Supplementary Appendix to "Intraday Periodic Volatility Curves"

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Abstract

This document consists of three parts. Section A presents a Monte Carlo study. Section B contains additional theoretical results that compliment and extend the ones in the main text. Proofs of the theorems and corollaries in the main article as well as those in Section B of this Appendix are provided in Section C.

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Appendix A Monte Carlo Simulations

This section explores the performance of the general calendar-effect estimator (5) through simulation experiments. The simulation setting is described in Section A.1, while Sections A.2, A.3 and A.4 are devoted to finite-sample analysis of the functional theory in Theorem 6, the test for nonstationarity in the intraday volatility curve over time, and the pointwise theory in Corollary 7, respectively.

A.1 The Simulation Setting

The log-price process X, the volatility process σ^2, the calendar effect f, and the stationary component $\breve{\sigma}^2$ of the volatility process σ^2 are given, respectively, by,

$$\begin{cases} X(t) \;=\; X(0) \,+\, \int_0^t \sigma(s)\, dW(s) \,+\, \sum_{j=1}^{\widetilde{N}(t)} Z_j, \qquad \sigma^2(t) \;=\; f\left(t - \lfloor t \rfloor\right) \breve{\sigma}^2(t), \\ f(t) \;=\; 3\left(t - \frac{1}{2}\right)^2 \,+\, \frac{3}{4}, \qquad t \in [0,1], \qquad \breve{\sigma}^2(t) = \breve{\sigma}_1^2(t) \,+\, \breve{\sigma}_2^2(t), \\ \breve{\sigma}_1^2(t) \;=\; \breve{\sigma}_1^2(0) \,+\, \int_0^t \lambda\left(\tilde{\eta} - \breve{\sigma}_1^2(s)\right) ds \,+\, \int_0^t \xi\, \breve{\sigma}_1(s)\, d\widetilde{W}(s), \\ \breve{\sigma}_2^2(t) \;=\; \exp(-\tilde{\lambda} t)\, \breve{\sigma}_2^2(0) \,+\, \int_0^t \exp\{-\tilde{\lambda}(t-s)\}\, dz(\tilde{\lambda} s), \end{cases} \tag{A.1}$$

where $\widetilde{N}(t)$ is a homogeneous Poisson process with constant intensity λ_J, $\{Z_j\}_{j\geq 1}$ is an iid sequence of $N(0, \sigma_J^2)$ distributed random variables, the quadratic covariation is given by $[W, \widetilde{W}](t) = \rho\, t$, and z is a nonnegative increasing Lévy process such that the (stationary) marginal distribution of $\breve{\sigma}_2^2$ is $\Gamma(\nu_{\mathrm{OU}},\, 1/\alpha_{\mathrm{OU}})$. In the simulation, we exploit the specification provided by [1], fixing the model parameters as follows,

$$(X(0), \lambda, \tilde{\eta}, \xi, \lambda_J, \sigma_J, \rho, \tilde{\lambda}, \nu_{\mathrm{OU}}, \alpha_{\mathrm{OU}}) = (1, 4, 0.4068, 1.8, 0.19, 0.9654, -0.5, 0.6930, 1, 0.1).$$

Throughout, we set $n = 2{,}730$, corresponding to a sampling frequency of 30 seconds across 22.75 hours, mimicking the trading day for the e-mini S&P 500 futures in our empirical analysis. For each simulation trial, we generate a series of 1,500-day thirty-second prices. The following results are based on 1,000 trajectories with $T \leq 1{,}500$. In truncating the price jumps, we employ the time-varying threshold $u_n = 3\sqrt{BV_i \wedge RV_i}\,\Delta^{3/8}$ with,

$$BV_i = \frac{\pi}{2}\sum_{j=2}^{n} \left|\Delta_{i,j-1}^n X\right| \left|\Delta_{i,j}^n X\right| \qquad \text{and} \qquad RV_i = \sum_{j=1}^{n} \left(\Delta_{i,j}^n X\right)^2.$$

A.2 Finite-Sample Evidence for Functional Inference

This section provides a simulation experiment to explore the workings of the feasible (functional) central limit theorem in the L^2 metric, i.e., Corollary 3 and Theorem 6. Figure 1 depicts the empirical distribution of $T \parallel \widehat{f}(\kappa) - f(\kappa) \parallel^2$ for $\ell = 10$ and T =1,500 based on 1,000 trials. Because one can not explicitly evaluate the integral $T \parallel \widehat{f}(\kappa) - f(\kappa) \parallel^2$, we approximate the integral using a Riemann sum with the interval $[0, 1]$ partitioned into 100 equidistant subintervals.

We compare the empirical distribution, obtained as indicated above, with the limiting distribution of $T \parallel \widehat{f}(\kappa) - f(\kappa) \parallel^2$ (and $\mathcal{Z}$ in Corollary 3). As discussed in Section 4, the distribution of $\mathcal{Z}$ may be approximated by that of $\widehat{\mathcal{Z}}$ in equation (21), with the latter obtained through Monte Carlo simulation. This involves computing eigenvalues of the limiting covariance matrix estimator $(\widehat{C}(\kappa_i, \kappa_j))_{1\leq i,j\leq 100}$ with the entries defined in equation (18). In this study, we use the average limiting covariance matrix estimates over 1,000 trajectories rather than relying on a single trajectory to compute the eigenvalues associated with equation (21) and, consistent with the properties of the limiting variable, we retain only the terms featuring positive eigenvalues. Figure 1 also displays the limiting distribution of $T \parallel \widehat{f}(\kappa) - f(\kappa) \parallel^2$ obtained in this manner for $L_n = 7$ (recall, L_n is defined below equation (18)). Note that, in Theorem 6, we require $L_n \asymp n^{\varrho}$ for a strictly positive ϱ satisfying equation (20). In our simulations, $\varpi = 3/8$. If one takes $b \approx 9/10$ and $c \approx 1/2$ for $T = 1{,}500$, $\ell = 10$, and $n = 2{,}730$, then condition (20) reduces to $\varrho < 1/4$. For simplicity, we implement $L_n = \lfloor \min\{T^{1/2}, n^{1/4}\} \rfloor$ in all our numerical illustrations, implying $L_n = 7$ for T =1,500 and n = 2,730. Figure 1 demonstrates that the limiting distribution (red curve) approximates the empirical distribution (histogram) of $T \parallel \widehat{f}(\kappa) - f(\kappa) \parallel^2$ quite well, corroborating the theory developed in Corollary 3 and Theorem 6.

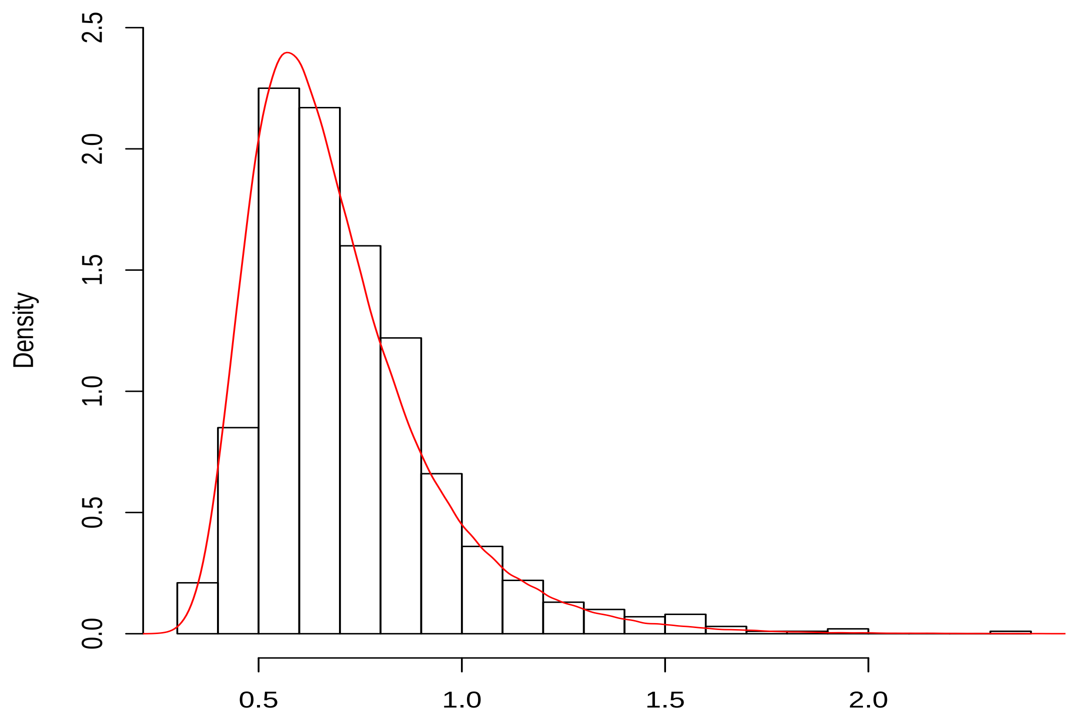

Figure 1. Approximating the distribution of the limiting variable $\mathcal{Z}$. The histogram represents the empirical distribution of $T \parallel \widehat{f}(\kappa) - f(\kappa) \parallel^2$ with $\ell = 10$ based on 1,000 simulation trials. The red curve indicates the density of $\widehat{\mathcal{Z}}$ in equation (21) obtained through Monte Carlo, as described in the main text.

A.3 Finite-Sample Test Performance

In this section, we seek to investigate the performance of our test (10) for a shift in the functional characterizing the average intraday volatility pattern for two non-overlapping periods P and P'. To ensure we capture the test performance within an empirically relevant setting, we calibrate the calendar effect to volatility curves obtained for two subsamples in our empirical study of the e-mini S&P 500 futures contract in Section 6. Specifically, we rely on the estimated volatility curves for periods covering 2005–2010 and 2015–2020, with the number of trading days equaling $T = 991$ and $T' = 1248$, respectively. We generate 1,000 simulated samples from model (A.1), but with the intraday volatility pattern modified to equal one of those estimated for the above subsamples. The shape of the curves is provided by the far left and right displays in the bottom panel of Figure 1 in Section 6.2.

The formal setup under the null hypothesis is $H_0 : f_P = f_{\mathrm{P}'} = f_{05-10}$, and it takes the form $H_A : f_{\mathrm{P}} = f_{05-10}$ and $f_{\mathrm{P}'} = f_{15-20}$ under the alternative, where f_{05-10} and f_{15-20} denote the estimated calendar effect functions for the e-mini futures over 2005-2010 and 2015-2020. That is, under the null hypothesis, we generate samples P and P' that all incorporate the volatility curve for 2005–2010. We then compute the test statistic, $T \parallel \widehat{f}_{\mathrm{P}}(\kappa) - \widehat{f}_{\mathrm{P}'}(\kappa) \parallel^2$, using a Riemann sum over the same grid employed for generating the volatility curve (with $\ell = 10$), i.e., the interval $[0, 1]$ is partitioned into 100 equal subintervals. We perform feasible inference following the procedure outlined in Section 4. Here again we use the average limiting covariance matrix estimates over 1,000 trajectories rather than relying on a single trajectory to compute the eigenvalues associated with equation (22) and, consistent with the properties of the limiting variable, we retain only the terms featuring positive eigenvalues.

Table 1 reports empirical rejection rates for the test under the null hypothesis at significance levels 1%, 5% and 10%. The top row of the table shows that the test is well sized.

Under the alternative hypothesis, the data are generated with different underlying volatility curves. Hence, the universal rejections reported in the second row of Table 1 reflect high power of the test in detecting the discrepancy between the two functions governing the respective intraday volatility patterns.

Table 1. Test size and power. The null hypothesis is $H_0 : f_P = f_{\mathrm{P}'} = f_{05-10}$, and the alternative is $H_A : f_{\mathrm{P}} = f_{05-10}$ and $f_{\mathrm{P}'} = f_{15-20}$. The test statistic is $T \parallel \widehat{f}_{\mathrm{P}}(\kappa) - \widehat{f}_{\mathrm{P}'}(\kappa) \parallel^2$, whose realization is computed using a Riemann sum over the same grid points employed in generating the calendar effects ($\ell = 10$). The limiting distribution of the test statistic under the null is approximated as described in Section 4. The table reports rejection rates for the test at significance levels 1%, 5% and 10% using 1000 trials.

Significance level	1%	5%	10%
Size under H_0	0.009	0.059	0.116
Power under H_A	1.000	1.000	1.000

A.4 Finite-Sample Evidence for Pointwise Inference

Figure 2 illustrates the pointwise feasible central limit theorem of Corollary 7, where we depict the empirical distribution of the standardized $\widehat{f}(\kappa)$ and associated Normal Q-Q plot for different values of κ for $\ell = 10$, n =2,730, and T = 1,500. The data are generated from model (A.1). Note that $\widehat{f}(\kappa)$ is standardized according to Corollary 7 with $L_n = 7$. It is apparent that the limiting distribution approximates the empirical distribution well.

We next explore how different values for ℓ and T affect the performance of the calendar-effect estimator in finite samples. Without loss of generality, we fix $\kappa = 0.2$. Table 2 reports the finite-sample bias, standard deviation (StDev), and root mean squared error (RMSE)

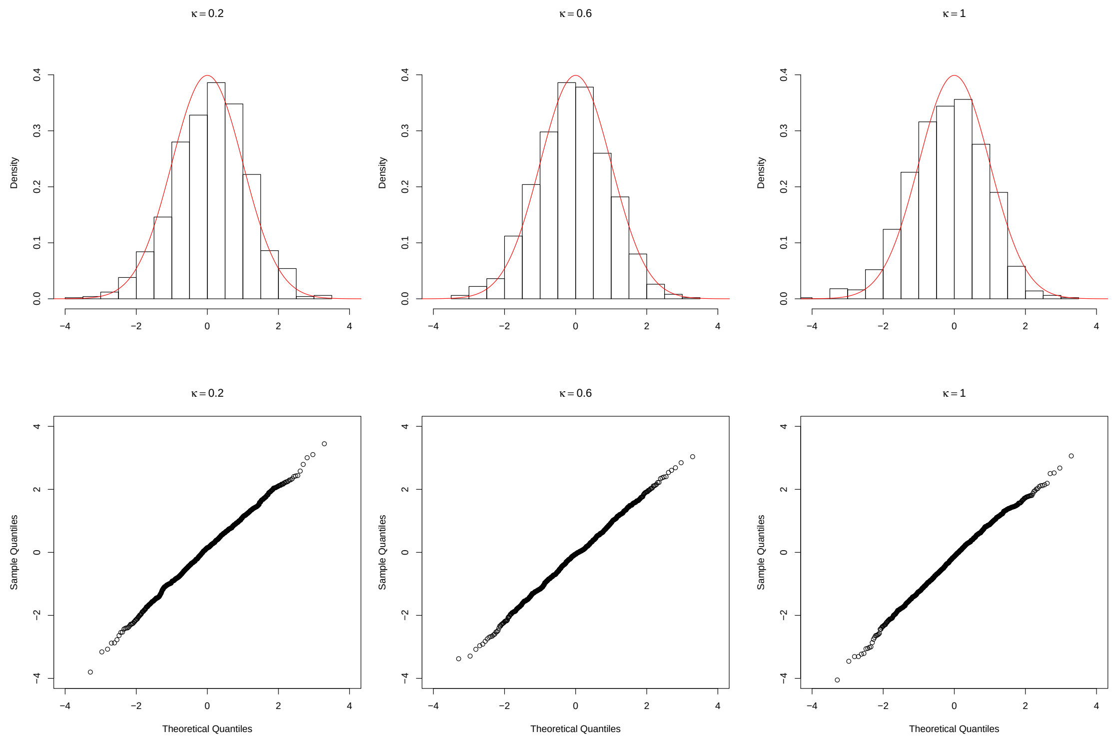

Figure 2. Empirical distribution of the standardized calendar-effect estimator. The empirical distribution of the standardized calendar-effect estimator $\widehat{f}(\kappa)$ and associated Normal Q-Q plots for different values of κ and $\ell = 10$ with data generated from model (A.1) by 1,000 simulation trials. $\widehat{f}(\kappa)$ is standardized according to Corollary 7 with $L_n = 7$.

of the estimator $\widehat{f}(\kappa)$ across different combinations of ℓ and T based on 1,000 replications. Two main conclusions emerge. First, when ℓ is fixed, larger T leads to a smaller standard deviation and root mean squared error. Note that the convergence rate of the calendar-effect estimator is $\sqrt{T}$. Second, for fixed T, the rows with ℓ ranging from 5 to 30 show that a large value of ℓ leads to a larger bias and smaller standard deviation. This finite-sample bias-variance tradeoff is also evident from the associated RMSE values. This is in line with our theoretical analysis in Section 5.

Table 2. Finite-sample performance of the calendar-effect estimator. Finite-sample statistics for the $\widehat{f}(\kappa)$ estimator with $\kappa = 0.2$. The data are generated from model (A.1) with $n = 2730$ over 1,000 trials. "Bias", "StDev" and "RMSE" refer to the bias, standard deviation and root mean squared error. The true value of $f(0.2)$ is 1.02.

ℓ	$T = 500$			$T = 1000$			$T = 1500$		
	Bias	StDev	RMSE	Bias	StDev	RMSE	Bias	StDev	RMSE
1	-0.0036	0.0873	0.0873	-0.0034	0.0630	0.0631	-0.0006	0.0497	0.0497
5	0.0022	0.0461	0.0461	0.0010	0.0325	0.0325	0.0016	0.0270	0.0270
10	0.0042	0.0368	0.0370	0.0023	0.0268	0.0268	0.0025	0.0222	0.0223
15	0.0047	0.0338	0.0341	0.0038	0.0247	0.0250	0.0042	0.0204	0.0208
20	0.0070	0.0316	0.0324	0.0060	0.0229	0.0236	0.0061	0.0189	0.0199
25	0.0084	0.0304	0.0315	0.0076	0.0221	0.0233	0.0078	0.0182	0.0198
30	0.0100	0.0300	0.0316	0.0093	0.0217	0.0236	0.0094	0.0179	0.0202

Appendix B Additional Theoretical Results

B.1 Accommodating rough volatility and infinite jump activity

This section provides an extension to our functional CLT for $\widehat{f}(\kappa)$ accommodating more general volatility and price jump settings. For the price process, we retain the setup of the main text, except that we replace the finite activity jump condition $F(\mathbb{R}) < \infty$ with,

$$\int_{\mathbb{R}} (|x|^r \vee |x|) F(dx) < \infty, \quad \text{for some } r \in [0, 1]. \tag{B.1}$$

We then define the jump component of the price process as follows,

$$X^J(t) \;:=\; X(t) - X^c(t) \;=\; \int_0^t \int_{\mathbb{R}} x\nu(ds, dx), \tag{B.2}$$

where X^c is the continuous part of the latent price process X, given by,

$$X^c(t) \;:=\; X(0) + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dW(s).$$

Condition (B.1) allows for $X^J(t)$ to be of infinite activity, with the parameter r controlling the concentration of small jumps.

Turning to the latent volatility process, we impose the following generic assumption,

$$E|\sigma^2(t) - \sigma^2(s)|^2 \leq C|t - s|^{2H}, \tag{B.3}$$

for any $s, t > 0$, some $0 < H \leq 1$, and a generic constant C. If $\sigma^2(t)$ is an Itô semimartingale, as stipulated in the main text, the above condition applies with $H = 1/2$. The assumption also allows for general volatility jump processes of infinite activity. When (B.3) holds with $H < 1/2$, our setup accommodates the so-called rough volatility models in which volatility is driven by fractional Brownian motion, see e.g., [4] and [12].

The next theorem presents the CLT for the calendar effect estimator under the above extended setup. It demonstrates explicitly how the jump activity index r and the volatility roughness index H affect the rate at which T, and hence ℓ, diverges.

Theorem 8. *Assume the same setup and assumptions as in Theorem 2, except for F being subject to condition* (B.1) *and $\sigma^2(t)$ being a rough process satisfying condition* (B.3). *Let $T \asymp n^b$ and $\ell \asymp n^c$ for some nonnegative exponents b and c subject to the conditions,*

$$0 < b < \min\{2 - 2\varpi, 1 + (1-r)\varpi, (4-2r)\varpi\} \quad \textit{and} \quad 1 - \varpi(4-r) < c < 1 - b/(2H), \tag{B.4}$$

where $0 < \varpi < 1/2$. *Then, as* $n \to \infty$,

$$\sqrt{T}\left(\widehat{f}(\kappa) - f(\kappa)\right) \xrightarrow{d} \mathcal{G}_{\mathcal{K}} \quad \textit{in} \quad \mathcal{L}^2,$$

where $\mathcal{G}_{\mathcal{K}}$ *is an* $\mathcal{L}^2$*-valued zero-mean Gaussian process with covariance operator* $\mathcal{K}$ *defined through the kernel* $C(\kappa, \kappa')$ *in equation* (8) *as follows,*

$$\mathcal{K}\, y(\kappa') \;=\; \int_{[0,1]} C(\kappa, \kappa')\, y(\kappa)\, d\kappa, \qquad \forall y \in \mathcal{L}^2 \,.$$

We close this section by investigating how the price jump activity index r and volatility roughness index H affect the bias-variance tradeoff for ℓ. The difference between Theorems 2 and 8 is that the feasible regions of b and c given by condition (9) is replaced with that given by condition (B.4). The ℓ-related terms II and III in Section 5 now have orders,

$$\text{term II} = O_P\left(\frac{1}{n^{(b+c)/2}}\right) \quad \text{and} \quad \text{term III} = O_P\left(\frac{1}{n^{(1-c)H}}\right).$$

We now provide the optimal choice of c, c_{opt}, for each configuration of ϖ, r, H and b, which minimizes the order of the sum of terms II and III. It is readily established that, for each configuration of ϖ, r and H, the feasible values of b are given by the interval $(0, b_U)$, where,

$$b_U := \min\{2 - 2\varpi, 1 + (1-r)\varpi, (4-2r)\varpi, 2H\};$$

and the feasible values of c consist of the interval (c_L, c_U), where.

$$c_L := \max\{1 - \varpi(4-r), 0\} \quad \text{and} \quad c_U := 1 - \frac{b}{2H}.$$

By similar arguments to those in Section C.8, we obtain the following exhaustive list of distinct cases with corresponding optimal c values,

$$c_{\text{opt}} = \begin{cases} \frac{2H}{2H+1} - \frac{b}{2H+1}, & \text{for } b \in (0, b_U) \text{ when } c_L = 0 \ , \\ \frac{2H}{2H+1} - \frac{b}{2H+1}, & \text{for } b \in (0, b_U) \text{ when } c_L \in (0, \frac{2H}{2H+1}) \text{ and } b_U \leq 2H - (2H+1)c_L, \\ \frac{2H}{2H+1} - \frac{b}{2H+1}, & \text{for } b \in (0, 2H - (2H+1)c_L] \text{ when } c_L \in (0, \frac{2H}{2H+1}) \\ & \hfill \text{and } b_U > 2H - (2H+1)c_L, \\ c_L+, & \text{for } b \in (2H - (2H+1)c_L, b_U) \text{ when } c_L \in (0, \frac{2H}{2H+1}) \\ & \hfill \text{and } 2H - (2H+1)c_L < b_U \leq 2H - 2Hc_L, \\ c_L+, & \text{for } b \in (2H - (2H+1)c_L, 2H - 2Hc_L] \text{ when } c_L \in (0, \frac{2H}{2H+1}) \\ & \hfill \text{and } b_U > 2H - 2Hc_L, \\ c_L+, & \text{for } b \in (0, 2H - 2Hc_L] \text{ when } c_L \geq \frac{2H}{2H+1} \text{ and } b_U > 2H - 2Hc_L, \\ c_L+, & \text{for } b \in (0, b_U) \text{ when } c_L \geq \frac{2H}{2H+1} \text{ and } b_U \leq 2H - 2Hc_L, \end{cases}$$

where c_L+ indicates a value of c as close to c_L as possible from above.

We note that both r and H affect the feasible choices for b and the optimal value of c. In particular, higher jump activity restricts the range of b. We note, however, that if r is close to 1 and the threshold parameter ϖ is taken very close to $1/2$, then a value of b slightly below 1 is feasible. Similarly, lower levels of $H < 1/2$, which correspond to rougher volatility paths, restrict the maximum possible value of b. This is intuitive as, for rougher volatility paths, the approximation error due to the discretization of the volatility path is higher.

B.2 Uniform confidence regions

This section provides a joint confidence region and band for the calendar effect function $f(\kappa)$. Confidence regions and bands for functional parameters are less studied than other core concepts and tools in the functional data analysis literature, because they, in general, are nontrivial to construct and visualize due to the infinite dimensional nature of the parameter. Nonetheless, under specific conditions on the covariance kernel of the limiting distribution, [2] develop and visualize confidence regions with the desired confidence level. We adapt their approach to our setting here.

Recall that $(\pi_i)_{i\geq 1}$ are the eigenvalues (in descending order) of the covariance operator $\mathcal{K}$ with kernel $C(\kappa,\kappa')$ in Theorem 2 and (8). Let $(\psi_i)_{i\geq 1}$ be the corresponding orthonormal eigenfunctions. We then have the following confidence region of hyper-ellipsoid form,

$$E_{\hat{f}} := \left\{ h \in \mathcal{L}^2 : \sum_{j=1}^{\infty} \frac{\langle \sqrt{T}(\hat{f}-h), \psi_j \rangle^2}{c_j^2} \leq \upsilon \right\},$$

where $(c_i)_{i\geq 1}$ are predefined weights depending on $(\pi_i)_{i\geq 1}$ and υ is a generic number. If one takes υ to be the $1-\alpha$ quantile of a weighted sum of chi-squared random variables $(\mathcal{X}_i^2)_{i\geq 1}$, i.e., $\sum_{j=1}^{\infty} \pi_j \mathcal{X}_j^2 / c_j^2$, it then follows immediately from Theorem 2 that

$$P(f \in E_{\hat{f}}) \longrightarrow 1-\alpha.$$

That is, the confidence region $E_{\hat{f}}$ has the desired asymptotic confidence level $1-\alpha$. To visualize the region $E_{\hat{f}}$, we propose the following symmetric confidence band due to [2],

$$B_{\hat{f}} := \left\{ h \in \mathcal{L}^2 : |h(\kappa) - \hat{f}(\kappa)| \leq r(\kappa), \text{ for } \kappa \in [0,1] \text{ almost everywhere} \right\}, \tag{B.5}$$

where,

$$r(\kappa) := \sqrt{\frac{\upsilon}{T}\sum_{j=1}^{\infty} c_j^2 \psi_j(\kappa)^2}. \tag{B.6}$$

The following proposition is a direct consequence of our Theorem 2 and Theorem 1 of [2].

Proposition 9. *Suppose all assumptions and conditions of Theorem 2 hold. If $\sum_{j=1}^{\infty} c_j^2 < \infty$ and $\sum_{j=1}^{\infty} \pi_j c_j^{-2} < \infty$, then $r(\kappa) \in \mathcal{L}^2$ and $E_{\hat{f}} \subset B_{\hat{f}}$. Therefore, $P(f \in B_{\hat{f}}) \geq 1-\alpha+o(1)$.*

This proposition shows that the simultaneous confidence band $B_{\hat{f}}$ has the desired level of coverage. However, these bands are infeasible, as $(\pi_j)_{j\geq 1}$ and $(\psi_j)_{j\geq 1}$ are unknown. We estimate them by the eigenvalues $(\widetilde{\pi}_j)_{j\geq 1}$ and orthonormal eigenfunctions $(\widetilde{\psi}_j)_{j\geq 1}$ of the integral operator $\widehat{\mathcal{K}}$ with kernel $\widehat{C}(\kappa, \kappa')$ defined in (19) and (18), respectively. Define $\mathcal{L}_J^2 := \operatorname{span}\{(\widetilde{\psi}_j)_{1\leq j\leq J}\} \subset \mathcal{L}^2$, where $J \leq T$. A feasible confidence region is then given by,

$$\widehat{E}_{\hat{f}} := \left\{ h \in \mathcal{L}_J^2 : \sum_{j=1}^{J} \frac{\langle \sqrt{T}(h-\hat{f}), \widetilde{\psi}_j\rangle^2}{c_j^2} \leq \upsilon \right\},$$

where one may take υ to be the $1-\alpha$ quantile of a weighted sum of chi-squared random variables with weights $(\widetilde{\pi}_j c_j^{-2})_{1\leq j\leq J}$. As argued by [2], because $\hat{f}$ lies outside the $\operatorname{span}\{\widetilde{\psi}_1, \ldots, \widetilde{\psi}_J\}$ almost surely, $\widehat{E}_{\hat{f}}$ has zero-coverage. However, we shall show below that $\widehat{E}_{\hat{f}}$ converges to $E_{\hat{f}}$, which has the desired asymptotic level of confidence, at a rate faster than $1/\sqrt{T}$ in Hausdroff distance. Note that the infeasible region $E_{\hat{f}}$ shrinks to a point at the rate $1/\sqrt{T}$, suggesting that $\widehat{E}_{\hat{f}}$ is a feasible proxy for $E_{\hat{f}}$.

