Appendix

1 Proofs of consistency and the central limit theorem

We begin by establishing some standard results regarding the asymptotic behavior of the singular values of $\mathbf{P}$, $\mathbf{A}$ and $\mathbf{A}-\mathbf{P}$. Recall that for $\mathbf{Z} \sim F$, the minimal dimensionality condition from Assumption 1 states that the random vector $\mathbf{X} = \phi(\mathbf{Z})$ has second moment matrix $\boldsymbol{\Delta} = \mathbb{E}(\mathbf{X}\mathbf{X}^\top)$ with full rank d. Therefore, a combination of a Hoeffding-style argument and a corollary of Weyl's inequalities (Horn and Johnson (2012), Corollary 7.3.5) shows that the d non-zero singular values $\sigma_i(\mathbf{P})$ satisfy $\sigma_i(\mathbf{P}) = \Omega(n)$ almost surely. By showing that the spectral norm of $\mathbf{A}-\mathbf{P}$ has smaller asymptotic growth, we can once again invoke Weyl's argument to show that the top d singular values $\sigma_i(\mathbf{A})$ also satisfy $\sigma_i(\mathbf{A}) = \Omega(n)$.

Proposition 1. $\|\mathbf{A}-\mathbf{P}\| = \mathrm{O}\left(n^{1/2}\log^{\alpha+1/2}(n)\right)$ almost surely.

Proof. We will make use of a matrix analogue of the Bernstein inequality (Tropp (2015), Theorem 1.6.2):

Theorem 3 (Matrix Bernstein)**.** Let $\mathbf{M}_1, \ldots, \mathbf{M}_n \in \mathbb{R}^{n\times n}$ be symmetric independent random matrices satisfying $\mathbb{E}(\mathbf{M}_k) = 0$ and $\|\mathbf{M}_k\| \le L$ for each $1 \le k \le n$, for some fixed value L.

Let $\mathbf{M} = \sum_{k=1}^{n} \mathbf{M}_k$ and let $v(\mathbf{M}) = \|\mathbb{E}(\mathbf{M}\mathbf{M}^\top)\|$ denote the matrix variance statistic of $\mathbf{M}$. Then for all $t \ge 0$:

$$\mathbb{P}(\|\mathbf{M}\| \ge t) \le 2n \exp\left(\frac{-t^2/2}{v(\mathbf{M}) + Lt/3}\right).$$

We apply this theorem as follows: for each $1 \leq i \leq j \leq n$, let $\mathbf{M}_{ij}$ be the $n \times n$ matrix with $(i,j)^{\text{th}}$ and $(j,i)^{\text{th}}$ entries equal to $\mathbf{A}_{ij} - \mathbf{P}_{ij}$, and all other entries equal to 0. Then

$$\|\mathbf{M}_{ij}\| = |\mathbf{A}_{ij} - \mathbf{P}_{ij}| < 2\beta \log^{\alpha}(n)$$

almost surely, and $\mathbb{E}(\mathbf{M}_{ij}) = 0$, and so the matrix $\mathbf{M} = \sum \mathbf{M}_{ij} = \mathbf{A} - \mathbf{P}$ satisfies the criteria for Bernstein's theorem.

To bound the matrix variance statistic $v(\mathbf{M})$, observe that

$$(\mathbf{M}\mathbf{M}^{\top})_{ij} = \sum_{k=1}^{n} (\mathbf{A}_{ik} - \mathbf{P}_{ik})(\mathbf{A}_{jk} - \mathbf{P}_{jk}),$$

and thus

$$\mathbb{E}\{(\mathbf{M}\mathbf{M}^{\top})_{ij}\} = \begin{cases} \displaystyle\sum_{k=1}^{n} \operatorname{Var}(\mathbf{A}_{ij}) & i = j \\ 0 & i \neq j \end{cases}$$

By Popoviciu's inequality, the variances $\operatorname{Var}(\mathbf{A}_{ij})$ are bounded in absolute value by $\beta^2 \log^{2\alpha}(n)$, and so, since the matrix $\mathbb{E}(\mathbf{M}\mathbf{M}^{\top})$ is diagonal, we see that

$$v(\mathbf{M}) \leq \beta^2 n \log^{2\alpha}(n)$$

almost surely, and after substituting into Theorem 3 we find that for any $t \geq 0$,

$$\mathbb{P}(\|\mathbf{A} - \mathbf{P}\| \geq t) \leq 2n \exp\left(\frac{-3t^2}{6\beta^2 n \log^{2\alpha}(n) + 4\beta \log^{\alpha}(n) t}\right)$$

almost surely.

