Nonparametric, Stochastic Frontier Models with Multiple Inputs and Outputs: Supplementary Material

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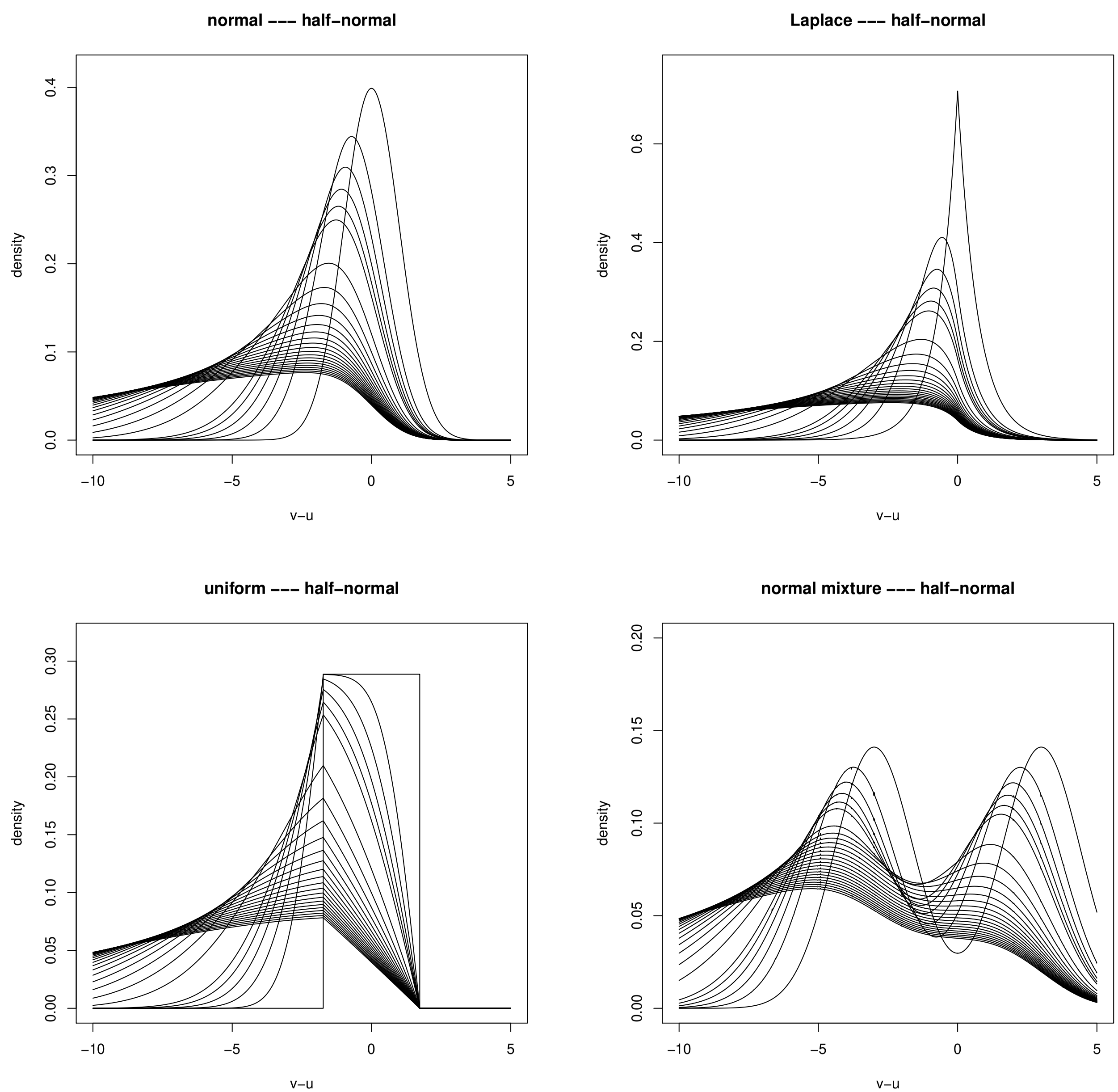
A Flexibility of the Stochastic Specification

As noted in the main paper, our stochastic specification consisting of a symmetric, two-sided noise process and a local half-normal inefficiency process is quite flexible. This is illustrated in Figure A.1 for the particular choice of the one-parameter scale density in Assumption 2.5, with various values of the scale parameter σ_η and various symmetric densities for ϵ (Normal, Laplace, Uniform and a Mixture of two Normal densities). Figure A.1 shows that the density of the convolution of noise and efficiency distributions can have a wide variety of shapes under our assumptions.

Figure A.1: Half-Normal Inefficiency with Various Symmetric Noise Distributions



B Handling the "Wrong Skewness" Problem

It is well-known that that the convolution of the density of an unbounded, symmetrically-distributed random variable and the density of a one-sided random variable may yield finite samples that are skewed in either direction. In both the basic and transformed models discussed above, one might expect the $\{\xi_i\}_{i=1}^n$ to be skewed to the left, but skewness to the right occurs with some probability depending on the noise-to-signal ratio $\sqrt{\mathrm{VAR}(\epsilon)/\mathrm{VAR}(\eta)}$. In samples of sizes often encountered in empirical work, and with reasonable noise-to-signal ratios, this phenomenon may occur with high probability (e.g., see Simar and Wilson, 2010, Table 1). This is sometimes called the "wrong skewness" problem, but as discussed by Simar and Wilson (2010), it can occur in the absence of any specification error or other mistakes; i.e., the problem may arise in cases there is nothing "wrong" with the model nor with anything else. In the context of our model, $r_3(z) \leq 0$ by construction, but in finite samples, one may find $\widehat{r}_3(z) > 0$. Since local estimation is used here, the effective sample size is of order $n^{4/(p+q+3)}$ due to the use of bandwidths with optimal order. Hence the problem is even more serious here than in the fully-parametric context discussed by Simar and Wilson (2010).

The SVKZ estimator deals with unexpected skewness by simply truncating the estimator $\widehat{\sigma}_\eta(z)$ at zero in (3.9). Alternatively, one might consider imposing the constraint $r_3(z) \leq 0$ in the local linear regression of the $\widehat{\xi}_i^3$ on Z_i as mentioned by SVKZ. Or, instead one might use the local exponential estimator proposed by Ziegelmann (2002). As discussed in Appendix C.2, constrained local linear estimation does not change the truncation in (3.9), but affects estimation of derivatives. As also discussed in Appendix C.2, local exponential estimation may be attractive from a theoretical viewpoint, but it loses the numerical flexibility of local linear methods and its computational burden is considerably larger than local linear methods. A third approach, adopted here, is to extend the ideas developed by HMS for fully parametric settings to the localized framework here.[1]

The idea is straightforward, and we provide only a brief sketch of the idea here. Details on derivations are given in HMS. To illustrate, we maintain the local half-normal specification for the distribution of the inefficiency. Let k_j denote the jth raw moment of the reference density $N^+(0,1)$. We have

$$E\left[(\eta - \mu_\eta(z))^3 \mid z\right] = a_3^+ \gamma^3(z) > 0 \tag{B.1}$$

where $a_3^+ = k_3 - 3k_1k_2 + 2k_1^3$. In the "regular" case, $\gamma(z) = \sigma_\eta(z) > 0$ and the skewness of $\eta \mid z$ is positive. Denote the density of $\eta \mid z$ in this case by $h^+(\eta \mid z) = \frac{1}{\gamma(z)} \widetilde{h}^+\left(\frac{\eta}{\gamma(z)}\right)$.

Now consider the density $h^-(\eta|z)$ with support on $\mathbb{R}_+$ having the same mean as h^+, but with negative skewness. This is obtained by reflecting the density h^+ around zero, shifting so that the support lies in $\mathbb{R}_+$, and then truncating on the right so that the resulting density has the same mean as h^+. This yields

$$h^-(\eta \mid z) = \frac{1}{|\gamma(z)|} \widetilde{h}^-\left(\frac{\eta}{|\gamma(z)|}\right), \tag{B.2}$$

where $\widetilde{h}^- \neq \widetilde{h}^+$ (see HMS for an explicit expression for $\widetilde{h}^-$ and for its properties; we use use only its first 3 moments). By construction, the mean given by the density $h^-(\eta|z)$ is equal to $\mu_\eta(z) \geq 0$ (same mean as $h^+(\eta|z)$), and so

$$E\left[(\eta - \mu_\eta(z))^3 \mid z\right] = a_3^- \gamma(z)^3 < 0, \tag{B.3}$$

where $\gamma(z) < 0$ and $\sigma_\eta(z) = |\gamma(z)| > 0$ For the half-normal family, $a_3^+ = \sqrt{2/\pi}\,((4-\pi)/\pi)$ and $a_3^- \approx 0.016741474$ (again, see HMS for further details).

Adapting equations (37)–(38) in HMS to our local version, we have

$$\widehat{\sigma}_\eta(z) = \begin{cases} \left[-\frac{1}{a_3^+}\widehat{r}_3(z)\right]^{1/3} & \text{if } \widehat{r}_3(z) \leq 0; \\ \left[\frac{1}{a_3^-}\widehat{r}_3(z)\right]^{1/3} & \text{otherwise.} \end{cases} \tag{B.4}$$

[1] Of course, the "wrong skewness" problems goes away asymptotically, but empirical researchers are typically confronted with samples of fixed sizes. All three methods mentioned here are asymptotically equivalent, since the unexpected skewness disappears with probability one as $n \to \infty$, but they may differ in finite samples.

Asymptotically, one should expect $\widehat{r}_3(z) \leq 0$, so the asymptotic properties are the same as for the SVKZ estimator. But in finite samples, the HMS estimator in (B.4) avoids excessive numbers of estimates equal to zero.

Similarly, if an estimator of the variance of ϵ is required, using the half-normal family for the inefficiency term we have

$$\widehat{\sigma}_\epsilon^2(z) = \begin{cases} \max\left[0,\ \widehat{r}_2(z) - a_2^+ \widehat{\sigma}_\eta^2(z)\right] & \text{if } \widehat{r}_3(z) \leq 0; \\ \max\left[0,\ \widehat{r}_2(z) - a_2^- \widehat{\sigma}_\eta^2(z)\right] & \text{otherwise,} \end{cases} \tag{B.5}$$

where $a_2^+ = (\pi - 2)/\pi$ and $a_2^- \approx 0.14471441$ (see HMS for details of derivations). Here, the estimator of $\widehat{\sigma}_\epsilon^2$ is truncated at zero but our experience with Monte Carlo experiments suggests this will rarely be needed.[2]

[2]Without truncation, negative estimates of σ_ϵ^2 can occur when the residuals $\widehat{\epsilon}_i$ have skewness greater than the maximum skewness that is possible with the convolution of normal and half-normal random variables. Olson et al. (1980) refer to this as "type-II failure" in the context of corrected ordinary least squares. See Papadopoulos and Parmeter (2021) for discussion.

C Alternative Approaches for Dealing with Unexpected Skewness

C.1 Sign-Constrained Local Linear Least Squares

The method is briefly mentioned in Remark 2.2 in SVKZ, where the authors suggest first computing the ordinary, local-linear least squares (LLLS) nonparametric estimator $\widehat{r}_3(z)$ of $r_3(z)$. If the resulting estimate has the wrong sign (i.e., if $\widehat{r}_3(z) > 0$), then a new, constrained LLLS estimate can be computed while imposing the constraint $r_3(z) \leq 0$ in a constrained optimization problem. We can use the same idea here, but we show that the constrained optimization problem can be easily solved by adapting results from Liew (1976) derived for parametric linear least squares.

Consider the LLLS estimator of the mean regression function $m(z) = \mathbb{E}(\zeta|Z = z)$ for a univariate response ζ and where z has dimension $(r-1)$. Here, $\zeta_i = \widehat{\varepsilon}_i^3$ for estimating $r_3(z)$, but it could also be $\widehat{\varepsilon}_i^2$ for estimating $r_2(z)$.

It is well known (see Fan and Gijbels, 1996, Section 7.8) that LLLS estimator can be written as

$$\widehat{\beta} = (\mathcal{Z}_D'\mathcal{W}\mathcal{Z}_D)^{-1}\mathcal{Z}_D'\mathcal{W}\mathcal{Y}, \tag{C.1}$$

where $\mathcal{Y} = (\zeta_1, \ldots, \zeta_n)'$, $\mathcal{Z}_D = \begin{bmatrix} i_n & (\mathcal{Z} - i_n z') \end{bmatrix}$ where $\mathcal{Z} = (Z_1, \; \ldots, \; Z_n)'$ and the usual weights matrix is $\mathcal{W} = \mathrm{diag}(K_h(Z_i - z))$. The LLLS estimator

$$\widehat{\beta} = \arg\min_{\beta}(\mathcal{Y} - \mathcal{Z}_D\beta)'\mathcal{W}(\mathcal{Y} - \mathcal{Z}_D\beta) \tag{C.2}$$

is the solution of a weighted least squares problem, and

$$\widehat{m}(z) = \widehat{\beta}_1 \tag{C.3}$$

and

$$\widehat{\frac{\partial m}{\partial z_j}}(z) = \widehat{\beta}_{j+1} \tag{C.4}$$

for $j = 1, \ldots, r-1$.

Following the arguments in Liew (1976), it can be shown that the solution of the weighted least squares problem under the constraint that $\beta_1 \leq 0$, when the constraint is binding, is given by the simple expression

$$\widehat{\beta}_C = \widehat{\beta} - A_1 \widehat{\beta}_1 / a_1, \tag{C.5}$$

where A_1 is the first column of the matrix $(\mathcal{Z}_D' \mathcal{W} \mathcal{Z}_D)^{-1}$ and a_1 is the first element of A_1. It can be shown that the same correction is valid for the case of the constraint $\beta_1 \geq 0$, when the latter is binding.

Another approach is to impose observation-specific bounds along the lines of Hall and Huang (2001) or Du et al. (2016). For instance, when estimating the variance $\sigma_\epsilon^2(Z_i)$, for observation i, one could impose the constraint

$$\widehat{r}_2(Z_i) \geq \begin{cases} a_2^+ \widehat{\sigma}_\eta^2(Z_i) > 0 & \text{if } \widehat{r}_3(Z_i) \leq 0; \\ a_2^- \widehat{\sigma}_\eta^2(Z_i) > 0 & \text{otherwise} \end{cases}$$

in order to ensure that $\widehat{\sigma}_\epsilon^2(Z_i) > 0$. However, this would involve high-dimensional nonlinear optimization. In our simulations, we prefer to use the simple estimator defined in (B.5) due to their numerical simplicity. In our empirical application, where we have almost a half-million observations, imposing observation-specific bounds would be computationally formidable, even using the powerful super-computer that we use.

C.2 Local Exponential smoothing

To be complete in our presentation, we briefly summarize the steps for obtaining the Local Exponential estimator proposed by Ziegelmann (2002) for regression models where the dependent variable is sign constrained (say positive). In our case (for the 3rd moment) it is obtained by solving

$$(\hat{\alpha}, \hat{\beta}) = \arg\min_{\alpha,\beta} \sum_{i=1}^{n} \left\{ \left[-\widehat{\varepsilon}_i^3 \right] - \exp\left[\alpha + \beta'(Z_i - z) \right] \right\}^2 K_h(Z_i - z) \tag{C.6}$$

for a given z. Then we have $\widehat{r}_3(z) = -\exp(\hat{\alpha}) \leq 0$. Ziegelmann shows that this estimator asymptotic properties similar to those of the usual local linear estimator. However, the numerical burden of the local exponential estimator is large, and numerical instabilities may arise due to the exponential transformation in (C.6). In practice, to eliminate the risk of ending with local minima, several starting values should be tried. We conducted some initial Monte Carlo experiments using this estimator, but even using robust nonlinear optimization techniques, e.g., the Nelder and Mead (1965) simplex method, the results indicate a high degree of numerical instability, frequently yielding unreliable results and in all cases incurring large computational cost.

D Some Alternative Approaches for Estimation

As discussed in Section 3 of the main paper, estimation by methods other than those of SVKZ and HMS could be used to estimate our transformed model. For example, one could specify the link function as a translog function of the elements of Z_i, and further specify the distribution of $\epsilon_i \mid Z_i$ as normal and the distribution of $\eta_i \mid Z_i$ as half-normal. If, in addition, specific functional forms are specified for $\sigma_\epsilon(Z_i)$ and $\sigma_\eta(Z_i)$, the resulting fully parametric model could then be estimated by maximum likelihood. However, one or more of the required specification assumptions are likely to be wrong.[3]

The goal here is to avoid restrictive parametric assumptions, but this raises questions regarding identification in the sense that we wish to be able to use knowledge about the conditional density $f(u \mid z)$ of U to recover information about triple $\{\phi(z),\ f(\epsilon|z),\ f(\eta|z)\}$ with ϵ and η being independent conditionally on z. For a given z, this amounts to a deconvolution problem since $u = \phi(z) + \epsilon - \eta$. So far, the discussions about identification has been in terms of the univariate dependent (output) variable case. But now, due to the change of coordinates, we have the transformed model (2.17)–(2.19) which also has a univariate dependent variable, namely U.

The results of Hall and Simar (2002) indicate that if the density of $\eta \mid z$ is completely unspecified and the density of $\epsilon \mid z$ is symmetric around zero, the model is not identified. For this reason, Hall and Simar (2002) consider a limiting case where the conditional variance of the noise $\sigma_\epsilon^2(z)$ converges to zero as the sample size n increases. Under this assumption, Hall and Simar derive consistent estimators of the model. Of course this technical assumption is rather restrictive, limiting its interest for practitioners.

Kumbhakar et al. (2007) propose a different approach, wherein the conditional densities of ϵ and η are specified *locally*, e.g., $\epsilon \mid z \sim N(0, \sigma_\epsilon^2(z))$ and $\eta \mid z \sim N^+(0, \sigma_\eta^2(z))$. The model

[3] While maximum likelihood estimators achieve root-n convergence in fully parametric models, this only ensures rapid convergence to something that is not a feature of the true model if the model is mis-specified. Robinson (1988) refers to this as "root-n *in*consistency."

is then clearly identified for each z, and the model can be estimated by local maximum likelihood. To maintain the nonparametric structure of the model, both scale functions are functional, and local polynomial approximations are used. Although the approach is appealing, estimation involves a formidable computational burden. Moreover, the local structure of the stochastic components is parametric, and asymptotic properties of the estimators rely on the specific parametric assumptions used to derive the local likelihood function.

Kneip et al. (2015) leave the inefficiency distribution unspecified, but require the noise $\epsilon \mid z$ to be Gaussian with unknown variance. Under the latter assumption, they prove the model is identified. For estimation, they use quasi-likelihood approaches with histogram estimators for the density of the inefficiency $\eta \mid z$. Here again the numerical burden is significant. Daouia et al. (2020) use an alternative approach based on nonparametric estimation of the conditional survival function of $U \mid Z$, but for identification must assume that the noise follows a fully known distribution, e.g., $\epsilon \mid z \sim N(0, \sigma_\epsilon^2(z))$ where $\sigma_\epsilon^2(z)$ is known.

