Sliced Full Factorial-Based Latin Hypercube Designs as a Framework for a Batch Sequential Design Algorithm

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Appendix A. Proof of Proposition 1

Proposition 1. *Let* $\overline{Y_b} = \frac{1}{L}\sum_{i=(b-1)L+1}^{bL} f(\mathbf{x}_{i\cdot})$*, the mean estimator using a single batch of sFFLHD. Then*

$$E(\overline{Y_b}) = \mu \quad \text{and} \quad E(\overline{Y}) = \mu. \tag{A.1}$$

For a continuous function f*, as* $L \to \infty$

$$Var(\overline{Y_b}) = \sum_{|u|\geq 2} S_b(u, |u|)L^{-2}Var(\alpha_u(\mathbf{x})) + o(L^{-1}). \tag{A.2}$$

At stages where the big grid design is an OA, as $L \to \infty$ *we also have*

$$Var(\overline{Y}) = \sum_{|u|\geq 3} S(u, |u|)n^{-2}Var(\alpha_u(\mathbf{x})) + o(n^{-1}). \tag{A.3}$$

Let L_b *be the number of levels of* $\mathbf{M}$*. At stages where* $n = L_b^D$*, the sequential design is an FFLHD, and as* $n \to \infty$ *we have*

$$Var(\overline{Y}) = O(L_b^{-D-2}). \quad \square \tag{A.4}$$

Proof. It can be verified in the algorithm that each possible batch in the space is equally likely to be sampled. With each batch being an LHD, the expectation of the mean estimator from a single run is an unbiased estimator for the true average. Therefore, the mean estimator from the sequential design is unbiased and (A.1) is true.

Whenever the big grid design is an OA, it is of the form $OA(\lambda L^2, D, L, 2)$, $\lambda \in \mathbb{N}_+$. He and Qian (2011) showed the variance structure as $L \to \infty$ when λ is fixed. (A.2) and (A.3) follow their proof.

To show (A.4), notice that when the intermediate grid design, $\mathbf{M}$, is a full factorial design with L_b levels, the variance of the mean estimator can be expressed as

$$Var(\overline{Y}) = \frac{1}{L_b^{2D}} \left[\sum_{i=1}^{L_b^D} Var(Y_i) + \sum_{i,j} Cov(Y_i, Y_j) \right].$$

The first part of the variance decomposition is the same as the lattice sampling variance under the uniform rectangle rule

$$x_{ij} = x_{ij}^c + (u_{ij} - \frac{1}{2})/L_b^D, i = 1, \ldots, L_b^D, j = 1, \ldots, D$$

where x_{ij} is the element in the ith row and the jth column of the design matrix, $\mathbf{X}$; x_{ij}^c's are the centers of the grid defined by the small grid design, $\mathbf{V}$; and the u_{ij}'s are independent random uniform $(0, 1]$ numbers. If $Y_i = f(\mathbf{x}_{i.})$, where $\mathbf{x}_{i.}$ is the ith row of $\mathbf{X}$, then Owen (1994) has shown the variance of lattice sampling is $O(L_b^{-D-2})$ under continuous f, which is smaller than that of random sampling $O(L_b^{-D})$.

Let m_{ih} denote the element in the ith row and hth column of $\mathbf{M}$. For i and j with $m_{ih} \neq m_{jh}, \forall h \in 1, \ldots, D$, we have $Cov(Y_i, Y_j) = 0$.

For i and j, with $m_{ik} = m_{jk}$ for a particular dimension k and $m_{ih} \neq m_{jh}, \forall h \in 1, \ldots, D, h \neq k$,

$$Cov(Y_i, Y_j) = E[(Y_i - \tau_i)(Y_j - \tau_j)] = E[(Y_i - \tau_i)E[(Y_j - \tau_j)|\mathbf{x}_{i.}]]$$

where τ_i and τ_j are $E(Y_i)$ and $E(Y_j)$, respectively.

$$\begin{aligned}E[(Y_j-\tau_j)|\mathbf{x}_{i.}] &= E[(Y_j-\tau_j)] - E[(f(\mathbf{x}_{j.})-\tau_j)|\lfloor L_b^D x_{jk}\rfloor = v_{ik}] \times P(\lfloor L_b^D x_{jk}\rfloor = v_{ik})\\ &= -\frac{L_b}{L_b^D - L_b} E[(f(\mathbf{x}_{j.})-\tau_j)|\lfloor L_b^D x_{jk}\rfloor = v_{ik}]\end{aligned}$$

where v_{ik} is the element in the ith row and the kth column of $\mathbf{V}$.