The numerator of the exponential term dominates for n sufficiently large if $t = cn^{1/2} \log^{\alpha+1/2}(n)$, and therefore $\|\mathbf{A} - \mathbf{P}\| = \mathrm{O}\left(n^{1/2} \log^{\alpha+1/2}(n)\right)$ almost surely, as required. $\square$

The following result follows from an identical argument as that used in the proof of Lemma 17 in Lyzinski et al. (2016):

Proposition 2. $\|\mathbf{U}_{\mathbf{P}}^\top(\mathbf{A}-\mathbf{P})\mathbf{U}_{\mathbf{P}}\|_F = \mathrm{O}\left(\log^{\alpha+1/2}(n)\right)$ almost surely.

Proposition 3. The following bounds hold almost surely:

i. $\|\mathbf{U}_{\mathbf{A}}\mathbf{U}_{\mathbf{A}}^\top - \mathbf{U}_{\mathbf{P}}\mathbf{U}_{\mathbf{P}}^\top\| = \mathrm{O}\left(n^{-1/2}\log^{\alpha+1/2}(n)\right)$;

ii. $\|\mathbf{U}_{\mathbf{A}} - \mathbf{U}_{\mathbf{P}}\mathbf{U}_{\mathbf{P}}^\top\mathbf{U}_{\mathbf{A}}\|_F = \mathrm{O}\left(n^{-1/2}\log^{\alpha+1/2}(n)\right)$;

iii. $\|\mathbf{U}_{\mathbf{P}}^\top\mathbf{U}_{\mathbf{A}}\mathbf{\Lambda}_{\mathbf{A}} - \mathbf{\Lambda}_{\mathbf{P}}\mathbf{U}_{\mathbf{P}}^\top\mathbf{U}_{\mathbf{A}}\|_F = \mathrm{O}\left(\log^{2\alpha+1}(n)\right)$;

iv. $\|\mathbf{U}_{\mathbf{P}}^\top\mathbf{U}_{\mathbf{A}}\mathbf{I}_{p,q} - \mathbf{I}_{p,q}\mathbf{U}_{\mathbf{P}}^\top\mathbf{U}_{\mathbf{A}}\|_F = \mathrm{O}\left(n^{-1}\log^{2\alpha+1}(n)\right)$

Proof.

i. Let $\sigma_1, \ldots, \sigma_d$ denote the singular values of $\mathbf{U}_{\mathbf{P}}^\top\mathbf{U}_{\mathbf{A}}$, and let $\theta_i = \cos^{-1}(\sigma_i)$ be the principal angles. It is a standard result that the non-zero eigenvalues of the matrix $\mathbf{U}_{\mathbf{A}}\mathbf{U}_{\mathbf{A}}^\top - \mathbf{U}_{\mathbf{P}}\mathbf{U}_{\mathbf{P}}^\top$ are precisely the $\sin(\theta_i)$ (each occurring twice) and so, by a variant of Davis-Kahan (Yu et al. (2015), Theorem 4) we have

$$\|\mathbf{U}_{\mathbf{A}}\mathbf{U}_{\mathbf{A}}^\top - \mathbf{U}_{\mathbf{P}}\mathbf{U}_{\mathbf{P}}^\top\| = \mathbf{A}x_{i\in\{1,\ldots,d\}}|\sin(\theta_i)| \leq \frac{2\sqrt{d}\,(2\sigma_1(\mathbf{P}) + \|\mathbf{A}-\mathbf{P}\|)\,\|\mathbf{A}-\mathbf{P}\|}{\sigma_d(\mathbf{P})^2}$$

for n sufficiently large.