More recently, Florens et al. (2020) specify a model where the noise is only assumed to be symmetric (as in our specification) and a flexible parametric density is chosen for the inefficiency. They also investigate identification issues (see their Section 2) and prove that if the density of $\eta \mid z$ is completely specified by its odd cumulants of order larger than 3, then the model is identified. This is a sufficient condition for identification when the noise is symmetric.

E Some Extensions

E.1 The Cases of Hyperbolic and Radial Distances

The directional distance function defined in (2.3) is *additive* in the sense that for a given point $(x, y) \in \Psi$, subtracting $\delta(x, y \mid d)d_x$ from x and adding $\delta(x, y \mid d)d_y$ to y projects (x, y) onto Ψ^∂ in the direction $d = (-d_x', d_y)$. Alternatively, one might be interested in the Farrell hyperbolic measure of efficiency

$$\gamma(x, y) := \inf\left\{\gamma \mid (\gamma x, \gamma^{-1}y) \in \Psi\right\} \tag{E.1}$$

described by Färe et al. (1985). The methods described above can easily be adapted to estimate such measures by working with logs of (X_i, Y_i) provided all of the input-output data are strictly positive.. Analogous to (2.4), we can write

$$\begin{bmatrix}\log X_i \\ \log Y_i\end{bmatrix} = \begin{bmatrix}\log X_i^\partial \\ \log Y_i^\partial\end{bmatrix} + \begin{bmatrix}-(\epsilon_i - \eta_i)i_p \\ (\epsilon_i - \eta_i)i_q\end{bmatrix} \tag{E.2}$$

where ϵ_i and η_i are as in Assumption 2.4. In the absence of noise, we have $\eta_i = \log \gamma(X_i, Y_i) \geq 0$, but noise can be accommodated exactly as before.

Alternatively, one might consider the Farrell (1957) output measure of efficiency given by

$$\lambda(x, y) := \sup\left\{\lambda \mid (x, \lambda y) \in \Psi\right\}. \tag{E.3}$$

Estimates can be obtained similarly using $d = (0_p', i_q')'$ and logged data. In the absence of noise, we have $\eta_i = \log \lambda(x, y)$, but noise can be allowed for as before. Yet another alternative is the Farrell (1957) input measure given by

$$\theta(x, y) := \inf\left\{\theta \mid (\theta x, y) \in \Psi\right\}. \tag{E.4}$$

This can be estimated by setting $d = (-i_p', 0_q')'$ and again using logged data, leading to $\eta_i = \log \theta(x, y)$ in the absence of noise.

Note that the links between additive, directional efficiency and the radial hyperbolic, input and output efficiencies discussed here involve use of direction vectors that consist of 0s and 1s. Consequently, the direction vector should not be normalized to have length $||d|| = 1$.

E.2 Introducing Exogenous Environmental Variables

The models developed in Sections 2.2 and 2.3 can be easily extended to accommodate environmental variables E_i that may influence either the frontier levels W_i^∂ or characteristics of the components of ξ_i. Conditioning the frontier points W_i^∂ on the E_i leads to $W_i^\partial(E_i) = (X_i^\partial(E_i), Y_i^\partial(E_i))$, and as a consequence the scale functions $\widetilde{\sigma}_\epsilon$ and $\widetilde{\sigma}_\eta$ must also be conditioned on the E_i. This amounts to replacing (2.17)–(2.19) with the conditional transformed model

$$U_i = \phi(Z_i, E_i) + ||d||\epsilon_i - ||d||\eta_i \tag{E.5}$$

where $\epsilon_i \mid Z_i, E_i \sim \text{Sym}(0, \sigma_\epsilon^2(Z_i, E_i))$ and $\eta_i \mid Z_i, E_i \sim D_+(\sigma_\eta(Z_i, E_i))$.

This is sufficient to allow one to disentangle to role of E on the frontier level as well as on the inefficiency distribution, thereby avoiding restrictive assumptions that are often made in in the case of no noise (see Simar and Wilson, 2007 for discussion). For estimation, one can condition everywhere on not just Z_i but instead (Z_i, E_i) and follow the route outlined above in Section 3. All of the results remain valid with the exception of the rate of convergence, which is slower when environmental variables are present. For $E \in \mathbb{R}^s$, the optimal rate is $n^{2/(p+q+s+3)}$, which makes clear that including environmental variables exacerbates the curse of dimensionality, leading to perhaps imprecise estimates unless n is large.

E.3 Stochastic Versions of FDH/DEA-Type Estimators

Both the SVKZ and HMS estimators discussed in Section 3 can be extended to impose assumptions of free disposability or convexity of the production set Ψ. It is sufficient to apply the FDH estimator (to impose free disposability) or the DEA estimator (to impose both free disposability and convexity) to the estimated frontier points $\widehat{W}_i^\partial$ obtained with either the SVKZ or HMS estimator as discussed above in Section 3. This amounts to a regularization step, as discussed Simar and Zelenyuk (2011). The usual convergence rates of the FDH and DEA estimators are $n^{1/(p+q)}$ and $n^{2/(p+q+1)}$, respectively, but here the rates can be improved.

To achieve this, frontier points on a grid of values $\{z_k\}_{k=1}^{K_n}$ should be estimated using the methods described in Section 3, with K_n large enough to control the size of the error of the regularization. For instance, if the FDH estimator is applied on these K_n points, the rate of error in the FDH regularization step is $K_n^{1/(p+q)}$. Hence, in order to obtain a free-disposal estimator of Ψ keeping the same properties as the initial, un-regularized estimator, one should select $K_n > n^{2(p+q)/(p+q+3)}$. Alternatively, if the DEA estimator is used, one should choose $K_n > n^{(p+q+1)/(p+q+3)}$ for the regularization step.

E.4 Data-Driven Direction

As observed following Assumption 2.2 in Section 2.2, the stochastic element ξ_i operates on frontier points in the direction d, and changing the direction vector results in a different model. Nonetheless, some papers allow the direction vector d to be determined by data according to some criterion function which is then optimized. Examples include Vardanyan and Noh (2006); Atkinson and Tsionas (2016); Färe et al. (2017); and and Layer et al. (2020) among others.

In some papers, the criterion involves minimizing inefficiency across firms in the sample, as in Färe et al. (2017). In others, the direction is optimized to be consistent with desirable economic behavior such as cost minimization or some other behavior, as in Atkinson and Tsionas (2016). In the latter case, whether firms minimize cost or some other economic criterion is an empirical question, and as such should be tested.

The former case, where direction is chosen to minimize some aggregate measure of efficiency, is even more problematic. Inefficiency is not *noise* in our model; instead, it is the *signal*, and the signal is potentially contaminated by noise. Choosing the direction vector d to minimize, for example, $\sum_{i=1}^{n} \xi_i^2$ would minimize not only the noise, but also the signal, and we see no reason why this is sensible. Moreover, this would make both the two-sided noise term ϵ_i and the one-sided inefficiency term η_i dependent on d, resulting in some uncertainty about the properties of estimators such as the ones we use. In addition, it is not clear what the underlying DGP might be or how one would simulate data from such a model. As such, the model would be incoherent. Allowing the choice of direction to be data-driven along the lines of Atkinson and Tsionas (2016) or Färe et al. (2017) require substantially more parametric structure than we impose in our model.

In many cases, researchers use a fixed direction vector with directions given by the medians or means of the corresponding input and output variables. It is important to note that if the directions are not given in the same units as the corresponding variables, estimated distances will depend on units of measurement. This is well-known.

F Monte Carlo Evidence

F.1 Experimental Framework

Here we describe results from a series of Monte Carlo experiments to gain insight on how well our estimation approach works in terms of recovering information about the true, underlying model. We consider various sample sizes n and various numbers of inputs (p) and outputs (q). We also consider various noise-to-signal ratios.

In order to simulate a frontier Ψ^{∂} in settings with multiple inputs and multiple outputs, consider a sphere of radius one centered at the origin in $\mathbb{R}^{p+q}$. Then reflect the negative part of the q output axes and the positive part of the p input axes around the origin so that only $1/2^{p+q}$ of the sphere lying in $\mathbb{R}^p_- \times \mathbb{R}^q_+$ remains. Then shift the remaining surface along the p input axes in the positive direction by one unit so that the frontier Ψ^{∂} lies in $\mathbb{R}^{p+q}_+$. In two dimensions with $p = q = 1$, this results in a frontier corresponding to the upper-left, northwest quarter of a circle centered at $(1, 0)$.

In each experiment, for given values of p and q, we first use the method of Muller (1959) and Marsaglia (1972) to draw $(p+q)$-tuples $a_i = (a_{p,i}, a_{q,i})$ uniformly distributed on the surface of a closed $(p+q)$-ball centered at the origin with radius one for $i = 1, \ldots, n$. The draws are independent, so that $a_i \perp\!\!\!\perp a_j$ for $i \neq j$. For each i we then set $X_i^{\partial} = (1 - |a_{p,i}|)$ and $Y_i^{\partial} = |a_{q,i}|$ to obtain efficient input-output pairs $(X_i^{\partial}, Y_i^{\partial})$. For the direction vector we use $d = \widetilde{d}/||\widetilde{d}||$ with $\widetilde{d} = \begin{bmatrix} -i_p' & i_q' \end{bmatrix}'$ where i_p and i_q are column vectors of ones with lengths p and q, respectively. We normalize d to have length equal to one, but this is not required and has no bearing on the results of our experiments.

We consider two sets of experiments. Section F.2 discusses experiments where inefficiency is distributed half-normal as in Assumption 2.5. Section F.3 discusses experiments where inefficiency is distributed exponential, but the half-normal specification in Assumption 2.5 is maintained for estimation. The second set of experiments discussed in Section F.3 serves as a robustness check.

F.2 Half-Normal Inefficiency

We first consider homoskedastic deviations from the frontier. We set $\sigma_\eta = 0.5$ and draw $\eta_i \sim N^+(0, \sigma_\eta^2)$ for each $i = 1, \ldots, n$. For a given value of $\rho \geq 0$, we set

$$\sigma_\epsilon = \rho \left[(\pi - 2)/\pi\right]^{1/2} \sigma_\eta \tag{F.1}$$

and then draw $\epsilon_i \sim N(0, \sigma_\epsilon^2)$. The value of ρ serves to control the noise-to-signal ratio, i.e., the ratio of the standard deviations of the efficiency η_i and the noise ϵ_i.

Using the direction vector d to construct the rotation matrix R_d defined in (2.10), the efficient input-output pairs $(X_i^\partial, Y_i^\partial)$ are transformed to (z, u)-space by computing $(Z_i^\partial, U_i^\partial)$ via (2.20). We compute then construct "observed values" (Z_i, U_i) by setting $Z_i = Z_i^\partial$ and computing $U_i = U_i^\partial + \xi_i$ where $\xi_i = \epsilon_i - \eta_i$. Note that this is identical to computing "observed" values (X_i, Y_i) in (x, y)-space from the $(X_i^\partial, Y_i^\partial)$ using (2.4) and then computing the (Z_i, U_i) using (2.11).

In each set of experiments, we consider sample sizes $n \in \{100,\ 200,\ 400,\ 800,\ 1600,\ 3200,\ 6400\}$ and noise-to-signal ratios $\rho \in \{0.0,\ 0.25,\ 0.50,\ 0.75,\ 1.0,\ 2.0\}$. In addition, we consider settings with two to six dimensions with $(p, q) \in \{(1,1),\ (1,2),\ (2,2),\ (2,3),\ (3,3)\}$. Each experiment with fixed values of n, ρ and (p, q) consists of 1,000 Monte Carlo trials. For our first set of experiments, with homoskedastic noise and inefficiency, we apply both the SVKZ or HMS estimator described above to obtain two sets of estimates $\widehat{U}_i^\partial$ of U_i^∂ for $i = 1, \ldots, n$. For each Monte Carlo trial, we compute an estimate of integrated mean square error (IMSE) given by

$$\widehat{\text{IMSE}}_m = \sum_{i=1}^{n} \left(\widehat{U}_i^\partial - U_i^\partial\right)^2, \tag{F.2}$$

$m = 1 \ldots, 1000$ for either set of estimates. Table F.1 reports the sample means of these estimates (for both the SVKZ and HMS estimators) over the 1,000 Monte Carlo trials in each of the resulting $6 \times 7 \times 5 = 210$ experiments.

The results in Table F.1 indicate that for both estimators, IMSE decreases with sample size, and increases with both number of dimensions and the noise-to-signal ratio as expected. Moreover, when $\rho \leq 1$, the HMS estimator results in smaller IMSE than the SVKZ estimator in all but a few cases. To check whether the differences in IMSE between the two estimators are significant, IMSE for the SVKZ estimator for given value of n, p and q is subtracted from the corresponding IMSE for the HMS estimator for each Monte Carlo trial, and this difference is divided by the square root of the variance of the differences across Monte Carlo trials divided by the number of trials. By the Lindeberg-Levy central limit theorem, the resulting quantity is approximately distributed $N(0, 1)$, allowing two-sided tests of significance. The differences are shown in Table F.2, with asterisks indicating significance at 90, 95 or 99 percent levels. The differences in F.2 are significant at 90-percent or better in 179 of 210 cases. In addition, all of the significant differences are negative when $\rho \leq 1$. For $\rho = 2$, the differences are positive in many cases. In all cases, the differences in Table F.2 become smaller as sample size increases. When $\rho = 2$, the differences eventually become negative and significant when $n = 6400$, except in the case with 6 dimensions.

The next round of experiments allow for heteroskedasticity in both the inefficiency as well as the noise processes. We consider the same values of the noise-to-signal ratio ρ, dimensions (p, q) and sample size n as in the previous set of experiments, We again perform 1,000 Monte Carlo trials for each experiment, and the true frontier remains the same as before. We again focus on IMSE of the frontier estimates. In these experiments, we generate efficient observations $(X_i^\partial, Y_i^\partial)$ in (x, y)-space and transform these to efficient observations (Z_i, U_i^∂) exactly as in the first set of experiments. But instead of drawing $\eta_i \sim N^+(0, \sigma_\eta^2)$ as before, we draw $\eta_i \sim N^+\left(0,\ (0.5/(1 + \exp(Z_i'\alpha)) + c_{pq})^2\right)$ where α is an $(r \times 1)$ vector of coefficients equal to 1 and c_{pq} is a constant depending on p and q as described below. Hence the mean and variance of η_i is made to depend on the elements of Z_i, and hence on all of the inputs and outputs in (x, y)-space. Similarly, for the noise we draw $\epsilon_i \sim N\left(0,\ (0.75 + (0.5/(1 + Z_1\alpha)^2)\sigma_\epsilon^2\right)$.

To facilitate comparison of results from these experiments comparable with the results with no heteroskedasticity in Table F.1, for each pair (p, q) we first estimate the expectation of the factor $0.5/(1 + \exp(Z_i\alpha))$ by simulating one billion draws of efficient input-output pairs $(X_i^\partial, Y_i^\partial)$, transforming these to (Z_i, U_i^∂) by multiplying by the rotation matrix R_d, and then computing the sample mean of the values $0.5/(1 + \exp(Z_1\alpha))$, $i = 1, \ldots, 10^9$. We then subtract this mean from 0.5, and use the resulting value for σ_η. This yields values $c_{pq} = 0.1684$, 0.1288, 0.1056, 0.0747 and 0.0611 corresponding to $(p, q) = (1, 1)$, (1,2), (2,2), (2,3) and (3,3), respectively, ensuring that $\mathbb{E}(\sigma_\eta(Z_i))$ is approximately 0.5 in each case. We then let σ_ϵ be determined by ρ as in (F.1). Finally, we use the draws of η_i and ϵ_i to construct "observed" data (X_i, Y_i) and (Z_i, U_i) as before.

Table F.3 shows our Monte Carlo estimates of IMSE for frontier estimates under heteroskedastic inefficiency and noise. Overall, we find the same patterns as before: for both estimators, IMSE decreases with increasing sample size, and increases with number of dimensions and with the noise-to-signal ratio ρ. Analogous to Table F.2, Table F.4 shows differences between average IMSE for the SVKZ and HMS estimators from Table F.3. The results indicate that the HMS estimator has significantly smaller IMSE than the SVKZ estimator in all but 15 of the 175 cases represented in Table F.3 where $\rho \leq 1$. Among these 15 cases, there are none where IMSE for the SVKZ estimator is significantly less than that for the HMS estimator. When $\rho = 2.0$, the IMSE estimator yields significantly smaller IMSE on average than the HMS estimator in 29 of 35 cases.

Comparing the results with heteroskedasticity in Table F.3 with the corresponding results in Table F.1 where there is no heteroskedasticity, we find mixed results. TableF.5 gives differences in average IMSE between Tables F.3 and F.1 for the SVKZ estimator, with positive signs indicating that IMSE is larger when there is no heteroskedasticity. Table F.6

gives similar differences for the HMS estimator. The results in both tables F.5 and F.6 indicate that some of the differences are statistically significant, but are numerically small.

Using the same experiments to produce the results in Tables F.1 and F.3, we also examine error in estimation of derivatives of expected inefficiency with respect to inputs and outputs in the original (x, y)-space. In our experiments, we consider marginal effects at a fixed point (x, y) on the frontier with the p elements of x determined by the median input level (which is the same for each of p dimensions) and the q elements of y equal to $[1-(x-1)'(x-1)]^{1/2}$. From (3.5) we have

$$\frac{\partial\mu_\eta(z)}{\partial z} = \sqrt{\frac{2}{\pi}}\frac{\partial\sigma_\eta(z)}{\partial z}. \tag{F.3}$$

For the SVKZ estimator of $\sigma_\eta(z)$ given in (3.9), we have

$$\frac{\partial\widehat{\sigma}_\eta(z)}{\partial z} = \begin{cases} \frac{1}{3}\left(\frac{\pi}{2}\right)^{1/2}\left(\frac{\pi}{\pi-4}\right)\left[\left(\frac{\pi}{2}\right)^{1/2}\left(\frac{pi}{\pi-4}\right)\widehat{r}_3(z)\right]^{-2/3}\frac{\partial\widehat{r}_3(z)}{\partial z} & \text{if } \widehat{r}_3(z)\leq 0; \\ 0 & \text{otherwise.}\end{cases} \tag{F.4}$$

For the HMS estimator of $\sigma_\eta(z)$ given in (B.4), we have

$$\frac{\partial\widehat{\sigma}_\eta(z)}{\partial z} = \begin{cases} -\left(\frac{1}{3a_3^+}\right)\left[-\frac{1}{a_3^+}\widehat{r}_3(z)\right]^{-2/3}\frac{\partial\widehat{r}_3(z)}{\partial z} & \text{if } \widehat{r}_3(z)\leq 0; \\ \left(\frac{1}{3a_3^-}\right)\left[\frac{1}{a_3^-}\widehat{r}_3(z)\right]^{-2/3}\frac{\partial\widehat{r}_3(z)}{\partial z} & \text{otherwise.}\end{cases} \tag{F.5}$$

Estimates $\widehat{r}_3(z)$ and $\partial\widehat{r}_3(z)/\partial z$ are obtained by LLLS estimation in the regression of cubed residuals $\widehat{\xi}_i$ on Z_i as discussed above in Section 3. Substituting these estimates into (F.3) and (F.5) give SVKZ and HMS estimates $\partial\widehat{\sigma}_\eta(z)/\partial z$, and substituting these into (2.32) give SVKZ and HMS estimates of the partial of expected inefficiency with respect to inputs and outputs in the original (x, y)-space.