We denote the confidence band associated with $\widehat{E}_{\hat{f}}$ as $\widehat{B}_{\hat{f}}$, which is defined analogously to (B.5) and (B.6), i.e.,

$$\widehat{B}_{\hat{f}} := \left\{ h \in \mathcal{L}_J^2 : |h(\kappa) - \hat{f}(\kappa)| \le r_J(\kappa), \text{ for } \kappa \in [0,1] \text{ almost everywhere} \right\}, \tag{B.7}$$

where

$$r_J(\kappa) := \sqrt{\frac{\upsilon}{T} \sum_{j=1}^{J} c_j^2 \widetilde{\psi}_j(\kappa)^2}. \tag{B.8}$$

We next present the convergence results about $\widehat{E}_{\hat{f}}$. To this end, we need some additional notation and an assumption. We denote the Hausdroff distance $d_H(S_1, S_2)$ between two subsets S_1 and S_2 of $\mathcal{L}^2$ as

$$d_H(S_1, S_2) := \max\{\Psi(S_1, S_2), \Psi(S_2, S_1)\}, \text{ where } \Psi(S_1, S_2) = \sup_{x \in S_1} \inf_{y \in S_2} \parallel x - y \parallel .$$

Moreover, define

$$\tilde{q} := \Big(\min\{c/2, (1-c)/4, 2\varpi - 3(1-c)/4, b/2, 2\varpi - 7/8 + 7(b+c)/8\} - \varrho \Big)^2 \wedge (9\varrho^2).$$

Under condition (20) of Theorem 6, it follows directly from Lemma 17 in Section C.6 that,

$$\parallel \widehat{\mathcal{K}} - \mathcal{K} \parallel_{\mathrm{HS}}^2 = O_P(n^{-\tilde{q}}), \tag{B.9}$$

where $\tilde{q} > 0$. The following explicit assumption (cf. Assumption 3 of [2]) on the eigenvalues π_j and weights c_j^2 is needed.

Assumption UB. *There exist constants $\epsilon > 1$ and $\vartheta > 0$ such that,*

$$\pi_j \asymp \frac{1}{j^{\epsilon}}, \quad \pi_j - \pi_{j+1} \asymp \frac{1}{j^{\epsilon+1}}, \quad \text{and} \quad c_j^2 \asymp \frac{1}{j^{2\vartheta}},$$

where $2\vartheta < \epsilon - 1$.

The first condition above controls the rate at which the eigenvalues π_j shrink to zero. The second one controls the gaps between the eigenvalues, $\pi_j - \pi_{j+1}$, which in turn affect the accuracy of eigenfunction estimation. The last condition controls the rate at which c_j^2 shrinks to zero, ensuring the convergence of $\sum_{i=j}^{\infty} \pi_j c_j^{-2}$ and the compactness of the confidence region. Similar assumption is imposed by [10].

We may now provide the convergence results for $\widehat{E}_{\hat{f}}$. The following proposition is a straightforward consequence of (B.9) and Theorem 4 and Corollary 1 of [2]. It shows that the distance between the feasible region $\widehat{E}_{\hat{f}}$ and the infeasible region $E_{\hat{f}}$ converges to zero at a rate faster than $1/\sqrt{T}$ under the conditions of our Theorem 6 and Assumption UB.

Proposition 10. *Suppose that all the assumptions and conditions of Theorem 6 hold. If Assumption UB and $J \asymp T^{\tilde{q}/(2\epsilon+2\vartheta+3)}$ hold, then*

$$d_H(\widehat{E}_{\hat{f}}, E_{\hat{f}})^2 = O_P\left(T^{-\left(2 - \frac{2\epsilon+3+2\vartheta(1-\tilde{q})}{2\epsilon+3+2\vartheta}\right)}\right).$$

We implement the feasible simultaneous band $\widehat{B}_{\hat{f}}$ defined in (B.7) and (B.8) with $c_j^2 = (\sum_{k=j}^{J} \widetilde{\pi}_k)^{1/2}$ in our numerical studies. This version of the confidence band is denoted $\widehat{B}_{E_c}$. It is in accordance with the ideal (infeasible) choice of $c_j^2 = (\sum_{k=j}^{\infty} \pi_k)^{1/2}$, which ensures that $\sum_{j=1}^{\infty} \pi_j c_j^{-2} < \infty$. In our implementation, we compute eigenvalues and eigenfunctions of $\widehat{C}(\kappa, \kappa')$ using the same matrix scheme as that adopted in Section 3.2 of [1].

Using the setup in Section A.1, we investigate the empirical coverage rate of the proposed uniform confidence band $\widehat{B}_{E_c}$ via simulations. The performance of $\widehat{B}_{E_c}$ is quite robust for $50 \le J \le 100$, so we provide results for $J = 50$ only. Based on 1000 trials, the simultaneous coverage rate of $\widehat{B}_{E_c}$ with 95% nominal level is 100%. By contrast, the confidence bounds, constructed using the pointwise theory in Corollary 7 with a 95% nominal level, only have a simultaneous coverage rate of 2%. The left panel of Figure 3 displays pointwise coverage rates for both the uniform confidence band and the pointwise confidence bounds. We also provide the calendar effect function estimate along with 95% confidence bounds for a particular simulation trial in the right panel of Figure 3. It illustrates the moderately wider width of the uniform confidence band $\widehat{B}_{E_c}$ relative to the pointwise bounds.

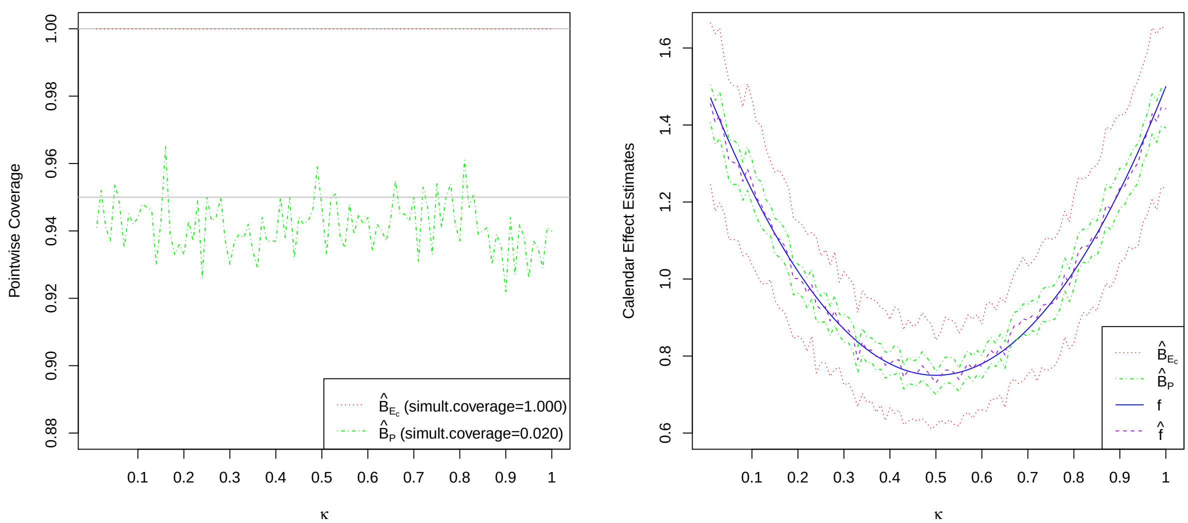

Figure 3. Comparison between $\widehat{B}_P$ and $\widehat{B}_{E_c}$ based on 1000 simulation trials. The results rely on data generated from the setup in Section A.1. $\widehat{B}_P$ indicates the pointwise 95% confidence intervals constructed using Corollary 7 with $L_n = 7$. $\widehat{B}_{E_c}$ refers to the simultaneous 95% confidence bands constructed using the theory in Section B.2. **Left panel**: Local empirical coverage rates. As indicated in the legend, the *simultaneous coverage rates* for $\widehat{B}_P$ and $\widehat{B}_{E_c}$ are 2% and 100%, respectively. **Right panel**: Calendar effect estimates together with 95% confidence bounds for a particular sample path.

An application of $\widehat{B}_{E_c}$ to the e-mini over the subsamples is presented in Figure 4. Not surprisingly, the simultaneous confidence bounds are wider than the pointwise ones. In practice, $\widehat{B}_{E_c}$ may occasionally produce negative lower bounds for real data, e.g., the subsamples covering periods 2005–2010 and 2015–2020, especially the period 2015–2020. If this is the case, we suggest using a log transformation and functional Delta method to ensure positiveness. To be precise, under mild conditions, by Theorem 2 and the functional Delta method, one would obtain

$$\sqrt{T}\left(\log\left(\widehat{f}(\kappa)\right) - \log\left(f(\kappa)\right)\right) \xrightarrow{d} \mathcal{G}_{\mathcal{K}_{\log}} \quad \text{in } \mathcal{L}^2,$$

where $\mathcal{G}_{\mathcal{K}_{\log}}$ is an $\mathcal{L}^2$-valued zero-mean Gaussian process with covariance operator $\mathcal{K}_{\log}$ that has $C(\kappa,\kappa')/(f(\kappa)f(\kappa'))$ as its kernel function. Therefore, the previous method of constructing uniform confidence bounds of $f(\kappa)$ applies straightforwardly for constructing that of $\log(f(\kappa))$. The simultaneous confidence bounds of $f(\kappa)$ then readily follows by applying the natural exponential function to both the upper and lower confidence bounds of $\log(f(\kappa))$. Figure 4 also displays simultaneous confidence bounds thus obtained for the three subsamples.

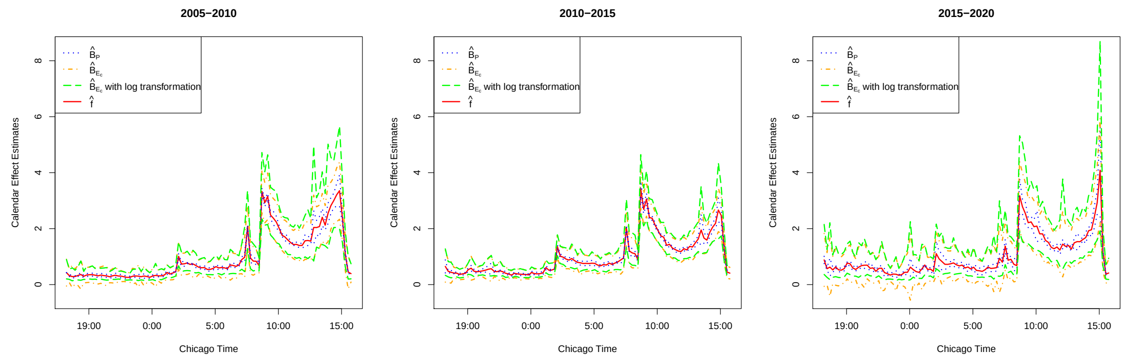

Figure 4. Intraday volatility curves for the e-mini over subsamples. $\widehat{B}_P$ indicates the pointwise 95% confidence intervals constructed using Corollary 7 with $L_n = 7$. $\widehat{B}_{E_c}$ indicates the simultaneous 95% confidence bands constructed as in Section B.2. $\widehat{B}_{E_c}$ with log transformation indicates the simultaneous confidence bands constructed based on log transformation which ensures positiveness of the lower confidence bounds.

Appendix C Proofs

Throughout this section, without further mention, we shall focus on $\kappa \in [\ell\Delta, 1]$ in the derivations of upper bounds for moments of various terms involving $\widehat{\sigma}^2_{i,\kappa}$. The same results and proofs as that for $\kappa \in [\ell\Delta, 1]$ obviously apply to the case $\kappa \in [0, \ell\Delta)$.

Recall that X^c is the continuous part of the latent price process X, defined as,

$$X^c(t) := X(0) + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dW(s).$$

In the proofs below, we rely on the calendar-effect estimator $\widehat{f}^c(\kappa)$ for X^c given by,

$$\widehat{f}^c(\kappa) := \frac{1}{T}\sum_{i=1}^{T} \widehat{\sigma}^{2,c}_{i,\kappa} \Big/ \widehat{\eta}^c \ ,$$

where,

$$\widehat{\sigma}^{2,c}_{i,\kappa} := \frac{1}{\ell\Delta} \sum_{k=j_\kappa-\ell+1}^{j_\kappa} \left(\Delta^n_{i,k} X^c\right)^2 \quad \text{and} \quad \widehat{\eta}^c := \frac{1}{T} \sum_{i=1}^{T} \sum_{j=1}^{n} \left(\Delta^n_{i,j} X^c\right)^2.$$

Throughout the proofs, C denotes a generic positive constant and $\varsigma > 0$ is an arbitrarily small number. Both may change value from line to line.

Furthermore, we will use the following notation throughout the proofs below,

$$\begin{cases} \zeta_1(\kappa) := \frac{1}{T\ell\Delta} \sum_{i=1}^{T} \sum_{k=j_\kappa-\ell+1}^{j_\kappa} \left[\int_{t_{i,k-1}}^{t_{i,k}} \mu(t)dt\right]^2, \\ \zeta_2(\kappa) := \frac{2}{T\ell\Delta} \sum_{i=1}^{T} \sum_{k=j_\kappa-\ell+1}^{j_\kappa} \int_{t_{i,k-1}}^{t_{i,k}} \mu(t)dt \int_{t_{i,k-1}}^{t_{i,k}} \sigma(t)dW(t), \\ \zeta_3(\kappa) := \frac{1}{T\ell\Delta} \sum_{i=1}^{T} \sum_{k=j_\kappa-\ell+1}^{j_\kappa} \left\{\left[\int_{t_{i,k-1}}^{t_{i,k}} \sigma(t)dW(t)\right]^2 - \int_{t_{i,k-1}}^{t_{i,k}} \sigma^2(t)dt\right\}, \\ \zeta_4(\kappa) := \frac{1}{T\ell\Delta} \sum_{i=1}^{T} \sum_{k=j_\kappa-\ell+1}^{j_\kappa} \int_{t_{i,k-1}}^{t_{i,k}} \left[\sigma^2(t) - \sigma^2(i-1+\kappa)\right] dt, \\ \zeta_5(\kappa) := \frac{1}{T} \sum_{i=1}^{T} \sigma^2(i-1+\kappa) - \frac{f(\kappa)}{T} \sum_{i=1}^{T} \int_{i-1}^{i} \sigma^2(t)dt, \quad \text{and} \\ \zeta_6(\kappa) := f(\kappa) \left[\frac{1}{T} \sum_{i=1}^{T} \int_{i-1}^{i} \sigma^2(t)dt - \widehat{\eta}^c\right], \end{cases} \tag{C.1}$$

where $\kappa \in [0,1]$ and note that, by the definition of $\widehat{\sigma}^2_{1,\kappa}$ for $\kappa \in [0, \ell\Delta)$, the inner summation variable k of the first four terms always takes values from 1 to ℓ when $\kappa \in [0, \ell\Delta)$ and the outer summation variable $i = 1$ (i.e., j_κ is fixed at ℓ in this case). Recalling the definition of $A_i(\kappa)$ in equation (7), one readily sees that $\zeta_5(\kappa) = \sum_{i=1}^{T} A_i(\kappa)/T$.

The following lemma will be repeatedly used in the proofs of Theorems 1 and 2.

Lemma 11. *Suppose that Assumption I(ii) holds. Then,*

$$E\left|\zeta_1(\kappa)\right|^m \leq \frac{C}{n^m}, \quad E\left|\zeta_2(\kappa)\right|^m \leq \frac{C}{n^{m/2}} \quad \textit{and} \quad E\left|\zeta_3(\kappa)\right|^m \leq \frac{C}{(T\ell)^{m/2}}$$

for any $m \geq 2$ and any $\kappa \in [0,1]$.

Proof of Lemma 11. For term $\zeta_1(\kappa)$, we have,

$$
\begin{aligned}
E\left|\zeta_1(\kappa)\right|^m &\leq E\left[\frac{1}{T\ell\Delta^m}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\left(\int_{t_{i,k-1}}^{t_{i,k}}\mu(t)dt\right)^{2m}\right] \\
&\leq \frac{\Delta^{m-1}}{T\ell}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,k-1}}^{t_{i,k}}E\left(\mu(t)\right)^{2m}dt \leq \frac{C}{n^m},
\end{aligned}
$$

where the first two inequalities follow from Jensen's inequality, and the last inequality is implied by Assumption I(ii).

For the term $\zeta_2(\kappa)$, by Cauchy-Schwarz inequality, Jensen's inequality, Itô isometry and Assumption I(ii), we obtain,

$$
\begin{aligned}
E\left|\zeta_2(\kappa)\right|^m &\leq \frac{1}{T\ell\Delta^m}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}E\left|\int_{t_{i,k-1}}^{t_{i,k}}\mu(t)dt\int_{t_{i,k-1}}^{t_{i,k}}\sigma(t)dW(t)\right|^m \\
&\leq \frac{1}{T\ell\Delta^m}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\left[E\left|\int_{t_{i,k-1}}^{t_{i,k}}\mu(t)dt\right|^{2m}E\left|\int_{t_{i,k-1}}^{t_{i,k}}\sigma(t)dW(t)\right|^{2m}\right]^{1/2} \\
&\leq \frac{\Delta^{m/2-1}}{T\ell}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\left[\int_{t_{i,k-1}}^{t_{i,k}}E\left|\mu(t)\right|^{2m}dt\int_{t_{i,k-1}}^{t_{i,k}}E\left(\sigma(t)\right)^{2m}dt\right]^{1/2} \leq \frac{C}{n^{m/2}}.
\end{aligned}
$$

We now deal with term $\zeta_3(\kappa)$. We first define the following continuous martingale,

$$
M_1(t) := \frac{1}{T\ell\Delta}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\left\{\left[\int_{t\wedge t_{i,k-1}}^{t\wedge t_{i,k}}\sigma(s)dW(s)\right]^2-\int_{t\wedge t_{i,k-1}}^{t\wedge t_{i,k}}\sigma^2(s)ds\right\}
$$

on the interval $[0,T]$. One readily sees $\zeta_3(\kappa)=M_1(T)$. The quadratic variation of $M_1(t)$ is takes the form,

$$[M_1, M_1](t) = \frac{4}{(T\ell\Delta)^2} \sum_{i=1}^{T} \sum_{k=j_\kappa-\ell+1}^{j_\kappa} \int_{t\wedge t_{i,k-1}}^{t\wedge t_{i,k}} \left(\sigma(u) \int_{t_{i,k-1}}^{u} \sigma(s) dW(s) \right)^2 du,$$

following the method in Section 2.3.3 on page 136 of [9]. Then we obtain,

$$\begin{aligned}
E\left|\zeta_3(\kappa)\right|^m &\leq CE\left[\frac{4}{(T\ell\Delta)^2} \sum_{i=1}^{T} \sum_{k=j_\kappa-\ell+1}^{j_\kappa} \int_{t_{i,k-1}}^{t_{i,k}} \left(\sigma(t) \int_{t_{i,k-1}}^{t} \sigma(s) dW(s) \right)^2 dt \right]^{m/2} \\
&\leq \frac{C}{(T\ell)^{m/2+1}\Delta^m} \sum_{i=1}^{T} \sum_{k=j_\kappa-\ell+1}^{j_\kappa} E\left[\int_{t_{i,k-1}}^{t_{i,k}} \left(\sigma(t) \int_{t_{i,k-1}}^{t} \sigma(s) dW(s) \right)^2 dt \right]^{m/2} \\
&\leq \frac{C}{(T\ell\Delta)^{m/2+1}} \sum_{i=1}^{T} \sum_{k=j_\kappa-\ell+1}^{j_\kappa} \int_{t_{i,k-1}}^{t_{i,k}} E\left| \sigma(t) \int_{t_{i,k-1}}^{t} \sigma(s) dW(s) \right|^m dt \\
&\leq \frac{C}{(T\ell\Delta)^{m/2+1}} \sum_{i=1}^{T} \sum_{k=j_\kappa-\ell+1}^{j_\kappa} \int_{t_{i,k-1}}^{t_{i,k}} \left[E\left(\int_{t_{i,k-1}}^{t} \sigma^2(s) ds \right)^m E\left(\sigma^{2m}(t)\right) \right]^{1/2} dt \\
&\leq \frac{C}{(T\ell\Delta)^{m/2+1}} \sum_{i=1}^{T} \sum_{k=j_\kappa-\ell+1}^{j_\kappa} \int_{t_{i,k-1}}^{t_{i,k}} \left[\Delta^{m-1} \int_{t_{i,k-1}}^{t} E\left(\sigma^{2m}(s)\right) ds \right]^{1/2} dt \leq \frac{C}{(T\ell)^{m/2}},
\end{aligned}$$

where the second and third inequalities follow from Jensen's inequality, the fourth inequality follows from the Cauchy-Schwarz inequality and the Burkholder-Davis-Gundy inequality, and the last inequality follows from Jensen's inequality and Assumption I(ii). □

In what follows, we use the shorthand notation

$$N(t) := \int_0^t \int_{\mathbb{R}} \nu(ds, dx) \quad \text{and} \quad \Delta_{i,j}^n N := \int_{t_{i,j-1}}^{t_{i,j}} \int_{\mathbb{R}} \nu(dt, dx).$$

The following two lemmas are repeatedly used in the proofs of Theorems 1, 2 and 6.

Lemma 12. *Let j be a positive integer. Suppose that $i_m \in \{1,2,\ldots,T\}$ and $k_m \in \{1,2,\ldots,n\}$ for $m \in \{1,2,\ldots,j\}$, and without loss of generality that $0 < t_{i_1,k_1} < t_{i_2,k_2} < \cdots < t_{i_j,k_j} \leq T$. Then, under Assumption I(ii), we have that $E\left(1_{\{\Delta^n_{i_1,k_1}N>0\}}\right) \leq C\Delta$ and,*

$$E\left(1_{\{\Delta^n_{i_1,k_1}N>0\}}1_{\{\Delta^n_{i_2,k_2}N>0\}}\cdots 1_{\{\Delta^n_{i_j,k_j}N>0\}}\right) \leq C\Delta^{j-\varsigma},$$

for $j \geq 2$ and arbitrarily small $\varsigma > 0$.

Proof of Lemma 12. First note that $N(t) = \int_0^t \int_{\mathbb{R}} \nu(dt,dx)$ is a counting process with intensity $\chi(t)F(\mathbb{R})$. Then $N(t) - \int_0^t \chi(s)dsF(\mathbb{R})$ is a martingale by Assumption I(ii).

When $j=1$, the result follows immediately from,

$$E\left(1_{\{\Delta^n_{i_1,k_1}N>0\}}\right) \leq E\left(\Delta^n_{i_1,k_1}N\right) = F(\mathbb{R})E\int_{t_{i_1,k_1-1}}^{t_{i_1,k_1}} \chi(s)ds \leq C\Delta.$$

When $j=2$, we have that, for any $\omega > 1$,

$$\begin{aligned}
&E\left(1_{\{\Delta^n_{i_1,k_1}N>0\}}1_{\{\Delta^n_{i_2,k_2}N>0\}}\right) = E\left[1_{\{\Delta^n_{i_1,k_1}N>0\}}E_{t_{i_2,k_2-1}}\left(1_{\{\Delta^n_{i_2,k_2}N>0\}}\right)\right] \\
&\leq CE\left[1_{\{\Delta^n_{i_1,k_1}N>0\}}E_{t_{i_2,k_2-1}}\left(\int_{t_{i_2,k_2-1}}^{t_{i_2,k_2}} \chi(s)ds\right)\right] \\
&= CE\left\{1_{\{\Delta^n_{i_1,k_1}N>0\}}\int_{t_{i_2,k_2-1}}^{t_{i_2,k_2}} \chi(s)ds\right\} \\
&\leq C\left[E\left(\int_{t_{i_2,k_2-1}}^{t_{i_2,k_2}} \chi(s)ds\right)^{\omega}\right]^{1/\omega}\left[E\left(1_{\{\Delta^n_{i_1,k_1}N>0\}}\right)\right]^{1-1/\omega} \\
&\leq C\left[\Delta^{\omega-1}\int_{t_{i_2,k_2-1}}^{t_{i_2,k_2}} E\chi(s)^{\omega}ds\right]^{1/\omega}\Delta^{1-1/\omega} \leq C\Delta^{2-1/\omega},
\end{aligned}$$

where the second inequality follows from Hölder's inequality, the third inequality follows from Jensen's inequality, and the last inequality follows from Assumption I(ii). Therefore, we obtain,

$$E\left(1_{\{\Delta^n_{i_1,k_1}N>0\}}1_{\{\Delta^n_{i_2,k_2}N>0\}}\right)\leq C\Delta^{2-\varsigma},$$

for arbitrarily small $\varsigma>0$.

By induction, the lemma holds for any positive integer $j>2$. □

Lemma 13. *Suppose that Assumption I(ii) holds. Then we have*

$$E\left|\frac{1}{T}\sum_{i=1}^{T}\left(\widehat{\sigma}^2_{i,\kappa}-\widehat{\sigma}^{2,c}_{i,\kappa}\right)\right|\leq C\Delta^{2\varpi}\quad \textit{and}\quad E\left|\widehat{\eta}-\widehat{\eta}^c\right|\leq C\Delta^{2\varpi}.$$

Proof of Lemma 13. First, for any $\omega>2$, we rewrite and calculate the stochastic order of $\sum_{i=1}^T\left(\widehat{\sigma}^2_{i,\kappa}-\widehat{\sigma}^{2,c}_{i,\kappa}\right)/T$ as follows,

$$\begin{aligned}
&E\left|\frac{1}{T}\sum_{i=1}^{T}\left(\widehat{\sigma}^2_{i,\kappa}-\widehat{\sigma}^{2,c}_{i,\kappa}\right)\right|\\
&\leq\frac{1}{T\ell\Delta}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\left[E\left|\left(\Delta^n_{i,k}X^c\right)^2 1_{\left\{\left|\Delta^n_{i,k}X^c\right|>u_n\right\}}\right|\right.\\
&\quad\left.+E\left|\left(\Delta^n_{i,k}X\right)^2 1_{\left\{\left|\Delta^n_{i,k}X\right|\leq u_n\right\}}1_{\left\{\Delta^n_{i,k}N>0\right\}}\right|+E\left|\left(\Delta^n_{i,k}X^c\right)^2 1_{\left\{\Delta^n_{i,k}N>0\right\}}\right|\right]\\
&\leq\frac{1}{T\ell\Delta}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\left[\left(E\left|\Delta^n_{i,k}X^c\right|^\omega\right)^{2/\omega}\left(E\left|1_{\left\{\left|\Delta^n_{i,k}X^c\right|>u_n\right\}}\right|\right)^{1-2/\omega}\right.\\
&\quad\left.+u_n^2P\left(\Delta^n_{i,k}N>0\right)+\left(E\left|\Delta^n_{i,k}X^c\right|^\omega\right)^{2/\omega}\left(P\left(\Delta^n_{i,k}N>0\right)\right)^{1-2/\omega}\right]\\
&\leq C\left(\Delta^{(\omega-2)(1/2-\varpi)}\vee\Delta^{2\varpi}\vee\Delta^{1-2/\omega}\right),
\end{aligned}\tag{C.2}$$

where the second inequality follows from Hölder's inequality and the last inequality follows from Markov inequality, Burkholder-Davis-Gundy inequality and Lemma 12. Note that $\sup_{t\in\mathbb{R}_+} E\left(e^{|\mu(t)|}\right) + \sup_{t\in\mathbb{R}_+} E\left(e^{|\sigma(t)|}\right) < \infty$ in Assumption I(ii) implies boundedness of moments of all orders for $|\mu(t)|$ and $\sigma(t)$. This in turn allows one to apply Hölder's inequality with arbitrarily large $\omega > 0$ in obtaining (C.2), where the constant C arises from the upper bound of higher order moments of $|\mu(t)|$ and $|\sigma(t)|$ and the applications of Burkholder-Davis-Gundy inequality, the elementary inequality $e^x \geq 1 + x + x^2/2 + \ldots + x^m/m!$ for $x \geq 0$ and any integer $m \geq 1$, and Lemma 12. Therefore, one can always choose a large enough ω such that,

$$E\left|\frac{1}{T}\sum_{i=1}^{T}\left(\widehat{\sigma}^2_{i,\kappa} - \widehat{\sigma}^{2,c}_{i,\kappa}\right)\right| \leq C\Delta^{2\varpi}.$$

Second, by substituting n for ℓ in the arguments for deriving the result in (C.2), we obtain,

$$E\left|\widehat{\eta} - \widehat{\eta}^c\right| \leq C\Delta^{2\varpi},$$

completing the proof. $\square$

C.1 Proof of Theorem 1

By triangle inequality, we have,

$$\| \widehat{f}(\kappa) - f(\kappa) \| \leq \| \widehat{f}^c(\kappa) - f(\kappa) \| + \| \widehat{f}(\kappa) - \widehat{f}^c(\kappa) \| .$$

We divide the proof into two steps. We prove $\| \widehat{f}^c(\kappa) - f(\kappa) \| \xrightarrow{P} 0$ in the first step and $\| \widehat{f}(\kappa) - \widehat{f}^c(\kappa) \| \xrightarrow{P} 0$ in the second.