The spectral norm bound from Proposition 1 then shows that

$$\begin{aligned}\|\mathbf{U}_{\mathbf{A}}\mathbf{U}_{\mathbf{A}}^\top - \mathbf{U}_{\mathbf{P}}\mathbf{U}_{\mathbf{P}}^\top\| &= \mathrm{O}\left(\frac{\left\{\sigma_1(\mathbf{P}) + n^{1/2}\log^{\alpha+1/2}(n)\right\}n^{1/2}\log^{\alpha+1/2}(n)}{\sigma_d(\mathbf{P})^2}\right)\\ &= \mathrm{O}\left(n^{-1/2}\log^{\alpha+1/2}(n)\right)\end{aligned}$$

since $\sigma_i(\mathbf{P}) = \Omega(n)$ almost surely.

ii. Using the bound from part ***i.***, we find that

$$\begin{aligned}\|\mathbf{U_A} - \mathbf{U_P}\mathbf{U_P}^\top\mathbf{U_A}\|_F = \|(\mathbf{U_A}\mathbf{U_A}^\top - \mathbf{U_P}\mathbf{U_P}^\top)\mathbf{U_A}\|_F &\leq \|\mathbf{U_A}\mathbf{U_A}^\top - \mathbf{U_P}\mathbf{U_P}^\top\|\|\mathbf{U_A}\|_F \\ &= \mathrm{O}\left(n^{-1/2}\log^{\alpha+1/2}(n)\right).\end{aligned}$$

iii. Observe that

$$\begin{aligned}\mathbf{U_P}^\top\mathbf{U_A}\mathbf{\Lambda_A} - \mathbf{\Lambda_P}\mathbf{U_P}^\top\mathbf{U_A}\mathbf{I}_{p,q} &= \mathbf{U_P}^\top(\mathbf{A}-\mathbf{P})\mathbf{U_A} \\ &= \mathbf{U_P}^\top(\mathbf{A}-\mathbf{P})(\mathbf{U_A} - \mathbf{U_P}\mathbf{U_P}^\top\mathbf{U_A}) + \mathbf{U_P}^\top(\mathbf{A}-\mathbf{P})\mathbf{U_P}\mathbf{U_P}^\top\mathbf{U_A},\end{aligned}$$

and so

$$\begin{aligned}&\|\mathbf{U_P}^\top\mathbf{U_A}\mathbf{\Lambda_A} - \mathbf{\Lambda_P}\mathbf{U_P}^\top\mathbf{U_A}\|_F \\ &\leq \|\mathbf{U_P}^\top\|\|\mathbf{A}-\mathbf{P}\|\|\mathbf{U_A} - \mathbf{U_P}\mathbf{U_P}^\top\mathbf{U_A}\|_F + \|\mathbf{U_P}^\top(\mathbf{A}-\mathbf{P})\mathbf{U_P}\|_F\|\mathbf{U_P}^\top\mathbf{U_A}\|_F \\ &= \mathrm{O}\left(n^{1/2}\log^{\alpha+\frac{1}{2}}(n)\cdot n^{-1/2}\log^{\alpha+1/2}(n)\right) + \mathrm{O}\left(\log^{\alpha+1/2}(n)\right) \\ &= \mathrm{O}\left(\log^{2\alpha+1}(n)\right),\end{aligned}$$

where we have used Propositions 1, 2 and the result from part ***ii.***.

iv. Note that

$$\begin{aligned}\mathbf{U_P}^\top\mathbf{U_A}\mathbf{I}_{p,q} - \mathbf{I}_{p,q}\mathbf{U_P}^\top\mathbf{U_A} = &\left\{(\mathbf{U_P}^\top\mathbf{U_A}\mathbf{\Lambda_A} - \mathbf{\Lambda_P}\mathbf{U_P}^\top\mathbf{U_A}) + (\mathbf{\Sigma_P}\mathbf{U_P}^\top\mathbf{U_A} - \mathbf{I}_{p,q}\mathbf{U_P}^\top\mathbf{U_A}\mathbf{\Lambda_A})\right\}\mathbf{\Sigma_A}^{-1} \\ &- \mathbf{\Sigma_P}(\mathbf{U_P}^\top\mathbf{U_A}\mathbf{I}_{p,q} - \mathbf{I}_{p,q}\mathbf{U_P}^\top\mathbf{U_A})\mathbf{\Sigma_A}^{-1}.\end{aligned}$$

where $\mathbf{\Sigma_A} = \mathbf{\Lambda_A}\mathbf{I}_{p,q}$ and $\mathbf{\Sigma_P} = \mathbf{\Lambda_P}\mathbf{I}_{p,q}$.