In each experiment, on each of 1,000 Monte Carlo trials, we compute $(p+q)$ estimates $\widehat{\mu}_\eta(w^\partial)/\partial w^\partial$, using either the SVKZ or HMS estimators, and then we compute the average squared error $\frac{1}{p+q}\left(\frac{\partial\widehat{\mu}_\eta(w^\partial)}{\partial w^\partial}-\frac{\partial\mu_\eta(w^\partial)}{\partial w^\partial}\right)'\left(\frac{\partial\widehat{\mu}_\eta(w^\partial)}{\partial w^\partial}-\frac{\partial\mu_\eta(w^\partial)}{\partial w^\partial}\right)$. Table F.7 reports medians of these average squared errors for each of our 210 Monte Carlo experiments.[4] The results

[4]Estimation of derivatives is necessarily more difficult than estimation of conditional means. Among 1,000

in Table F.7 are similar in magnitude to those in Tables F.1–F.3. As expected, squared error increases with increasing dimensionality, and diminishes with increasing sample size, although this effect is small when $p = q = 1$.

To examine marginal products, we consider the same fixed point as in the previous exercise. Recalling the discussion at the beginning of Section F, note that a point (x, y) on the frontier must satisfy $(x - 1)'(x - 1) + y'y = 1$. For a fixed point (x, y) and $j \neq k$, there are pq distinct marginal products $\frac{\partial y^k}{\partial x^j} = (1 - x^j)/y^k$, where x^j and y^k denote the jth and kth elements of x and j, respectively. To obtain estimates of marginal products, recall that estimates of $\partial\phi(z)/\partial z$ can be obtained as discussed in Section 3.1, and these can be used to estimate the Jacobian matrix in (2.24). Substituting estimates of the Jacobian into (2.28) and solving for the r-vector c allows marginal products to be estimated using (2.29).

On each Monte Carlo trial, we compute pq marginal products using either the SVKZ or HMS estimators, subtract the corresponding true values, and then compute the sum of squares of the resulting differences divided by $(p \times q)$. Table F.8 reports median average squared errors across 1,000 Monte Carlo trials for each of our 210 experiments, analogous to Table F.7. The medians in Table F.8 are necessarily larger than the corresponding values in Table F.7 due to the fact that $\widehat{r}_1(z)$ is added to $\widehat{\mu}_z(z)$ to obtain the frontier estimate $\widehat{\phi}(z)$ in (3.12). Nonetheless, the squared errors in Table F.8 are numerically small provided the number of dimensions is not too large.

F.3 Exponential Inefficiency

In order to check the robustness of our simulation results, we repeat both rounds of experiments described above in in Section F.2, replacing the half-normal distribution for inefficiency with an exponential distribution when simulating data, but maintaining As-

Monte Carlo trials, we sometimes find in a hand-full of cases derivative estimates that are implausibly large; in other words, the distribution of our squared errors across 1,000 trials is skewed to the right. Consequently, we report medians in Table F.7 to provide a more robust, stable measure than would be provided by the mean of the squared errors.

sumption 2.5 for purposes of estimation. One should expect to pay a price in terms of estimation error for mis-specifying the inefficiency process, but due to our local estimation, the results indicate the price is small as shown below.

In the first group of experiments, both inefficiency and noise are homoskedastic as in Section F.2. For each $i = 1, \ldots, n$ we draw η_i from the exponential density $f(\eta \mid \omega) = \omega \exp(-\omega\eta)$ instead of the half-normal distribution $N^+(0, \sigma_\eta^2)$ used in Section F.2. In order to facilitate comparison with the results in Table F.1, we set $\omega = \sqrt{\pi}(\sqrt{2}\sigma_\eta)^{-1}$, with $\sigma_\eta = 0.5$ as before, so that the exponential inefficiencies have the same mean as the half-normal inefficiencies in Section F.2. For the same values of ρ as in Section F.2, we then set $\sigma_\epsilon = \rho\sigma_\eta\sqrt{2}/\sqrt{\pi}$ and draw ϵ_i from $N(0, \sigma_\epsilon^2)$ for each $i = 1, \ldots, n$. As in Section F.2, ρ controls the noise-to-signal ratio in terms of the ratios of standard deviations of the noise and the inefficiency processes.

In the second group of experiments, both inefficiency and noise are heteroskedastic, exponentially-distributed. For each $i = 1, \ldots, n$, we compute $\sigma_\eta(Z_i) = (0.5/(1+\exp(Z_i\alpha))+ c_{p,q}$ and $\sigma_\epsilon(Z_i) = \rho\sigma_\eta(Z_i)\sqrt{2}/\sqrt{\pi}$ with ρ controlling the noise-to-signal ratio as before. The constants $c_{p,q}$ are described in Section F.2 of the paper, and ensure that the expectation of $\sigma_\eta(Z_i)$ equals 0.5 to facilitate comparison with the results in Tables F.1 and F.9. We then set $\omega(Z_i) = (\sqrt{2}/\sqrt{\pi})\sigma_\eta(Z_i)$ and draw η_i from the exponential density with parameter $\omega(Z_i)$, and draw ϵ_i from $N(0, \sigma_\epsilon(Z_i))$. The draws η_i and ϵ_i are used to construct (X_i, Y_i) and (Z_i, U_i) as described in Section F.2.

Results from the two sets of experiments with exponentially-distributed inefficiency are shown in Tables F.9–F.12, analogous to the results from experiments with half-normal inefficiency shown in Tables F.1–F.8 in the main paper. Of course, one should expect to pay a price for mis-specification of the inefficiency process. To facilitate comparison of the experiments where inefficiency is exponential with the corresponding experiment where inefficiency is half-normal, Tables F.13–F.16 show differences in IMSEs and median average square errors between Tables F.9–F.12 (where inefficiency is distributed exponential) and Tables F.1–F.8 (where inefficiency is distributed half-normal). Negative signs in Tables F.13–F.16 indicate smaller errors with the mis-specified inefficiency, but occur only in a few cases in each table. However, while most of the differences in Tables F.13–F.16 are positive, most are also of small magnitude. For example, the median and mean values of the differences (between Tables F.9 and F.1) shown in Table F.13 are 0.0447 and 0.0593, respectively. Similarly, the median and mean values of the differences (between Tables F.10 and F.3) shown in Table F.14 are 0.0474 and 0.0629, respectively. For purposes of estimating derivatives of expected efficiency, the differences between Tables F.11 and F.7 shown in Table F.15 are even smaller, with median and mean values of 0.0247 and 0.0411, respectively. For purposes of estimating marginal products, mis-specification of the inefficiency process is less costly still—the differences between Tables F.12 and F.8 shown in Table F.16 have median and mean values of 0.0047 and -0.0325 (respectively), and the differences are *negative* in 186 of 420 cases. While there is a price to pay for mis-specifying exponential inefficiency as half-normal, the results suggest that in most cases the price is not high, especially for purposes of estimating marginal products and derivatives of expected inefficiency.

Table F.1: IMSE for Frontier Estimates with No Heteroskedasticity

		SVKZ					HMS				
ρ	n	(1,1)	(1,2)	(2,2)	(2,3)	(3,3)	(1,1)	(1,2)	(2,2)	(2,3)	(3,3)
0.00	100	0.0193	0.0369	0.0461	0.0548	0.0632	0.0161	0.0274	0.0344	0.0388	0.0437
	200	0.0094	0.0227	0.0288	0.0337	0.0401	0.0082	0.0187	0.0228	0.0263	0.0293
	400	0.0043	0.0125	0.0176	0.0205	0.0239	0.0042	0.0113	0.0153	0.0170	0.0192
	800	0.0021	0.0067	0.0115	0.0131	0.0140	0.0021	0.0066	0.0111	0.0121	0.0125
	1600	0.0011	0.0036	0.0077	0.0094	0.0098	0.0010	0.0036	0.0077	0.0092	0.0094
	3200	0.0006	0.0020	0.0052	0.0073	0.0076	0.0006	0.0020	0.0051	0.0073	0.0075
	6400	0.0003	0.0012	0.0033	0.0059	0.0065	0.0003	0.0012	0.0033	0.0059	0.0065
0.25	100	0.0221	0.0407	0.0497	0.0578	0.0668	0.0174	0.0291	0.0366	0.0409	0.0462
	200	0.0104	0.0251	0.0322	0.0366	0.0427	0.0090	0.0199	0.0245	0.0279	0.0306
	400	0.0050	0.0142	0.0194	0.0224	0.0263	0.0047	0.0125	0.0164	0.0180	0.0205
	800	0.0023	0.0073	0.0126	0.0143	0.0149	0.0023	0.0072	0.0119	0.0129	0.0131
	1600	0.0012	0.0040	0.0084	0.0101	0.0104	0.0012	0.0039	0.0084	0.0099	0.0099
	3200	0.0006	0.0022	0.0056	0.0078	0.0081	0.0006	0.0022	0.0056	0.0078	0.0080
	6400	0.0003	0.0013	0.0036	0.0063	0.0070	0.0003	0.0013	0.0036	0.0063	0.0070
0.50	100	0.0323	0.0516	0.0601	0.0690	0.0776	0.0216	0.0346	0.0438	0.0508	0.0558
	200	0.0150	0.0340	0.0419	0.0459	0.0515	0.0118	0.0240	0.0293	0.0329	0.0366
	400	0.0071	0.0204	0.0261	0.0293	0.0336	0.0063	0.0159	0.0198	0.0214	0.0250
	800	0.0031	0.0100	0.0171	0.0185	0.0195	0.0031	0.0094	0.0147	0.0155	0.0159
	1600	0.0015	0.0053	0.0110	0.0130	0.0132	0.0015	0.0052	0.0106	0.0121	0.0119
	3200	0.0008	0.0029	0.0072	0.0096	0.0099	0.0008	0.0029	0.0072	0.0095	0.0097
	6400	0.0004	0.0016	0.0047	0.0078	0.0083	0.0004	0.0016	0.0047	0.0078	0.0083
0.75	100	0.0490	0.0672	0.0755	0.0838	0.0903	0.0315	0.0503	0.0636	0.0746	0.0808
	200	0.0257	0.0507	0.0574	0.0606	0.0663	0.0178	0.0328	0.0409	0.0471	0.0536
	400	0.0128	0.0337	0.0396	0.0431	0.0466	0.0097	0.0216	0.0265	0.0297	0.0339
	800	0.0053	0.0180	0.0281	0.0293	0.0300	0.0049	0.0137	0.0198	0.0210	0.0217
	1600	0.0024	0.0088	0.0181	0.0204	0.0207	0.0024	0.0081	0.0153	0.0164	0.0161
	3200	0.0012	0.0045	0.0112	0.0145	0.0148	0.0012	0.0045	0.0108	0.0134	0.0132
	6400	0.0006	0.0025	0.0071	0.0112	0.0116	0.0006	0.0025	0.0071	0.0111	0.0113
1.00	100	0.0675	0.0828	0.0896	0.0986	0.1055	0.0571	0.0856	0.1057	0.1212	0.1320
	200	0.0456	0.0691	0.0741	0.0770	0.0819	0.0303	0.0548	0.0686	0.0802	0.0883
	400	0.0270	0.0528	0.0567	0.0602	0.0636	0.0166	0.0337	0.0408	0.0488	0.0564
	800	0.0117	0.0357	0.0468	0.0472	0.0463	0.0088	0.0214	0.0286	0.0320	0.0342
	1600	0.0047	0.0192	0.0351	0.0357	0.0352	0.0042	0.0141	0.0222	0.0232	0.0238
	3200	0.0022	0.0090	0.0227	0.0270	0.0272	0.0022	0.0081	0.0178	0.0196	0.0192
	6400	0.0010	0.0043	0.0129	0.0200	0.0202	0.0010	0.0043	0.0123	0.0174	0.0168
2.00	100	0.1262	0.1416	0.1542	0.1691	0.1783	0.4107	0.4885	0.5573	0.5886	0.6284
	200	0.1079	0.1217	0.1317	0.1368	0.1435	0.2902	0.3862	0.4266	0.4731	0.4924
	400	0.0941	0.1087	0.1107	0.1164	0.1207	0.1980	0.2651	0.2907	0.3482	0.3780
	800	0.0811	0.1031	0.1045	0.1060	0.1052	0.1195	0.2014	0.2294	0.2542	0.2745
	1600	0.0669	0.1030	0.1041	0.0976	0.0941	0.0662	0.1474	0.1722	0.1728	0.1898
	3200	0.0431	0.0978	0.1085	0.0990	0.0944	0.0300	0.0959	0.1302	0.1298	0.1387
	6400	0.0227	0.0820	0.1168	0.1093	0.0967	0.0134	0.0538	0.1018	0.1051	0.0985

Table F.2: Differences in IMSE, HMS versus SVKZ, No Heteroskedasticity

ρ	n	(1,1)	(1,2)	(2,2)	(2,3)	(3,3)
0.00	100	−0.003244***	−0.009575***	−0.011749***	−0.016052***	−0.019432***
	200	−0.001154***	−0.004065***	−0.005942***	−0.007412***	−0.010825***
	400	−0.000105***	−0.001175***	−0.002346***	−0.003520***	−0.004755***
	800	0.000000	−0.000128***	−0.000441***	−0.001049***	−0.001549***
	1600	−0.000002	−0.000001	−0.000055***	−0.000177***	−0.000423***
	3200	0.000000	0.000000	−0.000004	−0.000018***	−0.000060***
	6400	0.000000	0.000000	−0.000001	−0.000001*	−0.000012**
0.25	100	−0.004672***	−0.011608***	−0.013121***	−0.016972***	−0.020602***
	200	−0.001361***	−0.005174***	−0.007784***	−0.008741***	−0.012084***
	400	−0.000270***	−0.001723***	−0.002976***	−0.004444***	−0.005807***
	800	−0.000014	−0.000158***	−0.000699***	−0.001375***	−0.001788***
	1600	−0.000006	−0.000013*	−0.000094***	−0.000253***	−0.000493***
	3200	0.000000	0.000000	−0.000002	−0.000015***	−0.000076***
	6400	0.000000	0.000000	−0.000002	−0.000001**	−0.000018**
0.50	100	−0.010666***	−0.016970***	−0.016289***	−0.018240***	−0.021777***
	200	−0.003191***	−0.009998***	−0.012592***	−0.013073***	−0.014949***
	400	−0.000742***	−0.004500***	−0.006288***	−0.007929***	−0.008649***
	800	−0.000057**	−0.000672***	−0.002367***	−0.002985***	−0.003615***
	1600	−0.000009	−0.000079***	−0.000413***	−0.000833***	−0.001323***
	3200	0.000000	−0.000001	−0.000020***	−0.000088***	−0.000241***
	6400	0.000000	0.000000	0.000000	−0.000017*	−0.000042***
0.75	100	−0.017536***	−0.016928***	−0.011953***	−0.009232***	−0.009488***
	200	−0.007948***	−0.017874***	−0.016524***	−0.013497***	−0.012658***
	400	−0.003082***	−0.012089***	−0.013062***	−0.013429***	−0.012683***
	800	−0.000345***	−0.004297***	−0.008249***	−0.008294***	−0.008326***
	1600	−0.000043***	−0.000674***	−0.002811***	−0.004056***	−0.004561***
	3200	0.000000	−0.000015**	−0.000366***	−0.001110***	−0.001606***
	6400	0.000000	0.000000	−0.000007***	−0.000177***	−0.000359***
1.00	100	−0.010354***	0.002823	0.016089***	0.022600***	0.026479***
	200	−0.015329***	−0.014297***	−0.005488***	0.003168**	0.006327***
	400	−0.010336***	−0.019091***	−0.015948***	−0.011386***	−0.007143***
	800	−0.002917***	−0.014280***	−0.018196***	−0.015210***	−0.012096***
	1600	−0.000456***	−0.005124***	−0.012921***	−0.012535***	−0.011378***
	3200	−0.000003	−0.000839***	−0.004972***	−0.007344***	−0.008035***
	6400	0.000000	−0.000027***	−0.000634***	−0.002547***	−0.003389***
2.00	100	0.284463***	0.346932***	0.403054***	0.419524***	0.450096***
	200	0.182323***	0.264428***	0.294969***	0.336303***	0.348912***
	400	0.103878***	0.156422***	0.179949***	0.231749***	0.257277***
	800	0.038412***	0.098306***	0.124905***	0.148143***	0.169266***
	1600	−0.000738	0.044403***	0.068119***	0.075144***	0.095688***
	3200	−0.013134***	−0.001912	0.021663***	0.030841***	0.044368***
	6400	−0.009266***	−0.028138***	−0.015012***	−0.004225***	0.001889

NOTE: One, two or three asterisks indicate siginifcant differences from zero at 90, 95 or 99-percent levels.