Step 1. By using the notation in (C.1) and triangle inequality, we can rewrite the estimation error of $\widehat{f}^c(\kappa)$, which is built on the continuous part of X, as follows,

$$\| \widehat{f}^c(\kappa) - f(\kappa) \| = \frac{1}{\widehat{\eta}^c} \| \frac{1}{T} \sum_{i=1}^{T} \widehat{\sigma}_{i,\kappa}^{2,c} - f(\kappa)\widehat{\eta}^c \| \leq \frac{1}{\widehat{\eta}^c} \sum_{i=1}^{6} \| \zeta_i(\kappa) \| . \tag{C.3}$$

We first show that $\widehat{\eta}^c$ converges in probability to η. The difference between $\widehat{\eta}^c$ and η is decomposed as follows,

$$\widehat{\eta}^c - \eta = \left(\frac{1}{T} \sum_{i=1}^{T} \sum_{j=1}^{n} \left(\Delta_{i,j}^n X^c\right)^2 - \frac{1}{T} \sum_{i=1}^{T} \int_{i-1}^{i} \sigma^2(t)dt \right) + \left(\frac{1}{T} \sum_{i=1}^{T} \int_{i-1}^{i} \sigma^2(t)dt - \eta \right). \tag{C.4}$$

It follows easily from Itô's lemma and the integrability assumption for μ and σ that the first term on the right hand side of (C.4) tends to zero in probability as $n, T \to \infty$. We next deal with the second term on the right hand side of (C.4). Let E_i be conditional expectation with respect to the sigma field $\mathscr{G}_i$ (see Assumption II for definition). Then for any $\omega > 2(1+\iota)/\iota$ where ι is given in Assumption II, we have that,

$$\begin{aligned}
& E\left(\frac{1}{T} \sum_{i=1}^{T} \int_{i-1}^{i} \sigma^2(t)dt - \eta \right)^2 \\
&= \frac{2}{T^2} \sum_{i=1}^{T} \sum_{j=i+1}^{T} E\left\{ \left(\int_{i-1}^{i} \sigma^2(t)dt - \eta \right) E_i \left(\int_{j-1}^{j} \sigma^2(t)dt - \eta \right) \right\} + \frac{1}{T^2} \sum_{i=1}^{T} E\left(\int_{i-1}^{i} \sigma^2(t)dt - \eta \right)^2 \\
&= \frac{2}{T^2} \sum_{i=1}^{T} \sum_{j=i+1}^{T} \left(E\left| \int_{i-1}^{i} \sigma^2(t)dt - \eta \right|^{\omega} \right)^{1/\omega} \left(E\left| E_i \left(\int_{j-1}^{j} \sigma^2(t)dt - \eta \right) \right|^{\omega/(\omega-1)} \right)^{1-1/\omega} + \frac{C}{T} \\
&\leq \frac{C}{T^2} \sum_{i=1}^{T} \sum_{j=i+1}^{T} \alpha_{j-i}^{1-2/\omega} + \frac{C}{T} \leq \frac{C}{T},
\end{aligned}$$

where the first inequality follows from Assumptions I(ii) and II with $q = 1$, Hölder's inequality and Lemma 3.102 on page 497 of [7]. Hence, we have that the second term on the right hand side of (C.4) goes to zero in probability. Therefore, $\widehat{\eta}^c \xrightarrow{P} \eta$.

It remains to show that,

$$\sum_{i=1}^{6} \parallel \zeta_i(\kappa) \parallel \xrightarrow{P} 0.$$

To this end, we treat terms $\zeta_i(\kappa)$, $i = 1, 2, ..., 6$, one by one. For terms $\zeta_1(\kappa)$, $\zeta_2(\kappa)$ and $\zeta_3(\kappa)$, applying Lemma 11 with $m = 2$ and Jensen's inequality, we obtain,

$$E \parallel \zeta_i(\kappa) \parallel \leq \left(\int_{[0,1]} E \left| \zeta_i(\kappa) \right|^2 d\kappa \right)^{1/2} \leq C \left(\frac{1}{\sqrt{n}} \vee \frac{1}{\sqrt{T\ell}} \right),$$

for $i = 1, 2, 3$. Thus,

$$\sum_{i=1}^{3} \parallel \zeta_i(\kappa) \parallel = O_P \left(\frac{1}{\sqrt{n}} \vee \frac{1}{\sqrt{T\ell}} \right). \tag{C.5}$$

Next, we deal with the term $\zeta_4(\kappa)$. By Proposition II.1.28 and Theorem II.1.33 on pages 72-73 of [7], $\int_{\mathbb{R}} |x|^2 \tilde{F}(dx) < \infty$ and $\sup_{t \in \mathbb{R}_+} E \left| \tilde{\chi}(t) \right|^2 < \infty$ in Assumption I(ii), we have that,

$$\int_0^t \int_{\mathbb{R}} x \tilde{\nu}(ds, dx) - \int_0^t \int_{\mathbb{R}} x \tilde{\chi}(s) ds \tilde{F}(dx)$$

is a locally square integrable martingale. Then, applying Theorem I.3.17 and Proposition II.1.28 on pages 32 and 72 of [7], $\int_{\mathbb{R}} |x|^2 \tilde{F}(dx) < \infty$ and $\sup_{t \in \mathbb{R}_+} E \left| \tilde{\chi}(t) \right|^2 < \infty$ in Assumption I(ii), we obtain,

$$E \left(\int_0^t \int_{\mathbb{R}} x^2 \tilde{\nu}(ds, dx) \right) = E \left(\int_0^t \int_{\mathbb{R}} x^2 \tilde{\chi}(s) ds \tilde{F}(dx) \right).$$

Using the above results, we can bound the first moment of $\parallel \zeta_4(\kappa) \parallel$ as follows,

$$
\begin{aligned}
&E \parallel \zeta_4(\kappa) \parallel \leq \left(\int_{[0,1]} E \left| \zeta_4(\kappa) \right|^2 d\kappa \right)^{1/2} \\
&\leq \left(\int_{[0,1]} \frac{1}{T\ell\Delta} \sum_{i=1}^{T} \int_{t_{i,j_\kappa - \ell}}^{t_{i,j_\kappa}} C \left[E \left| \int_t^{i-1+\kappa} \tilde{\mu}(s) ds \right|^2 + E \left| \int_t^{i-1+\kappa} \check{\sigma}(s) dW(s) \right|^2 \right. \right. \\
&\quad + E \left| \int_t^{i-1+\kappa} \tilde{\sigma}(s) d\widetilde{W}(s) \right|^2 + E \left| \int_t^{i-1+\kappa} \int_{\mathbb{R}} x \tilde{\nu}(ds, dx) - \int_t^{i-1+\kappa} \int_{\mathbb{R}} x \tilde{\chi}(s) ds \tilde{F}(dx) \right|^2 \\
&\quad \left. \left. + E \left| \int_t^{i-1+\kappa} \int_{\mathbb{R}} x \tilde{\chi}(s) ds \tilde{F}(dx) \right|^2 \right] dt d\kappa \right)^{1/2} \\
&\leq C \left(\int_{[0,1]} \frac{1}{T\ell\Delta} \sum_{i=1}^{T} \int_{t_{i,j_\kappa - \ell}}^{t_{i,j_\kappa}} \left[(\ell+1)\Delta \int_t^{i-1+\kappa} E \left| \tilde{\mu}(s) \right|^2 ds + \int_t^{i-1+\kappa} E \left(\check{\sigma}^2(s) + \tilde{\sigma}^2(s) \right) ds \right. \right. \\
&\quad \left. \left. + E \left(\int_t^{i-1+\kappa} \int_{\mathbb{R}} x^2 \tilde{\chi}(s) ds \tilde{F}(dx) \right) + (\ell+1)\Delta \int_t^{i-1+\kappa} E \left(\tilde{\chi}(t) \right)^2 dt \right] dt d\kappa \right)^{1/2} \\
&\leq C \sqrt{\frac{\ell}{n}},
\end{aligned}
$$

where the first inequality follows from Jensen's inequality, the third inequality follows from the Burkholder-Davis-Gundy inequality, and the last inequality follows from Assumption I(ii). Thus,

$$
\parallel \zeta_4(\kappa) \parallel = O_P \left(\sqrt{\frac{\ell}{n}} \right).
$$

Turning next to term $\zeta_5(\kappa)$, by (7), we can rewrite this term as,

$$
\zeta_5(\kappa) = \frac{1}{T} \sum_{i=1}^{T} A_i(\kappa).
$$

It follows easily from Assumptions I(ii), II with $q = 1$ and Corollary 14.3 on page 212 of [3] that,

$$E\left|\zeta_5(\kappa)\right|^2 \leq \frac{1}{T}\sum_{h=-\infty}^{\infty}\left|\phi_{\kappa,\kappa}(h)\right| \leq \frac{C}{T}\sum_{h=-\infty}^{\infty}\alpha_{|h|}^{1-1/a-1/r}\left(E\left|A_1(\kappa)\right|^a\right)^{1/a}\left(E\left|A_{1+|h|}(\kappa)\right|^r\right)^{1/r} \leq \frac{C}{T},$$

where $a, r > 0$ and $1/a + 1/r < \iota/(1+\iota)$. Then, we immediately obtain,

$$E \parallel \zeta_5(\kappa) \parallel \leq \left(\int_{[0,1]} E\left|\zeta_5(\kappa)\right|^2 d\kappa\right)^{1/2} \leq \frac{C}{\sqrt{T}}.$$

Therefore,

$$\parallel \zeta_5(\kappa) \parallel = O_P\left(\frac{1}{\sqrt{T}}\right).$$

Finally, for the last term $\zeta_6(\kappa)$, it follows from boundedness of $f(\kappa)$ as defined in (3) and exactly the same arguments as in calculating the upper bounds of terms $\zeta_1(\kappa)$, $\zeta_2(\kappa)$ and $\zeta_3(\kappa)$, that,

$$\parallel \zeta_6(\kappa) \parallel = O_P\left(\frac{1}{\sqrt{n}}\right).$$

To sum up, we have,

$$\sum_{i=1}^{6} \parallel \zeta_i(\kappa) \parallel \xrightarrow{P} 0.$$

Step 2. We now consider the difference between estimators $\widehat{f}(\kappa)$ and $\widehat{f}^c(\kappa)$ which are built based on X and X^c, respectively. By triangle inequality, we have,

$$\begin{aligned}\parallel \widehat{f}(\kappa) - \widehat{f}^c(\kappa) \parallel &= \parallel \frac{1}{\widehat{\eta}T}\sum_{i=1}^{T}\widehat{\sigma}_{i,\kappa}^2 - \frac{1}{\widehat{\eta}^c T}\sum_{i=1}^{T}\widehat{\sigma}_{i,\kappa}^{2,c} \parallel \\ &\leq \frac{1}{\widehat{\eta}} \parallel \frac{1}{T}\sum_{i=1}^{T}\left(\widehat{\sigma}_{i,\kappa}^2 - \widehat{\sigma}_{i,\kappa}^{2,c}\right) \parallel + \frac{|\widehat{\eta}^c - \widehat{\eta}|}{\widehat{\eta}\widehat{\eta}^c} \parallel \frac{1}{T}\sum_{i=1}^{T}\widehat{\sigma}_{i,\kappa}^{2,c} \parallel . \qquad \text{(C.6)}\end{aligned}$$

It then follows easily from (C.4) and Lemma 13 that,

$$\widehat{\eta}-\widehat{\eta}^c \xrightarrow{P} 0, \quad \frac{1}{\widehat{\eta}} \xrightarrow{P} \frac{1}{\eta}, \quad \text{and hence } \frac{\widehat{\eta}^c-\widehat{\eta}}{\widehat{\eta}} \xrightarrow{P} 0. \tag{C.7}$$

Recall that $\widehat{f}^c(\kappa)=\sum_{i=1}^T \widehat{\sigma}_{i,\kappa}^{2,c}/(\widehat{\eta}^c T)$ and $\| \widehat{f}^c(\kappa)-f(\kappa) \| \xrightarrow{P} 0$ by Step 1, we have thus proved that the second term on the right hand side of (C.6) goes to zero in probability.

It remains to calculate the stochastic order of $\| \frac{1}{T}\sum_{i=1}^T \left(\widehat{\sigma}_{i,\kappa}^2-\widehat{\sigma}_{i,\kappa}^{2,c}\right) \|$. On the one hand, define,

$$\mathcal{S}:=\{(i,i',k,k')|i\neq i' \text{ or } k\neq k', \text{ where } i,i'=1,2,...,T \text{ and } k,k'=j_\kappa-\ell+1,...,j_\kappa\},$$

we have that, for any $\omega>4$,

$$\begin{aligned}
&\frac{1}{(T\ell\Delta)^2}E\left\{\sum_{i=1}^T\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\left[\left(\Delta_{i,k}^n X\right)^2 1_{\left\{\left|\Delta_{i,k}^n X\right|\le u_n\right\}}-\left(\Delta_{i,k}^n X^c\right)^2\right]1_{\left\{\Delta_{i,k}^n N=0\right\}}\right\}^2\\
\le\ &\frac{1}{(T\ell\Delta)^2}\sum_{(i,i',k,k')\in\mathcal{S}}E\Big(\left(\Delta_{i,k}^n X^c\right)^2\left(\Delta_{i',k'}^n X^c\right)^2 1_{\left\{\left|\Delta_{i,k}^n X^c\right|>u_n\right\}}1_{\left\{\left|\Delta_{i',k'}^n X^c\right|>u_n\right\}}\Big)\\
&+\frac{1}{(T\ell\Delta)^2}\sum_{i=1}^T\sum_{k=j_\kappa-\ell+1}^{j_\kappa}E\left[\left(\Delta_{i,k}^n X^c\right)^4 1_{\left\{\left|\Delta_{i,k}^n X^c\right|>u_n\right\}}\right]\\
\le\ &\frac{1}{(T\ell\Delta)^2}\sum_{(i,i',k,k')\in\mathcal{S}}\left[E\left|\Delta_{i,k}^n X^c\right|^\omega\right]^{2/\omega}E\left[\left|\Delta_{i',k'}^n X^c\right|^\omega\right]^{2/\omega}\\
&\times\left[E\left(1_{\left\{\left|\Delta_{i,k}^n X^c\right|>u_n\right\}}\right)\right]^{(1-4/\omega)/2}\left[E\left(1_{\left\{\left|\Delta_{i',k'}^n X^c\right|>u_n\right\}}\right)\right]^{(1-4/\omega)/2}\\
&+\frac{1}{(T\ell\Delta)^2}\sum_{i=1}^T\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\left[E\left|\Delta_{i,k}^n X^c\right|^\omega\right]^{4/\omega}\left[E\left(1_{\left\{\left|\Delta_{i,k}^n X^c\right|>u_n\right\}}\right)\right]^{1-4/\omega}\\
\le\ &C\Delta^{(\omega-4)(1/2-\varpi)},
\end{aligned} \tag{C.8}$$

where the second and third inequalities follow by applying Hölder's, Burkholder-Davis-Gundy and Markov inequalities. On the other hand, we have that, for any $\omega > 4$,

$$
\begin{aligned}
&\frac{1}{(T\ell\Delta)^2} E\left\{\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\left[\left(\Delta_{i,k}^n X\right)^2 1_{\left\{\left|\Delta_{i,k}^n X\right|\leq u_n\right\}} - \left(\Delta_{i,k}^n X^c\right)^2\right] 1_{\left\{\Delta_{i,k}^n N>0\right\}}\right\}^2 \\
\leq\ & \frac{1}{(T\ell\Delta)^2}\sum_{(i,i',k,k')\in\mathcal{S}} E\left[u_n^4 1_{\left\{\Delta_{i,k}^n N>0\right\}} 1_{\left\{\Delta_{i',k'}^n N>0\right\}} + u_n^2\left(\Delta_{i',k'}^n X^c\right)^2 1_{\left\{\Delta_{i,k}^n N>0\right\}} 1_{\left\{\Delta_{i',k'}^n N>0\right\}}\right. \\
&\left. + u_n^2\left(\Delta_{i,k}^n X^c\right)^2 1_{\left\{\Delta_{i,k}^n N>0\right\}} 1_{\left\{\Delta_{i',k'}^n N>0\right\}} + \left(\Delta_{i,k}^n X^c\right)^2\left(\Delta_{i',k'}^n X^c\right)^2 1_{\left\{\Delta_{i,k}^n N>0\right\}} 1_{\left\{\Delta_{i',k'}^n N>0\right\}}\right] \\
& + \frac{1}{(T\ell\Delta)^2}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa} E\left|\left(\Delta_{i,k}^n X\right)^2 1_{\left\{\left|\Delta_{i,k}^n X\right|\leq u_n\right\}} - \left(\Delta_{i,k}^n X^c\right)^2\right|^2 1_{\left\{\Delta_{i,k}^n N>0\right\}} \\
\leq\ & \frac{1}{(T\ell\Delta)^2}\sum_{(i,i',k,k')\in\mathcal{S}}\left[u_n^4 E\left(1_{\left\{\Delta_{i,k}^n N>0\right\}} 1_{\left\{\Delta_{i',k'}^n N>0\right\}}\right) + u_n^2\left[E\left|\Delta_{i',k'}^n X^c\right|^\omega\right]^{2/\omega}\right. \\
& \times\left[E\big(1_{\left\{\Delta_{i,k}^n N>0\right\}} 1_{\left\{\Delta_{i',k'}^n N>0\right\}}\big)\right]^{1-2/\omega} + u_n^2\left[E\left|\Delta_{i,k}^n X^c\right|^\omega\right]^{2/\omega}\left[E\big(1_{\left\{\Delta_{i,k}^n N>0\right\}} 1_{\left\{\Delta_{i',k'}^n N>0\right\}}\big)\right]^{1-2/\omega} \\
&\left. + \left[E\left|\Delta_{i,k}^n X^c\right|^\omega\right]^{2/\omega}\left[E\left|\Delta_{i',k'}^n X^c\right|^\omega\right]^{2/\omega}\left(E\big(1_{\left\{\Delta_{i,k}^n N>0\right\}} 1_{\left\{\Delta_{i',k'}^n N>0\right\}}\big)\right)^{1-4/\omega}\right] \\
& + \frac{1}{(T\ell\Delta)^2}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\left[u_n^4 E\left(\Delta_{i,k}^n N\right) + \left(E\left|\Delta_{i,k}^n X^c\right|^\omega\right)^{4/\omega}\left[E\left(\Delta_{i,k}^n N\right)\right]^{1-4/\omega}\right] \\
\leq\ & \left(\Delta^{4\varpi-\varsigma}\vee\Delta^{2\varpi-4/\omega+1-\varsigma}\vee\Delta^{2-8/\omega-\varsigma}\vee\frac{\Delta^{4\varpi-1}}{T\ell}\vee\frac{\Delta^{1-4/\omega}}{T\ell}\right),
\end{aligned}
\tag{C.9}
$$

where the second inequality follows from Hölder's inequality and the last inequality follows from Burkholder-Davis-Gundy inequality and Lemma 12 for arbitrarily small $\varsigma > 0$. Based on the above results, we have that, for any $\omega > 4$ and arbitrarily small $\varsigma > 0$,

$$
E\parallel\frac{1}{T}\sum_{i=1}^{T}\left(\widehat{\sigma}_{i,\kappa}^2 - \widehat{\sigma}_{i,\kappa}^{2,c}\right)\parallel
$$

$$\leq \left(\int_{[0,1]} \frac{C}{(T\ell\Delta)^2} E\left\{\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\left[\left(\Delta_{i,k}^n X\right)^2 1_{\{|\Delta_{i,k}^n X|\leq u_n\}} - \left(\Delta_{i,k}^n X^c\right)^2\right] 1_{\{\Delta_{i,k}^n N=0\}}\right\}^2 d\kappa\right)^{1/2}$$

$$+ \left(\int_{[0,1]} \frac{C}{(T\ell\Delta)^2} E\left\{\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\left[\left(\Delta_{i,k}^n X\right)^2 1_{\{|\Delta_{i,k}^n X|\leq u_n\}} - \left(\Delta_{i,k}^n X^c\right)^2\right] 1_{\{\Delta_{i,k}^n N>0\}}\right\}^2 d\kappa\right)^{1/2}$$

$$\leq C\left(\Delta^{(\omega-4)(1/2-\varpi)} \vee \Delta^{4\varpi-\varsigma} \vee \Delta^{2\varpi-4/\omega+1-\varsigma} \vee \Delta^{2-8/\omega-\varsigma} \vee \frac{\Delta^{4\varpi-1}}{T\ell} \vee \frac{\Delta^{1-4/\omega}}{T\ell}\right)^{1/2}.$$

Because $\sup_{t\in\mathbb{R}_+} E\left(e^{|\mu(t)|}\right) + \sup_{t\in\mathbb{R}_+} E\left(e^{|\sigma(t)|}\right) < \infty$ in Assumption I(ii) and by the same arguments as that immediately following (C.2), one can always choose a large enough ω and a small enough ς such that,

$$E \parallel \frac{1}{T}\sum_{i=1}^{T}\left(\widehat{\sigma}_{i,\kappa}^2 - \widehat{\sigma}_{i,\kappa}^{2,c}\right) \parallel \leq C\left(\Delta^{4\varpi-\varsigma} \vee \frac{\Delta^{4\varpi-1}}{T\ell}\right)^{1/2}.$$

Therefore, by letting ς be sufficiently close to zero, we have that,

$$\parallel \widehat{f}(\kappa) - \widehat{f}^c(\kappa) \parallel = o_P(1),$$

under the conditions $0 < \varpi < 1/2$ and $b + c > 1 - 4\varpi$ in Theorem 1.

Combining Step 1 and Step 2 leads to $\parallel \widehat{f}(\kappa) - f(\kappa) \parallel \xrightarrow{P} 0$, completing the proof.

C.2 Proof of Theorem 2

First, we recall the definition of $A_i(\kappa_m)$,

$$A_i(\kappa_m) := \sigma^2(i-1+\kappa_m) - f(\kappa_m)\int_{i-1}^{i}\sigma^2(t)dt, \quad \text{for } i = 1, 2, ..., T \text{ and } m = 1, 2, ..., d,$$

where d is a positive integer. We introduce some additional notation that will be used in the proof. Denote with $E_i(\cdot)$ the conditional expectation with respect to the sigma field $\mathscr{G}_i$ (see Assumption II for definition). For each κ_m and a positive integer l, denote for $i = 1, 2, ..., T$,

$$\tilde{A}_{i,l}(\kappa_m) := \sum_{k=0}^{l-1} \left(E_i \left(A_{i+k}(\kappa_m) \right) - E_{i-1}\left(A_{i+k}(\kappa_m) \right) \right),$$

where $m = 1, 2, ..., d$ and d is a positive integer. We will show in the following that under the conditions of Theorem 2, the limit of $\tilde{A}_{i,l}(\kappa_m)$ as $l \to \infty$ exists a.s. It is denoted by,

$$\tilde{A}_{i,\infty}(\kappa_m) := \lim_{l \to \infty} \tilde{A}_{i,l}(\kappa_m). \tag{C.10}$$

Moreover, for each κ_m and a positive integer l, we define the following approximation errors,

$$R_{T,l}(\kappa_m) := \frac{1}{T} \sum_{i=1}^{T} \left(A_i(\kappa_m) - \tilde{A}_{i,l}(\kappa_m) \right) \text{ and } R_{T,\infty}(\kappa_m) := \frac{1}{T} \sum_{i=1}^{T} \left(A_i(\kappa_m) - \tilde{A}_{i,\infty}(\kappa_m) \right). \tag{C.11}$$

The following lemma is used in the proof of the limit result of Theorem 2.

Lemma 14. *Suppose that Assumptions I(ii) and II with $q = 3$ hold. Then,*

$$\left(\frac{1}{\sqrt{T}} \sum_{i=1}^{T} A_i(\kappa_1), \frac{1}{\sqrt{T}} \sum_{i=1}^{T} A_i(\kappa_2), \ldots \frac{1}{\sqrt{T}} \sum_{i=1}^{T} A_i(\kappa_d) \right)^{\top} \xrightarrow{d} \mathbf{N}_d(\mathbf{0}, \mathbf{\Lambda})$$

as $T \to \infty$, where $\mathbf{N}_d(\mathbf{0}, \mathbf{\Lambda})$ denotes the d-dimensional normal distribution with mean zero and covariance matrix $\mathbf{\Lambda}$ whose entries are given by $\Lambda_{mq} = \sum_{h=-\infty}^{\infty} \phi_{\kappa_m, \kappa_q}(h)$ for $m, q \in \{1, 2, ..., d\}$.

Proof of Lemma 14. The proof is based on approximating $A_i(\kappa_m)$ by $\tilde{A}_{i,\infty}(\kappa_m)$, where $\tilde{A}_{i,\infty}(\kappa_m)$ is defined in (C.10). Hence, the proof consists of two parts. In the first part, we show that the error due to the approximation of $A_i(\kappa_m)$ is asymptotically negligible. In the second part, we complete the proof by showing that,

$$\left(\frac{1}{\sqrt{T}}\sum_{i=1}^{T}\tilde{A}_{i,\infty}(\kappa_1), \frac{1}{\sqrt{T}}\sum_{i=1}^{T}\tilde{A}_{i,\infty}(\kappa_2), \ldots, \frac{1}{\sqrt{T}}\sum_{i=1}^{T}\tilde{A}_{i,\infty}(\kappa_d)\right)^{\top} \xrightarrow{d} \boldsymbol{N}_d(\mathbf{0}, \boldsymbol{\Lambda}). \quad \text{(C.12)}$$

Part 1. It follows from (B.49)-(B.50) in [1] that the average difference $R_{T,l}(\kappa_m)$ between $A_i(\kappa_m)$ and $\tilde{A}_{i,l}(\kappa_m)$ has the following decomposition,

$$R_{T,l}(\kappa_m) = \frac{1}{T}\sum_{i=0}^{T-1} E_i(A_{i+l}(\kappa_m)) - \frac{1}{T}\sum_{k=1}^{l-1}\left[E_T(A_{T+k}(\kappa_m)) - E_0(A_k(\kappa_m))\right] \quad \text{(C.13)}$$

for $m = 1, 2, ..., d$. Because of Assumptions I(ii) and II with $q = 3$, and using Lemma 3.102 on page 497 of [7], we have that, for any $\omega > (3+\iota)/(2+\iota)$ where ι is given in Assumption II,

$$E\left|E_i(A_{i+k}(\kappa_m))\right| \le C\alpha_k^{1-1/\omega}\left(E\left|A_{i+k}(\kappa_m)\right|^{\omega}\right)^{1/\omega}.$$

This further implies,

$$\begin{aligned} &E\left(\lim_{l\to\infty}\sum_{k=0}^{l-1}\left(\left|E_i(A_{i+k}(\kappa_m))\right| + \left|E_{i-1}(A_{i+k}(\kappa_m))\right|\right)\right) \\ &\le \liminf_{l\to\infty}\sum_{k=0}^{l-1}\left(E\left|E_i(A_{i+k}(\kappa_m))\right| + E\left|E_{i-1}(A_{i+k}(\kappa_m))\right|\right) \le C. \end{aligned}$$

Therefore,

$$\tilde{A}_{i,\infty}(\kappa_m) := \lim_{l\to\infty} \tilde{A}_{i,l}(\kappa_m) = \sum_{k=0}^{\infty} \left(E_i\left[A_{i+k}(\kappa_m)\right] - E_{i-1}\left[A_{i+k}(\kappa_m)\right] \right) \tag{C.14}$$

exists almost surely. It follows immediately that,

$$R_{T,\infty}(\kappa_m) = \lim_{l\to\infty} R_{T,l}(\kappa_m)$$

exists almost surely. Using similar arguments to those used above and decomposition (C.13), we obtain that,

$$\begin{aligned}
&E\left(\sqrt{T}\left|R_{T,\infty}(\kappa_m)\right|\right) = E\left(\sqrt{T}\left|\lim_{l\to\infty} R_{T,l}(\kappa_m)\right|\right) \\
&\leq \sqrt{T} \liminf_{l\to\infty} \left\{ \frac{1}{T}\sum_{i=0}^{T-1} E\left|E_i(A_{i+l}(\kappa_m))\right| + \frac{1}{T}\sum_{k=1}^{l-1}\left(E\left|E_T\left(A_{T+k}(\kappa_m)\right)\right| + E\left|E_0\left(A_k(\kappa_m)\right)\right|\right) \right\} \\
&\leq \frac{C}{\sqrt{T}}.
\end{aligned}$$

To sum up, the error from approximating $A_i(\kappa_m)$ by $\tilde{A}_{i,\infty}(\kappa_m)$ is asymptotically negligible. We only need to prove the central limit theorem with $A_i(\kappa_m)$ being replaced with $\tilde{A}_{i,\infty}(\kappa_m)$.