Given continuous f, $E[(f(\mathbf{x}_{j.})-\tau_j)|\lfloor L_b^D x_{jk}\rfloor = v_{ik}] = O(L_b^{-1})$. Therefore $E[(Y_j-\tau_j)|\mathbf{x}_{i.}] = O(L_b^{-D})$ and $Cov(Y_i, Y_j) = E[(Y_i-\tau_i)E[(Y_j-\tau_j)|\mathbf{x}_{i.}]] = O(L_b^{-1})O(L_b^{-D}) = O(L_b^{-D-1})$. So

$$\frac{1}{L_b^{2D}}\sum_{R_1} Cov(Y_i, Y_j) = \frac{1}{L_b^{2D}} D(L_b-1)^{D-1} L_b^D O(L_b^{-D-1}) = O(L_b^{-D-2})$$

where R_1 denotes the set of all i and j with $m_{ik} = m_{jk}$ for a particular dimension k and $m_{ih} \neq m_{jh}, \forall h \in 1, \ldots, D, h \neq k$. Using the above approach for i and j with $m_{ik} = m_{jk}$ on more than one dimension, we found the covariance is $o(L_b^{-D-2})$. Therefore we have proven the variance of the mean estimator at FFLHD stage is $O(L_b^{-D-2})$. □

Appendix B. Generating Non-overlapping OAs

Let $\mathbf{A}^1$ be an $OA(L^2, D+1, L, 2)$ $(L \geq D > 2)$ that serves as the initial OA and let $\mathbf{A}_{i.}$ denote the ith row of $\mathbf{A}^1$ with elements $a_{i1}, a_{i2}, \ldots, a_{iD}$. Let $\mathbf{v}$ be a $1 \times D$ vector, $\mathbf{v} = [v_1, v_2, \ldots, v_D]$. Set v_1 and v_2 equal to zero; the other elements can take values from $0, 1, 2, \ldots, L-1$. There are L^{D-2} unique vectors, $\mathbf{v}^1, \ldots, \mathbf{v}^{L^{D-2}}$. From these, we can generate L^{D-2} new orthogonal arrays $\mathbf{B}^j$, where the ith row of $\mathbf{B}^j$ has elements $b_{ik}^j = (a_{ik} + v_k^j)\text{mod}(L)$ $(k = 1, \ldots, D)$.

Proposition 2. *If $j_1 \neq j_2$, then $\mathbf{B}^{j_1}$ and $\mathbf{B}^{j_2}$ do not share a common row. Furthermore, the concatenation of $\mathbf{B}^1, \mathbf{B}^2, \ldots, \mathbf{B}^{L^{D-2}}$ forms a full factorial design.*

Proof. We have $b_{ik}^{j_1} = (a_{ik} + v_k^{j_1})\, mod\, L$ and $b_{ik}^{j_2} = (a_{ik} + v_k^{j_2})\, mod\, L$. Therefore $b_{ik}^{j_1} = b_{ik}^{j_2} + (v_k^{j_2} - v_k^{j_1}) \mod L$. By construction, the first two elements of row i_1 from $\mathbf{B}^{j_1}$ and row i_2 from $\mathbf{B}^{j_2}$ both agree (i.e., $b_{i_10}^{j_1} = b_{i_20}^{j_2}$ and $b_{i_11}^{j_1} = b_{i_21}^{j_2}$) only if $i_1 = i_2$, so the only way that $\mathbf{B}^{j_1}$ and $\mathbf{B}^{j_2}$ can have a row in common is if $(v_k^{j_2} - v_k^{j_1})\, mod\, L = 0\ \forall k > 2$. This cannot occur, since $v_k^j \in \{0, \ldots, L\}\ \forall \{j, k\}$ and $\mathbf{v}^{j_1} \neq \mathbf{v}^{j_2}$. Therefore, for any $j_1 \neq j_2$, $\mathbf{B}^{j_1}$ and $\mathbf{B}^{j_2}$ do not share a common row. Because this holds for all $j_1 \neq j_2$, and because the concatenation of $\mathbf{B}^1, \mathbf{B}^2, \ldots, \mathbf{B}^{L^{D-2}}$ has L^D rows, D columns, and L levels within each column, this concatenation forms a full factorial design. $\square$

Note 1: Without loss of generality, let $\mathbf{v}_j^1 = 0\ \forall j$, then $\mathbf{B}^1 = \mathbf{A}^1$ (the original OA). This means it is possible to generate $L^{D-2} - 1$ additional non-overlapping OAs from $\mathbf{A}^1$ while retaining the original design.

Note 2: Each remaining new OA, $\mathbf{B}^j$ for $j \neq 0$, can be partitioned into L slices of L rows each; these slices should be randomly reordered. Finally, the new OAs $\mathbf{A}^2, \mathbf{A}^3, \ldots, \mathbf{A}^{L^{D-2}}$ can be selected by randomly choosing (without replacement) from the set of reordered new OAs $\{\mathbf{B}^2, \ldots, \mathbf{B}^{L^{D-2}}\}$.

Note 3: For $D = 2$, the original OA ($\mathbf{A}^1$) is already a full factorial design, and no additional OAs need to be generated.