Table F.3: IMSE for Frontier Estimates with Heteroskedastic Inefficiency and Noise

		SVKZ					HMS				
ρ	n	(1,1)	(1,2)	(2,2)	(2,3)	(3,3)	(1,1)	(1,2)	(2,2)	(2,3)	(3,3)
0.00	100	0.0207	0.0356	0.0461	0.0546	0.0655	0.0179	0.0298	0.0349	0.0413	0.0444
	200	0.0099	0.0229	0.0292	0.0332	0.0400	0.0089	0.0196	0.0244	0.0263	0.0302
	400	0.0050	0.0135	0.0185	0.0207	0.0236	0.0045	0.0119	0.0162	0.0175	0.0192
	800	0.0023	0.0071	0.0120	0.0134	0.0144	0.0022	0.0067	0.0111	0.0123	0.0128
	1600	0.0012	0.0038	0.0079	0.0093	0.0097	0.0012	0.0037	0.0077	0.0090	0.0092
	3200	0.0006	0.0021	0.0051	0.0071	0.0074	0.0006	0.0021	0.0051	0.0071	0.0073
	6400	0.0003	0.0012	0.0033	0.0057	0.0063	0.0003	0.0012	0.0033	0.0057	0.0063
0.25	100	0.0230	0.0391	0.0488	0.0572	0.0693	0.0193	0.0312	0.0366	0.0429	0.0470
	200	0.0111	0.0255	0.0316	0.0364	0.0434	0.0100	0.0211	0.0254	0.0279	0.0323
	400	0.0055	0.0147	0.0199	0.0229	0.0257	0.0050	0.0130	0.0171	0.0189	0.0206
	800	0.0025	0.0080	0.0131	0.0147	0.0154	0.0024	0.0074	0.0120	0.0132	0.0135
	1600	0.0013	0.0042	0.0086	0.0100	0.0103	0.0013	0.0040	0.0083	0.0096	0.0097
	3200	0.0006	0.0023	0.0055	0.0076	0.0079	0.0006	0.0023	0.0055	0.0075	0.0078
	6400	0.0003	0.0013	0.0036	0.0061	0.0067	0.0003	0.0013	0.0036	0.0061	0.0067
0.50	100	0.0321	0.0503	0.0591	0.0670	0.0789	0.0238	0.0370	0.0444	0.0522	0.0582
	200	0.0157	0.0334	0.0405	0.0458	0.0537	0.0129	0.0258	0.0301	0.0337	0.0398
	400	0.0078	0.0204	0.0266	0.0302	0.0331	0.0067	0.0165	0.0209	0.0234	0.0256
	800	0.0035	0.0108	0.0171	0.0195	0.0203	0.0033	0.0096	0.0147	0.0161	0.0168
	1600	0.0017	0.0057	0.0112	0.0128	0.0132	0.0017	0.0054	0.0105	0.0117	0.0119
	3200	0.0008	0.0030	0.0071	0.0094	0.0098	0.0008	0.0030	0.0070	0.0092	0.0095
	6400	0.0004	0.0017	0.0046	0.0075	0.0081	0.0004	0.0017	0.0046	0.0075	0.0080
0.75	100	0.0486	0.0657	0.0749	0.0810	0.0930	0.0335	0.0530	0.0646	0.0773	0.0858
	200	0.0268	0.0485	0.0557	0.0609	0.0681	0.0194	0.0355	0.0421	0.0481	0.0565
	400	0.0135	0.0331	0.0391	0.0439	0.0475	0.0107	0.0233	0.0285	0.0335	0.0382
	800	0.0060	0.0184	0.0269	0.0305	0.0304	0.0052	0.0146	0.0203	0.0218	0.0230
	1600	0.0028	0.0095	0.0179	0.0200	0.0206	0.0027	0.0084	0.0151	0.0160	0.0162
	3200	0.0013	0.0049	0.0113	0.0141	0.0147	0.0013	0.0046	0.0105	0.0129	0.0131
	6400	0.0006	0.0026	0.0071	0.0109	0.0115	0.0006	0.0025	0.0070	0.0106	0.0110
1.00	100	0.0668	0.0818	0.0902	0.0967	0.1073	0.0580	0.0878	0.1043	0.1234	0.1355
	200	0.0450	0.0671	0.0733	0.0772	0.0840	0.0341	0.0586	0.0707	0.0799	0.0932
	400	0.0272	0.0518	0.0566	0.0610	0.0637	0.0182	0.0359	0.0444	0.0546	0.0613
	800	0.0127	0.0347	0.0453	0.0474	0.0465	0.0097	0.0230	0.0297	0.0332	0.0360
	1600	0.0057	0.0200	0.0342	0.0353	0.0346	0.0049	0.0147	0.0225	0.0230	0.0234
	3200	0.0023	0.0095	0.0223	0.0263	0.0260	0.0022	0.0083	0.0172	0.0191	0.0188
	6400	0.0011	0.0046	0.0132	0.0196	0.0196	0.0011	0.0044	0.0120	0.0167	0.0161
2.00	100	0.1269	0.1424	0.1563	0.1669	0.1822	0.4104	0.4964	0.5452	0.5997	0.6247
	200	0.1066	0.1252	0.1302	0.1358	0.1467	0.2975	0.3933	0.4452	0.4688	0.5242
	400	0.0930	0.1094	0.1120	0.1178	0.1223	0.1958	0.2762	0.3186	0.3509	0.3954
	800	0.0824	0.1032	0.1057	0.1055	0.1055	0.1277	0.2026	0.2331	0.2620	0.2855
	1600	0.0677	0.1019	0.1042	0.0968	0.0948	0.0724	0.1493	0.1713	0.1817	0.1895
	3200	0.0457	0.0955	0.1072	0.0991	0.0934	0.0346	0.0951	0.1324	0.1322	0.1392
	6400	0.0256	0.0791	0.1126	0.1073	0.0934	0.0161	0.0545	0.1013	0.1048	0.0976

Table F.4: Differences in IMSE, HMS versus SVKZ, Heteroskedastic Inefficiency and Noise

ρ	n	(1,1)	(1,2)	(2,2)	(2,3)	(3,3)
0.00	100	−0.002813***	−0.005766***	−0.011150***	−0.013373***	−0.021110***
	200	−0.000963***	−0.003373***	−0.004775***	−0.006889***	−0.009817***
	400	−0.000427***	−0.001582***	−0.002285***	−0.003193***	−0.004339***
	800	−0.000056***	−0.000377***	−0.000871***	−0.001132***	−0.001603***
	1600	0.000000	−0.000104***	−0.000254***	−0.000306***	−0.000496***
	3200	−0.000003	−0.000010***	−0.000040***	−0.000039***	−0.000131***
	6400	0.000000	0.000000	−0.000003***	−0.000010**	−0.000022***
0.25	100	−0.003743***	−0.007947***	−0.012171***	−0.014287***	−0.022296***
	200	−0.001113***	−0.004460***	−0.006191***	−0.008480***	−0.011055***
	400	−0.000540***	−0.001795***	−0.002853***	−0.004026***	−0.005116***
	800	−0.000070***	−0.000567***	−0.001114***	−0.001541***	−0.001904***
	1600	−0.000001	−0.000132***	−0.000325***	−0.000430***	−0.000581***
	3200	−0.000003	−0.000015***	−0.000058***	−0.000065***	−0.000176***
	6400	0.000000	0.000000	−0.000005***	−0.000008***	−0.000037***
0.50	100	−0.008305***	−0.013306***	−0.014689***	−0.014725***	−0.020692***
	200	−0.002850***	−0.007571***	−0.010445***	−0.012047***	−0.013845***
	400	−0.001091***	−0.003873***	−0.005626***	−0.006824***	−0.007486***
	800	−0.000232***	−0.001198***	−0.002376***	−0.003439***	−0.003492***
	1600	−0.000011***	−0.000321***	−0.000728***	−0.001019***	−0.001370***
	3200	−0.000006	−0.000048***	−0.000144***	−0.000209***	−0.000380***
	6400	0.000000	−0.000006***	−0.000019***	−0.000029***	−0.000076***
0.75	100	−0.015114***	−0.012690***	−0.010371***	−0.003768**	−0.007164***
	200	−0.007438***	−0.012982***	−0.013639***	−0.012773***	−0.011660***
	400	−0.002773***	−0.009827***	−0.010590***	−0.010387***	−0.009277***
	800	−0.000780***	−0.003817***	−0.006660***	−0.008681***	−0.007365***
	1600	−0.000160***	−0.001097***	−0.002841***	−0.003981***	−0.004421***
	3200	−0.000001	−0.000215***	−0.000736***	−0.001188***	−0.001617***
	6400	0.000000	−0.000033***	−0.000127***	−0.000265***	−0.000532***
1.00	100	−0.008824***	0.006066***	0.014108***	0.026786***	0.028212***
	200	−0.010938***	−0.008559***	−0.002612*	0.002615*	0.009202***
	400	−0.008980***	−0.015850***	−0.012159***	−0.006390***	−0.002354**
	800	−0.002965***	−0.011678***	−0.015565***	−0.014273***	−0.010475***
	1600	−0.000794***	−0.005358***	−0.011709***	−0.012254***	−0.011125***
	3200	−0.000085***	−0.001182***	−0.005098***	−0.007147***	−0.007214***
	6400	−0.000002	−0.000199***	−0.001147***	−0.002904***	−0.003488***
2.00	100	0.283525***	0.353977***	0.388882***	0.432786***	0.442439***
	200	0.190896***	0.268136***	0.315044***	0.332992***	0.377434***
	400	0.102743***	0.166809***	0.206615***	0.233049***	0.273093***
	800	0.045325***	0.099329***	0.127388***	0.156520***	0.179984***
	1600	0.004660**	0.047427***	0.067080***	0.084874***	0.094716***
	3200	−0.011095***	−0.000350	0.025187***	0.033054***	0.045881***
	6400	−0.009582***	−0.024610***	−0.011318***	−0.002496*	0.004231***

NOTE: One, two or three asterisks indicate siginifcant differences from zero at 90, 95 or 99-percent levels.

Table F.5: Differences in IMSE for SVKZ Estimator Due to Heteroskedasticity

ρ	n	(1,1)	(1,2)	(2,2)	(2,3)	(3,3)
0.00	100	0.001336	−0.001376	−0.000007	−0.000186	0.002361
	200	0.000544	0.000202	0.000420	−0.000503	−0.000127
	400	0.000625***	0.000947**	0.000850*	0.000214	−0.000348
	800	0.000164**	0.000394**	0.000516*	0.000251	0.000329
	1600	0.000120***	0.000152*	0.000160	−0.000049	−0.000074
	3200	0.000022	0.000073*	−0.000034	−0.000241***	−0.000124
	6400	−0.000001	0.000007	−0.000051	−0.000196***	−0.000253***
0.25	100	0.000940	−0.001569	−0.000981	−0.000589	0.002469
	200	0.000698	0.000436	−0.000677	−0.000225	0.000735
	400	0.000541**	0.000518	0.000549	0.000503	−0.000658
	800	0.000181**	0.000662***	0.000435	0.000445	0.000535
	1600	0.000140***	0.000224**	0.000150	−0.000075	−0.000115
	3200	0.000029	0.000085*	−0.000049	−0.000205**	−0.000119
	6400	0.000012	0.000020	−0.000058	−0.000235***	−0.000256***
0.50	100	−0.000149	−0.001250	−0.001005	−0.002062	0.001309
	200	0.000746	−0.000677	−0.001382	−0.000145	0.002146*
	400	0.000691*	−0.000006	0.000475	0.000898	−0.000503
	800	0.000391***	0.000749**	0.000014	0.001049*	0.000802
	1600	0.000208***	0.000431***	0.000258	−0.000202	−0.000005
	3200	0.000042	0.000137**	−0.000073	−0.000194	−0.000077
	6400	0.000029**	0.000039	−0.000051	−0.000281***	−0.000249**
0.75	100	−0.000476	−0.001518	−0.000583	−0.002797	0.002693
	200	0.001063	−0.002162	−0.001694	0.000261	0.001845
	400	0.000691	−0.000543	−0.000468	0.000751	0.000893
	800	0.000730***	0.000345	−0.001124	0.001201	0.000340
	1600	0.000437***	0.000734***	−0.000169	−0.000465	−0.000124
	3200	0.000040	0.000338***	0.000087	−0.000422	−0.000161
	6400	0.000066***	0.000087*	0.000053	−0.000366**	−0.000133
1.00	100	−0.000659	−0.000998	0.000585	−0.001918	0.001781
	200	−0.000604	−0.001933	−0.000804	0.000239	0.002112
	400	0.000200	−0.001044	−0.000160	0.000787	0.000110
	800	0.000946	−0.001029	−0.001476	0.000245	0.000229
	1600	0.000985***	0.000849	−0.000831	−0.000470	−0.000609
	3200	0.000158*	0.000496*	−0.000472	−0.000679	−0.001230*
	6400	0.000152***	0.000246**	0.000262	−0.000411	−0.000597
2.00	100	0.000667	0.000816	0.002061	−0.002140	0.003885
	200	−0.001299	0.003435	−0.001499	−0.000988	0.003278*
	400	−0.001039	0.000706	0.001246	0.001413	0.001534
	800	0.001258	0.000093	0.001168	−0.000585	0.000283
	1600	0.000791	−0.001125	0.000118	−0.000823	0.000745
	3200	0.002583	−0.002313	−0.001287	0.000127	−0.001018
	6400	0.002995**	−0.002856	−0.004210***	−0.002024	−0.003250**

NOTE: One, two or three asterisks indicate siginifcant differences from zero at 90, 95 or 99-percent levels.

Table F.6: Differences in IMSE for HMS Estimator Due to Heteroskedasticity

ρ	n	(1,1)	(1,2)	(2,2)	(2,3)	(3,3)
0.00	100	0.001767***	0.002434***	0.000592	0.002493**	0.000683
	200	0.000737**	0.000895	0.001588***	0.000019	0.000881
	400	0.000302*	0.000539*	0.000911**	0.000541	0.000068
	800	0.000108	0.000145	0.000086	0.000167	0.000275
	1600	0.000122***	0.000049	−0.000038	−0.000179	−0.000147
	3200	0.000019	0.000064	−0.000070	−0.000263***	−0.000196*
	6400	−0.000001	0.000007	−0.000053	−0.000206***	−0.000262***
0.25	100	0.001868***	0.002093**	−0.000032	0.002096**	0.000775
	200	0.000945***	0.001149**	0.000916	0.000036	0.001765**
	400	0.000271	0.000448	0.000672*	0.000921**	0.000033
	800	0.000125	0.000252	0.000019	0.000279	0.000419
	1600	0.000145***	0.000105	−0.000082	−0.000251	−0.000203
	3200	0.000026	0.000070	−0.000105	−0.000254**	−0.000219
	6400	0.000012	0.000019	−0.000062	−0.000242***	−0.000275***
0.50	100	0.002213**	0.002414**	0.000595	0.001452	0.002393*
	200	0.001087**	0.001750***	0.000765	0.000881	0.003250***
	400	0.000341	0.000621	0.001137**	0.002002***	0.000660
	800	0.000216*	0.000224	0.000005	0.000595*	0.000924**
	1600	0.000206***	0.000189	−0.000056	−0.000389*	−0.000052
	3200	0.000035	0.000090	−0.000197*	−0.000314**	−0.000216
	6400	0.000029**	0.000033	−0.000070	−0.000292***	−0.000283***
0.75	100	0.001946	0.002720*	0.001000	0.002667	0.005017**
	200	0.001573**	0.002731***	0.001191	0.000985	0.002843*
	400	0.001000**	0.001719***	0.002004***	0.003795***	0.004300***
	800	0.000295	0.000824**	0.000465	0.000815	0.001301**
	1600	0.000321***	0.000313	−0.000198	−0.000390	0.000016
	3200	0.000040	0.000138	−0.000283	−0.000499**	−0.000172
	6400	0.000066***	0.000054	−0.000067	−0.000455***	−0.000307*
1.00	100	0.000872	0.002244	−0.001397	0.002268	0.003515
	200	0.003787***	0.003806*	0.002072	−0.000315	0.004987**
	400	0.001554**	0.002197**	0.003630***	0.005784***	0.004899**
	800	0.000897**	0.001573***	0.001154	0.001183	0.001849*
	1600	0.000647***	0.000615*	0.000381	−0.000189	−0.000356
	3200	0.000077	0.000153	−0.000598*	−0.000482	−0.000408
	6400	0.000150***	0.000074	−0.000251	−0.000768***	−0.000695***
2.00	100	−0.000271	0.007861	−0.012111	0.011122	−0.003772
	200	0.007273	0.007144	0.018575	−0.004299	0.031801**
	400	−0.002174	0.011093	0.027912***	0.002714	0.017351*
	800	0.008172	0.001116	0.003650	0.007793	0.011001
	1600	0.006188*	0.001899	−0.000921	0.008907*	−0.000226
	3200	0.004622***	−0.000751	0.002237	0.002340	0.000494
	6400	0.002680***	0.000672	−0.000516	−0.000295	−0.000907

NOTE: One, two or three asterisks indicate siginifcant differences from zero at 90, 95 or 99-percent levels.