For any $i, j \in \{1, \ldots, d\}$, by rearranging and bounding the absolute value of the right-hand terms by the Frobenius norm, we find

$$
\begin{aligned}
&\left|(\mathbf{U}_{\mathbf{P}}^{\top}\mathbf{U}_{\mathbf{A}}\mathbf{I}_{p,q} - \mathbf{I}_{p,q}\mathbf{U}_{\mathbf{P}}^{\top}\mathbf{U}_{\mathbf{A}})_{ij}\right| \left(1 + \tfrac{\sigma_i(\mathbf{P})}{\sigma_j(\mathbf{A})}\right) \\
&\quad \leq \ \left(\|\mathbf{U}_{\mathbf{P}}^{\top}\mathbf{U}_{\mathbf{A}}\boldsymbol{\Lambda}_{\mathbf{A}} - \boldsymbol{\Lambda}_{\mathbf{P}}\mathbf{U}_{\mathbf{P}}^{\top}\mathbf{U}_{\mathbf{A}}\|_F + \|\boldsymbol{\Sigma}_{\mathbf{P}}\mathbf{U}_{\mathbf{P}}^{\top}\mathbf{U}_{\mathbf{A}} - \mathbf{I}_{p,q}\mathbf{U}_{\mathbf{P}}^{\top}\mathbf{U}_{\mathbf{A}}\boldsymbol{\Lambda}_{\mathbf{A}}\|_F\right) \|\boldsymbol{\Sigma}_{\mathbf{A}}^{-1}\|_F \\
&\quad = \ \left(\|\mathbf{U}_{\mathbf{P}}^{\top}\mathbf{U}_{\mathbf{A}}\boldsymbol{\Lambda}_{\mathbf{A}} - \boldsymbol{\Lambda}_{\mathbf{P}}\mathbf{U}_{\mathbf{P}}^{\top}\mathbf{U}_{\mathbf{A}}\|_F + \|\boldsymbol{\Lambda}_{\mathbf{P}}\mathbf{U}_{\mathbf{P}}^{\top}\mathbf{U}_{\mathbf{A}} - \mathbf{U}_{\mathbf{P}}^{\top}\mathbf{U}_{\mathbf{A}}\boldsymbol{\Lambda}_{\mathbf{A}}\|_F\right) \|\boldsymbol{\Sigma}_{\mathbf{A}}^{-1}\|_F \\
&\quad = \ \mathrm{O}\left(n^{-1}\log^{2\alpha+1}(n)\right),
\end{aligned}
$$

where we have used part ***iii.*** The result follows from the fact that $\left(1 + \frac{\sigma_i(\mathbf{P})}{\sigma_j(\mathbf{A})}\right) \geq 1$.

□

Proposition 4. Let $\mathbf{U}_{\mathbf{P}}^{\top}\mathbf{U}_{\mathbf{A}} + \mathbf{I}_{p,q}\mathbf{U}_{\mathbf{P}}^{\top}\mathbf{U}_{\mathbf{A}}\mathbf{I}_{p,q}$ admit the singular value decomposition

$$\mathbf{U}_{\mathbf{P}}^{\top}\mathbf{U}_{\mathbf{A}} + \mathbf{I}_{p,q}\mathbf{U}_{\mathbf{P}}^{\top}\mathbf{U}_{\mathbf{A}}\mathbf{I}_{p,q} = \mathbf{W}_1\boldsymbol{\Sigma}\mathbf{W}_2^{\top},$$

and let $\mathbf{W} = \mathbf{W}_1\mathbf{W}_2^{\top}$. Then $\mathbf{W} \in \mathbb{O}(d) \cap \mathbb{O}(p,q)$ and

$$\|\mathbf{U}_{\mathbf{P}}^{\top}\mathbf{U}_{\mathbf{A}} - \mathbf{W}\|_F,\ \|\mathbf{I}_{p,q}\mathbf{U}_{\mathbf{P}}^{\top}\mathbf{U}_{\mathbf{A}}\mathbf{I}_{p,q} - \mathbf{W}\|_F = \mathrm{O}\left(n^{-1}\log^{2\alpha+1}(n)\right)$$

almost surely.