Appendix C. Generating Non-overlapping Fractional Factorials

Let $\mathbf{X}$ denote an L-level, L^D-run sFFLHD for D factors, and suppose r satisfies $r^q = L, r \in \mathbb{N}_+$ for some $q \in \mathbb{N}_+$. Define $\mathbf{M} = \lfloor rL\mathbf{X} \rfloor$, i.e., $\mathbf{M}$ is the projection of $\mathbf{X}$ onto a grid of levels $0, \ldots, rL - 1$ and forms an L^D-run orthogonal array that is a fraction of a new sFFLHD with rL levels and $(rL)^D$ runs. Let $\mathbf{M}_{i.}$ denote the ith row of $\mathbf{M}$ with elements $m_{i1}, m_{i2}, \ldots, m_{iD}$. Let $\mathbf{v}$ be a $1 \times D$ vector, $\mathbf{v} = [v_1, v_2, \ldots, v_D]$, where each element can take values from $0, 1, 2, \ldots, r - 1$. There are r^D unique vectors, $\mathbf{v}^1, \ldots, \mathbf{v}^{r^D}$. From these, we can generate r^D new fractional factorials $\mathbf{F}^j$, where the ith row of $\mathbf{F}^j$ has elements $f_{ik}^j = r\lfloor m_{ik}/r \rfloor + [(m_{ik} + v_k^j) \text{ mod } r]$ ($i = 1, \ldots, r^D$ and $k = 1, \ldots, D$).

Proposition 3. *If $j_1 \neq j_2$, then $\mathbf{F}^{j_1}$ and $\mathbf{F}^{j_2}$ do not share a common row. Furthermore, the concatenation of $\mathbf{F}^1, \mathbf{F}^2, \ldots, \mathbf{F}^{r^D}$ forms a full factorial design.*

Proof. The proof is similar to that of Proposition 2.

Note 1: Without loss of generality, let $\mathbf{v}_j^1 = 0\ \forall j$, then $\mathbf{F}^1 = \mathbf{M}$ (the original OA). This means it is possible to generate $(rL)^{D-2} - 1$ additional non-overlapping OAs from $\mathbf{F}^1$ while retaining the original design.

$f_{ik}^{j_1} = f_{ik}^{j_2} + (v_k^{j_2} - v_k^{j_1}) \mod r$. The only way that $\mathbf{F}^{j_1}$ and $\mathbf{F}^{j_2}$ can have a row in common is if $(v_k^{j_2} - v_k^{j_1}) \mod r = 0\ \forall k$. This cannot occur, since $v_k^j \in \{0, \ldots, r\}\ \forall \{j, k\}$ and $\mathbf{v}^{j_1} \neq \mathbf{v}^{j_2}$. Therefore, for any $j_1 \neq j_2$, $\mathbf{F}^{j_1}$ and $\mathbf{F}^{j_2}$ do not share a common row. Because this holds for all $j_1 \neq j_2$, and because the concatenation of $\mathbf{F}^1, \mathbf{F}^2, \ldots, \mathbf{F}^{L^{D-2}}$ has $(rL)^D$ rows, D columns, and rL levels within each column, this concatenation forms a full factorial design. $\square$

Appendix D. Probability Mass Functions of the Small Grid Design Elements

Let l_b denote the number of levels in the small grid design, $\mathbf{V}$, at the stage when batch b is to be observed. Let Z_{l_b} denote the set $\{1, 2, \ldots, l_b\}$. If i is restricted to $(b-1)L+1 \leq i \leq bL$, then v_{ij} represents an element in the small grid design for batch, b and

$$P(v_{ij} = s) = \frac{1}{l_b},\ s \in Z_{l_b}. \tag{D.1}$$

For every i, k in batch b, such that $(b-1)L+1 \leq i \leq bL$ and $(b-1)L+1 \leq k \leq bL$, if $i \neq k$, then

$$P(v_{ij} = s, v_{kj} = t) = \begin{cases} \frac{L}{l_b^2(L-1)} & s, t \in Z_{l_b} \text{ and } \lfloor sL/l_b \rfloor \neq \lfloor tL/l_b \rfloor \\ 0 & \text{otherwise.} \end{cases} \tag{D.2}$$

Consider two batches b and b' and, without loss of generality, let $b > b'$. Now let $(b-1)L+1 \leq i \leq bL$ and $(b'-1)L+1 \leq k \leq b'L$ so that v_{ij} is in batch, b and v_{kj} is in batch, b', then

$$P(v_{ij} = s, v_{kj} = t) = \begin{cases} \frac{1}{l_b^2} & s, t \in Z_{l_b} \text{ and } \lfloor sL/l_b \rfloor \neq \lfloor tL/l_b \rfloor \\ \frac{1}{l_b(l_b-L)} & s, t \in Z_{l_b}\ ,\ s \neq t \text{ and } \lfloor sL/l_b \rfloor = \lfloor tL/l_b \rfloor \\ 0 & \text{otherwise.} \end{cases} \tag{D.3}$$

Proof. To prove (D.1), i is restricted to $(b-1)L+1 \leq i \leq bL$, so w_{ij} represents an element in the big grid design for batch b. For some integer $s \in Z_{l_b}, w_{ij} = \lfloor sL/l_b \rfloor$. Because the first OA is randomly shuffled and the algorithm selects every possible OA with equal probability, w_{ij} can take on any value from $\{0, 1, \ldots, L-1\}$ with equal likelihood. Furthermore, given w_{ij}, v_{ij} is randomly selected from all available choices. Therefore, (D.1) holds.

