks $\{A_t\}_{t=1}^n$ over time with m nodes. Here A_t is the $m \times m$ adjacency matrix at time t. We assume the edges in each network are generated from Bernoulli random variables and are un-directed. That is,

$$A_{ij,t} = 1 \text{ if nodes } i \text{ and } j \text{ are connected at time } t \text{ and } 0 \text{ otherwise.}$$

Let $A_t = (A_{ij,t})_{i,j=1}^m$ and assume $E(A_{ij,t}) = p_{ij,t}$. Let $E(A_t) = \Theta_t = (p_{ij,t})_{i,j=1}^m$.

Suppose that we are interested in testing

$$H_0 : \Theta_1 = \cdots = \Theta_n$$

versus certain change point alternatives. Here we can convert the adjacency matrix into a high-dimensional vector, and apply our test and estimation procedures. Note that a mean shift in $vech(\Theta_t)$ implies a shift in variance matrix of $vech(A_t)$, so the variance matrix is not constant under the alternative. However, the asymptotic distribution of our SN-based test statistics still holds under the null, and our change-point detection method is applicable. Note that our method allows the edges to be weakly dependent, which can be satisfied by many popular network models; see Wang et al. (2020).

To examine the finite sample performance of our change-point testing and estimation in the network framework, we consider the following stochastic block model as in Wang et al. (2020).

We generate A_t as a matrix with entries being i.i.d. Bernoulli variables with mean matrix $\Theta_t = \mu_t ZQZ^T - \text{diag}(\mu_t ZQZ^T)$ where $Z \in \mathbb{R}^{m \times r}$ is the membership matrix and $Q \in [0,1]^{r \times r}$ is the connectivity matrix. We set Z to be the first r columns of identity matrix I_m so that $\text{rank}(Z) = r$, and $Q = \mathbf{1}_r \cdot \mathbf{1}_r^T$ be a matrix of ones.

Table 6 presents the size with 1000 Monte Carlo repetitions. We take $r = cm, \mu_t \equiv 0.1/c$ with $c = 0.2, 1$.

DGP c	(n, m)	$\mathcal{H}_0$,5%					$\mathcal{H}_0$,10%				
		$q=2$	$q=4$	$q=6$	$q=2,4$	$q=2,6$	$q=2$	$q=4$	$q=6$	$q=2,4$	$q=2,6$
1	(200,10)	0.035	0.096	0.068	0.08	0.048	0.075	0.152	0.135	0.124	0.096
	(400,20)	0.054	0.084	0.049	0.071	0.048	0.097	0.142	0.094	0.135	0.099
0.2	(200,10)	0.065	0.117	0.08	0.116	0.062	0.095	0.153	0.151	0.147	0.121
	(400,20)	0.05	0.101	0.043	0.09	0.047	0.099	0.153	0.096	0.137	0.083

Table 6: Size for testing one change point of network time series

As regards the power simulation, we generate the network data with a change point located at $\lfloor n/2 \rfloor$, which leads to $\mu_t = \mu + \delta \mathbb{I}(t > n/2) \cdot \mu$. We take $\mu = 0.1/c, r = cm$ with $c = 0.2, 1$ and $\delta = 0.2, 0.5$. We obtain the empirical power based on 1000 Monte Carlo repetitions.

DGP (δ, c)	(n, m)	$\mathcal{H}_0$,5%					$\mathcal{H}_0$,10%				
		$q=2$	$q=4$	$q=6$	$q=2,4$	$q=2,6$	$q=2$	$q=4$	$q=6$	$q=2,4$	$q=2,6$
(0.2,1)	(200,10)	0.152	0.172	0.116	0.19	0.145	0.223	0.254	0.225	0.265	0.222
	(400,20)	0.83	0.309	0.238	0.787	0.775	0.908	0.411	0.364	0.865	0.85
(0.5,1)	(200,10)	0.93	0.628	0.527	0.917	0.904	0.963	0.723	0.666	0.952	0.937
	(400,20)	1	0.995	0.97	1	1	1	0.997	0.99	1	1
(0.2,0.2)	(200,10)	0.804	0.677	0.61	0.798	0.755	0.866	0.75	0.708	0.86	0.829
	(400,20)	1	0.994	0.991	1	1	1	0.997	0.999	1	1
(0.5,0.2)	(200,10)	1	1	1	1	1	1	1	1	1	1
	(400,20)	1	1	1	1	1	1	1	1	1	1

Table 7: Power for testing one change point of network time series

We can see that our method exhibits similar size behavior as compared to the setting for Gaussian distributed data in Section 4.1. The power also appears to be quite good and increases when the signal increases. Unfortunately, we are not aware of any particular testing method tailored for single network change-point so we did not include any other method into the comparison.

To estimate the change-points in the network time series, we also combine our method with WBS. We generate 100 samples of networks with connection probability μ_t and sparsity parameter r. The 3 change points are located at $30, 60$ and 90. We take $\mu_t = \mu + \delta \cdot \mathbb{I}(30 < t \leq 60 \text{ or } t > 90) \cdot \mu$. We report the MSE and ARI of 100 Monte Carlo simulations as before. We compare our method with modified neighborhood smoothing (MNBS) algorithm in Zhao et al. (2019) and the graph-based test in Chen and Zhang (2015) combined with the binary segmentation (denoted as CZ). We do not include a comparison with Wang et al. (2020) as their method requires two iid samples. We can see that CZ performs worse than the other two methods as our simulation involves non-monotonic changes in the mean that does not favor binary segmentation. When the network becomes sparse, i.e. $c = 0.3$, our method also has better performance than MNBS. Overall the performance of our method (e.g., $WBS\text{-}SN(2)$, $WBS\text{-}SN(2{,}6)$) seem quite stable. Of course, the scope of this simulation is quite limited, and we leave a more in-depth investigation of network change-point estimation to near future.

(μ, δ, c)		$\hat{N} - N$							MSE	ARI
		-3	-2	-1	0	1	2	3		
(0.2, 1,1)	WBS-SN(2)	0	1	14	74	10	1	0	0.32	0.865
	WBS-SN(4)	90	9	1	0	0	0	0	8.47	0.0373
	WBS-SN(6)	32	23	24	16	4	1	0	4.12	0.278
	WBS-SN(2,6)	1	2	18	39	32	8	0	0.99	0.728
	CZ	46	50	4	0	0	0	0	6.18	0.165
	MNBS	0	2	17	55	23	3	0	0.6	0.847
(0.1, 1,0.3)	WBS-SN(2)	0	0	4	82	14	0	0	0.18	0.893
	WBS-SN(4)	12	17	38	33	0	0	0	2.14	0.604
	WBS-SN(6)	28	27	27	14	4	0	0	3.91	0.383
	WBS-SN(2,6)	0	1	8	60	29	2	0	0.49	0.852
	CZ	55	33	6	4	1	1	0	6.38	0.156
	MNBS	97	0	2	1	0	0	0	8.75	0.019

Table 8: Multiple change point location estimations for network time series