Proof. A standard argument shows that a solution to the modified *one mode* orthogonal Procrustes problem

$$\widehat{\mathbf{W}} = \underset{\mathbf{Q}\in\mathbb{O}(d)}{\arg\min} \|\mathbf{P}_1^{\top}\mathbf{A}_1 - \mathbf{Q}\|_F^2 + \|\mathbf{P}_2^{\top}\mathbf{A}_2 - \mathbf{Q}\|_F^2$$

for matrices $\mathbf{A}_i, \mathbf{P}_i \in \mathbb{R}^{n\times d}$ is given by $\widehat{\mathbf{W}} = \widehat{\mathbf{W}}_1\widehat{\mathbf{W}}_2^{\top}$, where we have the singular value decomposition

$$\tfrac{1}{2}(\mathbf{P}_1^{\top}\mathbf{A}_1 + \mathbf{P}_2\mathbf{A}_2^{\top}) = \widehat{\mathbf{W}}_1\boldsymbol{\Sigma}\widehat{\mathbf{W}}_2^{\top}.$$

Setting $\mathbf{W}$ as in the statement of the proposition, we therefore observe that $\mathbf{W}$ satisfies

$$\mathbf{W} = \underset{\mathbf{Q} \in \mathbb{O}(d)}{\arg\min} \|\mathbf{U}_{\mathbf{P}}^\top \mathbf{U}_{\mathbf{A}} - \mathbf{Q}\|_F^2 + \|\mathbf{I}_{p,q} \mathbf{U}_{\mathbf{P}}^\top \mathbf{U}_{\mathbf{A}} \mathbf{I}_{p,q} - \mathbf{Q}\|_F^2.$$

Let $\mathbf{U}_{\mathbf{P}}^\top \mathbf{U}_{\mathbf{A}} = \mathbf{W}_{\mathbf{U},1} \mathbf{\Sigma}_{\mathbf{U}} \mathbf{W}_{\mathbf{U},2}^\top$ be the singular value decomposition of $\mathbf{U}_{\mathbf{P}}^\top \mathbf{U}_{\mathbf{A}}$, and define $\mathbf{W}_{\mathbf{U}} \in \mathbb{O}(d)$ by $\mathbf{W}_{\mathbf{U}} = \mathbf{W}_{\mathbf{U},1} \mathbf{W}_{\mathbf{U},2}^\top$. Then

$$\begin{aligned}
\|\mathbf{U}_{\mathbf{P}}^\top \mathbf{U}_{\mathbf{A}} - \mathbf{W}_{\mathbf{U}}\|_F = \|\mathbf{\Sigma} - \mathbf{I}\|_F = \left(\sum_{i=1}^d (1 - \sigma_i)^2 \right)^{1/2} &\leq \sum_{i=1}^d (1 - \sigma_i) \leq \sum_{i=1}^d (1 - \sigma_i^2) \\
&= \sum_{i=1}^d \sin^2(\theta_i) \leq d \|\mathbf{U}_{\mathbf{A}} \mathbf{U}_{\mathbf{A}}^\top - \mathbf{U}_{\mathbf{P}} \mathbf{U}_{\mathbf{P}}^\top\|^2 \\
&= \mathrm{O}\left(n^{-1} \log^{2\alpha+1}(n) \right).
\end{aligned}$$

Also,

$$\begin{aligned}
\|\mathbf{I}_{p,q} \mathbf{U}_{\mathbf{P}}^\top \mathbf{U}_{\mathbf{A}} \mathbf{I}_{p,q} - \mathbf{W}_{\mathbf{U}}\|_F &\leq \|\mathbf{I}_{p,q} \mathbf{U}_{\mathbf{P}}^\top \mathbf{U}_{\mathbf{A}} \mathbf{I}_{p,q} - \mathbf{U}_{\mathbf{P}}^\top \mathbf{U}_{\mathbf{A}}\|_F + \|\mathbf{U}_{\mathbf{P}}^\top \mathbf{U}_{\mathbf{A}} - \mathbf{W}_{\mathbf{U}}\|_F \\
&\leq \|\mathbf{U}_{\mathbf{P}}^\top \mathbf{U}_{\mathbf{A}} \mathbf{I}_{p,q} - \mathbf{I}_{p,q} \mathbf{U}_{\mathbf{P}}^\top \mathbf{U}_{\mathbf{A}}\|_F + \|\mathbf{U}_{\mathbf{P}}^\top \mathbf{U}_{\mathbf{A}} - \mathbf{W}_{\mathbf{U}}\|_F \\
&= \mathrm{O}\left(n^{-1} \log^{2\alpha+1}(n) \right)
\end{aligned}$$

by Proposition 3.