Table F.7: Median Average Square Error for Derivatives of Expected Efficiency $\mu_\eta(Z)$, with Heteroskedasticity

		SVKZ					HMS				
ρ	n	(1,1)	(1,2)	(2,2)	(2,3)	(3,3)	(1,1)	(1,2)	(2,2)	(2,3)	(3,3)
0.00	100	0.0182	0.0330	0.0458	0.0571	0.0663	0.0183	0.0350	0.0509	0.0642	0.0784
	200	0.0170	0.0284	0.0322	0.0364	0.0403	0.0170	0.0288	0.0325	0.0365	0.0418
	400	0.0163	0.0219	0.0223	0.0229	0.0256	0.0163	0.0219	0.0224	0.0230	0.0259
	800	0.0176	0.0185	0.0184	0.0168	0.0167	0.0176	0.0185	0.0184	0.0168	0.0167
	1600	0.0177	0.0176	0.0155	0.0124	0.0121	0.0177	0.0176	0.0155	0.0124	0.0121
	3200	0.0179	0.0164	0.0134	0.0100	0.0088	0.0179	0.0164	0.0134	0.0100	0.0088
	6400	0.0175	0.0164	0.0129	0.0087	0.0077	0.0175	0.0164	0.0129	0.0087	0.0077
0.25	100	0.0187	0.0393	0.0502	0.0624	0.0710	0.0197	0.0413	0.0565	0.0726	0.0954
	200	0.0179	0.0304	0.0355	0.0394	0.0468	0.0180	0.0307	0.0363	0.0405	0.0492
	400	0.0168	0.0229	0.0246	0.0256	0.0292	0.0168	0.0229	0.0246	0.0259	0.0295
	800	0.0176	0.0198	0.0191	0.0182	0.0181	0.0176	0.0198	0.0191	0.0183	0.0181
	1600	0.0172	0.0173	0.0158	0.0132	0.0130	0.0172	0.0173	0.0158	0.0132	0.0131
	3200	0.0177	0.0164	0.0137	0.0105	0.0092	0.0177	0.0164	0.0137	0.0105	0.0092
	6400	0.0177	0.0164	0.0130	0.0089	0.0081	0.0177	0.0164	0.0130	0.0089	0.0081
0.50	100	0.0219	0.0512	0.0659	0.0850	0.0843	0.0265	0.0635	0.0851	0.1130	0.1474
	200	0.0196	0.0400	0.0463	0.0545	0.0662	0.0200	0.0414	0.0494	0.0600	0.0764
	400	0.0178	0.0270	0.0332	0.0348	0.0399	0.0178	0.0273	0.0337	0.0360	0.0403
	800	0.0177	0.0218	0.0228	0.0242	0.0241	0.0177	0.0218	0.0228	0.0242	0.0243
	1600	0.0173	0.0181	0.0177	0.0159	0.0160	0.0173	0.0181	0.0177	0.0159	0.0160
	3200	0.0179	0.0170	0.0145	0.0124	0.0114	0.0179	0.0170	0.0145	0.0124	0.0114
	6400	0.0176	0.0163	0.0135	0.0098	0.0095	0.0176	0.0163	0.0135	0.0098	0.0095
0.75	100	0.0269	0.0573	0.0828	0.1034	0.1089	0.0462	0.1046	0.1483	0.2019	0.2829
	200	0.0264	0.0589	0.0644	0.0826	0.0928	0.0284	0.0731	0.0901	0.1056	0.1351
	400	0.0190	0.0362	0.0510	0.0558	0.0631	0.0193	0.0391	0.0558	0.0633	0.0726
	800	0.0180	0.0272	0.0324	0.0391	0.0403	0.0180	0.0273	0.0333	0.0403	0.0409
	1600	0.0170	0.0205	0.0231	0.0239	0.0245	0.0170	0.0205	0.0233	0.0242	0.0251
	3200	0.0179	0.0177	0.0173	0.0164	0.0170	0.0179	0.0177	0.0173	0.0164	0.0171
	6400	0.0177	0.0166	0.0147	0.0125	0.0130	0.0177	0.0166	0.0147	0.0125	0.0130
1.00	100	0.0259	0.0672	0.0901	0.1155	0.1190	0.0767	0.1846	0.2748	0.3875	0.4502
	200	0.0315	0.0741	0.0849	0.1093	0.1107	0.0528	0.1301	0.1812	0.2352	0.2923
	400	0.0264	0.0494	0.0738	0.0921	0.0962	0.0308	0.0744	0.1035	0.1265	0.1455
	800	0.0194	0.0417	0.0552	0.0622	0.0678	0.0197	0.0454	0.0677	0.0801	0.0821
	1600	0.0168	0.0287	0.0397	0.0447	0.0450	0.0168	0.0288	0.0416	0.0472	0.0491
	3200	0.0182	0.0213	0.0254	0.0294	0.0320	0.0182	0.0213	0.0257	0.0301	0.0327
	6400	0.0178	0.0178	0.0181	0.0196	0.0218	0.0178	0.0178	0.0181	0.0197	0.0223
2.00	100	0.0036	0.0032	0.0024	0.0013	0.0008	0.2859	0.7817	1.0884	1.4798	1.8207
	200	0.0036	0.0032	0.0024	0.0668	0.0008	0.2666	0.7382	1.0145	1.4084	1.8834
	400	0.0121	0.0211	0.0430	0.0920	0.0213	0.2010	0.5514	0.8411	1.1637	1.4428
	800	0.0106	0.0032	0.0024	0.0013	0.0740	0.1610	0.4011	0.7363	1.0270	1.4034
	1600	0.0185	0.0032	0.0024	0.0013	0.0802	0.1077	0.4311	0.5550	0.8538	0.8346
	3200	0.0267	0.0032	0.0024	0.0013	0.0008	0.0550	0.2867	0.4715	0.7489	0.8256
	6400	0.0248	0.0444	0.0024	0.0013	0.0008	0.0298	0.1729	0.3987	0.5710	0.6829

Table F.8: Median Average Square Error for Estimated Marginal Product, with Heteroskedasticity

		SVKZ					HMS				
ρ	n	(1,1)	(1,2)	(2,2)	(2,3)	(3,3)	(1,1)	(1,2)	(2,2)	(2,3)	(3,3)
0.00	100	0.0833	0.0737	0.1278	0.1469	0.1875	0.0836	0.0795	0.1475	0.1673	0.2359
	200	0.0472	0.0406	0.0761	0.0885	0.1434	0.0472	0.0422	0.0768	0.0886	0.1515
	400	0.0277	0.0242	0.0475	0.0527	0.0798	0.0277	0.0242	0.0478	0.0528	0.0814
	800	0.0192	0.0150	0.0350	0.0359	0.0528	0.0192	0.0150	0.0350	0.0359	0.0533
	1600	0.0138	0.0102	0.0189	0.0233	0.0325	0.0138	0.0102	0.0189	0.0233	0.0325
	3200	0.0099	0.0079	0.0115	0.0124	0.0213	0.0099	0.0079	0.0115	0.0124	0.0214
	6400	0.0071	0.0064	0.0083	0.0079	0.0148	0.0071	0.0064	0.0083	0.0079	0.0148
0.25	100	0.0882	0.0748	0.1470	0.1460	0.2188	0.0889	0.0835	0.1703	0.1683	0.2874
	200	0.0542	0.0500	0.0834	0.0997	0.1533	0.0547	0.0505	0.0859	0.1022	0.1633
	400	0.0310	0.0239	0.0586	0.0595	0.0859	0.0310	0.0239	0.0586	0.0605	0.0880
	800	0.0193	0.0156	0.0356	0.0365	0.0552	0.0193	0.0156	0.0356	0.0366	0.0572
	1600	0.0137	0.0103	0.0205	0.0219	0.0401	0.0137	0.0103	0.0205	0.0219	0.0405
	3200	0.0115	0.0086	0.0124	0.0135	0.0232	0.0115	0.0086	0.0124	0.0135	0.0232
	6400	0.0076	0.0070	0.0084	0.0078	0.0162	0.0076	0.0070	0.0084	0.0078	0.0162
0.50	100	0.1069	0.0972	0.2058	0.1847	0.2576	0.1219	0.1319	0.2573	0.2745	0.3936
	200	0.0729	0.0720	0.1199	0.1337	0.1979	0.0733	0.0749	0.1282	0.1546	0.2414
	400	0.0353	0.0359	0.0867	0.0851	0.1273	0.0353	0.0364	0.0880	0.0896	0.1311
	800	0.0206	0.0210	0.0468	0.0481	0.0748	0.0206	0.0210	0.0468	0.0485	0.0757
	1600	0.0153	0.0118	0.0255	0.0306	0.0534	0.0153	0.0118	0.0255	0.0308	0.0542
	3200	0.0132	0.0103	0.0138	0.0185	0.0298	0.0132	0.0103	0.0138	0.0185	0.0298
	6400	0.0079	0.0079	0.0094	0.0102	0.0207	0.0079	0.0079	0.0094	0.0102	0.0209
0.75	100	0.1519	0.1129	0.2258	0.2155	0.3269	0.2234	0.2226	0.4871	0.4396	0.6332
	200	0.1050	0.1365	0.1782	0.1851	0.2508	0.1150	0.1714	0.2775	0.2812	0.4314
	400	0.0516	0.0609	0.1377	0.1296	0.1922	0.0516	0.0679	0.1615	0.1678	0.2543
	800	0.0304	0.0335	0.0799	0.0891	0.1315	0.0306	0.0340	0.0819	0.0963	0.1360
	1600	0.0186	0.0178	0.0433	0.0457	0.0798	0.0186	0.0178	0.0436	0.0467	0.0835
	3200	0.0164	0.0118	0.0208	0.0293	0.0449	0.0164	0.0118	0.0208	0.0294	0.0457
	6400	0.0087	0.0093	0.0119	0.0156	0.0332	0.0087	0.0093	0.0119	0.0156	0.0332
1.00	100	0.1504	0.1649	0.2677	0.2354	0.3579	0.4225	0.3933	0.8651	0.6515	0.8998
	200	0.1708	0.1431	0.2045	0.2275	0.2889	0.2664	0.3430	0.6075	0.5562	0.7256
	400	0.0858	0.0890	0.1941	0.1897	0.2587	0.1026	0.1624	0.3796	0.3205	0.5062
	800	0.0529	0.0726	0.1469	0.1372	0.2200	0.0538	0.0836	0.2025	0.2322	0.2779
	1600	0.0252	0.0314	0.0905	0.0919	0.1386	0.0252	0.0314	0.1001	0.1096	0.1724
	3200	0.0194	0.0156	0.0421	0.0630	0.0840	0.0194	0.0156	0.0425	0.0672	0.0884
	6400	0.0110	0.0118	0.0195	0.0313	0.0646	0.0110	0.0118	0.0195	0.0314	0.0671
2.00	100	0.1810	0.1606	0.4240	0.3151	0.4970	1.5143	0.9216	1.6562	0.8609	0.9649
	200	0.1882	0.1200	0.2492	0.2869	0.3211	1.6679	1.0047	1.2736	0.8969	1.0565
	400	0.1535	0.1006	0.1963	0.1680	0.2475	1.1596	0.9809	1.5572	0.8425	1.1034
	800	0.1119	0.0553	0.1020	0.0986	0.1546	0.9428	0.7703	1.3756	0.8066	1.0385
	1600	0.1050	0.0365	0.0530	0.0516	0.1216	0.5052	0.6653	1.3412	0.7155	1.0097
	3200	0.0937	0.0465	0.0283	0.0217	0.0477	0.1938	0.4422	1.0167	0.8166	1.0893
	6400	0.0566	0.0536	0.0203	0.0104	0.0200	0.0698	0.3163	0.8885	0.6330	0.9004

Table F.9: IMSE for Frontier Estimates with No Heteroskedasticity (Exponential Inefficiency)

		SVKZ					HMS				
ρ	n	(1,1)	(1,2)	(2,2)	(2,3)	(3,3)	(1,1)	(1,2)	(2,2)	(2,3)	(3,3)
0.00	100	0.0569	0.0611	0.0727	0.0750	0.0839	0.0718	0.0925	0.1241	0.1328	0.1473
	200	0.0575	0.0575	0.0623	0.0664	0.0695	0.0645	0.0723	0.0892	0.1043	0.1113
	400	0.0557	0.0517	0.0575	0.0610	0.0663	0.0563	0.0560	0.0693	0.0785	0.0908
	800	0.0600	0.0505	0.0517	0.0585	0.0621	0.0602	0.0519	0.0548	0.0672	0.0727
	1600	0.0640	0.0518	0.0493	0.0523	0.0560	0.0642	0.0519	0.0501	0.0543	0.0588
	3200	0.0652	0.0560	0.0486	0.0498	0.0522	0.0652	0.0560	0.0487	0.0501	0.0527
	6400	0.0671	0.0589	0.0509	0.0482	0.0503	0.0671	0.0589	0.0509	0.0482	0.0504
0.25	100	0.0581	0.0626	0.0748	0.0776	0.0874	0.0736	0.0959	0.1287	0.1388	0.1518
	200	0.0574	0.0587	0.0642	0.0686	0.0719	0.0648	0.0749	0.0930	0.1086	0.1173
	400	0.0556	0.0525	0.0586	0.0615	0.0678	0.0564	0.0574	0.0706	0.0794	0.0946
	800	0.0595	0.0505	0.0523	0.0591	0.0631	0.0597	0.0518	0.0557	0.0684	0.0738
	1600	0.0637	0.0517	0.0493	0.0527	0.0562	0.0639	0.0519	0.0502	0.0546	0.0593
	3200	0.0650	0.0555	0.0482	0.0498	0.0523	0.0650	0.0556	0.0483	0.0501	0.0527
	6400	0.0671	0.0585	0.0503	0.0479	0.0502	0.0671	0.0585	0.0503	0.0480	0.0503
0.50	100	0.0626	0.0692	0.0826	0.0872	0.0981	0.0804	0.1086	0.1466	0.1565	0.1721
	200	0.0588	0.0629	0.0702	0.0749	0.0781	0.0676	0.0820	0.1043	0.1224	0.1336
	400	0.0555	0.0548	0.0617	0.0650	0.0724	0.0566	0.0608	0.0759	0.0862	0.1049
	800	0.0584	0.0506	0.0539	0.0610	0.0655	0.0587	0.0520	0.0583	0.0718	0.0781
	1600	0.0628	0.0508	0.0495	0.0538	0.0573	0.0630	0.0511	0.0506	0.0563	0.0608
	3200	0.0643	0.0541	0.0473	0.0500	0.0526	0.0643	0.0541	0.0474	0.0504	0.0533
	6400	0.0666	0.0572	0.0488	0.0471	0.0498	0.0666	0.0572	0.0489	0.0472	0.0499
0.75	100	0.0730	0.0841	0.0991	0.1042	0.1160	0.0963	0.1392	0.1859	0.1984	0.2202
	200	0.0627	0.0716	0.0802	0.0863	0.0909	0.0741	0.0981	0.1245	0.1509	0.1666
	400	0.0562	0.0591	0.0674	0.0721	0.0802	0.0578	0.0682	0.0860	0.1012	0.1232
	800	0.0572	0.0518	0.0568	0.0652	0.0703	0.0575	0.0539	0.0633	0.0802	0.0883
	1600	0.0613	0.0495	0.0499	0.0556	0.0596	0.0615	0.0500	0.0516	0.0589	0.0649
	3200	0.0631	0.0517	0.0460	0.0501	0.0533	0.0631	0.0517	0.0461	0.0509	0.0546
	6400	0.0658	0.0550	0.0464	0.0459	0.0491	0.0658	0.0550	0.0464	0.0460	0.0492
1.00	100	0.0900	0.1035	0.1216	0.1264	0.1413	0.1343	0.2001	0.2636	0.2753	0.3095
	200	0.0730	0.0852	0.0961	0.1045	0.1090	0.0912	0.1301	0.1686	0.2073	0.2286
	400	0.0590	0.0676	0.0777	0.0847	0.0942	0.0624	0.0838	0.1083	0.1330	0.1632
	800	0.0563	0.0551	0.0621	0.0719	0.0782	0.0568	0.0590	0.0730	0.0954	0.1081
	1600	0.0593	0.0485	0.0517	0.0587	0.0638	0.0595	0.0496	0.0548	0.0646	0.0729
	3200	0.0615	0.0486	0.0446	0.0505	0.0548	0.0615	0.0486	0.0448	0.0522	0.0573
	6400	0.0645	0.0517	0.0433	0.0443	0.0484	0.0645	0.0517	0.0433	0.0449	0.0488
2.00	100	0.2014	0.2256	0.2579	0.2768	0.2989	0.7316	0.9399	1.0986	1.1562	1.2255
	200	0.1632	0.1860	0.2051	0.2239	0.2321	0.4790	0.6829	0.7962	0.8903	0.9529
	400	0.1266	0.1497	0.1696	0.1822	0.1965	0.3126	0.4452	0.5245	0.6322	0.7184
	800	0.0958	0.1206	0.1330	0.1463	0.1569	0.1701	0.2788	0.3438	0.4053	0.4737
	1600	0.0687	0.0887	0.1042	0.1165	0.1254	0.0880	0.1597	0.2115	0.2579	0.3004
	3200	0.0580	0.0615	0.0753	0.0870	0.0950	0.0615	0.0827	0.1169	0.1458	0.1756
	6400	0.0560	0.0426	0.0526	0.0625	0.0715	0.0561	0.0446	0.0630	0.0805	0.0996

Table F.10: IMSE for Frontier Estimates with Heteroskedastic, Exponential Inefficiency and Noise

		SVKZ					HMS				
ρ	n	(1,1)	(1,2)	(2,2)	(2,3)	(3,3)	(1,1)	(1,2)	(2,2)	(2,3)	(3,3)
0.00	100	0.0582	0.0661	0.0687	0.0765	0.0793	0.0776	0.1112	0.1243	0.1392	0.1514
	200	0.0569	0.0623	0.0660	0.0701	0.0731	0.0685	0.0907	0.1020	0.1124	0.1244
	400	0.0578	0.0537	0.0574	0.0633	0.0658	0.0610	0.0639	0.0733	0.0855	0.0920
	800	0.0598	0.0525	0.0541	0.0583	0.0634	0.0610	0.0552	0.0606	0.0670	0.0778
	1600	0.0635	0.0548	0.0500	0.0537	0.0575	0.0636	0.0556	0.0516	0.0561	0.0619
	3200	0.0654	0.0570	0.0502	0.0507	0.0546	0.0655	0.0572	0.0505	0.0513	0.0563
	6400	0.0673	0.0603	0.0517	0.0494	0.0519	0.0673	0.0603	0.0519	0.0495	0.0522
0.25	100	0.0594	0.0680	0.0704	0.0798	0.0840	0.0805	0.1138	0.1279	0.1475	0.1607
	200	0.0572	0.0629	0.0673	0.0715	0.0746	0.0692	0.0915	0.1040	0.1158	0.1298
	400	0.0579	0.0545	0.0589	0.0642	0.0676	0.0612	0.0657	0.0752	0.0882	0.0960
	800	0.0598	0.0527	0.0549	0.0588	0.0642	0.0608	0.0557	0.0620	0.0681	0.0790
	1600	0.0633	0.0544	0.0500	0.0543	0.0578	0.0633	0.0552	0.0517	0.0569	0.0624
	3200	0.0652	0.0565	0.0500	0.0508	0.0549	0.0653	0.0568	0.0503	0.0515	0.0567
	6400	0.0671	0.0599	0.0512	0.0492	0.0518	0.0671	0.0599	0.0514	0.0493	0.0520
0.50	100	0.0640	0.0749	0.0781	0.0907	0.0946	0.0894	0.1253	0.1439	0.1689	0.1837
	200	0.0586	0.0667	0.0725	0.0775	0.0811	0.0715	0.0992	0.1133	0.1283	0.1453
	400	0.0579	0.0568	0.0623	0.0681	0.0719	0.0618	0.0702	0.0817	0.0972	0.1057
	800	0.0593	0.0530	0.0564	0.0608	0.0672	0.0602	0.0567	0.0647	0.0719	0.0852
	1600	0.0625	0.0535	0.0503	0.0555	0.0592	0.0626	0.0545	0.0523	0.0589	0.0645
	3200	0.0644	0.0551	0.0494	0.0509	0.0554	0.0646	0.0554	0.0500	0.0517	0.0576
	6400	0.0663	0.0585	0.0497	0.0484	0.0518	0.0663	0.0585	0.0499	0.0486	0.0521
0.75	100	0.0740	0.0881	0.0943	0.1081	0.1137	0.1074	0.1523	0.1808	0.2176	0.2360
	200	0.0628	0.0750	0.0826	0.0891	0.0932	0.0786	0.1145	0.1350	0.1555	0.1780
	400	0.0584	0.0616	0.0683	0.0753	0.0801	0.0636	0.0786	0.0945	0.1141	0.1258
	800	0.0587	0.0539	0.0595	0.0649	0.0719	0.0597	0.0591	0.0705	0.0805	0.0969
	1600	0.0613	0.0521	0.0509	0.0576	0.0614	0.0614	0.0535	0.0535	0.0626	0.0691
	3200	0.0633	0.0530	0.0484	0.0509	0.0562	0.0634	0.0534	0.0494	0.0522	0.0591
	6400	0.0652	0.0563	0.0472	0.0471	0.0511	0.0652	0.0563	0.0473	0.0473	0.0516
1.00	100	0.0912	0.1087	0.1179	0.1319	0.1404	0.1472	0.2114	0.2549	0.3030	0.3290
	200	0.0723	0.0894	0.0986	0.1074	0.1115	0.0953	0.1474	0.1809	0.2066	0.2445
	400	0.0614	0.0699	0.0791	0.0872	0.0946	0.0693	0.0955	0.1189	0.1456	0.1679
	800	0.0583	0.0567	0.0656	0.0714	0.0797	0.0599	0.0652	0.0823	0.0952	0.1173
	1600	0.0596	0.0514	0.0529	0.0613	0.0656	0.0599	0.0539	0.0573	0.0694	0.0776
	3200	0.0614	0.0502	0.0474	0.0516	0.0576	0.0616	0.0509	0.0492	0.0539	0.0616
	6400	0.0635	0.0531	0.0444	0.0457	0.0507	0.0635	0.0532	0.0446	0.0460	0.0514
2.00	100	0.2025	0.2361	0.2608	0.2781	0.2979	0.7914	0.9428	1.0901	1.1901	1.2334
	200	0.1632	0.1942	0.2133	0.2273	0.2342	0.5299	0.7197	0.8252	0.9139	0.9952
	400	0.1275	0.1599	0.1708	0.1851	0.1977	0.3153	0.4668	0.5763	0.6576	0.7611
	800	0.0961	0.1188	0.1384	0.1478	0.1585	0.1683	0.2940	0.3635	0.4359	0.4883
	1600	0.0718	0.0910	0.1048	0.1205	0.1262	0.1001	0.1712	0.2161	0.2743	0.3094
	3200	0.0582	0.0632	0.0776	0.0871	0.0988	0.0650	0.0915	0.1249	0.1532	0.1800
	6400	0.0542	0.0450	0.0533	0.0633	0.0723	0.0547	0.0510	0.0682	0.0848	0.1005

