Appendix for "Enforcing stationarity through the prior in vector autoregressions"

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Appendix

Forward mapping: VAR parameters to partial autocorrelations

This proceeds as described in Ansley and Newbold (1979). Using Cholesky factors in the matrix-square-roots below leads to the parameterization of Ansley and Kohn (1986), whilst using symmetric matrix-square-roots gives rise to the parameterization described in this paper.

The mapping from $(\Sigma, \Phi) \in \mathcal{S}_m^+ \times \mathcal{C}_{p,m}$ to $\{\Sigma, (P_1, \dots, P_p)\} \in \mathcal{S}_m^+ \times \mathcal{V}_m^p$, described in Ansley and Newbold (1979), proceeds in two main stages.

1. From (Σ, Φ), compute the autocovariances $\Gamma_i = \mathrm{Cov}(\boldsymbol{y}_t, \boldsymbol{y}_{t+i})$ $(i = 0, \dots, p)$. $\Gamma_0, \dots, \Gamma_{p-1}$ can be found by representing the autoregression as a $\mathrm{VAR}_{pm}(1)$ process and computing its stationary variance. The resulting discrete Lyapunov equation can be solved using vectorization and Kronecker product operators. The remaining autocovariance Γ_p can then be calculated using the Yule-Walker equations for the order p process. For further details, see, for example, Chapter 2 of Lütkepohl (2005).
2. From Φ and $(\Gamma_0, \dots, \Gamma_p)$ compute $(P_1, \dots, P_p)$:
   (a) Initialize: construct $\Sigma_0 = \Sigma_0^* = \Gamma_0$ and then calculate the matrix-square-root factorizations, $\Sigma_0 = \Sigma_0^* = S_0 S_0^{\mathrm{T}} = S_0^* S_0^{*\,T}$.
   (b) Recursion: for $s = 0, \dots, p-1$:
      (i) Compute $\phi_{s+1,s+1}$ and $\phi_{s+1,s+1}^*$ using

$$\begin{aligned}\phi_{s+1,s+1} &= \left(\Gamma_{s+1}^{\mathrm{T}} - \phi_{s1}\Gamma_s^{\mathrm{T}} - \cdots - \phi_{ss}\Gamma_1^{\mathrm{T}}\right) \Sigma_s^{*\,-1}, \\ \phi_{s+1,s+1}^* &= \left(\Gamma_{s+1} - \phi_{s1}^*\Gamma_s - \cdots - \phi_{ss}^*\Gamma_1\right) \Sigma_s^{-1},\end{aligned}$$

      where it is understood that when $s = 0$, these expressions simplify to $\phi_{11} = \Gamma_1^{\mathrm{T}} \Sigma_0^{*\,-1} = \Gamma_1^{\mathrm{T}} \Gamma_0^{-1}$, $\phi_{11}^* = \Gamma_1 \Gamma_0^{-1}$.
      (ii) If $s > 0$, for $i = 1, \dots, s$, compute $\phi_{s+1,i}$ and $\phi_{s+1,i}^*$ using

$$\phi_{s+1,i} = \phi_{si} - \phi_{s+1,s+1}\phi_{s,s-i+1}^*, \quad \phi_{s+1,i}^* = \phi_{si}^* - \phi_{s+1,s+1}^*\phi_{s,s-i+1}.$$

(iii) Compute P_{s+1} using one of

$$P_{s+1} = S_s^{-1}\phi_{s+1,s+1}S_s^*, \quad P_{s+1} = \left(S_s^{*\,-1}\phi_{s+1,s+1}^*S_s\right)^{\mathrm{T}}.$$

(iv) If $s < p-1$, compute $\varSigma_{s+1}$ and $\varSigma_{s+1}^*$ using

$$\begin{aligned}\varSigma_{s+1} &= \varGamma_0 - \phi_{s+1,1}\varGamma_1 - \cdots - \phi_{s+1,s+1}\varGamma_{s+1},\\ \varSigma_{s+1}^* &= \varGamma_0 - \phi_{s+1,1}^*\varGamma_1^{\mathrm{T}} - \cdots - \phi_{s+1,s+1}^*\varGamma_{s+1}^{\mathrm{T}},\end{aligned}$$

then calculate the matrix-square-roots, $\varSigma_{s+1} = S_{s+1}S_{s+1}^{\mathrm{T}}$, $\varSigma_{s+1}^* = S_{s+1}^*S_{s+1}^{*\,T}$.