Part 2. By the dominated convergence theorem, we have,

$$\begin{aligned}
E_{i-1}\left(\tilde{A}_{i,\infty}(\kappa_m)\right) &= E_{i-1}\left(\lim_{l\to\infty}\sum_{k=0}^{l-1}\left[E_i(A_{i+k}(\kappa_m)) - E_{i-1}(A_{i+k}(\kappa_m))\right]\right) \\
&= \lim_{l\to\infty}\sum_{k=0}^{l-1} E_{i-1}\left[E_i(A_{i+k}(\kappa_m)) - E_{i-1}(A_{i+k}(\kappa_m))\right] = 0.
\end{aligned}$$

Hence, $\tilde{A}_{i,\infty}(\kappa_m)$ is a martingale difference for fixed κ_m, $m \in \{1, 2, ..., d\}$.

We next show that the third moment of $\tilde{A}_{i,\infty}(\kappa_m)$ exists for each fixed $i \in \{1, 2, ..., T\}$ and $\kappa_m, m \in \{1, 2, ..., d\}$. To this end, define,

$$D_{i,k} := |E_i(A_{i+k}(\kappa_m)) - E_{i-1}(A_{i+k}(\kappa_m))|.$$

By Assumptions I(ii), II with $q = 3$ and applying Lemma 3.102 on page 497 of [7], we have that, for any $\omega > 3(3+\iota)/\iota$ where ι is given in Assumption II,

$$\begin{aligned}
&\lim_{l\to\infty} \sum_{k_1=0}^{l-1} \sum_{k_2=k_1+1}^{l-1} \sum_{k_3=k_2+1}^{l-1} E\left(D_{i,k_1} D_{i,k_2} D_{i,k_3}\right) \\
&\leq \lim_{l\to\infty} \sum_{k_1=0}^{l-1} \left[E\left(D_{i,k_1}\right)^3\right]^{1/3} \sum_{k_2=0}^{l-1} \left[E\left(D_{i,k_2}\right)^3\right]^{1/3} \sum_{k_3=0}^{l-1} \left[E\left(D_{i,k_3}\right)^3\right]^{1/3} \\
&\leq C \lim_{l\to\infty} \sum_{k_1=0}^{l-1} \left[\left(E\left|E_i(A_{i+k_1}(\kappa_m))\right|^3\right)^{1/3} + \left(E\left|E_{i-1}(A_{i+k_1}(\kappa_m))\right|^3\right)^{1/3}\right] \\
&\quad \times \sum_{k_2=0}^{l-1} \left[\left(E\left|E_i(A_{i+k_2}(\kappa_m))\right|^3\right)^{1/3} + \left(E\left|E_{i-1}(A_{i+k_2}(\kappa_m))\right|^3\right)^{1/3}\right] \\
&\quad \times \sum_{k_3=0}^{l-1} \left[\left(E\left|E_i(A_{i+k_3}(\kappa_m))\right|^3\right)^{1/3} + \left(E\left|E_{i-1}(A_{i+k_3}(\kappa_m))\right|^3\right)^{1/3}\right] \\
&\leq C \lim_{l\to\infty} \sum_{k_1=0}^{l-1} \left[\alpha_{k_1}^{(\omega-3)/3\omega} \left(E\left|A_{i+k_1}(\kappa_m)\right|^\omega\right)^{1/\omega} + \alpha_{k_1+1}^{(\omega-3)/3\omega} \left(E\left|A_{i+k_1}(\kappa_m)\right|^\omega\right)^{1/\omega}\right] \\
&\quad \times \sum_{k_2=0}^{l-1} \left[\alpha_{k_2}^{(\omega-3)/3\omega} \left(E\left|A_{i+k_2}(\kappa_m)\right|^\omega\right)^{1/\omega} + \alpha_{k_2+1}^{(\omega-3)/3\omega} \left(E\left|A_{i+k_2}(\kappa_m)\right|^\omega\right)^{1/\omega}\right] \\
&\quad \times \sum_{k_3=0}^{l-1} \left[\alpha_{k_3}^{(\omega-3)/3\omega} \left(E\left|A_{i+k_3}(\kappa_m)\right|^\omega\right)^{1/\omega} + \alpha_{k_3+1}^{(\omega-3)/3\omega} \left(E\left|A_{i+k_3}(\kappa_m)\right|^\omega\right)^{1/\omega}\right] \\
&\leq C \lim_{l\to\infty} \left(\sum_{k_1=1}^{l-1} \left(k_1^{-(\omega-3)(3+\iota)/3\omega} + (k_1+1)^{-(\omega-3)(3+\iota)/3\omega}\right) + C\right)
\end{aligned}$$

$$
\times \left(\sum_{k_2=1}^{l-1} \left(k_2^{-(\omega-3)(3+\iota)/3\omega} + (k_2+1)^{-(\omega-3)(3+\iota)/3\omega} \right) + C \right)
$$
$$
\times \left(\sum_{k_3=1}^{l-1} \left(k_3^{-(\omega-3)(3+\iota)/3\omega} + (k_3+1)^{-(\omega-3)(3+\iota)/3\omega} \right) + C \right) \leq C.
$$

By quite similar arguments, we also have,

$$
\lim_{l\to\infty} \sum_{k_1=0}^{l-1} E\left(D_{i,k_1}\right)^3 \leq C, \quad \lim_{l\to\infty} \sum_{k_1=0}^{l-1} \sum_{k_2=k_1+1}^{l-1} E\left[\left(D_{i,k_1}\right)^2 D_{i,k_2}\right] \leq C \quad \text{and}
$$
$$
\lim_{l\to\infty} \sum_{k_1=0}^{l-1} \sum_{k_2=k_1+1}^{l-1} E\left[D_{i,k_1}\left(D_{i,k_2}\right)^2\right] \leq C.
$$

Therefore, we have,

$$
E\left(\sum_{k=0}^{\infty} D_{i,k}\right)^3 \leq \lim_{l\to\infty} \sum_{k_1=0}^{l-1} E\left(D_{i,k_1}\right)^3 + \lim_{l\to\infty} C \sum_{p_1+p_2=3} \sum_{k_1=0}^{l-1} \sum_{k_2=k_1+1}^{l-1} E\left[\left(D_{i,k_1}\right)^{p_1}\left(D_{i,k_2}\right)^{p_2}\right]
$$
$$
+ \lim_{l\to\infty} C \sum_{k_1=0}^{l-1} \sum_{k_2=k_1+1}^{l-1} \sum_{k_3=k_2+1}^{l-1} E\left(D_{i,k_1} D_{i,k_2} D_{i,k_3}\right) \leq C < \infty,
$$

where $p_1, p_2 \in \mathbb{N}_+$. Now recalling (C.14), we have,

$$
E\left|\tilde{A}_{i,\infty}(\kappa_m)\right|^3 \leq E\left(\sum_{k=0}^{\infty} D_{i,k}\right)^3 < \infty. \tag{C.15}
$$

We are now ready to apply the martingale central limit theorem to obtain the limiting distribution of

$$
\left(\frac{1}{\sqrt{T}} \sum_{i=1}^{T} \tilde{A}_{i,\infty}\left(\kappa_1\right), \frac{1}{\sqrt{T}} \sum_{i=1}^{T} \tilde{A}_{i,\infty}\left(\kappa_2\right), \ldots, \frac{1}{\sqrt{T}} \sum_{i=1}^{T} \tilde{A}_{i,\infty}\left(\kappa_d\right)\right)^{\top}.
$$

The rest of the proof is divided into two steps. In the first step, we calculate the conditional covariances and find their limits. In the second step, we check the conditional Lyapunov condition.

Step 1. Conditional covariances. For $m, q \in \{1, 2, ..., d\}$, by Assumption II with $q = 3$, we have,

$$\frac{1}{T}\sum_{i=1}^{T} E_{i-1}\left[\tilde{A}_{i,\infty}(\kappa_m)\tilde{A}_{i,\infty}(\kappa_q)\right] \xrightarrow{P} E\left[\tilde{A}_{1,\infty}(\kappa_m)\tilde{A}_{1,\infty}(\kappa_q)\right].$$

Next, we derive the explicit formula for the limit $E\left[\tilde{A}_{1,\infty}(\kappa_m)\tilde{A}_{1,\infty}(\kappa_q)\right]$. Because of $E\left|\sum_{k=0}^{\infty} D_{1,k}\right|^3 < \infty$ as shown above and Assumption II with $q = 3$, we have,

$$\begin{aligned}
& E\left[\tilde{A}_{1,\infty}(\kappa_m)\tilde{A}_{1,\infty}(\kappa_q)\right] \\
&= E\left(\lim_{l\to\infty}\sum_{k=0}^{l-1}\left(E_1\left(A_{1+k}(\kappa_m)\right) - E_0\left(A_{1+k}(\kappa_m)\right)\right)\sum_{p=0}^{l-1}\left(E_1\left(A_{1+p}(\kappa_q)\right) - E_0\left(A_{1+p}(\kappa_q)\right)\right)\right) \\
&= \lim_{l\to\infty}\sum_{k=0}^{l-1}\sum_{p=0}^{l-1} E\left[A_{1+k}(\kappa_m)\left(E_1\left(A_{1+p}(\kappa_q)\right) - E_0\left(A_{1+p}(\kappa_q)\right)\right)\right] \\
&= \lim_{l\to\infty}\left(\sum_{k=0}^{l-1}\sum_{p=0}^{l-1} E\left[A_{1+k}(\kappa_m)E_1\left(A_{1+p}(\kappa_q)\right)\right] - \sum_{k=1}^{l}\sum_{p=1}^{l} E\left[A_{1+k}(\kappa_m)E_1\left(A_{1+p}(\kappa_q)\right)\right]\right) \\
&= \lim_{l\to\infty}\left(\sum_{k=0}^{l-1} E\left[A_{1+k}(\kappa_m)A_1(\kappa_q)\right] + \sum_{p=1}^{l-1} E\left[A_1(\kappa_m)A_{1+p}(\kappa_q)\right]\right. \\
&\quad \left. - \sum_{k=1}^{l} E\left[A_{1+k}(\kappa_m)E_1\left(A_{1+l}(\kappa_q)\right)\right] - \sum_{p=1}^{l-1} E\left[A_{1+l}(\kappa_m)E_1\left(A_{1+p}(\kappa_q)\right)\right]\right) \\
&= \sum_{k=0}^{\infty} E\left[A_{1+k}(\kappa_m)A_1(\kappa_q)\right] + \sum_{p=1}^{\infty} E\left[A_1(\kappa_m)A_{1+p}(\kappa_q)\right],
\end{aligned}$$

where the last equality follows from the Cauchy-Schwarz inequality and Assumption II with $q = 3$. Then, by applying definition (8) to terms on the right hand side of the last equality, we obtain,

$$E\left[\tilde{A}_{i,\infty}\left(\kappa_m\right)\tilde{A}_{i,\infty}\left(\kappa_q\right)\right] = \sum_{h=-\infty}^{\infty}\phi_{\kappa_m,\kappa_q}(h) =: \Lambda_{mq}.$$

Step 2. Conditional Lyapunov condition. In this step, we check the conditional Lyapunov condition. By (C.15) that $E\left(\tilde{A}_{i,\infty}\right)^3 < \infty$, we have that for each $m \in \{1, 2, ..., d\}$,

$$\frac{1}{T^{3/2}}\sum_{i=1}^{T} E_{i-1}\left|\tilde{A}_{i,\infty}\left(\kappa_m\right)\right|^3 \xrightarrow{P} 0,$$

as $T \to \infty$. The conditional Lyapunov condition is satisfied.

Lastly, the martingale central limit theorem (see, e.g., Corollary 3.1 on page 58 of [5] and Theorem A.1 of [13]) concludes the proof. $\square$

Proof of Theorem 2. We divide the proof into two steps. In the first step, we prove that $\sqrt{T} \parallel \widehat{f}(\kappa) - f(\kappa) - \zeta_5(\kappa)/\widehat{\eta}^c \parallel$ is asymptotically negligible. In the second step, we prove $\sqrt{T}\zeta_5(\kappa)/\widehat{\eta}^c \xrightarrow{d} N(0, \mathcal{K})$ in $\mathcal{L}^2$.

Step 1. By using the notation in (C.1) and triangle inequality, we have,

$$\sqrt{T} \parallel \widehat{f}(\kappa) - f(\kappa) - \frac{\zeta_5(\kappa)}{\widehat{\eta}^c} \parallel \leq \frac{\sqrt{T}}{\widehat{\eta}^c} \sum_{i\in\{1,2,3,4,6\}} \parallel \zeta_i(\kappa) \parallel + \sqrt{T} \parallel \widehat{f}(\kappa) - \widehat{f}^c(\kappa) \parallel .$$

Because $1/\widehat{\eta}^c \xrightarrow{P} 1/\eta$ by the arguments immediately following (C.4), it remains to show,

$$\sqrt{T} \sum_{i\in\{1,2,3,4,6\}} \parallel \zeta_i(\kappa) \parallel + \sqrt{T} \parallel \widehat{f}(\kappa) - \widehat{f}^c(\kappa) \parallel = o_P(1).$$

From Lemma 11 with $m = 2$ and Step 2 in Section C.1, we immediately obtain,

$$\sqrt{T}\left(\| \zeta_1(\kappa) \| + \| \zeta_3(\kappa) \| + \| \widehat{f}(\kappa) - \widehat{f}^c(\kappa) \|\right) = o_P(1),$$

under the condition (9).

We next determine the stochastic orders of terms $\sqrt{T} \| \zeta_2(\kappa) \|$, $\sqrt{T} \| \zeta_4(\kappa) \|$ and $\sqrt{T} \| \zeta_6(\kappa) \|$. Because for $i = 2, 4, 6$,

$$E\left(\| \zeta_i(\kappa) \|\right) \leq \left(\int_{[0,1]} E\left|\zeta_i(\kappa)\right|^2 d\kappa\right)^{1/2},$$

it remains to determine the bounds of $E\left|\zeta_2(\kappa)\right|^2$, $E\left|\zeta_4(\kappa)\right|^2$ and $E\left|\zeta_6(\kappa)\right|^2$.

Now we treat $\zeta_2(\kappa)$. First, note that, for any $\omega > 2$,

$$\begin{aligned}
& E\left|\frac{2}{T\ell\Delta}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,k-1}}^{t_{i,k}}\left[\mu(t)-\mu(t_{i,k-1})\right]dt\int_{t_{i,k-1}}^{t_{i,k}}\sigma(t)dW(t)\right|^2 \\
\leq{}& \frac{C}{T\ell\Delta^2}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\left(E\left|\int_{t_{i,k-1}}^{t_{i,k}}\left[\mu(t)-\mu(t_{i,k-1})\right]dt\right|^{\omega}\right)^{2/\omega}\left(E\left|\int_{t_{i,k-1}}^{t_{i,k}}\sigma(t)dW(t)\right|^{\frac{2\omega}{\omega-2}}\right)^{1-\frac{2}{\omega}} \\
\leq{}& \frac{C}{T\ell\Delta^2}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\left(\Delta^{\omega-1}\int_{t_{i,k-1}}^{t_{i,k}}E\left|\mu(t)-\mu(t_{i,k-1})\right|^{\omega}dt\right)^{2/\omega}\left(E\left(\int_{t_{i,k-1}}^{t_{i,k}}\sigma^2(t)dt\right)^{\frac{\omega}{\omega-2}}\right)^{1-\frac{2}{\omega}} \\
\leq{}& \frac{C}{T\ell\Delta^2}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\left(\Delta^{\omega+1}\right)^{2/\omega}\left(E\left(\int_{t_{i,k-1}}^{t_{i,k}}\sigma^2(t)dt\right)^{\frac{\omega}{\omega-2}}\right)^{1-\frac{2}{\omega}} \leq C\Delta^{1+2/\omega},
\end{aligned}$$

where the first inequality follows from Jensen's inequality and Hölder's inequality, the second inequality follows from Jensen's inequality and Burkholder-Davis-Gundy inequality, the third inequality follows from $E\left|\mu(t)-\mu(t_{i,k-1})\right|^{\omega} \leq C|t-t_{i,k-1}|$ which is a consequence of Assumption I(i) and $\sup_{t\in\mathbb{R}_+} E\exp(|\mu(t)|) \leq \infty$ in Assumption I(ii) (note that ω is a number close to 2 from above), and the last inequality follows from Jensen's inequality again. Then, by letting ω be arbitrarily close to 2 from above on both sides of the above inequality, we obtain that,

$$E\left|\frac{2}{T\ell\Delta}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,k-1}}^{t_{i,k}}\left[\mu(t)-\mu(t_{i,k-1})\right]dt\int_{t_{i,k-1}}^{t_{i,k}}\sigma(t)dW(t)\right|^2 \leq C\Delta^{2-\varsigma}$$

for arbitrarily small $\varsigma > 0$. Second, we define for $t \in [0,T]$,

$$M_2(t) := \frac{1}{T\ell}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\int_{t\wedge t_{i,k-1}}^{t\wedge t_{i,k}}\mu(t_{i,k-1})\sigma(s)dW(s),$$

which is a continuous martingale on the interval $[0,T]$. The quadratic variation of $M_2(t)$ is given by,

$$\left[M_2, M_2\right](t) = \frac{1}{(T\ell)^2}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\int_{t\wedge t_{i,k-1}}^{t\wedge t_{i,k}}\left[\mu(t_{i,k-1})\sigma(s)\right]^2 ds.$$

Then by Burkholder-Davis-Gundy inequality, we have,

$$\begin{aligned}
E\left|M_2(T)\right|^2 &\leq \frac{C}{(T\ell)^2}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,k-1}}^{t_{i,k}} E\left(\mu(t_{i,k-1})\sigma(t)\right)^2 dt \\
&\leq \frac{C}{(T\ell)^2}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,k-1}}^{t_{i,k}}\left[E\left(\mu(t_{i,k-1})\right)^4 E\left(\sigma(t)\right)^4\right]^{1/2} dt \leq \frac{C}{nT\ell},
\end{aligned}$$

where the last two inequalities follow from the Cauchy-Schwarz inequality and Assumption

I(ii). Therefore,

$$E\left|\zeta_2(\kappa)\right|^2 \leq CE\left|\frac{2}{T\ell\Delta}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,k-1}}^{t_{i,k}}\left[\mu(t)-\mu(t_{i,k-1})\right]dt\int_{t_{i,k-1}}^{t_{i,k}}\sigma(t)dW(t)\right|^2 + E\left|M_2(T)\right|^2$$
$$\leq C\left(\frac{1}{n^{2-\varsigma}} \vee \frac{1}{nT\ell}\right)$$

for arbitrarily small $\varsigma > 0$.

We next deal with $\zeta_4(\kappa)$. The following additional notations are needed,

$$\begin{cases} \zeta_{4,1}(\kappa) := -\frac{1}{T\ell\Delta}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,k-1}}^{t_{i,k}}\left[\int_{t}^{i-1+\kappa}\left(\tilde{\mu}(s)+\int_{\mathbb{R}}x\tilde{F}(dx)\tilde{\chi}(s)\right)ds\right]dt, \\ \zeta_{4,2}(\kappa) := \frac{1}{T\ell\Delta}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,k-1}}^{t_{i,k}}\left[\int_{t_{i,k-1}}^{t}\breve{\sigma}(s)dW(s)+\int_{t_{i,k-1}}^{t}\tilde{\sigma}(s)d\widetilde{W}(s)\right. \\ \qquad\qquad \left.+\int_{t_{i,k-1}}^{t}\int_{\mathbb{R}}x\left(\tilde{\nu}(ds,dx)-\tilde{F}(dx)\tilde{\chi}(s)ds\right)\right]dt, \\ \zeta_{4,3}(\kappa) := -\frac{1}{T\ell\Delta}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,k-1}}^{t_{i,k}}\left[\int_{t_{i,k-1}}^{i-1+\kappa}\breve{\sigma}(s)dW(s)+\int_{t_{i,k-1}}^{i-1+\kappa}\tilde{\sigma}(s)d\widetilde{W}(s)\right. \\ \qquad\qquad \left.+\int_{t_{i,k-1}}^{i-1+\kappa}\int_{\mathbb{R}}x\left(\tilde{\nu}(ds,dx)-\tilde{F}(dx)\tilde{\chi}(s)ds\right)\right]dt. \end{cases} \tag{C.16}$$

We can then rewrite $\zeta_4(\kappa)$ as,

$$\zeta_4(\kappa) = \sum_{i=1}^{3}\zeta_{4,i}(\kappa).$$

For $\zeta_{4,1}(\kappa)$, we have,

$$E\left|\zeta_{4,1}(\kappa)\right|^2 \leq \frac{1}{T\ell\Delta}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,k-1}}^{t_{i,k}}E\left[\int_{t}^{i-1+\kappa}\left(\tilde{\mu}(s)+\int_{\mathbb{R}}x\tilde{F}(dx)\tilde{\chi}(s)\right)ds\right]^2dt$$

$$\leq \frac{C}{T\ell\Delta}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,k-1}}^{t_{i,k}}\ell\Delta\int_{t}^{i-1+\kappa}\left(E\left|\tilde{\mu}(s)\right|^2+E\left|\tilde{\chi}(s)\right|^2\right)ds\leq\frac{C\ell^2}{n^2},$$

where the first two inequalities follow from Jensen's inequality and $\int_{\mathbb{R}}|x|^2\,\tilde{F}(dx)<\infty$ in Assumption I(ii), and the last inequality follows from Assumption I(ii) again. For $\zeta_{4,2}(\kappa)$, by applying exactly the same technique used in the proof of Lemma 2.22 on page 144 of [9], and again Burkholder-Davis-Gundy inequality, Jensen's inequality, and Assumption I(ii), we obtain that,

$$E\left|\zeta_{4,2}(\kappa)\right|^2\leq\frac{C}{nT\ell}.$$

Now turning to $\zeta_{4,3}(\kappa)$, we denote,

$$\begin{aligned}M_3(t) := \frac{1}{T\ell}\sum_{i=1}^{T}\Bigg[\sum_{k=j_\kappa-\ell+1}^{j_\kappa-1}\Bigg(&\int_{t\wedge t_{i,k-1}}^{t\wedge t_{i,k}}(k-j_\kappa+\ell)\check{\sigma}(s)dW(s)+\int_{t\wedge t_{i,k-1}}^{t\wedge t_{i,k}}(k-j_\kappa+\ell)\tilde{\sigma}(s)d\widetilde{W}(s)\\ &+\int_{t\wedge t_{i,k-1}}^{t\wedge t_{i,k}}(k-j_\kappa+\ell)x\left(\tilde{\nu}(ds,dx)-\tilde{F}(dx)\tilde{\chi}(s)ds\right)\Bigg)+\int_{t\wedge t_{i,j_\kappa-1}}^{t\wedge(i-1+\kappa)}\ell\check{\sigma}(s)dW(s)\\ &+\int_{t\wedge t_{i,j_\kappa-1}}^{t\wedge(i-1+\kappa)}\ell\tilde{\sigma}(s)d\widetilde{W}(s)+\int_{t\wedge t_{i,j_\kappa-1}}^{t\wedge(i-1+\kappa)}\ell x\left(\tilde{\nu}(ds,dx)-\tilde{F}(dx)\tilde{\chi}(s)ds\right)\Bigg],\end{aligned}$$

which is a continuous-time martingale over the interval $[0,T]$. The quadratic variation of $M_3(t)$ is,

$$\begin{aligned}\left[M_3,M_3\right](t) = \frac{1}{(T\ell)^2}\sum_{i=1}^{T}\Bigg[\sum_{k=j_\kappa-\ell+1}^{j_\kappa-1}\Bigg(&\int_{t\wedge t_{i,k-1}}^{t\wedge t_{i,k}}\left(\left|(k-j_\kappa+\ell)\check{\sigma}(s)\right|^2+\left|(k-j_\kappa+\ell)\tilde{\sigma}(s)\right|^2\right)ds\\ &+\int_{t\wedge t_{i,k-1}}^{t\wedge t_{i,k}}\left|(k-j_\kappa+\ell)x\right|^2\tilde{\nu}(ds,dx)\Bigg)+\int_{t\wedge t_{i,j_\kappa-1}}^{t\wedge(i-1+\kappa)}\left(\left|\ell\check{\sigma}(s)\right|^2+\left|\ell\tilde{\sigma}(s)\right|^2\right)ds\end{aligned}$$

$$+ \int_{t\wedge t_{i,j_\kappa-1}}^{t\wedge(i-1+\kappa)} |\ell x|^2 \, \tilde{\nu}(ds, dx)\Bigg].$$

It is easy to see that $\zeta_{4,3}(\kappa) = -M_3(T)$. Then by Burkholder-Davis-Gundy inequality, Assumption I(ii) and the arguments immediately following (C.5), we obtain that,

$$E\left|\zeta_{4,3}(\kappa)\right|^2 = E\left|M_3(T)\right|^2 \le CE\left[M_3, M_3\right](T) \le \frac{C\ell}{nT}.$$

Therefore, combining the results for terms $\zeta_{4,1}(\kappa)$, $\zeta_{4,2}(\kappa)$ and $\zeta_{4,3}(\kappa)$, we obtain that,

$$E\left|\zeta_4(\kappa)\right|^2 \le C\left[\left(\frac{\ell}{n}\right)^2 \vee \frac{\ell}{nT}\right].$$

Lastly, for term $\zeta_6(\kappa)$, by exactly the same method used in the proof of Lemma 11 for calculating the orders of $\zeta_1(\kappa)$ and $\zeta_3(\kappa)$ together with the same method used in calculating the order of $\zeta_2(\kappa)$, simply replacing ℓ with n in these derivations, we obtain that,

$$E\left|\zeta_6(\kappa)\right|^2 \le C\left(\frac{1}{n^{2-\varsigma}} \vee \frac{1}{nT}\right).$$

for arbitrarily small $\varsigma > 0$.

Based on the above bounds for terms $\zeta_2(\kappa)$, $\zeta_4(\kappa)$ and $\zeta_6(\kappa)$, by choosing a sufficiently small $\varsigma > 0$, we can easily obtain,

$$\sqrt{T}\left(\| \zeta_2(\kappa) \| + \| \zeta_4(\kappa) \| + \| \zeta_6(\kappa) \|\right) = O_P\left(\frac{\sqrt{T}\ell}{n} \vee \sqrt{\frac{\ell}{n}}\right) = o_p(1)$$

under the conditions $c < 1 - b/2$ in (9) and $\ell\Delta \to 0$.

Step 2. The only dominant term is $\sqrt{T}\zeta_5(\kappa)/\widehat{\eta}^c$. Recall the definitions of $\zeta_5(\kappa)$, $A_i(\kappa)$,

$\tilde{A}_{i,\infty}(\kappa)$, and $R_{T,\infty}$ in (C.1), (7), (C.10) and (C.11), respectively. It follows easily from the same arguments as that used in Part 1 of the proof of Lemma 14 that,

$$\sqrt{T} \parallel R_{T,\infty}(\kappa) \parallel = \left(\int_0^1 T R_{T,\infty}(\kappa)^2 d\kappa \right)^{1/2} \xrightarrow{P} 0.$$

Because of the above result and $\widehat{\eta}^c \xrightarrow{P} \eta$ which follows from the arguments immediately following (C.4), $\sqrt{T}\zeta_5(\kappa)/\widehat{\eta}^c$ and $1/(\eta\sqrt{T})\sum_{i=1}^T \tilde{A}_{i,\infty}(\kappa)$ have the same limiting law. From Part 2 of the proof of Lemma 14, $\{1/(\eta\sqrt{T})\tilde{A}_{i,\infty}(\kappa)\}_{i\geq 1}$ is a martingale difference array in the sense of [8]. We next check the three conditions of Theorem C of [8] which leads to the desired functional central limit theorem.