To prove D.2, since $i \neq k$ and both $(b-1)L+1 \leq i \leq bL$ and $(b-1)L+1 \leq k \leq bL$, then w_{ij} and w_{kj} represent elements in different rows of the same column in the big grid design for batch b. Note that $P(w_{ij} = w_{kj}) = 0$ because w_{ij} and w_{kj} are two distinct elements from the set $\{0, 1, \ldots, L-1\}$. All other possible pairs of (v_{ij}, v_{kj}) are equally likely to be obtained because of symmetry. Fixing j, there are l_b choices of v_{ij}. Given v_{ij}, since $w_{ij} \neq w_{kj}$, there are $l_b - l_b/L$ choices of v_{kj}. The number of all possible pairs of (v_{ij}, v_{kj}) is $l_b(l_b - l_b/L)$ and (D.2) holds.

To show D.3, i and k are such that $(b-1)L+1 \leq i \leq bL$ and $(b'-1)L+1 \leq k \leq b'L$ so that v_{ij} is in batch, b and v_{kj} is in batch, b'. Without loss of generality, $b > b'$. Consider the following three sets,

$S_1 = \{(v_{ij} = s, v_{kj} = t) | s, t \in Z_{l_b} \text{ and } \lfloor sL/l_b \rfloor \neq \lfloor tL/l_b \rfloor\}$

$S_2 = \{(v_{ij} = s, v_{kj} = t) | s, t \in Z_{l_b}\ ,\ s \neq t \text{ and } \lfloor sL/l_b \rfloor = \lfloor tL/l_b \rfloor\}$

$S_3 = \{(v_{ij} = s, v_{kj} = t) | s, t \in Z_{l_b}\ ,\ s = t\}$

S_1, S_2, and S_3 have $(l_b - l_b/L)l_b$, $(l_b/L - 1)l_b$, and l_b pairs respectively. The design algorithm prohibits S_3, and therefore $P(S_3) = 0$. For S_1, $\{v_{ij} = s\}$ and $\{v_{kj} = t\}$ are independent by the nature of the design algorithm. $P((v_{ij} = s, v_{kj} = t)|S_1) = P(v_{ij} = s)P(v_{kj} = t) = 1/1_b^2$. By symmetry, we have $P(S_1) = (L-1)/L$ and $P(S_2) = 1/L$. Thus, $P((v_{ij} = s, v_{kj} = t)|S_3) = 1/l_b(l_b - L)$, and (D.3) holds. □

Appendix E. Rastrigin Example

The Rastrigin function is:

$$f(x) = 10D + \sum_{i=1}^{D}[x_i^2 - 10cos(2\pi x_i)].$$

D is the dimensionality of the Rastrigin function and $x_i \in [-5.12, 5.12]$ in each dimension. We use $D = 4$, 6, and 8 to compare the two designs, and scale the Rastrigin function to fit $[0,1]^D$. Confidence intervals of *RMSE* differences are shown in Figure E.1. After the first batch, sFFLHD performs much better than MmDist in terms of *RMSE*s in the 6- and 8-dimensional cases, with the largest advantage around 6–7 batches. For the 4-dimensional case, differences between the two designs are not significant.

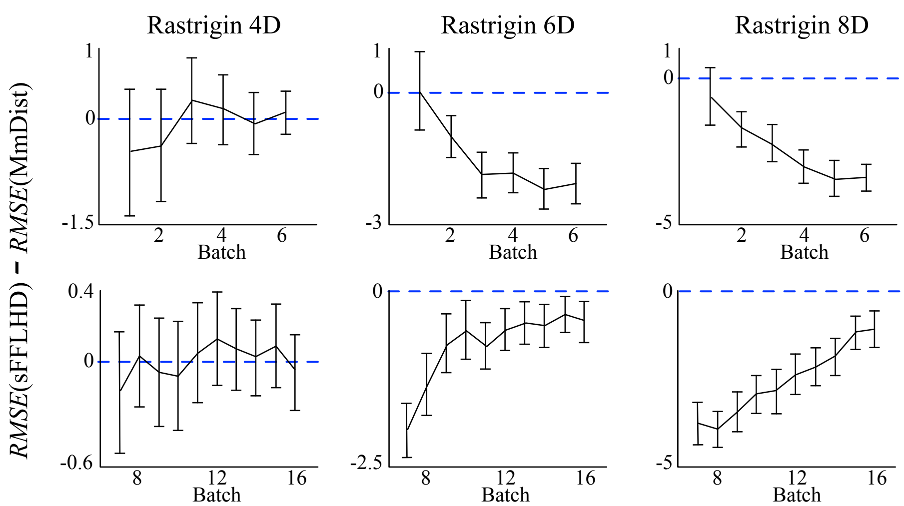

Figure E.1: Rastrigin Examples: 95% confidence intervals for *RMSE*(sFFLHD)−*RMSE*(MmDist) vs. number of batches completed, for batches 1–6 (top) and 7–16 (bottom). Dashed lines indicate differences of zero.