Table F.11: Median Average Square Error for Derivatives of Expected Efficiency $\mu_\eta(Z)$, with Heteroskedasticity

		SVKZ					HMS				
ρ	n	(1,1)	(1,2)	(2,2)	(2,3)	(3,3)	(1,1)	(1,2)	(2,2)	(2,3)	(3,3)
0.00	100	0.0410	0.0786	0.1088	0.1347	0.1612	0.0411	0.0789	0.1106	0.1393	0.1683
	200	0.0336	0.0702	0.0872	0.1090	0.1251	0.0336	0.0707	0.0879	0.1100	0.1293
	400	0.0299	0.0506	0.0619	0.0768	0.0889	0.0299	0.0509	0.0622	0.0772	0.0891
	800	0.0295	0.0432	0.0499	0.0552	0.0632	0.0295	0.0434	0.0499	0.0553	0.0632
	1600	0.0327	0.0362	0.0399	0.0388	0.0402	0.0327	0.0362	0.0399	0.0388	0.0402
	3200	0.0331	0.0340	0.0304	0.0288	0.0290	0.0331	0.0340	0.0304	0.0288	0.0290
	6400	0.0327	0.0320	0.0288	0.0214	0.0204	0.0327	0.0320	0.0288	0.0214	0.0204
0.25	100	0.0453	0.0812	0.1073	0.1508	0.1722	0.0455	0.0826	0.1110	0.1558	0.1854
	200	0.0347	0.0676	0.0899	0.1126	0.1389	0.0347	0.0688	0.0910	0.1134	0.1414
	400	0.0298	0.0526	0.0632	0.0789	0.0920	0.0298	0.0527	0.0638	0.0790	0.0928
	800	0.0292	0.0435	0.0495	0.0567	0.0664	0.0292	0.0438	0.0495	0.0568	0.0664
	1600	0.0320	0.0376	0.0405	0.0407	0.0418	0.0320	0.0376	0.0405	0.0407	0.0418
	3200	0.0327	0.0341	0.0314	0.0291	0.0289	0.0327	0.0341	0.0314	0.0291	0.0289
	6400	0.0322	0.0323	0.0290	0.0215	0.0206	0.0322	0.0323	0.0290	0.0215	0.0206
0.50	100	0.0485	0.0956	0.1258	0.1870	0.2212	0.0495	0.1006	0.1339	0.2040	0.2460
	200	0.0356	0.0744	0.1011	0.1291	0.1561	0.0357	0.0757	0.1022	0.1315	0.1601
	400	0.0308	0.0552	0.0740	0.0906	0.1046	0.0308	0.0555	0.0743	0.0909	0.1057
	800	0.0283	0.0454	0.0512	0.0613	0.0719	0.0283	0.0455	0.0514	0.0614	0.0719
	1600	0.0323	0.0384	0.0426	0.0431	0.0461	0.0323	0.0384	0.0426	0.0431	0.0462
	3200	0.0329	0.0338	0.0323	0.0310	0.0312	0.0329	0.0338	0.0323	0.0310	0.0313
	6400	0.0323	0.0321	0.0287	0.0227	0.0221	0.0323	0.0321	0.0287	0.0227	0.0221
0.75	100	0.0523	0.1188	0.1633	0.2433	0.2841	0.0628	0.1330	0.1998	0.3044	0.3610
	200	0.0410	0.0990	0.1305	0.1615	0.1970	0.0415	0.1036	0.1329	0.1705	0.2151
	400	0.0324	0.0635	0.0879	0.1068	0.1299	0.0324	0.0637	0.0882	0.1085	0.1315
	800	0.0290	0.0504	0.0597	0.0743	0.0850	0.0290	0.0504	0.0599	0.0744	0.0854
	1600	0.0328	0.0407	0.0454	0.0502	0.0515	0.0328	0.0407	0.0454	0.0502	0.0518
	3200	0.0327	0.0347	0.0348	0.0335	0.0336	0.0327	0.0347	0.0350	0.0335	0.0336
	6400	0.0327	0.0325	0.0294	0.0243	0.0244	0.0327	0.0325	0.0294	0.0243	0.0244
1.00	100	0.0621	0.1488	0.2134	0.2925	0.3332	0.0996	0.2125	0.3257	0.5003	0.6213
	200	0.0528	0.1293	0.1737	0.2251	0.2700	0.0599	0.1454	0.1980	0.2629	0.3237
	400	0.0370	0.0813	0.1207	0.1472	0.1736	0.0382	0.0830	0.1229	0.1539	0.1836
	800	0.0311	0.0596	0.0771	0.0940	0.1086	0.0311	0.0599	0.0777	0.0952	0.1108
	1600	0.0331	0.0443	0.0554	0.0623	0.0658	0.0331	0.0443	0.0555	0.0626	0.0664
	3200	0.0324	0.0366	0.0389	0.0397	0.0405	0.0324	0.0366	0.0390	0.0398	0.0405
	6400	0.0329	0.0338	0.0318	0.0276	0.0284	0.0329	0.0338	0.0318	0.0276	0.0284
2.00	100	0.0306	0.1721	0.2567	0.3573	0.3393	0.4477	1.0962	1.5814	2.4811	2.7370
	200	0.0544	0.2335	0.3046	0.4092	0.5095	0.3087	0.8262	1.3476	1.6593	2.4729
	400	0.0596	0.2030	0.3085	0.4492	0.4146	0.1898	0.5268	0.8158	1.0988	1.5071
	800	0.0649	0.1877	0.2363	0.3406	0.3768	0.1117	0.3497	0.4913	0.6653	0.7444
	1600	0.0534	0.1380	0.1876	0.2703	0.2838	0.0703	0.2070	0.2792	0.3694	0.4194
	3200	0.0420	0.0950	0.1355	0.1739	0.1998	0.0441	0.1121	0.1681	0.2009	0.2323
	6400	0.0372	0.0635	0.0966	0.1135	0.1208	0.0372	0.0649	0.1001	0.1205	0.1331

Table F.12: Median Average Square Error for Estimated Marginal Product, with Heteroskedasticity

		SVKZ					HMS				
ρ	n	(1,1)	(1,2)	(2,2)	(2,3)	(3,3)	(1,1)	(1,2)	(2,2)	(2,3)	(3,3)
0.00	100	0.1611	0.1483	0.2695	0.2616	0.3914	0.1611	0.1510	0.2752	0.2649	0.4131
	200	0.1071	0.1163	0.1858	0.2154	0.2865	0.1071	0.1169	0.1884	0.2163	0.2965
	400	0.0769	0.0853	0.1541	0.1733	0.2503	0.0769	0.0853	0.1547	0.1733	0.2525
	800	0.0576	0.0577	0.1178	0.1199	0.1782	0.0576	0.0579	0.1178	0.1201	0.1782
	1600	0.0418	0.0410	0.0839	0.0781	0.1132	0.0418	0.0410	0.0839	0.0781	0.1132
	3200	0.0421	0.0306	0.0462	0.0577	0.0808	0.0421	0.0306	0.0462	0.0577	0.0808
	6400	0.0336	0.0277	0.0360	0.0362	0.0534	0.0336	0.0277	0.0360	0.0362	0.0534
0.25	100	0.1720	0.1616	0.2657	0.2730	0.3777	0.1720	0.1632	0.2758	0.2793	0.4079
	200	0.1132	0.1196	0.2004	0.2212	0.3178	0.1132	0.1256	0.2022	0.2225	0.3194
	400	0.0745	0.0831	0.1747	0.1707	0.2548	0.0745	0.0833	0.1783	0.1707	0.2659
	800	0.0554	0.0633	0.1148	0.1347	0.1807	0.0554	0.0637	0.1157	0.1369	0.1807
	1600	0.0423	0.0439	0.0821	0.0862	0.1218	0.0423	0.0439	0.0821	0.0862	0.1218
	3200	0.0441	0.0327	0.0477	0.0589	0.0796	0.0441	0.0327	0.0477	0.0589	0.0796
	6400	0.0315	0.0283	0.0369	0.0389	0.0563	0.0315	0.0283	0.0369	0.0389	0.0563
0.50	100	0.2023	0.1821	0.3191	0.3537	0.4435	0.2059	0.1966	0.3532	0.3862	0.5297
	200	0.1247	0.1462	0.2371	0.2225	0.3630	0.1257	0.1482	0.2450	0.2269	0.3914
	400	0.0744	0.0867	0.1882	0.1885	0.2857	0.0744	0.0875	0.1906	0.1916	0.2892
	800	0.0637	0.0711	0.1287	0.1559	0.2191	0.0637	0.0714	0.1288	0.1572	0.2194
	1600	0.0419	0.0461	0.0963	0.0975	0.1298	0.0419	0.0461	0.0963	0.0975	0.1314
	3200	0.0496	0.0358	0.0557	0.0653	0.0869	0.0496	0.0358	0.0557	0.0653	0.0871
	6400	0.0317	0.0292	0.0403	0.0415	0.0591	0.0317	0.0292	0.0403	0.0415	0.0591
0.75	100	0.2555	0.2485	0.4034	0.4266	0.5228	0.2973	0.2952	0.4949	0.5301	0.6250
	200	0.1703	0.1862	0.3299	0.2923	0.4323	0.1732	0.2031	0.3444	0.3131	0.5366
	400	0.0954	0.1027	0.2389	0.2193	0.3383	0.0954	0.1032	0.2400	0.2283	0.3427
	800	0.0736	0.0831	0.1605	0.1794	0.2602	0.0736	0.0831	0.1620	0.1808	0.2639
	1600	0.0497	0.0496	0.1225	0.1151	0.1613	0.0497	0.0496	0.1225	0.1153	0.1635
	3200	0.0527	0.0386	0.0657	0.0760	0.1012	0.0527	0.0386	0.0657	0.0762	0.1012
	6400	0.0330	0.0313	0.0464	0.0472	0.0659	0.0330	0.0313	0.0464	0.0472	0.0659
1.00	100	0.3398	0.3178	0.5147	0.4138	0.5521	0.4786	0.4126	0.8098	0.6485	0.7341
	200	0.2220	0.2615	0.4100	0.3645	0.4973	0.2495	0.3198	0.4983	0.4418	0.6420
	400	0.1319	0.1440	0.3357	0.2917	0.4269	0.1347	0.1463	0.3543	0.3230	0.4521
	800	0.0911	0.1060	0.2223	0.2524	0.3377	0.0911	0.1060	0.2252	0.2641	0.3453
	1600	0.0568	0.0623	0.1486	0.1386	0.1940	0.0568	0.0623	0.1493	0.1400	0.1993
	3200	0.0536	0.0460	0.0840	0.0956	0.1230	0.0536	0.0460	0.0840	0.0956	0.1230
	6400	0.0365	0.0352	0.0572	0.0584	0.0760	0.0365	0.0352	0.0572	0.0584	0.0760
2.00	100	0.3916	0.4025	0.7532	0.4992	0.7055	2.5102	1.2524	1.5276	0.9308	0.9905
	200	0.3673	0.3693	0.5471	0.4675	0.5127	2.1563	1.3826	1.4545	0.8951	0.9614
	400	0.3448	0.2839	0.5841	0.4649	0.6011	1.0959	0.9290	1.3798	0.8451	0.9953
	800	0.3194	0.2910	0.5647	0.4399	0.5710	0.6137	0.8006	1.2553	0.9122	0.9232
	1600	0.2308	0.2521	0.4977	0.4450	0.5550	0.3051	0.4315	0.9295	0.6785	0.8654
	3200	0.1506	0.2004	0.3797	0.3834	0.5072	0.1600	0.2524	0.5303	0.5031	0.6552
	6400	0.0892	0.0943	0.2753	0.2419	0.3257	0.0892	0.0992	0.3090	0.2808	0.4229

Table F.13: Differences in IMSE for Frontier Estimates, Exponential versus Half-Normal Inefficiency, No Heteroskedasticity

		SVKZ					HMS				
ρ	n	(1,1)	(1,2)	(2,2)	(2,3)	(3,3)	(1,1)	(1,2)	(2,2)	(2,3)	(3,3)
0.00	100	0.0376	0.0241	0.0266	0.0202	0.0208	0.0558	0.0652	0.0898	0.0940	0.1035
	200	0.0481	0.0347	0.0335	0.0327	0.0294	0.0563	0.0537	0.0664	0.0780	0.0821
	400	0.0513	0.0392	0.0399	0.0405	0.0424	0.0521	0.0447	0.0540	0.0615	0.0716
	800	0.0579	0.0438	0.0402	0.0454	0.0481	0.0581	0.0453	0.0438	0.0551	0.0603
	1600	0.0630	0.0481	0.0416	0.0429	0.0462	0.0631	0.0483	0.0425	0.0451	0.0494
	3200	0.0646	0.0539	0.0434	0.0424	0.0446	0.0646	0.0540	0.0435	0.0428	0.0452
	6400	0.0668	0.0578	0.0475	0.0422	0.0438	0.0668	0.0578	0.0475	0.0423	0.0439
0.25	100	0.0360	0.0219	0.0250	0.0197	0.0205	0.0562	0.0668	0.0920	0.0979	0.1056
	200	0.0471	0.0336	0.0320	0.0320	0.0293	0.0558	0.0550	0.0686	0.0807	0.0867
	400	0.0506	0.0383	0.0393	0.0391	0.0415	0.0517	0.0449	0.0542	0.0614	0.0741
	800	0.0571	0.0432	0.0396	0.0448	0.0482	0.0573	0.0446	0.0438	0.0555	0.0607
	1600	0.0626	0.0477	0.0409	0.0426	0.0458	0.0627	0.0479	0.0418	0.0448	0.0493
	3200	0.0644	0.0533	0.0426	0.0420	0.0442	0.0644	0.0533	0.0427	0.0424	0.0447
	6400	0.0668	0.0573	0.0467	0.0416	0.0432	0.0668	0.0573	0.0467	0.0416	0.0433
0.50	100	0.0303	0.0177	0.0225	0.0182	0.0205	0.0588	0.0740	0.1028	0.1057	0.1163
	200	0.0438	0.0289	0.0282	0.0289	0.0266	0.0558	0.0579	0.0750	0.0896	0.0971
	400	0.0485	0.0344	0.0356	0.0357	0.0387	0.0503	0.0450	0.0561	0.0648	0.0799
	800	0.0553	0.0406	0.0368	0.0425	0.0460	0.0556	0.0426	0.0436	0.0563	0.0623
	1600	0.0612	0.0455	0.0385	0.0409	0.0440	0.0614	0.0459	0.0401	0.0441	0.0489
	3200	0.0635	0.0512	0.0401	0.0403	0.0427	0.0635	0.0512	0.0402	0.0409	0.0436
	6400	0.0662	0.0556	0.0441	0.0393	0.0415	0.0662	0.0556	0.0442	0.0394	0.0416
0.75	100	0.0239	0.0169	0.0236	0.0204	0.0258	0.0648	0.0890	0.1223	0.1238	0.1394
	200	0.0370	0.0209	0.0227	0.0256	0.0246	0.0563	0.0653	0.0837	0.1037	0.1129
	400	0.0434	0.0255	0.0278	0.0290	0.0336	0.0481	0.0467	0.0594	0.0716	0.0893
	800	0.0519	0.0338	0.0287	0.0359	0.0403	0.0526	0.0402	0.0435	0.0592	0.0666
	1600	0.0589	0.0407	0.0319	0.0351	0.0388	0.0592	0.0419	0.0364	0.0426	0.0488
	3200	0.0619	0.0472	0.0348	0.0356	0.0384	0.0619	0.0472	0.0353	0.0375	0.0413
	6400	0.0652	0.0525	0.0393	0.0346	0.0374	0.0652	0.0525	0.0394	0.0349	0.0380
1.00	100	0.0225	0.0207	0.0320	0.0278	0.0359	0.0772	0.1145	0.1579	0.1541	0.1776
	200	0.0274	0.0161	0.0220	0.0275	0.0271	0.0609	0.0753	0.1000	0.1272	0.1403
	400	0.0321	0.0147	0.0210	0.0245	0.0307	0.0458	0.0501	0.0675	0.0842	0.1068
	800	0.0446	0.0194	0.0154	0.0247	0.0319	0.0480	0.0376	0.0444	0.0634	0.0739
	1600	0.0546	0.0293	0.0166	0.0229	0.0286	0.0553	0.0355	0.0327	0.0414	0.0491
	3200	0.0594	0.0396	0.0219	0.0236	0.0275	0.0594	0.0405	0.0270	0.0326	0.0381
	6400	0.0635	0.0474	0.0304	0.0243	0.0282	0.0635	0.0474	0.0310	0.0274	0.0320
2.00	100	0.0752	0.0840	0.1037	0.1077	0.1206	0.3209	0.4514	0.5414	0.5676	0.5971
	200	0.0554	0.0643	0.0734	0.0871	0.0886	0.1888	0.2968	0.3696	0.4172	0.4606
	400	0.0325	0.0411	0.0589	0.0658	0.0757	0.1146	0.1801	0.2339	0.2840	0.3404
	800	0.0147	0.0174	0.0285	0.0402	0.0517	0.0506	0.0773	0.1144	0.1511	0.1992
	1600	0.0017	−0.0143	0.0001	0.0188	0.0313	0.0219	0.0124	0.0393	0.0851	0.1106
	3200	0.0149	−0.0363	−0.0332	−0.0120	0.0006	0.0315	−0.0132	−0.0133	0.0159	0.0369
	6400	0.0333	−0.0394	−0.0642	−0.0468	−0.0252	0.0427	−0.0093	−0.0388	−0.0246	0.0011