First, we have that,

$$\begin{aligned}
&\frac{1}{\eta^2 T}\sum_{i=1}^T E_{i-1}\left(\parallel \tilde{A}_{i,\infty}(\kappa) \parallel^2\right) - \mathrm{Trace}(\mathcal{K}) \\
&= \frac{1}{T}\sum_{i=1}^T \int_0^1 \left(1/\eta^2 E_{i-1}\tilde{A}_{i,\infty}(\kappa)^2 - C(\kappa,\kappa)\right) d\kappa \\
&= o_P(1),
\end{aligned}$$

where the last equality follows from the ergodicity Assumption II and the fact that,

$$E\left|\int_0^1 \left(1/\eta^2 E_{i-1}\tilde{A}_{i,\infty}(\kappa)^2 - C(\kappa,\kappa)\right) d\kappa\right| < \infty,$$

and $C(\kappa,\kappa)$ is defined in (8). Hence, we obtain that the first condition of Theorem C of [8], i.e.,

$$\frac{1}{\eta^2 T}\sum_{i=1}^T E_{i-1}\left(\parallel \tilde{A}_{i,\infty}(\kappa) \parallel^2\right) \xrightarrow{P} \mathrm{Trace}(\mathcal{K}),$$

is satisfied.

Second, we have,

$$E\left[\frac{1}{\eta^3 T^{3/2}}\sum_{i=1}^{T} E_{i-1}\left(\| \tilde{A}_{i,\infty}(\kappa) \|^3\right)\right] \leq E\left[\frac{1}{\eta^3 T^{3/2}}\sum_{i=1}^{T} E_{i-1}\int_0^1 |\tilde{A}_{i,\infty}(\kappa)|^3 d\kappa\right]$$
$$= \frac{1}{\eta^3 T^{1/2}}\int_0^1 E|\tilde{A}_{i,\infty}(\kappa)|^3 d\kappa \leq C/\sqrt{T},$$

where the last inequality follows from the result that $E|\tilde{A}_{i,\infty}(\kappa)|^3 \leq C$ as proved in Part 2 of the proof of Lemma 14. Hence, the conditional Lyapunov condition that is stronger than the second condition of Theorem C of [8], i.e.,

$$\frac{1}{\eta^3 T^{3/2}}\sum_{i=1}^{T} E_{i-1}\left(\| \tilde{A}_{i,\infty}(\kappa) \|^3\right) \xrightarrow{P} 0,$$

is satisfied.

Third, we have that, for an orthonormal basis $\{e_i\}_{i\in\mathbb{N}_+}$ in $\mathcal{L}^2$,

$$\frac{1}{\eta^2 T}\sum_{i=1}^{T} E_{i-1}\left(\langle\tilde{A}_{i,\infty}(\kappa), e_j\rangle\langle\tilde{A}_{i,\infty}(\kappa), e_k\rangle\right) - \langle\mathcal{K}e_j, e_k\rangle$$
$$= \frac{1}{T}\sum_{i=1}^{T}\int_0^1\int_0^1\left[\left(\frac{1}{\eta^2}E_{i-1}\tilde{A}_{i,\infty}(u)\tilde{A}_{i,\infty}(v)\right)e_j(u)e_k(v) - C(u,v)e_j(u)e_k(v)\right]dudv$$
$$= o_P(1),$$

where the last equality follows from again the ergodicity Assumption II and the fact that,

$$E\left|\int_0^1\int_0^1\left[\left(\frac{1}{\eta^2}E_{i-1}\tilde{A}_{i,\infty}(u)\tilde{A}_{i,\infty}(v)\right)e_j(u)e_k(v) - C(u,v)e_j(u)e_k(v)\right]dudv\right| < \infty.$$

Hence, the third condition of Theorem C of [8], i.e.,

$$\frac{1}{\eta^2 T}\sum_{i=1}^{T} E_{i-1}\left(\langle \tilde{A}_{i,\infty}(\kappa), e_j\rangle\langle \tilde{A}_{i,\infty}(\kappa), e_k\rangle\right) \xrightarrow{P} \langle \mathcal{K} e_j, e_k\rangle, \quad \forall j,k \in \mathbb{N}_+,$$

is satisfied.

An application of Theorem C of [8] then yields,

$$1/(\eta\sqrt{T})\sum_{i=1}^{T} \tilde{A}_{i,\infty}(\kappa) \xrightarrow{d} N(0,\mathcal{K}) \quad \text{in } \mathcal{L}^2.$$

Combining Step 1 and Step 2 leads to the desired result of Theorem 2, completing the proof. □

C.3 Proof of Corollary 3

The result follows from the arguments in Section A.5 of the supplementary appendix to [1].

C.4 Proof of Theorem 4

Without loss of generality, we assume that P and P′ are two consecutive time periods with trading days being labeled as $1, 2, \ldots, T, T+1, T+2, \ldots, T+T'$. It suffices to prove that random vectors defined in (C.12) for the two periods P and P′ are uncorrelated which is obvious. This implies that $\sqrt{T}\,(\widehat{f}_{\mathrm{P}}(\kappa) - f_{\mathrm{P}}(\kappa))$ and $\sqrt{T'}\,(\widehat{f}_{\mathrm{P'}}(\kappa) - f_{\mathrm{P'}}(\kappa))$ are asymptotically independent. The results of the theorem then follow straightforwardly from Theorem 2 and the continuous mapping theorem.

C.5 Proof of Theorem 5

We shall adopt the same notation as that used at the beginning of Section C. Moreover, define,

$$k_1^* := \lfloor (\tau - \epsilon_n)T \rfloor \quad \text{and} \quad k_2^* := \lfloor (\tau + \epsilon_n)T \rfloor.$$

It is readily seen that $k_1^*/T \to \tau$ and $(T - k_2^*)/T \to 1 - \tau$ since $\sqrt{T}\epsilon_n \to 0$.

We first establish the consistency of $\widehat{\eta}^c$ which is repeatedly used in the following. This can be easily seen as follows. First, by the assumption that $\breve{\sigma}^2(t)$ is ergodic, α-mixing with $q = 3$ and has finite moments of all orders, it follows that,

$$\frac{1}{T}\sum_{i=1}^{k_1^*}\left(\int_{i-1}^{i} g(t - \lfloor t \rfloor)\breve{\sigma}^2(t)dt - \int_0^1 g(u)du\right) = o_P(1)$$

and

$$\frac{1}{T}\sum_{i=k_2^*+1}^{T}\left(\int_{i-1}^{i} (g(t - \lfloor t \rfloor) + \gamma(t - \lfloor t \rfloor))\breve{\sigma}^2(t)dt - \int_0^1 (g(u) + \gamma(u))du\right) = o_P(1).$$

Second,

$$\begin{aligned}
&E\left|\frac{1}{T}\sum_{i=k_1^*+1}^{k_2^*}\int_{i-1}^{i} (g(t - \lfloor t \rfloor) + h_{\tau,n}(i/T)\gamma(t - \lfloor t \rfloor))\breve{\sigma}^2(t)dt\right| \\
&\leq C\frac{k_2^* - k_1^*}{T} = O(\epsilon_n).
\end{aligned}$$

Third,

$$\frac{k_1^*}{T}\int_0^1 g(u)du + \frac{T - k_2^*}{T}\int_0^1 (g(u) + \gamma(u))du - \eta$$

$$
\begin{aligned}
&= \left(\frac{k_1^*}{T} + \frac{T-k_2^*}{T} - 1 \right) \int_0^1 g(u)du + \left(\frac{T-k_2^*}{T} - (1-\tau) \right) \int_0^1 \gamma(u)du \\
&= O(\epsilon_n).
\end{aligned}
$$

Therefore, by the above results, we have that,

$$
\begin{aligned}
&\frac{1}{T}\sum_{i=1}^{T}\int_{i-1}^{i}\sigma^2(t)dt - \eta \\
&= \frac{1}{T}\sum_{i=1}^{k_1^*}\left(\int_{i-1}^{i} g(t-\lfloor t\rfloor)\breve{\sigma}^2(t)dt - \int_0^1 g(u)du\right) \\
&\quad + \frac{1}{T}\sum_{i=k_1^*+1}^{k_2^*}\int_{i-1}^{i}(g(t-\lfloor t\rfloor) + h_{\tau,n}(i/T)\gamma(t-\lfloor t\rfloor))\breve{\sigma}^2(t)dt \\
&\quad + \frac{1}{T}\sum_{i=k_2^*+1}^{T}\left(\int_{i-1}^{i}(g(t-\lfloor t\rfloor) + \gamma(t-\lfloor t\rfloor))\breve{\sigma}^2(t)dt - \int_0^1 (g(u)+\gamma(u))du\right) \\
&\quad + \frac{k_1^*}{T}\int_0^1 g(u)du + \frac{T-k_2^*}{T}\int_0^1 (g(u)+\gamma(u))du - \eta \\
&= o_P(1).
\end{aligned}
$$

Lastly,

$$
\begin{aligned}
\widehat{\eta}^c - \eta &= \frac{1}{T}\sum_{i=1}^{T}\sum_{j=1}^{n}\left(\Delta_{i,j}^n X^c\right)^2 - \eta \\
&= \frac{1}{T}\sum_{i=1}^{T}\sum_{j=1}^{n}\left(\Delta_{i,j}^n X^c\right)^2 - \frac{1}{T}\sum_{i=1}^{T}\int_{i-1}^{i}\sigma^2(t)dt + \frac{1}{T}\sum_{i=1}^{T}\int_{i-1}^{i}\sigma^2(t)dt - \eta \\
&= o_P(1),
\end{aligned}
$$

which follows from the classical theory of quadratic variation and the results derived above.

We have thus proved $\widehat{\eta}^c \xrightarrow{P} \eta$.

Now note that,

$$\sqrt{T}\left(\widehat{f}(\kappa) - f(\kappa)\right) = \sqrt{T}\left(\widehat{f}(\kappa) - \widehat{f}^c(\kappa)\right) + \sqrt{T}\left(\widehat{f}^c(\kappa) - f(\kappa)\right)$$

and

$$\sqrt{T}\left(\widehat{f}^c(\kappa) - f(\kappa)\right) = \frac{\sqrt{T}}{\widehat{\eta}^c}\sum_{j=1}^{6}\zeta_j(\kappa).$$

By using the same arguments as that used in Step 2 of the proof of Theorem 1, we obtain that $\sqrt{T} \parallel \widehat{f}(\kappa) - \widehat{f}^c(\kappa) \parallel = o_P(1)$ under the conditions of Theorem 5. Similarly, by the same arguments as that used in the proof of Theorem 2, we also have that $\sqrt{T}(\parallel \zeta_1(\kappa) \parallel + \parallel \zeta_2(\kappa) \parallel + \parallel \zeta_3(\kappa) \parallel + \parallel \zeta_6(\kappa) \parallel) = o_P(1)$ under the conditions of Theorem 5. It remains to deal with $\sqrt{T}\zeta_4(\kappa)/\widehat{\eta}^c$ and $\sqrt{T}\zeta_5(\kappa)/\widehat{\eta}^c$, whose treatments differ from those in Section C.2.

We treat $\sqrt{T}\zeta_4(\kappa)/\widehat{\eta}^c$ first. Because $\widehat{\eta}^c \xrightarrow{P} \eta$, it suffices to consider $\sqrt{T}\zeta_4(\kappa)$. Recall that,

$$\zeta_4(\kappa) := \frac{1}{T\ell\Delta}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,k-1}}^{t_{i,k}}\left[\sigma^2(t) - \sigma^2(i-1+\kappa)\right] dt.$$

By Assumption I-NS, we have that, instead of $\sigma^2(t)$, $\breve{\sigma}^2(t)$ follows,

$$\breve{\sigma}^2(t) = \breve{\sigma}^2(0) + \int_0^t \widetilde{\mu}(s)\, ds + \int_0^t \breve{\sigma}(s)\, dW(s) + \int_0^t \widetilde{\sigma}(s)\, d\widetilde{W}(s) + \int_0^t\int_{\mathbb{R}} x\, \widetilde{\nu}(ds, dx),$$

where the rest notations have the same interpretations as that in the main text. Using the shorthand notation $\widetilde{g}(t) = g(t - \lfloor t \rfloor)$ and $\widetilde{\gamma}(t) := \gamma(t - \lfloor t \rfloor)$ as given in Assumption I-NS, for $0 \le \lceil t \rceil \le k_1^*$, by Itô's formula, we have,

$$
\begin{aligned}
d\sigma^2(t) &= d(\widetilde{g}(t)\breve{\sigma}^2(t)) = \widetilde{g}'(t)\breve{\sigma}^2(t)dt + \widetilde{g}(t)d\breve{\sigma}^2(t) \\
&= (\widetilde{g}'(t)\breve{\sigma}^2(t) + \widetilde{g}(t)\widetilde{\mu}(t))\,dt \;+\; \widetilde{g}(t)\breve{\sigma}(t)\,dW(t) \;+\; \widetilde{g}(t)\widetilde{\sigma}(t)\,d\widetilde{W}(t) \;+\; \widetilde{g}(t)\int_{\mathbb{R}} x\,\widetilde{\nu}(dt,dx).
\end{aligned}
$$

Similarly, we have, for $k_2^* + 1 \le \lceil t \rceil \le T$,

$$
\begin{aligned}
d\sigma^2(t) = ((\widetilde{g}'(t) + \widetilde{\gamma}'(t))\breve{\sigma}^2(t) + (\widetilde{g}(t) + \widetilde{\gamma}(t))\widetilde{\mu}(t))\,dt \;+\; (\widetilde{g}(t) + \widetilde{\gamma}(t))\breve{\sigma}(t)\,dW(t) \\
+ (\widetilde{g}(t) + \widetilde{\gamma}(t))\widetilde{\sigma}(t)\,d\widetilde{W}(t) \;+\; (\widetilde{g}(t) + \widetilde{\gamma}(t))\int_{\mathbb{R}} x\,\widetilde{\nu}(dt,dx).
\end{aligned}
$$

Moreover, by the same arguments as above, when $t \in (i-1, i]$ for each $k_1^* + 1 \le i \le k_2^*$, $\sigma^2(t)$ follows,

$$
\begin{aligned}
d\sigma^2(t) =& ((\widetilde{g}'(t) + h_{\tau,n}(i/T)\widetilde{\gamma}'(t))\breve{\sigma}^2(t) + (\widetilde{g}(t) + h_{\tau,n}(i/T)\widetilde{\gamma}(t))\widetilde{\mu}(t))\,dt \\
&+ (\widetilde{g}(t) + h_{\tau,n}(i/T)\widetilde{\gamma}(t))\breve{\sigma}(t)\,dW(t) \;+\; (\widetilde{g}(t) + h_{\tau,n}(i/T)\widetilde{\gamma}(t))\widetilde{\sigma}(t)\,d\widetilde{W}(t) \\
&+ (\widetilde{g}(t) + h_{\tau,n}(i/T)\widetilde{\gamma}(t))\int_{\mathbb{R}} x\,\widetilde{\nu}(dt,dx).
\end{aligned}
$$

Therefore, because of the boundedness of $h_{\tau,n}$, $\widetilde{g}$, $\widetilde{\gamma}$, $\widetilde{g}'$ and $\widetilde{\gamma}'$, $\sigma^2(t)$ follows an Itô semi-martingale of the same type as (2) whenever $1 \le \lceil t \rceil \le k_1^*$ or $k_2^* + 1 \le \lceil t \rceil \le T$ or $t \in (i-1, i]$ for each $k_1^* + 1 \le i \le k_2^*$. Hence, by the same arguments as that used in dealing with term $\zeta_4(\kappa)$ in Step 1 of the proof of Theorem 2, we obtain that,

$$
E|\zeta_4(\kappa)|^2 \le C\left(\left(\frac{\ell}{n}\right)^2 \vee \frac{\ell}{nT}\right).
$$

This implies that $\sqrt{T} \parallel \zeta_4(\kappa) \parallel /\widehat{\eta}^c = o_P(1)$ under the conditions of Theorem 5.

We now turn to the dominant term $\sqrt{T}\zeta_5(\kappa)/\widehat{\eta}^c$. Recall that,

$$\zeta_5(\kappa) := \frac{1}{T}\sum_{i=1}^{T}\sigma^2(i-1+\kappa) - \frac{f(\kappa)}{T}\sum_{i=1}^{T}\int_{i-1}^{i}\sigma^2(t)dt.$$

First, note that,

$$E\left|\frac{1}{T}\sum_{i=k_1^*+1}^{k_2^*}\sigma^2(i-1+\kappa)\right|^2 \leq C\epsilon_n^2$$

and

$$E\left|\frac{f(\kappa)}{T}\sum_{i=k_1^*+1}^{k_2^*}\int_{i-1}^{i}\sigma^2(t)dt\right|^2 \leq C\epsilon_n^2.$$

Second, it is easy to see that,

$$\left|\frac{k_1^*}{T}g(\kappa) + \frac{T-k_2^*}{T}(g(\kappa)+\gamma(\kappa)) - (g(\kappa)+(1-\tau)\gamma(\kappa))\right| \leq C\epsilon_n.$$

Therefore, we obtain that

$$\begin{aligned}
&\zeta_5(\kappa)\\
&= \frac{1}{T}\sum_{i=1}^{k_1^*}(\sigma^2(i-1+\kappa)-g(\kappa)) + \frac{1}{T}\sum_{i=k_2^*+1}^{T}(\sigma^2(i-1+\kappa)-(g(\kappa)+\gamma(\kappa)))\\
&+\left(\frac{k_1^*}{T}g(\kappa)+\frac{T-k_2^*}{T}(g(\kappa)+\gamma(\kappa))-(g(\kappa)+(1-\tau)\gamma(\kappa))\right)+g(\kappa)+(1-\tau)\gamma(\kappa)-\\
&f(\kappa)\left[\frac{1}{T}\sum_{i=1}^{k_1^*}\left(\int_{i-1}^{i}\sigma^2(t)dt-\int_0^1 g(u)du\right)+\frac{1}{T}\sum_{i=k_2^*+1}^{T}\left(\int_{i-1}^{i}\sigma^2(t)dt-\int_0^1(g(u)+\gamma(u))du\right)\right.
\end{aligned}$$

$$+\left(\frac{k_1^*}{T}\int_0^1 g(u)du + \frac{T-k_2^*}{T}\int_0^1 (g(u)+\gamma(u))du - \int_0^1 (g(u)+(1-\tau)\gamma(u))du\right)$$

$$+\int_0^1 (g(u)+(1-\tau)\gamma(u))du\Bigg] + \frac{1}{T}\sum_{i=k_1^*+1}^{k_2^*}\sigma^2(i-1+\kappa) + \frac{f(\kappa)}{T}\sum_{i=k_1^*+1}^{k_2^*}\int_{i-1}^{i}\sigma^2(t)dt$$

$$=\frac{1}{T}\sum_{i=1}^{k_1^*}A_i(\kappa) + \frac{1}{T}\sum_{i=k_2^*+1}^{T}B_i(\kappa) + O(\epsilon_n) + O_P(\epsilon_n),$$

where $O(\epsilon_n)$ and $O_P(\epsilon_n)$ are in the L^2 norm, and $A_i(\kappa)$ and $B_i(\kappa)$ are given by (15) and (16), respectively. Because of $\widehat{\eta}^c \xrightarrow{P} \eta$, $\sqrt{T}\epsilon_n \to 0$, and the above results, $\sqrt{T}\zeta_5(\kappa)/\widehat{\eta}^c$ has the same limiting distribution as that of

$$\frac{1}{\eta\sqrt{T}}\sum_{i=1}^{k_1^*}A_i(\kappa) + \frac{1}{\eta\sqrt{T}}\sum_{i=k_2^*+1}^{T}B_i(\kappa).$$

Following the same method as that used in Section C.2, we approximate $A_i(\kappa)$ and $B_i(\kappa)$ by $\tilde{A}_{i,\infty}(\kappa)$ and $\tilde{B}_{i,\infty}(\kappa)$, where $\tilde{A}_{i,\infty}(\kappa) = \lim_{l\to\infty}\tilde{A}_{i,l}(\kappa)$, $\tilde{B}_{i,\infty}(\kappa) = \lim_{l\to\infty}\tilde{B}_{i,l}(\kappa)$,

$$\tilde{A}_{i,l}(\kappa) := \sum_{k=0}^{l-1}(E_iA_{i+k}(\kappa) - E_{i-1}A_{i+k}(\kappa)), \quad \text{and} \quad \tilde{B}_{i,l}(\kappa) := \sum_{k=0}^{l-1}(E_iB_{i+k}(\kappa) - E_{i-1}B_{i+k}(\kappa)).$$

By the same arguments as that used in Part 1 of the proof of Lemma 14, we easily obtain that,

$$E\left(\left|\frac{1}{\eta\sqrt{T}}\sum_{i=1}^{k_1^*}(A_i(\kappa) - \tilde{A}_{i,\infty}(\kappa))\right|^2 + \left|\frac{1}{\eta\sqrt{T}}\sum_{i=k_2^*+1}^{T}(B_i(\kappa) - \tilde{B}_{i,\infty}(\kappa))\right|^2\right) \le C/T.$$

Therefore, we only need to establish the L^2 functional limit theorem for

$$\frac{1}{\eta\sqrt{T}}\sum_{i=1}^{k_1^*}\tilde{A}_{i,\infty}(\kappa)+\frac{1}{\eta\sqrt{T}}\sum_{i=k_2^*+1}^{T}\tilde{B}_{i,\infty}(\kappa),$$

where $\tilde{A}_{i,\infty}(\kappa)$ and $\tilde{B}_{i,\infty}(\kappa)$ are martingale differences by again the same arguments as that used in Part 2 of the proof of Lemma 14. It follows from the same arguments as that used in Step 1 of Part 2 of the proof for Lemma 14 that,

$$\frac{1}{\eta^2 T}\sum_{i=1}^{k_1^*}E_{i-1}\left[\tilde{A}_{i,\infty}(\kappa)\tilde{A}_{i,\infty}(\kappa')\right]\xrightarrow{P}\frac{\tau}{\eta^2}E\left[\tilde{A}_{1,\infty}(\kappa)\tilde{A}_{1,\infty}(\kappa')\right],$$

where

$$\frac{1}{\eta^2}E\left[\tilde{A}_{1,\infty}(\kappa)\tilde{A}_{1,\infty}(\kappa')\right]=C_A(\kappa,\kappa');$$

and

$$\frac{1}{\eta^2 T}\sum_{i=k_2^*+1}^{T}E_{i-1}\left[\tilde{B}_{i,\infty}(\kappa)\tilde{B}_{i,\infty}(\kappa')\right]\xrightarrow{P}\frac{1-\tau}{\eta^2}E\left[\tilde{B}_{1,\infty}(\kappa)\tilde{B}_{1,\infty}(\kappa')\right],$$

where

$$\frac{1}{\eta^2}E\left[\tilde{B}_{1,\infty}(\kappa)\tilde{B}_{1,\infty}(\kappa')\right]=C_B(\kappa,\kappa').$$

An application of the same arguments as that used in Step 2 of the proof of Theorem 2 concludes the proof of the theorem.

C.6 Proof of Theorem 6

We first present a series of lemmas that are used in proving Theorem 6.

Lemma 15. *If Assumptions I(ii) and II with $q=4$ hold. Moreover, suppose $u_n=\beta\Delta^{\varpi}$*

for some $\beta > 0$ *and* $0 < \varpi < 1/2$. *Then, we have,*

$$E\left|\frac{1}{T}\sum_{i=1}^{T}A_i(\kappa)\right|^8 \leq \frac{C}{T^4}, \tag{C.17}$$

$$E\left|\frac{1}{T}\sum_{i=1}^{T}\left(\widehat{\sigma}_{i,\kappa}^2 - \widehat{\sigma}_{i,\kappa}^{2,c}\right)\right|^8 \leq C\left(\frac{u_n^{16}\Delta^{-7}}{(T\ell)^7} \vee u_n^{16}\Delta^{-\varsigma}\right), \tag{C.18}$$

$$E\left|\widehat{\eta} - \widehat{\eta}^c\right|^8 \leq Cu_n^{16}\Delta^{-\varsigma}, \tag{C.19}$$

for sufficiently small $\varsigma > 0$.

Proof of Lemma 15. First, we show (C.17). Note that summands of $(\sum_{i=1}^T A_i(\kappa))^8$ in general take the following form,

$$A_{i_1}^{e_1}(\kappa)A_{i_2}^{e_2}(\kappa)\cdots A_{i_j}^{e_j}(\kappa),$$

where $j \in \{1, 2, \ldots, 8\}$, $e_m \in \{1, 2, \ldots, 8\}$ for $m = 1, 2, \ldots, j$ such that $\sum_{m=1}^j e_m = 8$, and $i_m \in \{1, 2, \ldots, T\}$ for $m = 1, 2, \ldots, j$ such that $i_1 < i_2 < \ldots < i_j$.