Appendix F. Computer Model Mean Estimation Example

Consider a computer model given by

$$f = x_1 + x_2 + x_1x_2 + x_1^2 + x_2^2 + min(e^{3x_2}, 10) - 1.5x_1x_2x_3 + x_3^2 \tag{F.1}$$

$$x_1 \sim \text{Unif}[-2, 0], x_2 \sim \text{Unif}[0, 1], x_3 \sim \text{Unif}[0.5, 1.5].$$

We adopt a final run size of 64 using batches of size 4 and calculate $\hat{\mu}$ for each scheme at each batch stage over 2,000 replications. *RMSE*s of $\hat{\mu}$ at selected batch stages are shown in Table 1. The result shows that in terms of *RMSE*, sFFLHD has the best mean estimator at all batch stages except for the very first batch when bLHS and MmDist have essentially equivalent *RMSE*. After batch one, the mean estimator of sFFLHD is significantly superior to all the designs except rsFFLHD. At stages where the sFFLHD is an OALHD (shown in bold), sFFLHD and rsFFLHD are equivalent designs.

Batch	1	**4**	8	12	**16**
Design Points	4	**16**	32	48	**64**
sFFLHD	0.213	**0.004**	0.001	0.003	**0.00012**
bMmLHD	3.512	**0.704**	0.238	0.077	**0.00047**
MmDist	0.206	**0.146**	0.049	0.031	**0.02372**
rsFFLHD	3.006	**0.004**	0.001	0.003	**0.00012**
bLHD	0.208	**0.051**	0.026	0.017	**0.01305**

Table F.1: Computer Model Example: Comparison of *RMSE* of $\hat{\mu}$ for each design scheme.

Appendix G. *RMSE* Performance in Several Examples

Table G.2: A negative entry indicates better performance by sFFLHD, bold indicates significance at 0.05 level