Table F.14: Differences in IMSE for Frontier Estimates, Exponential versus Half-Normal Inefficiency, Heteroskedastic Inefficiency and Noise

		SVKZ					HMS				
ρ	n	(1,1)	(1,2)	(2,2)	(2,3)	(3,3)	(1,1)	(1,2)	(2,2)	(2,3)	(3,3)
0.00	100	0.0376	0.0305	0.0226	0.0219	0.0138	0.0598	0.0814	0.0893	0.0979	0.1070
	200	0.0470	0.0394	0.0368	0.0368	0.0331	0.0596	0.0711	0.0775	0.0861	0.0942
	400	0.0528	0.0402	0.0390	0.0426	0.0422	0.0564	0.0521	0.0571	0.0679	0.0728
	800	0.0576	0.0454	0.0421	0.0449	0.0490	0.0587	0.0485	0.0494	0.0547	0.0650
	1600	0.0623	0.0510	0.0421	0.0443	0.0478	0.0624	0.0519	0.0440	0.0471	0.0526
	3200	0.0648	0.0549	0.0451	0.0436	0.0472	0.0650	0.0551	0.0454	0.0442	0.0490
	6400	0.0671	0.0592	0.0485	0.0436	0.0456	0.0671	0.0592	0.0486	0.0438	0.0459
0.25	100	0.0364	0.0288	0.0216	0.0225	0.0147	0.0612	0.0826	0.0913	0.1045	0.1137
	200	0.0461	0.0374	0.0358	0.0351	0.0312	0.0592	0.0704	0.0786	0.0879	0.0975
	400	0.0523	0.0397	0.0390	0.0413	0.0420	0.0562	0.0528	0.0581	0.0693	0.0754
	800	0.0572	0.0447	0.0418	0.0441	0.0488	0.0583	0.0483	0.0500	0.0550	0.0655
	1600	0.0620	0.0502	0.0414	0.0442	0.0475	0.0620	0.0512	0.0434	0.0473	0.0527
	3200	0.0645	0.0542	0.0445	0.0432	0.0470	0.0646	0.0545	0.0448	0.0439	0.0489
	6400	0.0667	0.0586	0.0477	0.0431	0.0451	0.0667	0.0586	0.0478	0.0432	0.0453
0.50	100	0.0318	0.0246	0.0190	0.0238	0.0157	0.0656	0.0883	0.0995	0.1166	0.1255
	200	0.0428	0.0334	0.0319	0.0317	0.0274	0.0586	0.0734	0.0832	0.0946	0.1055
	400	0.0501	0.0365	0.0358	0.0379	0.0387	0.0551	0.0537	0.0608	0.0738	0.0801
	800	0.0558	0.0422	0.0393	0.0413	0.0469	0.0569	0.0471	0.0500	0.0558	0.0684
	1600	0.0608	0.0478	0.0391	0.0427	0.0460	0.0609	0.0491	0.0418	0.0472	0.0526
	3200	0.0636	0.0521	0.0423	0.0414	0.0456	0.0637	0.0524	0.0430	0.0424	0.0481
	6400	0.0659	0.0569	0.0451	0.0409	0.0437	0.0659	0.0569	0.0453	0.0411	0.0441
0.75	100	0.0254	0.0224	0.0194	0.0271	0.0208	0.0739	0.0993	0.1162	0.1404	0.1502
	200	0.0360	0.0265	0.0269	0.0282	0.0250	0.0592	0.0790	0.0929	0.1074	0.1215
	400	0.0449	0.0285	0.0292	0.0314	0.0326	0.0529	0.0553	0.0660	0.0806	0.0876
	800	0.0527	0.0355	0.0326	0.0344	0.0415	0.0545	0.0446	0.0503	0.0587	0.0739
	1600	0.0584	0.0426	0.0329	0.0376	0.0408	0.0587	0.0451	0.0384	0.0466	0.0529
	3200	0.0620	0.0481	0.0371	0.0368	0.0416	0.0621	0.0487	0.0388	0.0393	0.0460
	6400	0.0645	0.0537	0.0401	0.0362	0.0396	0.0645	0.0538	0.0403	0.0367	0.0406
1.00	100	0.0244	0.0269	0.0278	0.0352	0.0332	0.0893	0.1236	0.1506	0.1795	0.1936
	200	0.0273	0.0223	0.0254	0.0302	0.0275	0.0612	0.0888	0.1102	0.1267	0.1512
	400	0.0343	0.0181	0.0225	0.0262	0.0309	0.0511	0.0596	0.0745	0.0910	0.1066
	800	0.0457	0.0220	0.0203	0.0240	0.0332	0.0502	0.0422	0.0525	0.0620	0.0812
	1600	0.0539	0.0314	0.0186	0.0260	0.0310	0.0550	0.0392	0.0347	0.0464	0.0542
	3200	0.0591	0.0408	0.0252	0.0254	0.0316	0.0593	0.0426	0.0321	0.0348	0.0428
	6400	0.0624	0.0485	0.0312	0.0261	0.0311	0.0624	0.0488	0.0326	0.0293	0.0354
2.00	100	0.0756	0.0937	0.1046	0.1112	0.1157	0.3810	0.4464	0.5450	0.5904	0.6087
	200	0.0566	0.0690	0.0832	0.0916	0.0875	0.2324	0.3264	0.3800	0.4452	0.4710
	400	0.0345	0.0505	0.0589	0.0672	0.0754	0.1195	0.1906	0.2578	0.3067	0.3658
	800	0.0137	0.0155	0.0327	0.0424	0.0530	0.0406	0.0914	0.1305	0.1739	0.2028
	1600	0.0041	−0.0109	0.0005	0.0237	0.0314	0.0277	0.0219	0.0448	0.0926	0.1199
	3200	0.0125	−0.0322	−0.0296	−0.0120	0.0055	0.0304	−0.0036	−0.0075	0.0211	0.0408
	6400	0.0285	−0.0341	−0.0593	−0.0440	−0.0211	0.0386	−0.0035	−0.0331	−0.0200	0.0028

Table F.15: Differences in Median Average Square Error for Derivatives of Expected Efficiency $\mu_\eta(Z)$, Exponential versus Half-Normal Inefficiency, with Heteroskedasticity

		SVKZ					HMS				
ρ	n	(1,1)	(1,2)	(2,2)	(2,3)	(3,3)	(1,1)	(1,2)	(2,2)	(2,3)	(3,3)
0.00	100	0.0228	0.0455	0.0629	0.0776	0.0950	0.0228	0.0439	0.0598	0.0750	0.0899
	200	0.0166	0.0418	0.0550	0.0725	0.0848	0.0166	0.0419	0.0554	0.0735	0.0875
	400	0.0136	0.0287	0.0396	0.0538	0.0633	0.0136	0.0290	0.0398	0.0543	0.0632
	800	0.0119	0.0247	0.0315	0.0384	0.0465	0.0119	0.0249	0.0315	0.0385	0.0464
	1600	0.0150	0.0186	0.0243	0.0264	0.0281	0.0150	0.0186	0.0243	0.0264	0.0281
	3200	0.0152	0.0176	0.0170	0.0188	0.0202	0.0152	0.0176	0.0170	0.0188	0.0202
	6400	0.0152	0.0156	0.0159	0.0127	0.0127	0.0152	0.0156	0.0159	0.0127	0.0127
0.25	100	0.0265	0.0418	0.0571	0.0884	0.1011	0.0258	0.0413	0.0545	0.0831	0.0900
	200	0.0168	0.0372	0.0545	0.0732	0.0922	0.0168	0.0381	0.0547	0.0729	0.0922
	400	0.0130	0.0297	0.0386	0.0533	0.0628	0.0130	0.0298	0.0393	0.0531	0.0633
	800	0.0116	0.0237	0.0304	0.0385	0.0483	0.0116	0.0240	0.0304	0.0385	0.0483
	1600	0.0148	0.0202	0.0246	0.0275	0.0287	0.0148	0.0202	0.0246	0.0275	0.0287
	3200	0.0149	0.0176	0.0177	0.0186	0.0197	0.0149	0.0176	0.0177	0.0186	0.0197
	6400	0.0145	0.0159	0.0160	0.0126	0.0125	0.0145	0.0159	0.0160	0.0126	0.0125
0.50	100	0.0266	0.0444	0.0600	0.1020	0.1369	0.0231	0.0371	0.0488	0.0911	0.0986
	200	0.0160	0.0344	0.0548	0.0746	0.0899	0.0157	0.0343	0.0528	0.0716	0.0837
	400	0.0130	0.0282	0.0408	0.0558	0.0647	0.0130	0.0281	0.0405	0.0549	0.0654
	800	0.0106	0.0236	0.0285	0.0371	0.0478	0.0106	0.0237	0.0286	0.0372	0.0476
	1600	0.0150	0.0203	0.0249	0.0272	0.0301	0.0150	0.0203	0.0249	0.0272	0.0302
	3200	0.0150	0.0168	0.0178	0.0186	0.0198	0.0150	0.0168	0.0179	0.0186	0.0199
	6400	0.0147	0.0158	0.0152	0.0129	0.0126	0.0147	0.0158	0.0152	0.0129	0.0126
0.75	100	0.0255	0.0615	0.0805	0.1398	0.1752	0.0165	0.0284	0.0515	0.1025	0.0781
	200	0.0146	0.0401	0.0661	0.0789	0.1042	0.0131	0.0305	0.0428	0.0650	0.0801
	400	0.0134	0.0272	0.0369	0.0510	0.0668	0.0132	0.0246	0.0324	0.0452	0.0589
	800	0.0110	0.0232	0.0273	0.0352	0.0447	0.0110	0.0231	0.0265	0.0341	0.0446
	1600	0.0158	0.0201	0.0223	0.0262	0.0270	0.0158	0.0201	0.0222	0.0260	0.0266
	3200	0.0148	0.0169	0.0175	0.0171	0.0166	0.0148	0.0169	0.0177	0.0170	0.0165
	6400	0.0150	0.0159	0.0148	0.0118	0.0114	0.0150	0.0159	0.0148	0.0118	0.0114
1.00	100	0.0362	0.0816	0.1233	0.1770	0.2143	0.0229	0.0278	0.0509	0.1128	0.1711
	200	0.0213	0.0551	0.0888	0.1158	0.1593	0.0071	0.0152	0.0168	0.0277	0.0315
	400	0.0106	0.0319	0.0469	0.0551	0.0774	0.0074	0.0086	0.0194	0.0274	0.0381
	800	0.0116	0.0179	0.0219	0.0318	0.0408	0.0113	0.0146	0.0099	0.0151	0.0288
	1600	0.0163	0.0156	0.0157	0.0176	0.0209	0.0163	0.0155	0.0140	0.0154	0.0173
	3200	0.0142	0.0154	0.0134	0.0103	0.0086	0.0142	0.0154	0.0134	0.0097	0.0079
	6400	0.0151	0.0160	0.0137	0.0080	0.0066	0.0151	0.0160	0.0137	0.0079	0.0061
2.00	100	0.0270	0.1689	0.2543	0.3560	0.3384	0.1617	0.3145	0.4930	1.0013	0.9163
	200	0.0508	0.2303	0.3022	0.3424	0.5087	0.0421	0.0880	0.3331	0.2509	0.5895
	400	0.0475	0.1819	0.2655	0.3572	0.3933	−0.0112	−0.0246	−0.0253	−0.0649	0.0642
	800	0.0543	0.1845	0.2339	0.3393	0.3028	−0.0492	−0.0514	−0.2450	−0.3617	−0.6590
	1600	0.0349	0.1348	0.1852	0.2690	0.2036	−0.0375	−0.2241	−0.2758	−0.4844	−0.4152
	3200	0.0154	0.0917	0.1332	0.1726	0.1990	−0.0108	−0.1745	−0.3034	−0.5480	−0.5934
	6400	0.0124	0.0191	0.0943	0.1122	0.1200	0.0074	−0.1080	−0.2986	−0.4505	−0.5498

Table F.16: Differences in Median Average Square Error for Estimated Marginal Product, Exponential versus Half-Normal Inefficiency, with Heteroskedasticity

		SVKZ					HMS				
ρ	n	(1,1)	(1,2)	(2,2)	(2,3)	(3,3)	(1,1)	(1,2)	(2,2)	(2,3)	(3,3)
0.00	100	−0.0423	0.0049	−0.0190	−0.0121	−0.0263	−0.0425	−0.0007	−0.0369	−0.0281	−0.0676
	200	−0.0136	0.0297	0.0111	0.0205	−0.0183	−0.0136	0.0285	0.0111	0.0214	−0.0222
	400	0.0022	0.0264	0.0144	0.0240	0.0091	0.0022	0.0267	0.0144	0.0244	0.0077
	800	0.0103	0.0282	0.0148	0.0193	0.0103	0.0103	0.0284	0.0148	0.0194	0.0099
	1600	0.0189	0.0260	0.0210	0.0155	0.0077	0.0189	0.0260	0.0210	0.0155	0.0077
	3200	0.0233	0.0261	0.0189	0.0164	0.0077	0.0233	0.0261	0.0189	0.0164	0.0076
	6400	0.0256	0.0256	0.0205	0.0134	0.0056	0.0256	0.0256	0.0205	0.0134	0.0056
0.25	100	−0.0430	0.0064	−0.0397	0.0048	−0.0467	−0.0434	−0.0009	−0.0593	−0.0125	−0.1020
	200	−0.0195	0.0176	0.0065	0.0129	−0.0144	−0.0199	0.0183	0.0051	0.0112	−0.0219
	400	−0.0012	0.0286	0.0046	0.0195	0.0060	−0.0012	0.0287	0.0053	0.0185	0.0048
	800	0.0099	0.0278	0.0138	0.0202	0.0112	0.0099	0.0281	0.0139	0.0202	0.0092
	1600	0.0183	0.0273	0.0200	0.0188	0.0016	0.0183	0.0273	0.0200	0.0188	0.0012
	3200	0.0212	0.0254	0.0190	0.0156	0.0057	0.0212	0.0254	0.0190	0.0156	0.0057
	6400	0.0246	0.0252	0.0207	0.0137	0.0043	0.0246	0.0252	0.0207	0.0137	0.0043
0.50	100	−0.0585	−0.0016	−0.0800	0.0023	−0.0364	−0.0724	−0.0313	−0.1234	−0.0705	−0.1476
	200	−0.0373	0.0024	−0.0188	−0.0046	−0.0418	−0.0376	0.0008	−0.0260	−0.0230	−0.0813
	400	−0.0045	0.0193	−0.0128	0.0055	−0.0227	−0.0045	0.0191	−0.0137	0.0013	−0.0254
	800	0.0077	0.0244	0.0044	0.0132	−0.0029	0.0077	0.0245	0.0045	0.0129	−0.0038
	1600	0.0170	0.0266	0.0171	0.0125	−0.0073	0.0170	0.0266	0.0171	0.0123	−0.0080
	3200	0.0197	0.0235	0.0185	0.0125	0.0014	0.0197	0.0235	0.0185	0.0125	0.0016
	6400	0.0245	0.0243	0.0193	0.0125	0.0014	0.0245	0.0243	0.0193	0.0125	0.0012
0.75	100	−0.0996	0.0059	−0.0625	0.0278	−0.0429	−0.1606	−0.0896	−0.2872	−0.1351	−0.2722
	200	−0.0640	−0.0375	−0.0477	−0.0237	−0.0537	−0.0735	−0.0679	−0.1446	−0.1107	−0.2162
	400	−0.0192	0.0026	−0.0498	−0.0227	−0.0622	−0.0192	−0.0042	−0.0733	−0.0593	−0.1229
	800	−0.0014	0.0169	−0.0202	−0.0148	−0.0465	−0.0016	0.0164	−0.0221	−0.0219	−0.0506
	1600	0.0142	0.0228	0.0021	0.0044	−0.0282	0.0142	0.0228	0.0018	0.0036	−0.0317
	3200	0.0163	0.0229	0.0141	0.0042	−0.0113	0.0163	0.0229	0.0143	0.0041	−0.0121
	6400	0.0240	0.0232	0.0176	0.0087	−0.0088	0.0240	0.0232	0.0176	0.0087	−0.0088
1.00	100	−0.0883	−0.0161	−0.0543	0.0571	−0.0246	−0.3229	−0.1808	−0.5394	−0.1512	−0.2786
	200	−0.1180	−0.0139	−0.0308	−0.0024	−0.0189	−0.2065	−0.1977	−0.4095	−0.2933	−0.4018
	400	−0.0488	−0.0077	−0.0734	−0.0425	−0.0851	−0.0644	−0.0795	−0.2567	−0.1666	−0.3226
	800	−0.0218	−0.0130	−0.0698	−0.0432	−0.1114	−0.0228	−0.0236	−0.1249	−0.1370	−0.1671
	1600	0.0079	0.0129	−0.0350	−0.0295	−0.0728	0.0079	0.0129	−0.0445	−0.0470	−0.1060
	3200	0.0130	0.0210	−0.0033	−0.0233	−0.0435	0.0130	0.0210	−0.0035	−0.0273	−0.0479
	6400	0.0218	0.0220	0.0123	−0.0037	−0.0362	0.0218	0.0220	0.0123	−0.0038	−0.0387
2.00	100	−0.1503	0.0115	−0.1673	0.0422	−0.1577	−1.0666	0.1746	−0.0748	1.6202	1.7721
	200	−0.1337	0.1135	0.0553	0.1223	0.1884	−1.3591	−0.1784	0.0740	0.7624	1.4163
	400	−0.0939	0.1024	0.1122	0.2812	0.1672	−0.9697	−0.4541	−0.7414	0.2563	0.4037
	800	−0.0471	0.1324	0.1343	0.2420	0.2221	−0.8310	−0.4206	−0.8843	−0.1412	−0.2941
	1600	−0.0515	0.1015	0.1346	0.2187	0.1622	−0.4349	−0.4583	−1.0620	−0.3462	−0.5904
	3200	−0.0516	0.0484	0.1072	0.1522	0.1521	−0.1497	−0.3301	−0.8486	−0.6157	−0.8571
	6400	−0.0195	0.0099	0.0764	0.1031	0.1008	−0.0326	−0.2514	−0.7884	−0.5125	−0.7674

G Additional Results on the Efficiency of US Banks

G.1 Results Mentioned in Section 4

As noted in Section 4, summary statistics for the original $p = 5$ input variables X_j, $j \in \{1,\ 2,\ 3,\ 4,\ 5\}$ and $q = 3$ output variables Y_j, $j \in \{1,\ 2,\ 3\}$ used in our empirical illustration appear below in Table G.1. As expected with banking data, the marginal distributions in the original (x, y)-space show evidence of substantial skewness to the right. Also shown in Table G.1 are summary statistics for the standardized and reduced-dimensional inputs X_1^*, X_2^* and X_3^* as well as the standardized and reduced-dimensional output variables Y_1^* and Y_2^* discussed in Section 4. The last three rows of Table G.1 show summary statistics for Z and U after applying the rotation matrix R_d to the reduced dimensional data $(X_1^*,\ X_2^*,\ X_3^*,\ \ Y_1^*,\ Y_2^*)$ as described in Section 4.