For $1 \leq j \leq 4$, by Hölder's inequality and Assumption I(ii), we immediately obtain that, for any $\omega > 7$,

$$\begin{aligned}
&\frac{1}{T^8}\sum_{i_1=1}^{T}\sum_{i_2=i_1+1}^{T}\cdots\sum_{i_j=i_{j-1}+1}^{T}E\left|A_{i_1}^{e_1}(\kappa)A_{i_2}^{e_2}(\kappa)\cdots A_{i_j}^{e_j}(\kappa)\right| \\
\leq &\frac{1}{T^8}\sum_{i_1=1}^{T}\sum_{i_2=i_1+1}^{T}\cdots\sum_{i_j=i_{j-1}+1}^{T}\left(E\left|A_{i_1}(\kappa)^{\omega}\right|\right)^{e_1/\omega}\left(E\left|A_{i_2}(\kappa)^{\omega}\right|\right)^{e_2/\omega} \\
&\times\cdots\times\left(E\left|A_{i_j}(\kappa)^{\omega e_j/(\omega-\sum_{k=1}^{j-1}e_k)}\right|\right)^{(\omega-\sum_{k=1}^{j-1}e_k)/\omega} \leq \frac{C}{T^4}.
\end{aligned}$$

We shall only provide detailed calculations for the representative case $j = 6$ in what follows. The rest of the cases can be treated similarly. When $j = 6$, $(e_1, e_2, ..., e_6)$ can only be one of the two combinations $(1, 1, 1, 1, 1, 3)$ and $(1, 1, 1, 1, 2, 2)$ and their permutations. We shall show that,

$$\frac{1}{T^8} \sum_{i_1=1}^{T} \sum_{i_2=i_1+1}^{T} \cdots \sum_{i_6=i_5+1}^{T} \left| E\left(A_{i_1}^{e_{i_1}}(\kappa) A_{i_2}^{e_{i_2}}(\kappa) \cdots A_{i_6}^{e_6}(\kappa) \right) \right| \leq \frac{C}{T^4}. \tag{C.20}$$

Again, we only deal with two representative cases and the remaining cases can be treated by quite similar arguments. We first consider the case where $(e_1, e_2, ..., e_6) = (1, 1, 1, 1, 1, 3)$, by applying Hölder's inequality, Lemma 3.102 on page 497 of [7] and Assumption I(ii), we have that, for any $\omega > 2(4 + \iota)/(2 + \iota)$ where ι is given in Assumption II,

$$\begin{aligned}
&\left| E\left(A_{i_1}(\kappa) A_{i_2}(\kappa) A_{i_3}(\kappa) A_{i_4}(\kappa) A_{i_5}(\kappa) A_{i_6}^3(\kappa) \right) \right| \\
&= \left| E\left\{ A_{i_1}(\kappa) \left[E_{i_1}\left(A_{i_2}(\kappa) A_{i_3}(\kappa) A_{i_4}(\kappa) A_{i_5}(\kappa) A_{i_6}^3(\kappa) \right) - E\left(A_{i_2}(\kappa) A_{i_3}(\kappa) A_{i_4}(\kappa) A_{i_5}(\kappa) A_{i_6}^3(\kappa) \right) \right] \right\} \right| \\
&\leq \left(E\left| A_{i_1}(\kappa) \right|^{\omega} \right)^{1/\omega} \Bigg(E\Big| E_{i_1}\left(A_{i_2}(\kappa) A_{i_3}(\kappa) A_{i_4}(\kappa) A_{i_5}(\kappa) A_{i_6}^3(\kappa) \right) \\
&\quad - E\left(A_{i_2}(\kappa) A_{i_3}(\kappa) A_{i_4}(\kappa) A_{i_5}(\kappa) A_{i_6}^3(\kappa) \right) \Big|^{\omega/(\omega-1)} \Bigg)^{1-1/\omega} \\
&\leq C \alpha_{i_2 - i_1}^{1-2/\omega}
\end{aligned}$$

and

$$\begin{aligned}
&\left| E\left(A_{i_1}(\kappa) A_{i_2}(\kappa) A_{i_3}(\kappa) A_{i_4}(\kappa) A_{i_5}(\kappa) A_{i_6}^3(\kappa) \right) \right| \\
&\leq \left| E\left\{ A_{i_1}(\kappa) A_{i_2}(\kappa) A_{i_3}(\kappa) A_{i_4}(\kappa) A_{i_5}(\kappa) \left[E_{i_5}\left(A_{i_6}^3(\kappa) \right) - E\left(A_{i_6}^3(\kappa) \right) \right] \right\} \right| \\
&\quad + C \left| E\left(A_{i_1}(\kappa) A_{i_2}(\kappa) A_{i_3}(\kappa) A_{i_4}(\kappa) A_{i_5}(\kappa) \right) \right| \\
&\leq C \left(\alpha_{i_6 - i_5}^{1-2/\omega} + \alpha_{i_5 - i_4}^{1-2/\omega} \right).
\end{aligned}$$

Then, we have that, for any $\omega > 2(4+\iota)/(2+\iota)$ where ι is given in Assumption II,

$$\begin{aligned}
&\left|E\left(A_{i_1}(\kappa)A_{i_2}(\kappa)A_{i_3}(\kappa)A_{i_4}(\kappa)A_{i_5}(\kappa)A_{i_6}^3(\kappa)\right)\right| \\
\leq\ & C\left(\alpha_{i_2-i_1}^{1-2/\omega} \wedge \left(\alpha_{i_6-i_5}^{1-2/\omega} + \alpha_{i_5-i_4}^{1-2/\omega}\right)\right) \leq C\left(\alpha_{i_2-i_1}^{(1-2/\omega)/2}\alpha_{i_6-i_5}^{(1-2/\omega)/2} + \alpha_{i_2-i_1}^{(1-2/\omega)/2}\alpha_{i_5-i_4}^{(1-2/\omega)/2}\right).
\end{aligned}$$

Thus, by Assumption II with $q = 4$, we obtain that, for any $\omega > 2(4+\iota)/(2+\iota)$ where ι is given in Assumption II,

$$\begin{aligned}
&\frac{1}{T^8}\sum_{i_1=1}^{T}\sum_{i_2=i_1+1}^{T}\cdots\sum_{i_6=i_5+1}^{T}\left|E\left(A_{i_1}(\kappa)A_{i_2}(\kappa)A_{i_3}(\kappa)A_{i_4}(\kappa)A_{i_5}(\kappa)A_{i_6}^3(\kappa)\right)\right| \\
\leq\ & \frac{C}{T^6}\sum_{i_1=1}^{T}\sum_{i_2=i_1+1}^{T}\alpha_{i_2-i_1}^{(1-2/\omega)/2}\sum_{i_5=1}^{T}\sum_{i_6=i_5+1}^{T}\alpha_{i_6-i_5}^{(1-2/\omega)/2} + \frac{C}{T^6}\sum_{i_1=1}^{T}\sum_{i_2=i_1+1}^{T}\alpha_{i_2-i_1}^{(1-2/\omega)/2}\sum_{i_4=1}^{T}\sum_{i_5=i_4+1}^{T}\alpha_{i_5-i_4}^{(1-2/\omega)/2} \\
=\ & \frac{2C}{T^4}\left(\frac{T-1}{T}\alpha_1^{(1-2/\omega)/2} + \frac{T-2}{T}\alpha_2^{(1-2/\omega)/2} + \cdots + \frac{1}{T}\alpha_{T-1}^{(1-2/\omega)/2}\right)^2 \leq \frac{C}{T^4}.
\end{aligned}$$

As to the second case where $(e_1, e_2, ..., e_6) = (2, 1, 1, 1, 1, 2)$, we have that, for any $\omega > 2(4+\iota)/(2+\iota)$ where ι is given in Assumption II,

$$\begin{aligned}
&\left|E\left(A_{i_1}^2(\kappa)A_{i_2}(\kappa)A_{i_3}(\kappa)A_{i_4}(\kappa)A_{i_5}(\kappa)A_{i_6}^2(\kappa)\right)\right| \\
\leq\ & \left|E\left\{A_{i_1}^2(\kappa)A_{i_2}(\kappa)A_{i_3}(\kappa)A_{i_4}(\kappa)A_{i_5}(\kappa)\left[E_{i_5}\left(A_{i_6}^2(\kappa)\right) - E\left(A_{i_6}^2(\kappa)\right)\right]\right\}\right| \\
&+ C\left|E\left(A_{i_1}^2(\kappa)A_{i_2}(\kappa)A_{i_3}(\kappa)A_{i_4}(\kappa)A_{i_5}(\kappa)\right)\right| \\
\leq\ & C\left(\alpha_{i_6-i_5}^{1-2/\omega} + \alpha_{i_5-i_4}^{1-2/\omega}\right)
\end{aligned}$$

and

$$\left|E\left(A_{i_1}^2(\kappa)A_{i_2}(\kappa)A_{i_3}(\kappa)A_{i_4}(\kappa)A_{i_5}(\kappa)A_{i_6}^2(\kappa)\right)\right|$$

$$
\begin{aligned}
&\leq \left| E\left\{ A_{i_1}^2(\kappa) \left[E_{i_1}\left(A_{i_2}(\kappa) A_{i_3}(\kappa) A_{i_4}(\kappa) A_{i_5}(\kappa) A_{i_6}^2(\kappa) \right) - E\left(A_{i_2}(\kappa) A_{i_3}(\kappa) A_{i_4}(\kappa) A_{i_5}(\kappa) A_{i_6}^2(\kappa) \right) \right] \right\} \right| \\
&\quad + C \left| E\left(A_{i_2}(\kappa) A_{i_3}(\kappa) A_{i_4}(\kappa) A_{i_5}(\kappa) A_{i_6}^2(\kappa) \right) \right| \\
&\leq C \left(\alpha_{i_2-i_1}^{1-2/\omega} + \alpha_{i_3-i_2}^{1-2/\omega} \right).
\end{aligned}
$$

Then, we have that, for any $\omega > 2(4+\iota)/(2+\iota)$ where ι is given in Assumption II,

$$
\begin{aligned}
&\left| E\left(A_{i_1}^2(\kappa) A_{i_2}(\kappa) A_{i_3}(\kappa) A_{i_4}(\kappa) A_{i_5}(\kappa) A_{i_6}^2(\kappa) \right) \right| \\
&\leq C \left[\left(\alpha_{i_6-i_5}^{1-2/\omega} + \alpha_{i_5-i_4}^{1-2/\omega} \right) \wedge \left(\alpha_{i_2-i_1}^{1-2/\omega} + \alpha_{i_3-i_2}^{1-2/\omega} \right) \right] \\
&\leq C \left[\left(\alpha_{i_6-i_5}^{(1-2/\omega)/2} + \alpha_{i_5-i_4}^{(1-2/\omega)/2} \right) \left(\alpha_{i_2-i_1}^{(1-2/\omega)/2} + \alpha_{i_3-i_2}^{(1-2/\omega)/2} \right) \right].
\end{aligned}
$$

Thus, by Assumption II with $q = 4$, we obtain that, for any $\omega > 2(4+\iota)/(2+\iota)$ where ι is given in Assumption II,

$$
\begin{aligned}
&\frac{1}{T^8} \sum_{i_1=1}^{T} \sum_{i_2=i_1+1}^{T} \cdots \sum_{i_6=i_5+1}^{T} \left| E\left(A_{i_1}^2(\kappa) A_{i_2}(\kappa) A_{i_3}(\kappa) A_{i_4}(\kappa) A_{i_5}(\kappa) A_{i_6}^2(\kappa) \right) \right| \\
&\leq \frac{C}{T^6} \sum_{i_1=1}^{T} \sum_{i_2=i_1+1}^{T} \alpha_{i_2-i_1}^{(1-2/\omega)/2} \sum_{i_4=1}^{T} \sum_{i_5=i_4+1}^{T} \alpha_{i_5-i_4}^{(1-2/\omega)/2} + \frac{1}{T^6} \sum_{i_1=1}^{T} \sum_{i_2=i_1+1}^{T} \alpha_{i_2-i_1}^{(1-2/\omega)/2} \sum_{i_5=1}^{T} \sum_{i_6=i_5+1}^{T} \alpha_{i_6-i_5}^{(1-2/\omega)/2} \\
&\quad + \frac{C}{T^6} \sum_{i_2=1}^{T} \sum_{i_3=i_2+1}^{T} \alpha_{i_3-i_2}^{(1-2/\omega)/2} \sum_{i_4=1}^{T} \sum_{i_5=i_4+1}^{T} \alpha_{i_5-i_4}^{(1-2/\omega)/2} + \frac{1}{T^6} \sum_{i_2=1}^{T} \sum_{i_3=i_2+1}^{T} \alpha_{i_3-i_2}^{(1-2/\omega)/2} \sum_{i_5=1}^{T} \sum_{i_6=i_5+1}^{T} \alpha_{i_6-i_5}^{(1-2/\omega)/2} \\
&\leq \frac{C}{T^4}.
\end{aligned}
$$

The same arguments as above lead to that (C.20) holds for $(e_1, e_2, \ldots, e_6)$ taking any permutation of the six numbers in each of the two vectors $(1,1,1,1,1,3)$ and $(1,1,1,1,2,2)$.

By applying similar arguments to that used in dealing with the case $j = 6$, we obtain that, under Assumptions I(ii) and II with $q = 4$, for any $\omega > 2(4+\iota)/\iota$ where ι is given in

Assumption II,

$$\frac{1}{T^8}\sum_{i_1=1}^{T}\sum_{i_2=i_1+1}^{T}\cdots\sum_{i_5=i_4+1}^{T}\left|E\left(A_{i_1}^{e_1}(\kappa)A_{i_2}^{e_2}(\kappa)\cdots A_{i_5}^{e_5}(\kappa)\right)\right|\leq\frac{C}{T^4},$$
$$\frac{1}{T^8}\sum_{i_1=1}^{T}\sum_{i_2=i_1+1}^{T}\cdots\sum_{i_7=i_6+1}^{T}\left|E\left(A_{i_1}^{e_1}(\kappa)A_{i_2}^{e_2}(\kappa)\cdots A_{i_7}^{e_7}(\kappa)\right)\right|\leq\frac{C}{T^4},$$

and in particular,

$$\begin{aligned}
&\frac{1}{T^8}\sum_{i_1=1}^{T}\sum_{i_2=i_1+1}^{T}\cdots\sum_{i_8=i_7+1}^{T}\left|E\left(A_{i_1}(\kappa)A_{i_2}(\kappa)\cdots A_{i_8}(\kappa)\right)\right| \\
&\leq\frac{C}{T^6}\sum_{i_1=1}^{T}\sum_{i_2=i_1+1}^{T}\alpha_{i_2-i_1}^{(1-2/\omega)/4}\sum_{i_3=i_2+1}^{T}\alpha_{i_3-i_2}^{(1-2/\omega)/4}\sum_{i_6=1}^{T}\sum_{i_7=i_6+1}^{T}\alpha_{i_7-i_6}^{(1-2/\omega)/4}\sum_{i_8=i_7+1}^{T}\alpha_{i_8-i_7}^{(1-2/\omega)/4} \\
&\quad+\frac{C}{T^7}\sum_{i_1=1}^{T}\sum_{i_2=i_1+1}^{T}\alpha_{i_2-i_1}^{(1-2/\omega)/4}\sum_{i_3=i_2+1}^{T}\alpha_{i_3-i_2}^{(1-2/\omega)/4}\sum_{i_5=1}^{T}\sum_{i_6=i_5+1}^{T}\alpha_{i_6-i_5}^{(1-2/\omega)/4}\sum_{i_7=1}^{T}\sum_{i_8=i_7+1}^{T}\alpha_{i_8-i_7}^{(1-2/\omega)/4} \\
&\quad+\frac{C}{T^7}\sum_{i_1=1}^{T}\sum_{i_2=i_1+1}^{T}\alpha_{i_2-i_1}^{(1-2/\omega)/4}\sum_{i_3=1}^{T}\sum_{i_4=i_3+1}^{T}\alpha_{i_4-i_3}^{(1-2/\omega)/4}\sum_{i_6=1}^{T}\sum_{i_7=i_6+1}^{T}\alpha_{i_7-i_6}^{(1-2/\omega)/4}\sum_{i_8=i_7+1}^{T}\alpha_{i_8-i_7}^{(1-2/\omega)/4} \\
&\quad+\frac{C}{T^8}\sum_{i_1=1}^{T}\sum_{i_2=i_1+1}^{T}\alpha_{i_2-i_1}^{(1-2/\omega)/4}\sum_{i_3=1}^{T}\sum_{i_4=i_3+1}^{T}\alpha_{i_4-i_3}^{(1-2/\omega)/4}\sum_{i_5=1}^{T}\sum_{i_6=i_5+1}^{T}\alpha_{i_6-i_5}^{(1-2/\omega)/4}\sum_{i_7=1}^{T}\sum_{i_8=i_7+1}^{T}\alpha_{i_8-i_7}^{(1-2/\omega)/4} \\
&\leq\frac{C}{T^4}.
\end{aligned}$$

Overall, we have obtained the first result (C.17) of the lemma, i.e.,

$$E\left|\frac{1}{T}\sum_{i=1}^{T}A_i(\kappa)\right|^8\leq\frac{C}{T^4}.$$

Second, we prove inequality (C.18). Note that,

$$
\begin{aligned}
& E\left|\frac{1}{T}\sum_{i=1}^{T}\left(\widehat{\sigma}_{i,\kappa}^{2}-\widehat{\sigma}_{i,\kappa}^{2,c}\right)\right|^{8} \\
\leq\; & E\left|\frac{1}{T\ell\Delta}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\left[\left(\Delta_{i,k}^{n}X\right)^{2}1_{\left\{\left|\Delta_{i,k}^{n}X\right|\leq u_n\right\}}-\left(\Delta_{i,k}^{n}X^{c}\right)^{2}\right]1_{\left\{\Delta_{i,k}^{n}N=0\right\}}\right|^{8} \\
& + E\left|\frac{1}{T\ell\Delta}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\left[\left(\Delta_{i,k}^{n}X\right)^{2}1_{\left\{\left|\Delta_{i,k}^{n}X\right|\leq u_n\right\}}-\left(\Delta_{i,k}^{n}X^{c}\right)^{2}\right]1_{\left\{\Delta_{i,k}^{n}N>0\right\}}\right|^{8} \\
\leq\; & E\left|\frac{1}{T\ell\Delta}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\left(\Delta_{i,k}^{n}X^{c}\right)^{2}1_{\left\{\left|\Delta_{i,k}^{n}X^{c}\right|\geq u_n\right\}}\right|^{8} \\
& + E\left|\frac{1}{T\ell\Delta}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\left[\left(\Delta_{i,k}^{n}X\right)^{2}1_{\left\{\left|\Delta_{i,k}^{n}X\right|\leq u_n\right\}}-\left(\Delta_{i,k}^{n}X^{c}\right)^{2}\right]1_{\left\{\Delta_{i,k}^{n}N>0\right\}}\right|^{8}.
\end{aligned}
$$

We treat the two terms on the right hand side of the last inequality one by one. For notational convenience, we set,

$$
\mathcal{B}_{i,k}:=\left(\Delta_{i,k}^{n}X^{c}\right)^{2}1_{\left\{\left|\Delta_{i,k}^{n}X\right|\geq u_n\right\}}\quad\text{and}\quad\mathcal{B}_{i,k}^{J}:=\left[\left(\Delta_{i,k}^{n}X\right)^{2}1_{\left\{\left|\Delta_{i,k}^{n}X\right|\leq u_n\right\}}-\left(\Delta_{i,k}^{n}X^{c}\right)^{2}\right]1_{\left\{\Delta_{i,k}^{n}N>0\right\}}.
$$

Similar to the situation where we prove (C.17), summands of $\left(\sum_{i}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\mathcal{B}_{i,k}\right)^{8}$ and $\left(\sum_{i}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\mathcal{B}_{i,k}^{J}\right)^{8}$ in general take the forms,

$$
\mathcal{B}_{i_1,k_1}^{e_1}\mathcal{B}_{i_2,k_2}^{e_2}\cdots\mathcal{B}_{i_j,k_j}^{e_j}\quad\text{and}\quad(\mathcal{B}_{i_1,k_1}^{J})^{e_1}(\mathcal{B}_{i_2,k_2}^{J})^{e_2}\cdots(\mathcal{B}_{i_j,k_j}^{J})^{e_j},
$$

respectively, where $j\in\{1,2,\ldots,8\}$, $e_m\in\{1,2,\ldots,8\}$ for $m=1,2,\ldots,j$ such that $\sum_{m=1}^{j}e_m=8$, $i_m\in\{1,2,\ldots,T\}$ and $k_m\in\{j_\kappa-\ell+1,j_\kappa-\ell+2,\ldots,j_\kappa\}$ for $m=1,2,\ldots,j$. First, by Hölder's inequality, Burkholder-Davis-Gundy inequality, Markov inequality and

Assumption I(ii), we have that for any j, e_m's, i_m's, k_m's and $\omega > 16$,

$$
\begin{aligned}
& E\left|\mathcal{B}_{i_1,k_1}^{e_1}\mathcal{B}_{i_2,k_2}^{e_2}\cdots\mathcal{B}_{i_j,k_j}^{e_j}\right| \\
\leq & \left(E\left|\Delta_{i_1,k_1}^n X^c\right|^{\omega}\right)^{2e_1/\omega}\left(E\left|\Delta_{i_2,k_2}^n X^c\right|^{\omega}\right)^{2e_2/\omega}\cdots\left(E\left|\Delta_{i_j,k_j}^n X^c\right|^{\omega}\right)^{2e_j/\omega} \\
& \times\left(E1_{\left\{\left|\Delta_{i_1,k_1}^n X^c\right|\geq u_n\right\}}\right)^{(1-16/\omega)/j}\left(E1_{\left\{\left|\Delta_{i_2,k_2}^n X^c\right|\geq u_n\right\}}\right)^{(1-16/\omega)/j}\cdots\left(E1_{\left\{\left|\Delta_{i_j,k_j}^n X^c\right|\geq u_n\right\}}\right)^{(1-16/\omega)/j} \\
\leq & C\Delta^{8+(\omega-16)(1/2-\varpi)}.
\end{aligned}
$$

Thus, we have that, for any $\omega > 16$,

$$
E\left|\frac{1}{T\ell\Delta}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\left(\Delta_{i,k}^n X^c\right)^2 1_{\left\{\left|\Delta_{i,k}^n X^c\right|\geq u_n\right\}}\right|^8 \leq C\Delta^{(\omega-16)(1/2-\varpi)}.
$$

Second, by Hölder's inequality, Burkholder-Davis-Gundy inequality and Lemma 12, for $j > 1$, we have that, for any $\omega > 16$ and arbitrarily small $\varsigma > 0$,

$$
\begin{aligned}
& E\left|(\mathcal{B}_{i_1,k_1}^J)^{e_1}(\mathcal{B}_{i_2,k_2}^J)^{e_2}\cdots(\mathcal{B}_{i_j,k_j}^J)^{e_j}\right| \\
\leq & CE\Bigg\{\left|u_n^{2e_1}+\left(\Delta_{i_1,k_1}^n X^c\right)^{2e_1}\right|\left|u_n^{2e_2}+\left(\Delta_{i_2,k_2}^n X^c\right)^{2e_2}\right|\cdots\left|u_n^{2e_j}+\left(\Delta_{i_j,k_j}^n X^c\right)^{2e_j}\right| \\
& \times 1_{\left\{\Delta_{i_1,k_1}^n N\Delta_{i_2,k_2}^n N\cdots\Delta_{i_j,k_j}^n N>0\right\}}\Bigg\} \leq C\left(u_n^{16}\Delta^{j-\varsigma}\vee\Delta^{8+j(1-16/\omega)-\varsigma}\right).
\end{aligned}
$$

When $j = 1$, the above inequality holds by setting $\varsigma = 0$. Thus, by letting ς be sufficiently close to zero, we have that, for any $\omega > 16$,

$$
E\left|\frac{1}{T\ell\Delta}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\left[\left(\Delta_{i,k}^n X\right)^2 1_{\left\{\left|\Delta_{i,k}^n X\right|\leq u_n\right\}}-\left(\Delta_{i,k}^n X^c\right)^2\right]1_{\left\{\Delta_{i,k}^n N>0\right\}}\right|^8
$$

$$
\begin{aligned}
&\leq C\left(\frac{u_n^{16}\Delta^{-7}}{(T\ell)^7} \vee \frac{\Delta^{(1-16/\omega)}}{(T\ell)^7} \vee \frac{u_n^{16}\Delta^{-6-\varsigma}}{(T\ell)^6} \vee \frac{\Delta^{2(1-16/\omega)-\varsigma}}{(T\ell)^6} \vee \frac{u_n^{16}\Delta^{-5-\varsigma}}{(T\ell)^5}\right. \\
&\quad \vee \frac{\Delta^{3(1-16/\omega)-\varsigma}}{(T\ell)^5} \vee \frac{u_n^{16}\Delta^{-4-\varsigma}}{(T\ell)^4} \vee \frac{\Delta^{4(1-16/\omega)-\varsigma}}{(T\ell)^4} \vee \frac{u_n^{16}\Delta^{-3-\varsigma}}{(T\ell)^3} \vee \frac{\Delta^{5(1-16/\omega)-\varsigma}}{(T\ell)^3} \\
&\quad \left.\vee \frac{u_n^{16}\Delta^{-2-\varsigma}}{(T\ell)^2} \vee \frac{\Delta^{6(1-16/\omega)-\varsigma}}{(T\ell)^2} \vee \frac{u_n^{16}\Delta^{-1-\varsigma}}{T\ell} \vee \frac{\Delta^{7(1-16/\omega)-\varsigma}}{T\ell} \vee u_n^{16}\Delta^{-\varsigma} \vee \Delta^{8(1-16/\omega)-\varsigma}\right) \\
&\leq C\left(\frac{u_n^{16}\Delta^{-7}}{(T\ell)^7} \vee \frac{\Delta^{(1-16/\omega)}}{(T\ell)^7} \vee u_n^{16}\Delta^{-\varsigma} \vee \Delta^{8(1-16/\omega)-\varsigma}\right).
\end{aligned}
$$

To summarize, because $\sup_{t\in\mathbb{R}_+} E\left(e^{|\mu(t)|}\right)+\sup_{t\in\mathbb{R}_+} E\left(e^{|\sigma(t)|}\right)<\infty$ in Assumption I(ii) and by the same arguments as that immediately following (C.2), one can always choose a large enough ω and a small enough $\varsigma>0$ such that,

$$
E\left|\frac{1}{T}\sum_{i=1}^{T}\left(\widehat{\sigma}_{i,\kappa}^2-\widehat{\sigma}_{i,\kappa}^{2,c}\right)\right|^8 \leq C\left(\frac{u_n^{16}\Delta^{-7}}{(T\ell)^7} \vee u_n^{16}\Delta^{-\varsigma}\right).
$$

Third, the same arguments as above lead to the bound of $E\left|\widehat{\eta}-\widehat{\eta}^c\right|^8$ and are hence omitted. □

Denote

$$
\sigma_i^{2*} := \int_{i-1}^{i}\sigma^2(t)dt \quad \text{and} \quad \widehat{\sigma_i^{2*}} := \sum_{j=1}^{n}\left(\Delta_{i,j}^n X\right)^2 1_{\left\{\left|\Delta_{i,j}^n X\right|\leq u_n\right\}}.
$$

Lemma 16. *Suppose that all Assumptions in Lemma 15 and $\int_{\mathbb{R}}|x|^8\tilde{F}(dx)<\infty$ hold. Moreover, $T\asymp n^b$ and $\ell\asymp n^c$ for some positive exponents b and c which satisfy $c<1-b/2$ and $16\varpi+7(b+c)-7>0$, where $0<\varpi<1/2$. Then for any $\kappa\in[0,1]$, we have that, for sufficiently small $\varsigma>0$,*

$$
E\left|\widehat{f}(\kappa)\widehat{\eta}\widehat{\sigma_i^{2*}}-f(\kappa)\widehat{\eta}\sigma_i^{2*}\right|^4 \leq C\left(\frac{u_n^8\Delta^{-7/2}}{(T\ell)^{7/2}} \vee u_n^8\Delta^{-\varsigma} \vee \Delta^2 \vee \frac{1}{T^2}\right), \tag{C.21}
$$

$$E\left|\widehat{f}(\kappa)\widehat{\eta}\widehat{\sigma_i^{2*}}\right|^4 \leq C. \tag{C.22}$$

Proof of Lemma 16. We prove (C.21) first. By Cauchy-Schwarz inequality and triangular inequality, we have,

$$\begin{aligned}
E\left|\widehat{f}(\kappa)\widehat{\eta}\widehat{\sigma_i^{2*}} - f(\kappa)\widehat{\eta}\sigma_i^{2*}\right|^4 &\leq C\left(E\left|\widehat{f}(\kappa)\widehat{\eta} - f(\kappa)\widehat{\eta}\right|^8 E\left|\sigma_i^{2*}\right|^8\right)^{1/2} \\
&\quad + C\left(E\left|\widehat{\sigma_i^{2*}} - \sigma_i^{2*}\right|^8 E\left|\widehat{f}(\kappa)\widehat{\eta}\right|^8\right)^{1/2}.
\end{aligned}$$

Next, we treat the terms on the right hand side of the last inequality one by one. By Lemmas 11 and 15, similar arguments to that used in Step 1 of the proof of Theorem 2 in Section C.2, boundedness of f, Assumption I(ii), and conditions $c < 1 - b/2$ and $\int_{\mathbb{R}} |x|^8 \tilde{F}(dx) < \infty$, we have that, for sufficiently small $\varsigma > 0$,

$$\begin{aligned}
&E\left|\widehat{f}(\kappa)\widehat{\eta} - f(\kappa)\widehat{\eta}\right|^8 \\
&\leq CE\left|\frac{1}{T}\sum_{i=1}^{T}\widehat{\sigma}_{i,\kappa}^2 - \frac{1}{T}\sum_{i=1}^{T}\widehat{\sigma}_{i,\kappa}^{2,c}\right|^8 + CE\left|\frac{1}{T}\sum_{i=1}^{T}\widehat{\sigma}_{i,\kappa}^{2,c} - f(\kappa)\widehat{\eta}^c\right|^8 + CE\left|f(\kappa)\left(\widehat{\eta}^c - \widehat{\eta}\right)\right|^8 \\
&\leq CE\left|\frac{1}{T}\sum_{i=1}^{T}\widehat{\sigma}_{i,\kappa}^2 - \frac{1}{T}\sum_{i=1}^{T}\widehat{\sigma}_{i,\kappa}^{2,c}\right|^8 + C\sum_{i=1}^{6} E\left|\zeta_i(\kappa)\right|^8 + CE\left|\widehat{\eta}^c - \widehat{\eta}\right|^8 \\
&\leq C\left(\frac{u_n^{16}\Delta^{-7}}{(T\ell)^7} \vee u_n^{16}\Delta^{-\varsigma} \vee \Delta^4 \vee \frac{1}{T^4}\right).
\end{aligned}$$

By the same arguments as above, we obtain that, for sufficiently small $\varsigma > 0$,

$$E\left|\widehat{\sigma_i^{2*}} - \sigma_i^{2*}\right|^8 \leq C\left(u_n^{16}\Delta^{-\varsigma} \vee \Delta^4\right). \tag{C.23}$$

Next, applying Lemma 15 and condition $16\varpi + 7(b+c) - 7 > 0$, we immediately have,

$$E\left|\widehat{f}(\kappa)\widehat{\eta}\right|^8 \le CE\left|\frac{1}{T}\sum_{i=1}^{T}\left(\widehat{\sigma}_{i,\kappa}^2 - \widehat{\sigma}_{i,\kappa}^{2,c}\right)\right|^8 + CE\left|\frac{1}{T}\sum_{i=1}^{T}\widehat{\sigma}_{i,\kappa}^{2,c}\right|^8 \le C. \tag{C.24}$$

Lastly, by Jensen's inequality and Assumption I(ii), we have,

$$E\left|\sigma_i^{2*}\right|^8 \le E\left|\int_{i-1}^{i}\sigma^{16}(t)dt\right| = \int_{i-1}^{i}E\left(\sigma^{16}(t)\right)dt \le C.$$

Second, we derive (C.22). This result is trivial by the Cauchy-Schwarz inequality, (C.23), (C.24) and condition $16\varpi + 7(b+c) - 7 > 0$. □

The next lemma provides an estimate for the order of magnitude of the Hilbert-Schmidt norm of the difference between $\mathcal{K}$ and $\widehat{\mathcal{K}}$.