Example	Batch 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Borehole 4D																
sFFLHD-MmDist	0.2167	**-0.8769**	**-0.0239**	0.0042	0.0023	**-0.0018**	**-0.0057**	**-0.0062**	**-0.0059**	**-0.0042**	**-0.0046**	**-0.0033**	**-0.0028**	**-0.0026**	**-0.0018**	**-0.0020**
sFFLHD-bMmLHD	**-2.2089**	-0.0223	**-0.0203**	**-0.0113**	**-0.0094**	**-0.0038**	**-0.0030**	**-0.0031**	**-0.0026**	**-0.0019**	**-0.0015**	**-0.0014**	-0.0003	-0.0006	-0.0002	-0.0004
sFFLHD-rsFFLHD	**-3.2831**	-0.0872	**-0.0188**	-0.0038	**-0.0022**	-0.0005	-0.0007	-0.0004	-0.0001	0.0007	0.0007	0.0006	0.0007	0.0003	0.0006	0.0005
sFFLHD-bLHD	0.0985	0.0160	-0.0079	-0.0018	**-0.0061**	0.0006	-0.0012	**-0.0011**	**-0.0014**	-0.0008	0.0000	-0.0003	-0.0001	-0.0005	-0.0008	-0.0006
Borehole 6D																
sFFLHD-MmDist	0.1670	**-1.7631**	**-2.3901**	**-0.5417**	**-0.0408**	**-0.0216**	**-0.0308**	-0.0067	-0.0057	**-0.0148**	**-0.0112**	**-0.0041**	-0.0006	0.0001	-0.0003	-0.0008
sFFLHD-bMmLHD	**-4.4655**	-0.7585	**-0.4883**	**-0.1987**	**-0.0651**	**-0.0340**	**-0.0312**	**-0.0201**	**-0.0110**	**-0.0041**	**-0.0038**	**-0.0036**	**-0.0030**	**-0.0024**	**-0.0019**	**-0.0023**
sFFLHD-rsFFLHD	**-3.7856**	0.2073	-0.4100	**-0.0658**	**-0.0192**	-0.0042	**-0.0055**	-0.0023	0.0024	0.0001	0.0003	-0.0003	0.0002	0.0005	-0.0001	-0.0006
sFFLHD-bLHD	-1.7317	0.1518	-0.4441	-0.1661	**-0.0285**	**-0.0258**	**-0.0203**	**-0.0173**	-0.0040	**-0.0035**	**-0.0029**	**-0.0026**	**-0.0026**	**-0.0017**	**-0.0019**	**-0.0022**
Borehole 8D																
sFFLHD-MmDist	0.9058	**-1.7326**	**-3.8926**	**-2.3692**	**-1.1305**	**-0.7615**	**-0.8555**	**-0.8742**	**-0.5922**	**-0.5502**	**-0.5535**	**-0.4721**	**-0.4222**	**-0.3377**	**-0.3399**	**-0.2227**
sFFLHD-bMmLHD	**-2.8725**	**-2.5794**	**-1.5455**	**-0.7859**	-0.3669	**-0.4084**	**-0.3723**	**-0.2060**	**-0.1292**	**-0.0485**	-0.0287	**-0.0544**	**-0.0443**	**-0.0262**	**-0.0353**	**-0.0264**
sFFLHD-rsFFLHD	-5.7156	**-1.3069**	-0.5506	0.0637	-0.0561	0.0750	-0.0779	-0.0594	**-0.0670**	0.0085	-0.0134	-0.0128	-0.0136	-0.0021	-0.0104	0.0022
sFFLHD-bLHD	0.3362	-1.6434	**-2.3916**	**-1.3897**	**-0.6331**	**-0.3043**	**-0.2701**	**-0.2388**	**-0.0993**	-0.0280	-0.0070	-0.0291	-0.0186	-0.0110	-0.0161	**-0.0223**
Rastrigin 4D																
sFFLHD-MmDist	-0.4894	-0.4424	0.2556	0.0704	-0.0158	0.0892	-0.1193	0.0746	-0.0346	-0.0599	0.0961	0.1526	0.0941	0.0623	0.1124	-0.0459
sFFLHD-bMmLHD	0.0929	-0.0881	-0.1048	-0.3097	**-0.5777**	**-0.5121**	**-0.5184**	**-0.4114**	**-0.4534**	-0.2803	-0.2254	-0.2426	-0.1311	**-0.2151**	-0.2019	**-0.3001**
sFFLHD-rsFFLHD	0.6360	0.0227	-0.2609	-0.2782	**-0.3728**	-0.3103	-0.2360	-0.2297	-0.2317	-0.1342	-0.1085	-0.0420	0.0254	-0.0785	-0.0514	-0.1618
sFFLHD-bLHD	0.0758	-0.1410	-0.2483	-0.3774	**-0.6460**	**-0.6851**	**-0.6728**	**-0.6501**	**-0.6257**	**-0.5930**	**-0.5371**	**-0.4294**	**-0.4010**	**-0.4405**	**-0.4415**	**-0.5731**
Rastrigin 6D																
sFFLHD-MmDist	-0.5810	**-0.9849**	**-1.3288**	**-2.0820**	**-2.5413**	**-3.3498**	**-3.3046**	**-2.7339**	**-2.1492**	**-1.8426**	**-1.1970**	**-0.8713**	**-0.5618**	**-0.3803**	**-0.2826**	-0.1279
sFFLHD-bMmLHD	**-0.7031**	-0.1642	0.0452	-0.1595	0.2935	0.0967	-0.2810	**-0.3024**	-0.0925	0.0529	0.0814	-0.0436	0.0142	0.0927	0.0418	0.0516
sFFLHD-rsFFLHD	**-0.8445**	-0.1091	0.0915	0.1524	0.1590	0.2525	-0.1455	-0.1996	-0.0718	**-0.6548**	**-0.6280**	**-0.6345**	**-0.6578**	**-0.7006**	**-0.6891**	-0.0423
sFFLHD-bLHD	-0.6137	0.0398	-0.0874	-0.3351	-0.2169	-0.1186	**-0.3439**	**-0.6950**	**-0.3798**	**-1.0419**	**-0.8528**	**-0.9183**	**-0.8713**	**-0.9577**	**-0.9017**	**-0.9311**
Rastrigin 8D																
sFFLHD-MmDist	-0.6066	**-1.7347**	**-2.2364**	**-2.9977**	**-3.4235**	**-3.3879**	**-3.7468**	**-3.9275**	**-3.4273**	**-2.9117**	**-2.8339**	**-2.3444**	**-2.1446**	**-1.8555**	**-1.1634**	**-1.0677**
sFFLHD-bMmLHD	-0.3068	-0.1033	-0.1762	**-0.4237**	-0.1684	-0.0880	-0.0357	0.0571	-0.1984	-0.1676	-0.0441	0.1472	0.1469	0.1097	0.1298	-0.0332
sFFLHD-rsFFLHD	-0.0767	-0.2563	-0.0978	-0.3288	0.0624	-0.0598	0.0327	-0.0164	-0.0728	-0.0366	-0.0640	0.0024	0.0300	-0.0191	-0.0396	-0.1751
sFFLHD-bLHD	0.2069	0.0001	-0.2959	**-0.5488**	**-0.3739**	**-0.4707**	-0.2712	-0.2437	**-0.4599**	**-0.3709**	-0.2741	-0.1929	-0.2043	-0.2233	-0.1686	**-0.3272**

Table G.3: A negative entry indicates better performance by sFFLHD, bold indicates significance at 0.05 level