Table G.2 gives bandwidths optimized via leave-one-out cross validation for the regression of U on Z as well as the regression of squared and cubed residuals from this regression on Z. Table G.2 also shows bounds placed on the bandwidths as discussed in Section 4.

Table G.1: Summary Statistics for Data Used in Empirical Example

	Minimum	1st Quartile	Median	Mean	3rd Quartile	Maximum
X_1	4.2558×10^{2}	6.0741×10^{4}	1.2509×10^{5}	1.6003×10^{6}	2.8237×10^{5}	2.2089×10^{9}
X_2	1.0000	1.8500×10^{1}	3.6500×10^{1}	2.9020×10^{2}	7.9500×10^{1}	2.3955×10^{5}
X_3	8.6466×10^{-1}	7.6941×10^{2}	2.3102×10^{3}	1.7352×10^{4}	5.8730×10^{3}	2.0098×10^{7}
X_4	1.6680×10^{2}	7.5382×10^{3}	1.4930×10^{4}	1.8985×10^{5}	3.2386×10^{4}	2.2968×10^{8}
X_5	0.0000	3.1239×10^{1}	5.6761×10^{2}	2.0718×10^{4}	2.6015×10^{3}	8.6506×10^{7}
Y_1	4.0000	3.6882×10^{4}	8.3700×10^{4}	9.8186×10^{5}	2.0076×10^{5}	1.0041×10^{9}
Y_2	0.0000	1.3129×10^{4}	2.9528×10^{4}	5.2828×10^{5}	6.8619×10^{4}	1.2366×10^{9}
Y_3	0.0000	2.9200×10^{2}	7.8000×10^{2}	3.0686×10^{4}	2.3040×10^{3}	5.9248×10^{7}
X_1^*	3.3825×10^{-4}	5.9934×10^{-3}	1.3128×10^{-2}	1.0885×10^{-1}	2.9924×10^{-2}	1.1517×10^{2}
X_2^*	4.7286×10^{-5}	2.1369×10^{-3}	4.2325×10^{-3}	5.3820×10^{-2}	9.1809×10^{-3}	6.5109×10^{1}
X_3^*	0.0000	5.1046×10^{-5}	9.2750×10^{-4}	3.3853×10^{-2}	4.2509×10^{-3}	1.4135×10^{2}
Y_1^*	2.3581×10^{-7}	2.1743×10^{-3}	4.9344×10^{-3}	5.7884×10^{-2}	1.1836×10^{-2}	5.9194×10^{1}
Y_2^*	0.0000	1.2020×10^{-3}	2.6221×10^{-3}	6.0747×10^{-2}	6.3668×10^{-3}	1.2225×10^{2}
Z_1	-7.2311	-4.7671×10^{-1}	1.1314×10^{-1}	-1.7842×10^{-16}	6.3375×10^{-1}	3.5368
Z_2	-4.4695×10^{1}	7.8296×10^{-2}	1.0825×10^{-1}	3.7897×10^{-15}	1.1617×10^{-1}	1.1663×10^{-1}
Z_3	-6.6617	-5.2698×10^{-1}	8.0653×10^{-2}	1.6523×10^{-16}	6.5071×10^{-1}	3.0989
Z_4	-3.0897×10^{1}	9.0272×10^{-2}	1.2826×10^{-1}	3.7145×10^{-17}	1.4340×10^{-1}	1.5688×10^{-1}
U	-6.4415×10^{1}	-2.2061×10^{-2}	-9.5839×10^{-3}	-6.8971×10^{-2}	-4.2000×10^{-3}	1.7741×10^{-1}

Table G.2: Bandwidths used for Local-Linear Regressions in Banking Application

	bandwidth	Lower Bound	Upper Bound
Regression of U on Z:			
h_1	1.6522	1.5067×10^{-2}	2.2372×10^{1}
h_2	4.6293×10^{1}	9.5718×10^{-3}	7.4936×10^{1}
h_3	1.7525	1.5687×10^{-2}	2.1272×10^{1}
h_4	1.8579×10^{1}	9.1608×10^{-3}	6.3652×10^{1}
h_5	2.6488×10^{-1}	0.0000	1.0000
h_6	1.2395×10^{-1}	0.0000	1.0000
Regression of Squared Residuals on Z:			
h_1	1.1869×10^{1}	1.5067×10^{-2}	2.2372×10^{1}
h_2	1.5396×10^{1}	9.5718×10^{-3}	7.4936×10^{1}
h_3	9.6649	1.5687×10^{-2}	2.1272×10^{1}
h_4	4.8877×10^{1}	9.1608×10^{-3}	6.3652×10^{1}
h_5	5.0575×10^{-1}	0.0000	1.0000
h_6	5.4680×10^{-1}	0.0000	1.0000
Regression of Cubed Residuals on Z:			
h_1	8.6870	1.5067×10^{-2}	2.2372×10^{1}
h_2	2.5866×10^{1}	9.5718×10^{-3}	7.4936×10^{1}
h_3	9.6712	1.5687×10^{-2}	2.1272×10^{1}
h_4	2.7969×10^{1}	9.1608×10^{-3}	6.3652×10^{1}
h_5	5.1500×10^{-1}	0.0000	1.0000
h_6	5.4759×10^{-1}	0.0000	1.0000

Table G.3: Summary Statistics for Estimated Derivatives (SVKZ)

	.05 Quantile	1st Quartile	Median	Mean	3rd Quartile	.95 Quantile	Pct
$\partial X_2^*/\partial X_1^*$	−0.0735	−0.0232	−0.0147	−0.0155	−0.0093	0.1855	86.2232
$\partial X_3^*/\partial X_1^*$	−2.7029	0.0393	0.1086	−0.3380	0.2957	0.8907	14.5539
$\partial X_3^*/\partial X_2^*$	1.5695	3.9892	9.3864	60.7204	19.1904	44.4816	0.8194
$\partial Y_1^*/\partial X_1^*$	0.3973	0.4062	0.4168	0.4787	0.4405	0.6899	99.8436
$\partial Y_1^*/\partial X_2^*$	−5.9077	11.5886	22.4781	2.3402	33.2353	50.5607	86.1078
$\partial Y_1^*/\partial X_3^*$	−12.3783	−5.2030	−2.3778	−3.6729	−0.6839	0.3847	14.7103
$\partial Y_2^*/\partial X_1^*$	0.1786	0.6603	0.6917	0.6243	0.7371	0.8367	96.3479
$\partial Y_2^*/\partial X_2^*$	−5.1261	19.4873	37.2362	17.3778	55.5525	85.5500	89.8744
$\partial Y_2^*/\partial X_3^*$	−22.5386	−9.2115	−3.7292	−6.4038	−1.1424	0.2825	10.9022
$\partial Y_2^*/\partial Y_1^*$	−1.9264	−1.7274	−1.6558	−1.4963	−1.5567	−0.2551	96.1919
Scale Elasticity	1.6858	3.1036	4.3803	6.4313	6.2496	11.5679	98.1166

NOTE: The column labeled "Pct" gives the percentage of observations yielding estimates with the expected signs. As discussed in Section 4, X_3^* was initially viewed as a quasi-fixed input, but the estimates for $\partial X_3^*/\partial X_1^*$, $\partial X_3^*/\partial X_2^*$, $\partial Y_1^*/\partial X_3^*$ and $\partial Y_2^*/\partial X_3^*$ suggest that X_3^* behaves as an undesirable output. Viewed as such, the derivatives listed above have the expected signs in 85.4461, 99.1806, 85.2897 and 89.0978 percent (respectively) of the sample observations.

Table G.4: Percentage of Estimated Derivatives of $\mu_\eta(W_i^\partial)$ Less than, Equal to or Greater than Zero (SVKZ Estimates)

	Pct < 0	Pct $= 0$	Pct > 0
X_1^*	21.4235	39.6702	38.9063
X_2^*	51.3412	39.6702	8.9886
X_3^*	2.0557×10^{-4}	39.6702	60.3296
Y_1^*	3.0952	39.6702	57.2346
Y_2^*	47.6071	39.6702	12.7227

G.2 Results Obtained from HMS Estimators

We use the modification of the SVKZ estimators proposed by HMS and described in Appendix B to estimate bank efficiency, parallel to the estimation described in Section 4 of the main paper. As discussed in Appendix B, the HMS estimators avoid estimates of $\sigma_\eta(Z_i)$ equal to zero, which as seen in Section 4 occurs in more than a third of the sample observations due to positive skewness of estimated residuals in a neighborhood of Z_i. Results obtained with the HMS estimators are described here.

Figure G.1 plots means of HMS estimates of conditional expected inefficiency$\mu_\eta(Z_i)$ in each quarter 2001Q1–2020Q4 as a function of time, similar to Figure 2 in the main paper. Comparing the curves in Figure G.1 with those in Figure 2, it is clear that the means of conditional expected inefficiency obtained with the HMS estimators follow the same pattern as those obtained with the SVKZ estimators. However, because the HMS estimators eliminate the zero estimates for $\mu_\eta(Z_i)$, the means (and medians) shown in Figure G.1 are larger than the corresponding means and medians shown in Figure 2, as one should expect. In addition, the means in Figure G.1 are larger than the medians in each quarter, while the reverse is true in Figure 2. As noted in Section 4, the magnitudes of the mean and medians of the estimates $\widehat{\mu}_\eta(Z_i)$ are proportional to the length (0.01427) of the direction vector.

Analogous to Table G.3, Table G.5 gives summary statistics for estimates of marginal rates of substitution, marginal products, marginal rates of transformation and scale elasticities obtained using the HMS estimators. As in Table G.3, the last column in Table G.5 gives the percentage of observations with signs one should expect, i.e., negative signs for marginal rates of substitution and transformation, and positive signs for marginal products and scale elasticities. However, as noted in Section 4, given our use of nonparametric, local estimation, one should expect some noise in the estimates, and some to have wrong signs. Nonetheless, the estimates shown in Table G.5 are qualitatively similar to those described in Table G.3 in the sense that the percentage of estimates with the expected signs are similar across the two tables. In addition, the derivatives involving non-performing loans (X_3^*) here too indicate that this variable should be viewed as an undesirable output. As noted in Section 4 of the main paper, using a zero direction for this variable means that our initial view of the non-performing loans variable as a fixed input has no bearing on the estimation. Treating the variable as a fixed, undesirable output merely amounts to changing the label on the variable from X_3^* to Y_3^*.

Table G.6 gives percentages of estimated derivatives of the conditional expectation of inefficiency given in (2.32) and obtained with the HMS estimators. In contrast to Table G.4 giving similar percentages obtained with the SVKZ estimators, the results in Table G.6 reveal that none of the estimated derivatives are equal to zero. The number of negative estimates ranges from about 43.8 to 60.3 percent, while the number of positive estimates ranges from approximately 41.2 to 56.2 percent. As discussed in Section 4, the evidence suggests heteroskedasticity in the inefficiency process, and our model and estimation strategy allow for this.

Turning to the HMS estimates $\widehat{\sigma}_\epsilon^2(Z_i)$, 485,664 (more than 99.8 percent of the sample observations) are equal to to zero, suggesting almost no noise in the DGP. This is a consequence of eliminating zero values for $\widehat{\sigma}_\eta(Z_i)$. Under the assumptions of the model, $\eta_i = 0$ is a measure-zero event, and hence in this sense the HMS estimates more closely accord with the model assumptions than the SVKZ estimates.

Table G.5: Summary Statistics for Estimated Derivatives (HMS)

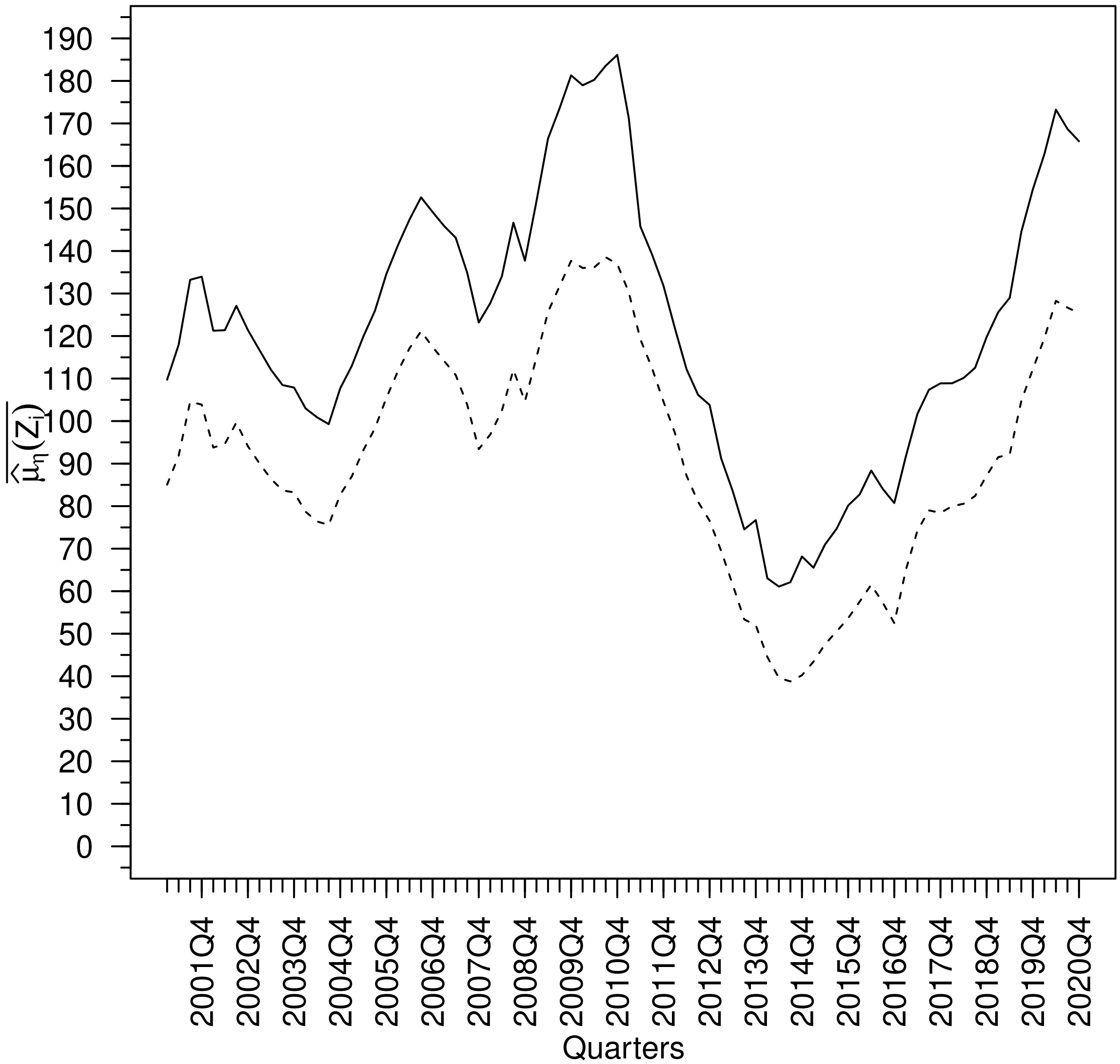
	.05 Quantile	1st Quartile	Median	Mean	3rd Quartile	.95 Quantile	Pct
$\partial X_2^*/\partial X_1^*$	−0.5584	−0.0379	−0.0183	0.0579	−0.0117	0.9190	90.0171
$\partial X_3^*/\partial X_1^*$	−13.2208	0.0456	0.1821	−1.2340	0.6206	6.6530	10.3568
$\partial X_3^*/\partial X_2^*$	1.7431	4.6060	9.5673	14.2599	21.1442	42.6848	0.4159
$\partial Y_1^*/\partial X_1^*$	−0.1624	0.4034	0.4143	0.5515	0.4316	1.5574	92.1959
$\partial Y_1^*/\partial X_2^*$	−1.9019	1.2784	18.9205	18.5220	29.0405	45.6179	82.2541
$\partial Y_1^*/\partial X_3^*$	−12.3805	−4.1308	−1.5472	−3.2049	−0.1482	0.1240	18.1601
$\partial Y_2^*/\partial X_1^*$	−1.4380	0.6631	0.6982	0.4193	0.7674	2.1193	92.3073
$\partial Y_2^*/\partial X_2^*$	0.7422	5.4664	31.4806	31.7445	47.8855	75.2565	96.3193
$\partial Y_2^*/\partial X_3^*$	−22.5524	−7.0502	−2.4810	−5.6024	−0.6808	−0.0299	4.0447
$\partial Y_2^*/\partial Y_1^*$	−7.4932	−1.7299	−1.6494	−2.4336	−1.5401	5.0394	85.3641
Scale Elasticity	−61.9995	−3.8396	5.9717	2.8298	16.3658	56.4053	67.3330

NOTE: The column labeled "Pct" gives the percentage of observations yielding estimates with the expected signs. As discussed in Section 4, X_3^* was initially viewed as a quasi-fixed input, but the estimates for $\partial X_3^*/\partial X_1^*$, $\partial X_3^*/\partial X_2^*$, $\partial Y_1^*/\partial X_3^*$ and $\partial Y_2^*/\partial X_3^*$ suggest that X_3^* behaves as an undesirable output. Viewed as such, the derivatives listed above have the expected signs in 89.6432, 99.5841, 81.8399 and 95.9553 percent (respectively) of the sample observations.

Table G.6: Percentage of Estimated Derivatives of $\mu_\eta(W_i^\partial)$ Less than, Equal to or Greater than Zero (HMS Estimates)

	Pct < 0	Pct $= 0$	Pct > 0
X_1^*	53.4203	0.0000	46.5797
X_2^*	44.5076	0.0000	55.4924
X_3^*	60.3306	0.0000	39.6694
Y_1^*	58.8417	0.0000	41.1583
Y_2^*	43.8448	0.0000	56.1552

Figure G.1: Sample Average and Median Values of $\widehat{\mu}_\eta(Z_i)$, HMS Estimates, 2001Q1–2020Q4



NOTE: Averages are indicated by the solid curve, while medians are represented by the dashed curve.

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