Lemma 17. *Suppose that all Assumptions in Lemma 16 hold, we have that, for sufficiently small* $\varsigma > 0$*,*

$$\begin{aligned}\|\widehat{\mathcal{K}} - \mathcal{K}\|_{\mathrm{HS}} &:= \left[\int_{[0,1]}\int_{[0,1]}\left(\widehat{C}(\kappa,\kappa') - C(\kappa,\kappa')\right)^2 d\kappa d\kappa'\right]^{1/2} \\ &= O_P\left[\left(\frac{1}{\ell^{1/2}} \vee \left(\frac{\ell}{n}\right)^{1/4} \vee \frac{u_n^2\Delta^{-3/4}}{\ell^{3/4}} \vee u_n^2\Delta^{-\varsigma} \vee \frac{u_n^2\Delta^{-7/8}}{(T\ell)^{7/8}} \vee \frac{1}{\sqrt{T}}\right)L_n \vee \frac{1}{L_n^3}\right].\end{aligned}$$

Proof of Lemma 17. Observe that,

$$\begin{aligned}&\int_{[0,1]}\int_{[0,1]}\left|C(\kappa,\kappa') - \widehat{C}(\kappa,\kappa')\right|^2 d\kappa d\kappa' \\ &\le C\int_{[0,1]}\int_{[0,1]}\frac{1}{\eta^4}\left[\left(E\left(A_1(\kappa)A_1(\kappa')\right) + \sum_{h=1}^{\infty}E\left(A_1(\kappa)A_{1+h}(\kappa') + A_{1+h}(\kappa)A_1(\kappa')\right)\right)\right.\end{aligned}$$

$$
\begin{aligned}
&-\left(\frac{1}{T}\sum_{i=1}^{T} A_i(\kappa)A_i(\kappa') + \sum_{h=1}^{L_n}\frac{1}{T-h}\sum_{i=1}^{T} A_i(\kappa)\left(A_{i+h}(\kappa') + A_{i-h}(\kappa')\right)\right)\Bigg]^2 d\kappa d\kappa' \\
&+\frac{C}{\widehat{\eta}^4}\int_{[0,1]}\int_{[0,1]}\Bigg[\frac{1}{T}\sum_{i=1}^{T} A_i(\kappa)A_i(\kappa') + \sum_{h=1}^{L_n}\frac{1}{T-h}\sum_{i=1}^{T} A_i(\kappa)\left(A_{i+h}(\kappa') + A_{i-h}(\kappa')\right) \\
&-\frac{1}{T}\sum_{i=1}^{T} \widehat{A}_i(\kappa)\widehat{A}_i(\kappa') - \sum_{h=1}^{L_n}\frac{1}{T-h}\sum_{i=1}^{T} \widehat{A}_i(\kappa)\left(\widehat{A}_{i+h}(\kappa') + \widehat{A}_{i-h}(\kappa')\right)\Bigg]^2 d\kappa d\kappa' \\
&+\frac{C(\eta^2-\widehat{\eta}^2)^2}{(\eta\widehat{\eta})^4}\int_{[0,1]}\int_{[0,1]}\Bigg[\frac{1}{T}\sum_{i=1}^{T} A_i(\kappa)A_i(\kappa') \\
&+\sum_{h=1}^{L_n}\frac{1}{T-h}\sum_{i=1}^{T} A_i(\kappa)\left(A_{i+h}(\kappa') + A_{i-h}(\kappa')\right)\Bigg]^2 d\kappa d\kappa' \\
&=: I + II + III.
\end{aligned}
$$

In the following, we estimate the order of each of the three terms on the right hand side of the last equality.

For term I, we have,

$$
\begin{aligned}
E\left|I\right| \leq\; & C\int_{[0,1]}\int_{[0,1]} E\left|\frac{1}{T}\sum_{i=1}^{T} A_i(\kappa)A_i(\kappa') - E\left(A_1(\kappa)A_1(\kappa')\right)\right|^2 d\kappa d\kappa' \\
&+ C\int_{[0,1]}\int_{[0,1]} L_n\sum_{h=1}^{L_n} E\Bigg[\frac{1}{T-h}\sum_{i=1}^{T} A_i(\kappa)\left(A_{i+h}(\kappa') + A_{i-h}(\kappa')\right) \\
&\quad - E\left(A_1(\kappa)A_{1+h}(\kappa') + A_{1+h}(\kappa)A_1(\kappa')\right)\Bigg]^2 d\kappa d\kappa' \\
&+ C\int_{[0,1]}\int_{[0,1]}\left[\sum_{h=L_n+1}^{\infty} E\left(A_1(\kappa)A_{1+h}(\kappa') + A_{1+h}(\kappa)A_1(\kappa')\right)\right]^2 d\kappa d\kappa' \\
=:\; & I_1 + I_2 + I_3,
\end{aligned}
$$

where the first inequality follows from Jensen's inequality. We shall treat the three terms on the right hand side of the last equality one by one. First, for term I_1, note that, for any $\omega > 2(4+\iota)/(3+\iota)$ where ι is given in Assumption II,

$$\begin{aligned}
& E\left|\frac{1}{T}\sum_{i=1}^{T}\left[A_i(\kappa)A_i(\kappa') - E\left(A_i(\kappa)A_i(\kappa')\right)\right]\right|^2 \\
\leq & \frac{C}{T^2}\sum_{i=1}^{T}\sum_{j=i}^{T}\left|Cov\left(A_i(\kappa)A_i(\kappa'), A_j(\kappa)A_j(\kappa')\right)\right| \\
\leq & \frac{C}{T^2}\sum_{i=1}^{T}\sum_{j=i}^{T}\alpha_{j-i}^{(\omega-2)/\omega}\left(E\left|A_i(\kappa)A_i(\kappa')\right|^{\omega}\right)^{1/\omega}\left(E\left|A_j(\kappa)A_j(\kappa')\right|^{\omega}\right)^{1/\omega} \leq \frac{C}{T},
\end{aligned}$$

where the second inequality follows from Assumption II and applying, e.g., Corollary 14.3 on page 212 of [3], and the last inequality follows from Assumptions I(ii) and II with $q = 4$. Thus, $I_1 = O(1/T)$. Similarly, we have $I_2 = O(L_n^2/T)$. We now turn to term I_3. By Assumptions I(ii) and II with $q = 4$, Hölder's inequality and applying Lemma 3.102 on page 497 of [7], we have that, for any $\omega > 2(4+\iota)/\iota$ where ι is given in Assumption II,

$$\begin{aligned}
& \left|\sum_{h=L_n+1}^{\infty} E\left(A_1(\kappa)A_{1+h}(\kappa') + A_{1+h}(\kappa)A_1(\kappa')\right)\right| \\
\leq & \sum_{h=L_n+1}^{\infty}\left[\left(E\left|A_1(\kappa)\right|^{\omega}\right)^{1/\omega}\left(E\left|E_1\left(A_{1+h}(\kappa')\right)\right|^{\omega/(\omega-1)}\right)^{1-1/\omega}\right. \\
& \left. + \left(E\left|A_1(\kappa')\right|^{\omega}\right)^{1/\omega}\left(E\left|E_1\left(A_{1+h}(\kappa)\right)\right|^{\omega/(\omega-1)}\right)^{1-1/\omega}\right] \\
\leq & C\sum_{h=L_n+1}^{\infty}\alpha_h^{1-2/\omega} \leq \frac{C}{L_n^3}.
\end{aligned}$$

Thus, $I_3 = O(1/L_n^6)$. To summarize, we obtain $I = O_P\left(L_n^2/T \vee 1/L_n^6\right)$.

Turning next to term II, by (C.7), we have $1/\widehat{\eta}^4 \xrightarrow{P} 1/\eta^4$. Hence, it remains to compute the order of the double integral component of II. To this end, by Jensen's inequality, we have,

$$
\begin{aligned}
II\widehat{\eta}^4 \leq C \int_{[0,1]} \int_{[0,1]} \Bigg\{ & \frac{1}{T} \sum_{i=1}^{T} \left(\widehat{A}_i(\kappa)\widehat{A}_i(\kappa') - A_i(\kappa)A_i(\kappa') \right)^2 \\
& + \sum_{h=1}^{L_n} \frac{L_n}{T-h} \sum_{i=1}^{T} \Bigg[\widehat{A}_i(\kappa) \left(\widehat{A}_{i+h}(\kappa') + \widehat{A}_{i-h}(\kappa') \right) \\
& - A_i(\kappa) \left(A_{i+h}(\kappa') + A_{i-h}(\kappa') \right) \Bigg]^2 \Bigg\} d\kappa d\kappa' =: II_1 + II_2.
\end{aligned}
$$

For term II_1, we have,

$$
\begin{aligned}
II_1 &\leq \int_{[0,1]} \int_{[0,1]} \frac{C}{T} \sum_{i=1}^{T} \Bigg[\left(\widehat{A}_i(\kappa) - A_i(\kappa) \right)^2 (A_i(\kappa'))^2 \\
&\quad + \left(\widehat{A}_i(\kappa') - A_i(\kappa') \right)^2 \left(\widehat{A}_i(\kappa) \right)^2 \Bigg] d\kappa d\kappa' \\
&\leq C \int_{[0,1]} \int_{[0,1]} \frac{1}{T} \sum_{i=1}^{T} \left(\widehat{\sigma}^2_{i,\kappa} - \sigma^2(i-1+\kappa) \right)^2 A_i^2(\kappa') d\kappa d\kappa' \\
&\quad + C \int_{[0,1]} \int_{[0,1]} \frac{1}{T} \sum_{i=1}^{T} \left(\widehat{f}(\kappa)\widehat{\sigma_i^{2*}} - f(\kappa)\sigma_i^{2*} \right)^2 A_i^2(\kappa') d\kappa d\kappa' \\
&\quad + C \int_{[0,1]} \int_{[0,1]} \frac{1}{T} \sum_{i=1}^{T} \left(\widehat{\sigma}^2_{i,\kappa'} - \sigma^2(i-1+\kappa') \right)^2 \left(\widehat{\sigma}^2_{i,\kappa} \right)^2 d\kappa d\kappa' \\
&\quad + C \int_{[0,1]} \int_{[0,1]} \frac{1}{T} \sum_{i=1}^{T} \left(\widehat{f}(\kappa')\widehat{\sigma_i^{2*}} - f(\kappa')\sigma_i^{2*} \right)^2 \left(\widehat{\sigma}^2_{i,\kappa} \right)^2 d\kappa d\kappa' \\
&\quad + C \int_{[0,1]} \int_{[0,1]} \frac{1}{T} \sum_{i=1}^{T} \left(\widehat{\sigma}^2_{i,\kappa'} - \sigma^2(i-1+\kappa') \right)^2 \left(\widehat{f}(\kappa)\widehat{\sigma_i^{2*}} \right)^2 d\kappa d\kappa'
\end{aligned}
$$

$$
\begin{aligned}
&+ C\int_{[0,1]}\int_{[0,1]}\frac{1}{T}\sum_{i=1}^{T}\left(\widehat{f}(\kappa')\widehat{\sigma_i^{2*}} - f(\kappa')\sigma_i^{2*}\right)^2\left(\widehat{f}(\kappa)\widehat{\sigma_i^{2*}}\right)^2 d\kappa d\kappa' \\
=:\ & II_{1,1} + II_{1,2} + \cdots + II_{1,6}.
\end{aligned}
$$

In what follows, we only treat two representative cases of the six terms on the right hand side of the last equality and the remaining terms can be dealt with quite similarly. For term $II_{1,1}$, we have that, for sufficiently small $\varsigma > 0$,

$$
\begin{aligned}
E\left|II_{1,1}\right| &\leq C\int_{[0,1]}\int_{[0,1]}\frac{1}{T}\sum_{i=1}^{T}\left(E\left|\widehat{\sigma}_{i,\kappa}^2 - \sigma^2(i-1+\kappa)\right|^4 E\left|A_i(\kappa')\right|^4\right)^{1/2} d\kappa d\kappa' \\
&\leq C\left(\frac{1}{\ell}\vee\left(\frac{\ell}{n}\right)^{1/2}\vee\frac{u_n^4\Delta^{-3/2}}{\ell^{3/2}}\vee u_n^4\Delta^{-\varsigma}\right),
\end{aligned}
$$

where the last inequality follows from decomposition (C.1) and similar arguments to that used in Step 1 of the proof of Theorem 2 in Section C.2 and in showing (C.18) in the proof of Lemma 15. For term $II_{1,2}$, by Lemma 16 and (C.7) that $\widehat{\eta} \xrightarrow{P} \eta$, we obtain that, for sufficiently small $\varsigma > 0$,

$$
\begin{aligned}
II_{1,2} &= \frac{C}{\widehat{\eta}^2}\int_{[0,1]}\int_{[0,1]}\frac{1}{T}\sum_{i=1}^{T}\left(\widehat{f}(\kappa)\widehat{\eta}\widehat{\sigma_i^{2*}} - f(\kappa)\widehat{\eta}\sigma_i^{2*}\right)^2 A_i^2(\kappa')d\kappa d\kappa' \\
&= O_P\left(\frac{u_n^4\Delta^{-7/4}}{(T\ell)^{7/4}}\vee u_n^4\Delta^{-\varsigma}\vee\Delta\vee\frac{1}{T}\right).
\end{aligned}
$$

By the same arguments as above, one obtains that terms $II_{1,3}$ and $II_{1,5}$ have the same order estimates as that of term $II_{1,1}$ and that terms $II_{1,4}$ and $II_{1,6}$ have the same order estimates as that of term $II_{1,2}$. To summarize, for sufficiently small $\varsigma > 0$,

$$II_1 = O_P\left(\frac{1}{\ell} \vee \left(\frac{\ell}{n}\right)^{1/2} \vee \frac{u_n^4\Delta^{-3/2}}{\ell^{3/2}} \vee \frac{u_n^4\Delta^{-7/4}}{(T\ell)^{7/4}} \vee u_n^4\Delta^{-\varsigma} \vee \frac{1}{T}\right).$$

By the same arguments as that used above in dealing with II_1, we have that, for sufficiently small $\varsigma > 0$,

$$II_2 = O_P\left[\left(\frac{1}{\ell} \vee \left(\frac{\ell}{n}\right)^{1/2} \vee \frac{u_n^4\Delta^{-3/2}}{\ell^{3/2}} \vee \frac{u_n^4\Delta^{-7/4}}{(T\ell)^{7/4}} \vee u_n^4\Delta^{-\varsigma} \vee \frac{1}{T}\right)L_n^2\right].$$

Therefore, for sufficiently small $\varsigma > 0$,

$$II = O_P\left[\left(\frac{1}{\ell} \vee \left(\frac{\ell}{n}\right)^{1/2} \vee \frac{u_n^4\Delta^{-3/2}}{\ell^{3/2}} \vee \frac{u_n^4\Delta^{-7/4}}{(T\ell)^{7/4}} \vee u_n^4\Delta^{-\varsigma} \vee \frac{1}{T}\right)L_n^2\right].$$

Lastly, we deal with term III. On the one hand, by the arguments immediately following (C.4), (C.7), (C.8) and (C.9) with ℓ being replaced with n, we have that, for sufficiently small $\varsigma > 0$,

$$\frac{(\widehat{\eta} - \eta)^2 (\widehat{\eta} + \eta)^2}{(\eta\widehat{\eta})^4} = O_P\left(\Delta^{4\varpi-\varsigma} \vee \Delta \vee \frac{1}{T}\right).$$

On the other hand, by Assumption I(ii) and applying Jensen's inequality, we obtain,

$$\begin{aligned}\int_{[0,1]}\int_{[0,1]} E\Bigg[&\frac{1}{T}\sum_{i=1}^{T} A_i(\kappa)A_i(\kappa') \\ &+ \sum_{h=1}^{L_n}\frac{1}{T-h}\sum_{i=1}^{T} A_i(\kappa)\left(A_{i+h}(\kappa') + A_{i-h}(\kappa')\right)\Bigg]^2 d\kappa d\kappa'\end{aligned}$$

$$\leq \int_{[0,1]}\int_{[0,1]}\left[\frac{C}{T}\sum_{i=1}^{T} E\left|A_i(\kappa)A_i(\kappa')\right|^2 + \sum_{h=1}^{L_n}\frac{CL_n}{T-h}\sum_{i=1}^{T} E\left|A_i(\kappa)\left(A_{i+h}(\kappa')+A_{i-h}(\kappa')\right)\right|^2\right]$$
$$d\kappa d\kappa'$$
$$\leq CL_n^2.$$

Therefore, for sufficiently small $\varsigma > 0$,

$$III = O_P\left[\left(\Delta^{4\varpi-\varsigma}\vee\Delta\vee\frac{1}{T}\right)L_n^2\right],$$

and this completes the proof. □

We are now ready to present the proof of Theorem 6.

Proof of Theorem 6. We shall apply Theorem 1.8.4 of [11] to prove Theorem 6. This proof consists of two parts. In the first part, we establish finite-dimensional convergence in law. In the second part, asymptotic finite-dimensionality (or tightness) is established.

Part I. Finite-dimensional convergence in law.

We shall show that $\langle\mathcal{G}(\widehat{\mathcal{K}}), h\rangle \xrightarrow{d} \langle\mathcal{G}(\mathcal{K}), h\rangle$ for every $h \in \mathcal{L}^2$. For notational convenience, define,

$$G_n := \langle\mathcal{G}(\widehat{\mathcal{K}}), h\rangle \quad \text{and} \quad G := \langle\mathcal{G}(\mathcal{K}), h\rangle.$$

Then, G_n's are, conditionally on $\mathscr{F}$, zero-mean normal random variables with conditional variances,

$$\sigma_{G_n}^2 := \int_0^1\int_0^1 \widehat{C}(u,v)h(u)h(v)dudv.$$

Similarly, G is a zero-mean normal random variable with variance,

$$\sigma_G^2 := \int_0^1 \int_0^1 C(u,v)h(u)h(v)dudv.$$

Hence, it suffices to show $\sigma_{G_n}^2 \xrightarrow{P} \sigma_G^2$. This follows directly from the following fact,

$$\begin{aligned}
|\sigma_{G_n}^2 - \sigma_G^2| &\leq \int_0^1 \int_0^1 |\widehat{C}(u,v) - C(u,v)||h(u)||h(v)|dudv \\
&\leq C \left(\int_0^1 \int_0^1 |\widehat{C}(u,v) - C(u,v)|^2 dudv \right)^{1/2} \\
&= C \parallel \widehat{\mathcal{K}} - \mathcal{K} \parallel_{\mathrm{HS}} = o_P(1),
\end{aligned}$$

where the second inequality follows from the Cauchy-Schwarz inequality and that $h \in \mathcal{L}^2$, and the last equality follows from Lemma 17 and the conditions, particularly (20), of Theorem 6.

Part II. Asymptotical finite-dimensionality.

Let $(e_j)_{j\geq 1}$ be the orthonormal basis of $\mathcal{L}^2$ that consists of all eigenfunctions of $\mathcal{K}$. We shall show that, for all $\delta, \varepsilon > 0$, there exists a J such that,

$$\limsup_n P\left(\sum_{j\geq J} \langle \mathcal{G}(\widehat{\mathcal{K}}), e_j \rangle^2 > \delta \right) < \varepsilon.$$

In what follows, we need to deal with the integrability of $1/\widehat{\eta}$. We take care of this issue by decomposing the above probability as follows,

$$P\left(\sum_{j\geq J} \langle \mathcal{G}(\widehat{\mathcal{K}}), e_j \rangle^2 > \delta \right) \leq P\left(\sum_{j\geq J} \langle \mathcal{G}(\widehat{\mathcal{K}}), e_j \rangle^2 > \delta, 1/\widehat{\eta} < 2/\eta \right) + P\left(1/\widehat{\eta} \geq 2/\eta\right).$$

The second term on the right hand side of the above inequality can be made arbitrarily small by letting n be sufficiently large because $1/\widehat{\eta} \xrightarrow{P} 1/\eta$. Hence, we only need to focus on the first term on the right hand side of the above inequality. We treat it as follows.

Note that,

$$\begin{aligned}
&P\left(\sum_{j\geq J}\langle \mathcal{G}(\widehat{\mathcal{K}}), e_j\rangle^2 > \delta, 1/\widehat{\eta} < 2/\eta\right) = E\left\{1_{\{1/\widehat{\eta}<2/\eta\}} P\left(\sum_{j\geq J}\langle \mathcal{G}(\widehat{\mathcal{K}}), e_j\rangle^2 > \delta \middle| \mathscr{F}\right)\right\} \\
&\leq \frac{1}{\delta} E\left(1_{\{1/\widehat{\eta}<2/\eta\}} E\left(\sum_{j\geq J}\langle \mathcal{G}(\widehat{\mathcal{K}}), e_j\rangle^2 \middle| \mathscr{F}\right)\right) \\
&\leq \underbrace{\frac{1}{\delta} E\left(1_{\{1/\widehat{\eta}<2/\eta\}} E\left(\sum_{j\geq J}\langle \mathcal{G}(\widehat{\mathcal{K}}), e_j\rangle^2 - \sum_{j\geq J}\langle \mathcal{G}(\mathcal{K}), e_j\rangle^2 \middle| \mathscr{F}\right)\right)}_{\text{(I)}} + \underbrace{\frac{1}{\delta} E\left(\sum_{j\geq J}\langle \mathcal{G}(\mathcal{K}), e_j\rangle^2\right)}_{\text{(II)}}.
\end{aligned}$$

Because

$$E\left(\sum_{j=1}^{\infty}\langle \mathcal{G}(\mathcal{K}), e_j\rangle^2\right) = E\left(\| \mathcal{G}(\mathcal{K}) \|^2\right) = \int_0^1 C(u,u)du < \infty,$$

term (II) can be made arbitrarily small by letting J be sufficiently large. It remains to treat term (I). This can be done as follows,

$$\begin{aligned}
&\frac{1}{\delta} E\left(1_{\{1/\widehat{\eta}<2/\eta\}} E\left(\sum_{j\geq J}\langle \mathcal{G}(\widehat{\mathcal{K}}), e_j\rangle^2 - \sum_{j\geq J}\langle \mathcal{G}(\mathcal{K}), e_j\rangle^2 \middle| \mathscr{F}\right)\right) \\
&= \frac{1}{\delta} E\left(1_{\{1/\widehat{\eta}<2/\eta\}} E\left(\| \mathcal{G}(\widehat{\mathcal{K}}) \|^2 - \| \mathcal{G}(\mathcal{K}) \|^2 \middle| \mathscr{F}\right)\right) + \\
&\quad \frac{1}{\delta} E\left(1_{\{1/\widehat{\eta}<2/\eta\}} E\left(\sum_{1\leq j<J}\langle \mathcal{G}(\mathcal{K}), e_j\rangle^2 - \sum_{1\leq j<J}\langle \mathcal{G}(\widehat{\mathcal{K}}), e_j\rangle^2 \middle| \mathscr{F}\right)\right).
\end{aligned}$$

We deal with the two terms on the right hand side of the above equality one by one. First,

$$\begin{aligned}&\left|\frac{1}{\delta}E\left(1_{\{1/\widehat{\eta}<2/\eta\}}E\left(\|\ \mathcal{G}(\widehat{\mathcal{K}})\ \|^2-\|\ \mathcal{G}(\mathcal{K})\ \|^2\middle|\ \mathscr{F}\right)\right)\right|\\ \leq\ &\frac{1}{\delta}E\left(1_{\{1/\widehat{\eta}<2/\eta\}}\int_0^1|\widehat{C}(u,u)-C(u,u)|du\right)=o(1),\end{aligned}$$

because by exactly the same calculations in the proof of Lemma 17 and noting that $1_{\{1/\widehat{\eta}<2/\eta\}}(\widehat{\eta}^2-\eta^2)^2/(\eta^4\widehat{\eta}^4)\leq 9/\eta^4$, one obtains that, under the conditions of Theorem 6,

$$E\left(|\widehat{C}(u,v)-C(u,v)|1_{\{1/\widehat{\eta}<2/\eta\}}\right)\leq C/n^{\iota},$$

where $\iota>0$ and does not depend on u and v. Second,

$$\begin{aligned}&\left|\frac{1}{\delta}E\left(1_{\{1/\widehat{\eta}<2/\eta\}}E\left(\sum_{1\leq j<J}\langle\mathcal{G}(\mathcal{K}),e_j\rangle^2-\sum_{1\leq j<J}\langle\mathcal{G}(\widehat{\mathcal{K}}),e_j\rangle^2\middle|\ \mathscr{F}\right)\right)\right|\\ =&\left|\frac{1}{\delta}E\left(1_{\{1/\widehat{\eta}<2/\eta\}}\sum_{1\leq j<J}\int_0^1\int_0^1(C(u,v)-\widehat{C}(u,v))e_j(u)e_j(v)dudv\right)\right|\\ \leq&\ \frac{C}{\delta}\sum_{j=1}^{J-1}E\int_0^1\int_0^1 1_{\{1/\widehat{\eta}<2/\eta\}}|C(u,v)-\widehat{C}(u,v)|^2dudv\leq\frac{(J-1)C}{\delta}\times o(1),\end{aligned}$$

where the first inequality follows from the Cauchy-Schwarz inequality and the second inequality follows because by again the same calculations in the proof of Lemma 17 and noting that $1_{\{1/\widehat{\eta}<2/\eta\}}(\widehat{\eta}^2-\eta^2)^2/(\eta^4\widehat{\eta}^4)\leq 9/\eta^4$, one obtains that, under the conditions of Theorem 6,

$$E\left(|\widehat{C}(u,v)-C(u,v)|^2 1_{\{1/\widehat{\eta}<2/\eta\}}\right)\leq C/n^{\iota},$$

where $\iota > 0$ and does not depend on u and v. We have thus proved that asymptotical finite-dimensionality holds.

To sum up, Theorem 1.8.4 of [11] leads to the conclusion of Theorem 6, completing the proof. $\square$

C.7 Proof of Corollary 7

The result follows straightforwardly from Lemma 17 and Theorem 2.

C.8 Derivation of the optimal order of ℓ in Section 5

Recall that $T \asymp n^b$ and $\ell \asymp n^c$. As discussed in Section 5, there is a bias-variance tradeoff related to the choice of the order of ℓ by minimizing the order of the sum of terms II and III. To this end, we provide the solution to the optimal choice of c for each fixed b. Note that,

$$\text{term II} = O_P\left(\frac{1}{n^{(b+c)/2}}\right) \quad \text{and} \quad \text{term III} = O_P\left(\frac{1}{n^{1-c}} \vee \frac{1}{n^{(1+b-c)/2}}\right).$$

One readily finds,

$$\text{term III} = \begin{cases} O_P\left(\frac{1}{n^{1-c}}\right), & \text{if } b+c \geq 1, \\ O_P\left(\frac{1}{n^{(1+b-c)/2}}\right), & \text{if } b+c < 1. \end{cases}$$

We first give an initial analysis of the optimal choice of c for minimizing the order of the sum of terms II and III when there is no additional constraint other than $b+c \geq 1$ or $b+c < 1$. When $b+c \geq 1$, the order of the sum of terms II and III could achieve its minimum value at $c = (2-b)/3$, which occurs only if $b \geq 1/2$ because $b+c \geq 1$. When $b+c \geq 1$ and $b < 1/2$ which imply $(2-b)/3 < 1-b$, one should choose c as close to $1-b$

as possible to minimize the order of the sum of terms II and III. Similarly, when $b+c<1$, the order of the sum of terms II and III could achieve its minimum value at $c=1/2$, which occurs only if $b<1/2$ because $b+c<1$. When $b+c<1$ and $b\geq 1/2$ which imply $1-b\leq 1/2$, one should choose c as close to $1-b$ as possible to minimize the order of the sum of terms II and III.

We are now ready to present a formal analysis of the optimal choice of c for a given b under the conditions of Theorem 2. We shall take ϖ as given. Then feasible values of b are given by the interval $(0,4\varpi)$, and feasible values of c are given by the interval (c_L,c_U), where,

$$c_L := \max\{1-4\varpi, 0\} \quad \text{and} \quad c_U := 1-\frac{b}{2}.$$

It is easy to verify that $1-b<c_U$ for any fixed $b\in(0,4\varpi)$; $1-b\leq(2-b)/3$ and $c_L<(2-b)/3<c_U$ for any fixed $b\geq 1/2$ if possible; and $1-b>c_L$ for any fixed $b<1$. Based on the previous initial analysis, we make the following exhaustive list of different situations and analyze them one by one.

- Case I. When $b\geq 1/2$.