Example	Batch 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Gaussian 2D, $\theta = 5$																
sFFLHD-MmDist	-1.48E-02	**-2.01E-02**	**-6.87E-03**	-1.59E-03	3.87E-04	-7.03E-04	**-6.21E-04**	-1.48E-04	-8.57E-06	3.44E-06	-2.01E-05	-6.19E-06	1.42E-05	1.45E-05	8.28E-06	6.01E-06
sFFLHD-bMmLHD	7.05E-03	**-6.45E-02**	-2.08E-02	-6.68E-03	**-4.93E-03**	**-4.96E-03**	**-3.28E-03**	**-1.66E-03**	**-1.10E-03**	**-4.86E-04**	**-3.39E-04**	**-2.29E-04**	**-6.91E-05**	**-5.18E-05**	**-3.85E-05**	**-2.88E-05**
sFFLHD-rsFFLHD	1.81E-02	**-3.15E-02**	-4.76E-03	2.73E-03	2.86E-04	4.85E-05	-1.55E-04	3.77E-05	-1.17E-04	-9.02E-05	-1.93E-05	3.62E-06	-1.72E-05	-6.81E-06	1.30E-06	3.67E-06
sFFLHD-bLHD	4.69E-02	4.06E-03	3.08E-03	-8.73E-04	**-2.74E-03**	**-2.71E-03**	**-2.34E-03**	**-1.43E-03**	**-5.60E-04**	**-4.44E-04**	**-2.84E-04**	**-1.26E-04**	**-1.02E-04**	**-4.78E-05**	**-3.90E-05**	**-3.95E-05**
Gaussian 4D, $\theta = 5$																
sFFLHD-MmDist	-0.0029	-0.0103	-0.0083	-0.0106	**-0.0241**	-0.0054	-0.0002	**-0.0402**	**-0.0304**	**-0.0210**	**-0.0165**	**-0.0110**	**-0.0093**	**-0.0083**	**-0.0067**	-0.0048
sFFLHD-bMmLHD	-0.0310	-0.0243	-0.0068	-0.0215	**-0.0317**	-0.0197	**-0.0180**	**-0.0275**	**-0.0214**	**-0.0158**	**-0.0223**	**-0.0200**	**-0.0158**	**-0.0185**	**-0.0132**	-0.0075
sFFLHD-rsFFLHD	-0.0230	-0.0182	0.0157	0.0187	0.0005	-0.0085	-0.0047	-0.0009	-0.0089	-0.0109	**-0.0157**	-0.0072	-0.0041	-0.0045	-0.0016	0.0028
sFFLHD-bLHD	0.0150	-0.0110	0.0040	-0.0094	-0.0109	-0.0172	**-0.0168**	**-0.0219**	**-0.0172**	**-0.0134**	**-0.0181**	**-0.0112**	**-0.0094**	**-0.0139**	**-0.0147**	**-0.0147**
Gaussian 6D, $\theta = 5$																
sFFLHD-MmDist	0.0132	0.0021	-0.0039	-0.0179	**-0.0439**	**-0.0461**	**-0.0439**	**-0.0651**	**-0.0669**	**-0.0823**	**-0.0703**	**-0.0535**	**-0.0568**	**-0.0462**	**-0.0497**	**-0.0432**
sFFLHD-bMmLHD	0.0092	-0.0008	-0.0086	-0.0133	-0.0116	-0.0070	0.0030	-0.0147	**-0.0166**	**-0.0159**	-0.0116	-0.0085	-0.0075	-0.0071	-0.0096	**-0.0142**
sFFLHD-rsFFLHD	0.0175	0.0016	0.0026	-0.0031	-0.0032	0.0058	0.0124	0.0100	-0.0025	-0.0026	-0.0041	0.0066	0.0062	0.0048	0.0038	0.0020
sFFLHD-bLHD	-0.0069	0.0023	0.0010	-0.0093	-0.0035	0.0056	0.0020	-0.0025	-0.0025	-0.0026	-0.0014	0.0031	0.0017	0.0012	-0.0013	-0.0042
Gaussian 8D, $\theta = 5$																
sFFLHD-MmDist	0.0054	-0.0021	-0.0122	-0.0093	-0.0095	-0.0119	**-0.0147**	**-0.0126**	**-0.0212**	**-0.0258**	**-0.0280**	**-0.0346**	**-0.0300**	**-0.0381**	**-0.0375**	**-0.0355**
sFFLHD-bMmLHD	-0.0077	-0.0072	-0.0051	-0.0026	-0.0003	0.0021	0.0063	0.0080	0.0016	0.0031	0.0009	-0.0006	0.0042	-0.0012	0.0064	0.0046
sFFLHD-rsFFLHD	-0.0124	-0.0150	-0.0035	-0.0118	-0.0006	0.0005	0.0035	0.0055	-0.0039	-0.0033	0.0006	-0.0042	0.0003	-0.0033	0.0005	0.0007
sFFLHD-bLHD	-0.0081	-0.0130	-0.0093	-0.0070	-0.0003	-0.0031	0.0026	0.0033	-0.0009	-0.0009	-0.0005	-0.0003	0.0059	0.0003	0.0052	0.0020