Reverse mapping: partial autocorrelations to VAR parameters

The inverse mapping from $\{\varSigma, (P_1, \ldots, P_p)\} \in \mathcal{S}_m^+ \times \mathcal{V}_m^p$ to $(\varSigma, \varPhi) \in \mathcal{S}_m^+ \times \mathcal{C}_{p,m}$, comprises two recursions; the first is new and the second is based on Lemma 2.1 of Ansley and Kohn (1986).

1. From $\{\varSigma, (P_1, \ldots, P_p)\}$ compute the stationary variance matrix $\varGamma_0$:
   (a) Initialize: let $\varSigma_p = \varSigma$ with corresponding matrix-square-root factorization, $\varSigma_p = S_pS_p^{\mathrm{T}}$.
   (b) Recursion: for $s = p-1, \ldots, 0$ construct the symmetric (or lower triangular) matrix S_s such that

   $$\varSigma_{s+1} = S_s(I_m - P_{s+1}P_{s+1}^{\mathrm{T}})S_s^{\mathrm{T}},$$

   then compute $\varSigma_s = S_sS_s^{\mathrm{T}}$.
   (c) Output: take $\varGamma_0 = \varSigma_0$.
2. From $(P_1, \ldots, P_p)$ and $\varGamma_0$ compute the matrices in $\varPhi$:
   (a) Initialize: let $\varSigma_0 = \varSigma_0^* = \varGamma_0$ with corresponding matrix-square-root factorization, $\varSigma_0 = \varSigma_0^* = S_0S_0^{\mathrm{T}} = S_0^*S_0^{*\,T}$.
   (b) Recursion: for $s = 0, \ldots, p-1$:
      (i) Compute $\phi_{s+1,s+1}$ and $\phi_{s+1,s+1}^*$ using

      $$\phi_{s+1,s+1} = S_sP_{s+1}S_s^{*\,-1}, \quad \phi_{s+1,s+1}^* = S_s^*P_{s+1}^{\mathrm{T}}S_s^{-1}.$$

      (ii) If $s > 0$, for $i = 1, \ldots, s$, compute $\phi_{s+1,i}$ and $\phi_{s+1,i}^*$ using

      $$\phi_{s+1,i} = \phi_{si} - \phi_{s+1,s+1}\phi_{s,s-i+1}^*, \quad \phi_{s+1,i}^* = \phi_{si}^* - \phi_{s+1,s+1}^*\phi_{s,s-i+1}.$$

      (iii) Compute $\varSigma_{s+1}$ and $\varSigma_{s+1}^*$ using

      $$\varSigma_{s+1} = \varSigma_s - \phi_{s+1,s+1}\varSigma_s^*\phi_{s+1,s+1}^{\mathrm{T}}, \quad \varSigma_{s+1}^* = \varSigma_s^* - \phi_{s+1,s+1}^*\varSigma_s\phi_{s+1,s+1}^{*\,T},$$

      then calculate the matrix-square-roots, $\varSigma_{s+1} = S_{s+1}S_{s+1}^{\mathrm{T}}$ and $\varSigma_{s+1}^* = S_{s+1}^*S_{s+1}^{*\,T}$.
      (iv) Compute $\varGamma_{s+1}$ using

      $$\varGamma_{s+1}^{\mathrm{T}} = \phi_{s+1,s+1}\varSigma_s^* + \phi_{s1}\varGamma_s^{\mathrm{T}} + \cdots + \phi_{ss}\varGamma_1^{\mathrm{T}}.$$

   (c) Output: take $\phi_i = \phi_{pi}$ $(i = 1, \ldots, p)$. By construction, $\varSigma = \varSigma_p$.