  This case can be further divided into the following subcases based on whether $(1-b)$ lies in the interval (c_L,c_U), i.e., $1-b\leq c_L$, $1-b\in(c_L,c_U)$ and $1-b\geq c_U$. This is due to that the relation between $b+c$ and 1 is required to determine the order of the sum of terms II and III. Nonetheless, only $1-b\leq c_L$ when $b\geq 1$ and $c_L<1-b<c_U$ when $b\in[1/2,1)$ are possible according to the above analysis. Therefore, it is easy to find that $c_{opt}=(2-b)/3$ in this case.

- Case II. When $b<1/2$.

Similarly, this case is further divided into the following subcases: $1-b \leq c_L$, $1-b \geq c_U$, and $c_L < 1-b < c_U$. Nonetheless, only $c_L < 1-b < c_U$ is possible according to the above analysis. We need to further consider whether $c_L < 1/2$ or $c_L \geq 1/2$, because $1-b$ and hence c_U are always greater than $1/2$ when $b < 1/2$. Therefore, it is easy to find that $c_{opt} = 1/2$ when $c_L < 1/2$; and $c_{opt} = c_L+$ when $c_L \geq 1/2$, where c_L+ means that one should choose a value of c that is as close to c_L as possible from above.

In summary, we have the following exhaustive list of different cases with the corresponding optimal c values,

$$c_{\text{opt}} = \begin{cases} \frac{2-b}{3}, \text{ when } b \geq 1/2 \ , \\ \frac{1}{2}, \quad \text{ when } b < 1/2 \text{ and } c_L < 1/2, \\ c_L+, \quad \text{ when } b < 1/2 \text{ and } c_L \geq 1/2. \end{cases}$$

C.9 Proof of Theorem 8

To prove the theorem, we first introduce some additional notation and establish some preliminary results. We define,

$$\begin{aligned} d_{i,k}^n &:= \left(\Delta_{i,k}^n X\right)^2 1_{\left\{|\Delta_{i,k}^n X| \leq u_n\right\}} - \left(\Delta_{i,k}^n X^c\right)^2 \\ &= |\Delta_{i,k}^n X^c + \Delta_{i,k}^n X^J|^2 1_{\left\{|\Delta_{i,k}^n X| \leq u_n\right\}} - \left(\Delta_{i,k}^n X^c\right)^2, \end{aligned}$$

where the last equality follows from (B.2). By applying the elementary inequality $||x+y|^2 - |x|^2| \leq C(|y|^2 + |x||y|)$ to cases $|\Delta_{i,k}^n X^c| \geq u_n/2$ and $|\Delta_{i,k}^n X^c| < u_n/2$ separately, we obtain that, for any $m, l > 0$,

$$
\begin{aligned}
|d_{i,k}^n| \le\ & C\frac{|\Delta_{i,k}^n X^c|^{2+l}}{u_n^l} + C\frac{|\Delta_{i,k}^n X^c|^2|\Delta_{i,k}^n X^J|^m}{u_n^m} \\
& + C(|\Delta_{i,k}^n X^J| \wedge u_n)^2 + C|\Delta_{i,k}^n X^c|(|\Delta_{i,k}^n X^J| \wedge u_n) \\
=: & |d_{i,k}^{n,1}| + |d_{i,k}^{n,2}| + |d_{i,k}^{n,3}| + |d_{i,k}^{n,4}|. \qquad \text{(C.25)}
\end{aligned}
$$

We next present two lemmas which are used in proving Theorem 8.

Lemma 18. *Under model* (1)*, assume that Assumption I and condition* (B.1) *hold. Then,*

$$E_{t_{i,k-1}}|\Delta_{i,k}^n X^J| \le C\int_{t_{i,k-1}}^{t_{i,k}} E_{t_{i,k-1}}\chi(s)ds, \qquad \text{(C.26)}$$

$$E_{t_{i,k-1}}(|\Delta_{i,k}^n X^J| \wedge u_n) \le Cu_n^{1-s}\int_{t_{i,k-1}}^{t_{i,k}} E_{t_{i,k-1}}\chi(s)ds, \text{ for } r \le s \le 1. \qquad \text{(C.27)}$$

Hence, $E|\Delta_{i,k}^n X^J| \le C\Delta$ *and* $E(|\Delta_{i,k}^n X^J| \wedge u_n) \le C\Delta u_n^{1-s}$ *for* $r \le s \le 1$.

Proof of Lemma 18. First, because $E\int_0^t\int_{\mathbb{R}}|x|\chi(t)dtF(dx) \le \infty$ for each $t > 0$, by Proposition II.1.28 of [7], we have that $\int_0^t\int_{\mathbb{R}}|x|\nu(ds,dx) - \int_0^t\int_{\mathbb{R}}|x|\chi(t)dtF(dx)$ is a local martingale. Furthermore, since $\int_0^t\int_{\mathbb{R}}|x|\nu(ds,dx)$ and $\int_0^t\int_{\mathbb{R}}|x|\chi(t)dtF(dx)$ are increasing and locally integrable, $\int_0^t\int_{\mathbb{R}}|x|\nu(ds,dx) - \int_0^t\int_{\mathbb{R}}|x|\chi(t)dtF(dx)$ is in fact a martingale. Hence, we obtain that,

$$
\begin{aligned}
E_{t_{i,k-1}}(|\Delta_{i,k}^n X^J|) = E_{t_{i,k-1}}\left|\int_{t_{i,k-1}}^{t_{i,k}}\int_{\mathbb{R}} x\nu(ds,dx)\right| &\le E_{t_{i,k-1}}\int_{t_{i,k-1}}^{t_{i,k}}\int_{\mathbb{R}}|x|\chi(t)dtF(dx) \\
&\le C\int_{t_{i,k-1}}^{t_{i,k}} E_{t_{i,k-1}}\chi(t)dt.
\end{aligned}
$$

Therefore, (C.26) follows.

Second, (C.27) follows from the same arguments as above and the same calculations used in proving (6.26) of [6] with $\alpha_n = u_n/\sqrt{\Delta}$. □

Lemma 19. *Under model* (1)*, assume that Assumption I and condition* (B.1) *hold. Then, for any $r \leq s \leq 1$ and arbitrarily small $\varsigma > 0$,*

$$E|d_{i,k}^n|^2 \leq C\left(\frac{\Delta^{3-\varsigma}}{u_n} + \Delta u_n^{4-s} + \Delta^{2-\varsigma}u_n^{2-s}\right), \tag{C.28}$$

$$E|d_{i,k}^n||d_{i',k'}^n| \leq C\left(\frac{\Delta^{4-\varsigma}}{u_n^2} + \Delta^{3-\varsigma}u_n^{1-s-\varsigma} + \Delta^{2-\varsigma}u_n^{4-2s-\varsigma}\right), \tag{C.29}$$

where $i \neq i'$ or $k \neq k'$.

Proof of Lemma 19. We prove (C.28) and (C.29) in Parts I and II, respectively.

Part I.

We first prove (C.28). From (C.25), we have that,

$$E|d_{i,k}^n|^2 \leq C(E|d_{i,k}^{n,1}|^2 + E|d_{i,k}^{n,2}|^2 + E|d_{i,k}^{n,3}|^2 + E|d_{i,k}^{n,4}|^2).$$

We next treat the four terms on the right hand side of the above inequality one by one.

For $E|d_{i,k}^{n,1}|^2$, by Burkholder-Davis-Gundy inequality, we have,

$$E|d_{i,k}^{n,1}|^2 = CE\frac{|\Delta_{i,k}^n X^c|^{4+2l}}{u_n^{2l}} \leq C_l\frac{\Delta^{2+l}}{u_n^{2l}},$$

which can be made arbitrarily small by letting l be sufficiently large.

For $E|d_{i,k}^{n,2}|^2$, by (C.26), Hölder's inequality, Burkholder-Davis-Gundy inequality and Jensen's inequality, we have that, for any $m \in (0, 1/2)$ and any $\omega > 1/(1-2m)$ such that

$2m\frac{\omega}{\omega-1} < 1$,

$$
\begin{aligned}
E|d_{i,k}^{n,2}|^2 &= CE\frac{|\Delta_{i,k}^n X^c|^4|\Delta_{i,k}^n X^J|^{2m}}{u_n^{2m}} \\
&\leq \frac{C}{u_n^{2m}}\left(E|\Delta_{i,k}^n X^c|^{4\omega}\right)^{\frac{1}{\omega}}\left(E|\Delta_{i,k}^n X^J|^{2m\frac{\omega}{\omega-1}}\right)^{\frac{\omega-1}{\omega}} \leq C_m\frac{\Delta^{2+2m}}{u_n^{2m}},
\end{aligned}
$$

where C_m is a constant that depends on m. Note that $C_m \to \infty$ as $m \to 1/2$. Therefore, for arbitrarily small $\varsigma > 0$, we can always find a m close to $1/2$ enough such that,

$$
E|d_{i,k}^{n,2}|^2 \leq C\frac{\Delta^{3-\varsigma}}{u_n}.
$$

As to $E|d_{i,k}^{n,3}|^2$, by using the inequality $(|x| \wedge u_n)^p \leq u_n^{p-1}(|x| \wedge u_n)$ for $p \geq 1$ and (C.27), we have,

$$
E|d_{i,k}^{n,3}|^2 = CE(|\Delta_{i,k}^n X^J| \wedge u_n)^4 \leq C\Delta u_n^{4-s}.
$$

Lastly, for $E|d_{i,k}^{n,4}|^2$, by Hölder's inequality and the same arguments as above, we have that, for any $r \leq s \leq 1$ and any $\omega > 1$,

$$
\begin{aligned}
E|d_{i,k}^{n,4}|^2 &= CE|\Delta_{i,k}^n X^c|^2(|\Delta_{i,k}^n X^J| \wedge u_n)^2 \\
&\leq C\left(E|\Delta_{i,k}^n X^c|^{2\omega}\right)^{\frac{1}{\omega}}\left(E(|\Delta_{i,k}^n X^J| \wedge u_n)^{2\frac{\omega}{\omega-1}}\right)^{\frac{\omega-1}{\omega}} \\
&\leq C_\omega\Delta(u_n^{2\frac{\omega}{\omega-1}-1}\Delta u_n^{1-s})^{\frac{\omega-1}{\omega}},
\end{aligned}
$$

where C_ω is a constant that depends on ω. Note that $C_\omega \to \infty$ as $\omega \to \infty$. Hence, for arbitrarily small $\varsigma > 0$, we can always find a sufficiently large ω such that,

$$E|d_{i,k}^{n,4}|^2 \leq C\Delta^{2-\varsigma}u_n^{2-s}$$

for any $r \leq s \leq 1$

To sum up, we have that, for arbitrarily small $\varsigma > 0$ and any $r \leq s \leq 1$,

$$E|d_{i,k}^{n}|^2 \leq C\left(\frac{\Delta^{3-\varsigma}}{u_n} + \Delta u_n^{4-s} + \Delta^{2-\varsigma}u_n^{2-s}\right).$$

Part II.

We now prove (C.29). In Part I, we have proved that $d_{i,k}^{n,1}$ can be made arbitrarily small. Hence, in determining the order of $|d_{i,k}^{n}||d_{i',k'}^{n}|$, one only needs to consider the following nine cross-product terms $|d_{i,k}^{n,2}||d_{i',k'}^{n,2}|$, $|d_{i,k}^{n,2}||d_{i',k'}^{n,3}|$, $|d_{i',k'}^{n,2}||d_{i,k}^{n,3}|$, $|d_{i,k}^{n,2}||d_{i',k'}^{n,4}|$, $|d_{i',k'}^{n,2}||d_{i,k}^{n,4}|$, $|d_{i,k}^{n,3}||d_{i',k'}^{n,3}|$, $|d_{i,k}^{n,3}||d_{i',k'}^{n,4}|$, $|d_{i',k'}^{n,3}||d_{i,k}^{n,4}|$, and $|d_{i,k}^{n,4}||d_{i',k'}^{n,4}|$. Without loss of generality, we assume that $t_{i,k} \leq t_{i',k'-1}$ and deal with these terms one by one.

We deal with $|d_{i,k}^{n,2}||d_{i',k'}^{n,2}|$ first. For any $0 < m < 1$, any $\omega_1 > \frac{1}{1-m}$ and any $\omega_2 > \frac{1}{1-\omega_1 m/(\omega_1-1)}$ such that $m\frac{\omega_1}{\omega_1-1}\frac{\omega_2}{\omega_2-1} < 1$, by Hölder's inequality, Burkholder-Davis-Gundy inequality, Jensen's inequality and (C.26), we have,

$$\begin{aligned}
&E|d_{i,k}^{n,2}||d_{i',k'}^{n,2}| = CE\frac{|\Delta_{i,k}^n X^c|^2|\Delta_{i,k}^n X^J|^m}{u_n^m}\frac{|\Delta_{i',k'}^n X^c|^2|\Delta_{i',k'}^n X^J|^m}{u_n^m} \\
&\leq \frac{C}{u_n^{2m}}\left(E|\Delta_{i,k}^n X^c|^{2\omega_1}|\Delta_{i',k'}^n X^c|^{2\omega_1}\right)^{\frac{1}{\omega_1}}\left(E|\Delta_{i,k}^n X^J|^{m\frac{\omega_1}{\omega_1-1}}|\Delta_{i',k'}^n X^J|^{m\frac{\omega_1}{\omega_1-1}}\right)^{\frac{\omega_1-1}{\omega_1}} \\
&\leq \frac{C\Delta^2}{u_n^{2m}}\left(E\left\{|\Delta_{i,k}^n X^J|^{m\frac{\omega_1}{\omega_1-1}}E_{t_{i',k'-1}}|\Delta_{i',k'}^n X^J|^{m\frac{\omega_1}{\omega_1-1}}\right\}\right)^{\frac{\omega_1-1}{\omega_1}} \\
&\leq \frac{C\Delta^2\Delta^m}{u_n^{2m}}\left(E\left\{|\Delta_{i,k}^n X^J|^{m\frac{\omega_1}{\omega_1-1}}\left|\frac{1}{\Delta}\int_{t_{i',k'-1}}^{t_{i',k'}}E_{t_{i',k'-1}}\chi(s)ds\right|^{m\frac{\omega_1}{\omega_1-1}}\right\}\right)^{\frac{\omega_1-1}{\omega_1}}
\end{aligned}$$

$$\leq \frac{C\Delta^2\Delta^m}{u_n^{2m}} \left(\left(E|\Delta_{i,k}^n X^J|^{m\frac{\omega_1}{\omega_1-1}\frac{\omega_2}{\omega_2-1}} \right)^{\frac{\omega_2-1}{\omega_2}} \left(E\left| \frac{1}{\Delta}\int_{t_{i',k'-1}}^{t_{i',k'}} E_{t_{i',k'-1}}\chi(s)ds \right|^{m\frac{\omega_1\omega_2}{\omega_1-1}} \right)^{\frac{1}{\omega_2}} \right)^{\frac{\omega_1-1}{\omega_1}}$$
$$\leq \frac{C\Delta^2\Delta^{2m}}{u_n^{2m}}.$$

Therefore, for arbitrarily small $\varsigma > 0$, one can always find a m close to 1 enough such that,

$$E|d_{i,k}^{n,2}||d_{i',k'}^{n,2}| \leq C\frac{\Delta^{4-\varsigma}}{u_n^2}.$$

As to $|d_{i,k}^{n,2}||d_{i',k'}^{n,3}|$, for any $0 < m < 1$, any $r \leq s \leq 1$ and any $\omega > \frac{1}{1-m}$ such that $m\frac{\omega}{\omega-1} < 1$, by (C.26) and (C.27), we have,

$$E|d_{i,k}^{n,2}||d_{i',k'}^{n,3}| = CE\frac{|\Delta_{i,k}^n X^c|^2|\Delta_{i,k}^n X^J|^m}{u_n^m}(|\Delta_{i',k'}^n X^J| \wedge u_n)^2$$
$$\leq CE\left\{|\Delta_{i,k}^n X^c|^2|\Delta_{i,k}^n X^J|^m E_{t_{i',k'-1}}(|\Delta_{i',k'}^n X^J| \wedge u_n)\right\}$$
$$\leq C\Delta u_n^{1-s}E\left\{\left(\frac{1}{\Delta}\int_{t_{i',k'-1}}^{t_{i',k'}} E_{t_{i',k'-1}}\chi(s)ds\right)|\Delta_{i,k}^n X^c|^2|\Delta_{i,k}^n X^J|^m\right\}$$
$$\leq C\Delta u_n^{1-s}\left(E\left(\frac{1}{\Delta}\int_{t_{i',k'-1}}^{t_{i',k'}} E_{t_{i',k'-1}}\chi(s)ds\right)^{\omega}|\Delta_{i,k}^n X^c|^{2\omega}\right)^{\frac{1}{\omega}}\left(E|\Delta_{i,k}^n X^J|^{m\frac{\omega}{\omega-1}}\right)^{\frac{\omega-1}{\omega}}$$
$$\leq C\Delta^{2+m}u_n^{1-s}.$$

Therefore, for arbitrarily small $\varsigma > 0$, one can always find a m close to 1 enough such that,

$$E|d_{i,k}^{n,2}||d_{i',k'}^{n,3}| \leq C\Delta^{3-\varsigma}u_n^{1-s}$$

for any $r \leq s \leq 1$.

As to $|d^{n,2}_{i',k'}||d^{n,3}_{i,k}|$, for any $r \leq s \leq 1$, any $0 < m < 1$ and any $\omega_2 > \omega_1 > \frac{1}{1-m}$ such that $m\frac{\omega_2}{\omega_2-1} < m\frac{\omega_1}{\omega_1-1} < 1$, by (C.26) and (C.27), we have,

$$
\begin{aligned}
&E|d^{n,2}_{i',k'}||d^{n,3}_{i,k}| = CE\frac{|\Delta^n_{i',k'}X^c|^2|\Delta^n_{i',k'}X^J|^m}{u_n^m}(|\Delta^n_{i,k}X^J| \wedge u_n)^2 \\
&\leq CE\left\{(|\Delta^n_{i,k}X^J| \wedge u_n)^m E_{t_{i',k'-1}}|\Delta^n_{i',k'}X^c|^2|\Delta^n_{i',k'}X^J|^m\right\} \\
&\leq CE\left\{(|\Delta^n_{i,k}X^J| \wedge u_n)^m \left(E_{t_{i',k'-1}}|\Delta^n_{i',k'}X^c|^{2\omega_1}\right)^{\frac{1}{\omega_1}} \left(E_{t_{i',k'-1}}|\Delta^n_{i',k'}X^J|^{m\frac{\omega_1}{\omega_1-1}}\right)^{\frac{\omega_1-1}{\omega_1}}\right\} \\
&\leq C\Delta^m E\left\{(|\Delta^n_{i,k}X^J| \wedge u_n)^m \left(E_{t_{i',k'-1}}|\Delta^n_{i',k'}X^c|^{2\omega_1}\right)^{\frac{1}{\omega_1}} \left|\frac{1}{\Delta}\int_{t_{i',k'-1}}^{t_{i',k'}} E_{t_{i',k'-1}}\chi(s)ds\right|^m\right\} \\
&\leq C\Delta^m \left(E\left(E_{t_{i',k'-1}}|\Delta^n_{i',k'}X^c|^{2\omega_1}\right)^{\frac{\omega_2}{\omega_1}} \left|\frac{1}{\Delta}\int_{t_{i',k'-1}}^{t_{i',k'}} E_{t_{i',k'-1}}\chi(s)ds\right|^{m\omega_2}\right)^{\frac{1}{\omega_2}} \\
&\qquad\qquad\qquad \times \left(E(|\Delta^n_{i,k}X^J| \wedge u_n)^{m\frac{\omega_2}{\omega_2-1}}\right)^{\frac{\omega_2-1}{\omega_2}} \\
&\leq C\Delta^{1+2m}u_n^{(1-s)m}.
\end{aligned}
$$

Therefore, for arbitrarily small $\varsigma > 0$, one can always find a m close to 1 enough such that,

$$
E|d^{n,2}_{i',k'}||d^{n,3}_{i,k}| \leq C\Delta^{3-\varsigma}u_n^{1-s-\varsigma},
$$

for any $r \leq s \leq 1$.

By repeatedly using the same arguments as above, we obtain that, for $0 < m < 1$ being sufficiently close to 1,

$$
\begin{aligned}
E|d^{n,2}_{i,k}||d^{n,4}_{i',k'}| &= CE\frac{|\Delta^n_{i,k}X^c|^2|\Delta^n_{i,k}X^J|^m}{u_n^m}|\Delta^n_{i',k'}X^c|(|\Delta^n_{i',k'}X^J| \wedge u_n) \leq C\Delta^{3+\frac{1}{2}-\varsigma}u_n^{-s-\varsigma}, \\
E|d^{n,2}_{i',k'}||d^{n,4}_{i,k}| &= CE\frac{|\Delta^n_{i',k'}X^c|^2|\Delta^n_{i',k'}X^J|^m}{u_n^m}|\Delta^n_{i,k}X^c|(|\Delta^n_{i,k}X^J| \wedge u_n) \leq C\Delta^{3+\frac{1}{2}-\varsigma}u_n^{-s-\varsigma},
\end{aligned}
$$

$$E|d_{i,k}^{n,3}||d_{i',k'}^{n,3}| = CE(|\Delta_{i,k}^n X^J| \wedge u_n)^2(|\Delta_{i',k'}^n X^J| \wedge u_n)^2 \leq C\Delta^{2-\varsigma}u_n^{4-2s-\varsigma},$$
$$E|d_{i,k}^{n,3}||d_{i',k'}^{n,4}| = CE(|\Delta_{i,k}^n X^J| \wedge u_n)^2|\Delta_{i',k'}^n X^c|(|\Delta_{i',k'}^n X^J| \wedge u_n) \leq C\Delta^{2+\frac{1}{2}-\varsigma}u_n^{3-2s-\varsigma}$$
$$E|d_{i',k'}^{n,3}||d_{i,k}^{n,4}| = CE(|\Delta_{i',k'}^n X^J| \wedge u_n)^2|\Delta_{i,k}^n X^c|(|\Delta_{i,k}^n X^J| \wedge u_n) \leq C\Delta^{2+\frac{1}{2}-\varsigma}u_n^{3-2s-\varsigma}$$
$$E|d_{i,k}^{n,4}||d_{i',k'}^{n,4}| = CE|\Delta_{i,k}^n X^c|(|\Delta_{i,k}^n X^J| \wedge u_n)|\Delta_{i',k'}^n X^c|(|\Delta_{i',k'}^n X^J| \wedge u_n) \leq C\Delta^{3-\varsigma}u_n^{2-2s-\varsigma}$$

for any $r \leq s \leq 1$ and arbitrarily small $\varsigma > 0$.

To sum up, we have proved that, for arbitrarily small $\varsigma > 0$ and any $r \leq s \leq 1$,

$$E|d_{i,k}^n||d_{i',k'}^n| \leq C\left(\frac{\Delta^{4-\varsigma}}{u_n^2} + \Delta^{3-\varsigma}u_n^{1-s-\varsigma} + \Delta^{2-\varsigma}u_n^{4-2s-\varsigma}\right)$$

for $i \neq i'$ or $k \neq k'$.

$\square$

We are now ready to present the proof of Theorem 8. The proof differs from that of Theorem 2 in the calculation of the order of term $\| \frac{1}{T}\sum_{i=1}^T \left(\widehat{\sigma}_{i,\kappa}^2 - \widehat{\sigma}_{i,\kappa}^{2,c}\right) \|$ in Step 2 of Section C.1 and in the calculation of the order of term $\zeta_4(\kappa)$. We deal with them separately in two parts.

Part I. Term $\| \frac{1}{T}\sum_{i=1}^T \left(\widehat{\sigma}_{i,\kappa}^2 - \widehat{\sigma}_{i,\kappa}^{2,c}\right) \|$.

Using the same notation as that in Step 2 of Section C.1, we define,

$$\mathcal{S} := \{(i,i',k,k')|i \neq i' \text{ or } k \neq k', \text{ where } i,i' = 1,2,...,T \text{ and } k,k' = j_\kappa - \ell + 1,...,j_\kappa\}.$$

Applying Lemma 19, we have that, for arbitrarily small $\varsigma > 0$ and any $r \leq s \leq 1$,

$$
\begin{aligned}
& E\left\{\frac{1}{T}\sum_{i=1}^{T}\left(\widehat{\sigma}_{i,\kappa}^{2}-\widehat{\sigma}_{i,\kappa}^{2,c}\right)\right\}^{2} \\
= & \frac{1}{(T\ell\Delta)^{2}} E\left\{\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\left[\left(\Delta_{i,k}^{n}X\right)^{2} 1_{\{|\Delta_{i,k}^{n}X|\leq u_n\}}-\left(\Delta_{i,k}^{n}X^{c}\right)^{2}\right]\right\}^{2} \\
\leq & \frac{1}{(T\ell\Delta)^{2}} \sum_{(i,i',k,k')\in\mathcal{S}} E(|d_{i,k}^{n}||d_{i',k'}^{n}|)+\frac{1}{(T\ell\Delta)^{2}}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa} E(|d_{i,k}^{n}|^{2}) \\
\leq & \frac{C}{\Delta^{2}}\left(\frac{\Delta^{4-\varsigma}}{u_n^{2}}+\Delta^{3-\varsigma}u_n^{1-s-\varsigma}+\Delta^{2-\varsigma}u_n^{4-2s-\varsigma}\right)+\frac{C}{T\ell\Delta^{2}}\left(\frac{\Delta^{3-\varsigma}}{u_n}+\Delta u_n^{4-s}+\Delta^{2-\varsigma}u_n^{2-s}\right).
\end{aligned}
$$

By letting $s = r$, we obtain that, for arbitrarily small $\varsigma > 0$,

$$
\begin{aligned}
& E\left\{\frac{1}{T}\sum_{i=1}^{T}\left(\widehat{\sigma}_{i,\kappa}^{2}-\widehat{\sigma}_{i,\kappa}^{2,c}\right)\right\}^{2} \\
\leq & C\left(n^{-2+2\varpi+\varsigma}+n^{-1-(1-r)\varpi+\varsigma}+n^{-(4-2r)\varpi+\varsigma}\right)+\frac{C}{T\ell}\left(n^{(\varpi-1)+\varsigma}+n^{1-\varpi(4-r)}+n^{-\varpi(2-r)+\varsigma}\right).
\end{aligned}
$$

The same bound is obtained for $E|\widehat{\eta}^{c}-\widehat{\eta}|^{2}$. Therefore, for the same results in Theorem 2 to hold, in addition to other conditions, one needs to ensure that, for arbitrarily small $\varsigma > 0$,

$$
T\left(n^{-2+2\varpi+\varsigma}+n^{-1-(1-r)\varpi+\varsigma}+n^{-(4-2r)\varpi+\varsigma}\right)+\frac{1}{\ell}\left(n^{(\varpi-1)+\varsigma}+n^{1-\varpi(4-r)}+n^{-\varpi(2-r)+\varsigma}\right)=o(1).
$$

This amounts to

$$
c > 1-\varpi(4-r),
$$

and for arbitrarily small $\varsigma > 0$,

$$
b < \min\{2-2\varpi-\varsigma, 1+(1-r)\varpi-\varsigma, (4-2r)\varpi-\varsigma\}.
$$

Hence, it suffices to require that,

$$b < \min\{2 - 2\varpi, 1 + (1-r)\varpi, (4-2r)\varpi\} \quad \text{and} \quad c > 1 - \varpi(4-r).$$

Part II. Term $\zeta_4(\kappa)$.

Recall the definition of $\zeta_4(\kappa)$,

$$\zeta_4(\kappa) := \frac{1}{T\ell\Delta}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,k-1}}^{t_{i,k}}(\sigma^2(t) - \sigma^2(i-1+\kappa))dt.$$

Under the assumption that the volatility process $\sigma^2(t)$ is rough, we calculate the order of term $\zeta_4(\kappa)$ as follows,

$$\begin{aligned}
E|\zeta_4(\kappa)|^2 &\leq E\left(\frac{1}{T\ell\Delta}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,k-1}}^{t_{i,k}}(\sigma^2(t) - \sigma^2(i-1+\kappa))dt\right)^2 \\
&\leq \frac{1}{T\ell\Delta}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,k-1}}^{t_{i,k}}E(\sigma^2(t) - \sigma^2(i-1+\kappa))^2dt \\
&\leq C|i-1+\kappa - t_{i,j_\kappa-\ell}|^{2H} \leq C\left(\frac{\ell}{n}\right)^{2H},
\end{aligned}$$

where the third inequality follows from (B.3). Therefore, under the conditions (B.4), we have $\sqrt{T} \parallel \zeta_4(\kappa) \parallel = o_P(1)$.

Combining the above results and the results for the rest terms in the proof of Theorem 2 completes the proof.

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