Gaussian 2D, $\theta = 15$																
sFFLHD-MmDist	0.0054	**-0.0313**	**-0.0315**	-0.0125	**-0.0129**	**-0.0133**	**-0.0075**	**-0.0135**	**-0.0060**	**-0.0060**	**-0.0031**	**-0.0014**	-0.0004	0.0001	0.0002	0.0001
sFFLHD-bMmLHD	-0.0524	-0.0183	-0.0124	**-0.0475**	**-0.0349**	**-0.0335**	**-0.0251**	**-0.0227**	**-0.0134**	**-0.0121**	**-0.0106**	**-0.0063**	**-0.0043**	**-0.0030**	**-0.0016**	**-0.0022**
sFFLHD-rsFFLHD	-0.0474	0.0013	0.0170	-0.0202	**-0.0209**	**-0.0053**	**-0.0077**	-0.0009	0.0009	**-0.0023**	**-0.0031**	**-0.0025**	-0.0011	-0.0011	0.0001	0.0001
sFFLHD-bLHD	-0.0040	0.0153	0.0234	-0.0051	**-0.0223**	**-0.0169**	**-0.0275**	**-0.0255**	**-0.0158**	**-0.0170**	**-0.0095**	**-0.0068**	**-0.0053**	**-0.0045**	**-0.0034**	**-0.0023**
Gaussian 4D, $\theta = 15$																
sFFLHD-MmDist	-0.0026	-0.0114	-0.0123	**-0.0234**	**-0.0412**	**-0.0474**	**-0.0614**	**-0.0588**	**-0.0714**	**-0.0814**	**-0.0908**	**-0.0934**	**-0.0909**	**-0.0887**	**-0.0886**	**-0.0903**
sFFLHD-bMmLHD	0.0149	0.0020	0.0113	0.0034	-0.0037	0.0045	-0.0024	0.0049	0.0058	-0.0040	-0.0052	**-0.0084**	**-0.0111**	**-0.0154**	**-0.0160**	**-0.0178**
sFFLHD-rsFFLHD	0.0006	0.0076	0.0063	0.0038	-0.0051	-0.0007	0.0007	0.0080	0.0086	0.0049	0.0088	0.0025	0.0021	-0.0005	-0.0028	0.0015
sFFLHD-bLHD	0.0051	-0.0144	0.0053	0.0027	**-0.0121**	-0.0072	-0.0074	0.0002	0.0010	-0.0035	-0.0037	**-0.0130**	**-0.0164**	**-0.0196**	**-0.0188**	**-0.0214**
Gaussian 6D, $\theta = 15$																
sFFLHD-MmDist	0.0057	0.0022	0.0045	0.0052	-0.0037	**-0.0056**	**-0.0081**	**-0.0089**	**-0.0119**	**-0.0151**	**-0.0143**	**-0.0179**	**-0.0192**	**-0.0218**	**-0.0264**	**-0.0276**
sFFLHD-bMmLHD	0.0058	0.0047	-0.0124	-0.0038	-0.0075	-0.0027	-0.0017	0.0020	0.0011	0.0004	0.0005	0.0013	-0.0029	-0.0020	0.0004	-0.0016
sFFLHD-rsFFLHD	0.0032	-0.0049	0.0034	0.0029	-0.0033	-0.0030	-0.0058	-0.0016	-0.0004	-0.0006	0.0050	0.0022	0.0024	0.0047	0.0018	0.0025
sFFLHD-bLHD	-0.0003	-0.0006	**-0.0140**	-0.0087	-0.0048	0.0000	0.0015	0.0006	-0.0001	0.0025	0.0019	-0.0009	-0.0002	-0.0005	0.0029	-0.0003
Gaussian 8D, $\theta = 15$																
sFFLHD-MmDist	0.0016	0.0018	0.0012	0.0014	0.0022	0.0007	0.0008	0.0003	0.0003	-0.0001	-0.0009	**-0.0018**	**-0.0015**	**-0.0020**	**-0.0021**	**-0.0018**
sFFLHD-bMmLHD	-0.0045	-0.0031	-0.0017	-0.0004	0.0016	-0.0003	0.0005	0.0003	0.0010	0.0006	0.0007	-0.0001	-0.0006	0.0000	0.0000	-0.0001
sFFLHD-rsFFLHD	-0.0016	-0.0022	-0.0002	-0.0003	0.0005	-0.0005	0.0010	-0.0003	-0.0001	-0.0008	-0.0003	**-0.0013**	-0.0009	**-0.0013**	-0.0010	-0.0008
sFFLHD-bLHD	0.0048	0.0021	0.0026	0.0016	0.0008	0.0002	0.0013	0.0006	0.0004	0.0004	-0.0003	**-0.0014**	**-0.0013**	-0.0008	-0.0003	-0.0003
Gaussian 3D, $\theta = 5$																
sFFLHD-binbased	-0.0104	0.0052	-0.0147	-0.0162	-0.0030	0.0034	0.0039	-0.0034								

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