Combining these shows that

$$\begin{aligned}
\|\mathbf{U}_{\mathbf{P}}^\top \mathbf{U}_{\mathbf{A}} - \mathbf{W}\|_F^2 + \|\mathbf{I}_{p,q} \mathbf{U}_{\mathbf{P}}^\top \mathbf{U}_{\mathbf{A}} \mathbf{I}_{p,q} - \mathbf{W}\|_F^2 &\leq \|\mathbf{U}_{\mathbf{P}}^\top \mathbf{U}_{\mathbf{A}} - \mathbf{W}_{\mathbf{U}}\|_F^2 + \|\mathbf{I}_{p,q} \mathbf{U}_{\mathbf{P}}^\top \mathbf{U}_{\mathbf{A}} \mathbf{I}_{p,q} - \mathbf{W}_{\mathbf{U}}\|_F^2 \\
&= \mathrm{O}\left(n^{-2} \log^{4\alpha+2}(n) \right),
\end{aligned}$$

which gives the desired bound.

Finally, we observe that the matrix $\mathbf{U}_{\mathbf{P}}^\top \mathbf{U}_{\mathbf{A}} + \mathbf{I}_{p,q} \mathbf{U}_{\mathbf{P}}^\top \mathbf{U}_{\mathbf{A}} \mathbf{I}_{p,q} \in \mathbb{R}^{p \times p} \oplus \mathbb{R}^{q \times q}$, and thus the matrices $\mathbf{W}_1, \mathbf{W}_2 \in \mathbb{O}(p) \oplus \mathbb{O}(q)$, so in particular $\mathbf{W} \in \mathbb{O}(d) \cap \mathbb{O}(p, q)$. □

Proposition 5. The following bounds hold almost surely:

i. $\|\mathbf{W}\boldsymbol{\Sigma}_{\mathbf{A}} - \boldsymbol{\Sigma}_{\mathbf{P}}\mathbf{W}\|_F = \mathrm{O}\left(\log^{2\alpha+1}(n)\right)$;

ii. $\|\mathbf{W}\boldsymbol{\Sigma}_{\mathbf{A}}^{1/2} - \boldsymbol{\Sigma}_{\mathbf{P}}^{1/2}\mathbf{W}\|_F = \mathrm{O}\left(n^{-1/2}\log^{2\alpha+1}(n)\right)$;

iii. $\|\mathbf{W}\boldsymbol{\Sigma}_{\mathbf{A}}^{-1/2} - \boldsymbol{\Sigma}_{\mathbf{P}}^{-1/2}\mathbf{W}\|_F = \mathrm{O}\left(n^{-3/2}\log^{2\alpha+1}(n)\right)$.

Proof. ***i.*** Observe that

$$\begin{aligned}\mathbf{W}\boldsymbol{\Sigma}_{\mathbf{A}} - \boldsymbol{\Sigma}_{\mathbf{P}}\mathbf{W} &= (\mathbf{W} - \mathbf{U}_{\mathbf{P}}^{\top}\mathbf{U}_{\mathbf{A}})\boldsymbol{\Sigma}_{\mathbf{A}} + \mathbf{U}_{\mathbf{P}}^{\top}\mathbf{U}_{\mathbf{A}}\boldsymbol{\Sigma}_{\mathbf{A}} - \boldsymbol{\Sigma}_{\mathbf{P}}\mathbf{W} \\ &= (\mathbf{W} - \mathbf{U}_{\mathbf{P}}^{\top}\mathbf{U}_{\mathbf{A}})\boldsymbol{\Sigma}_{\mathbf{A}} + (\mathbf{U}_{\mathbf{P}}^{\top}\mathbf{U}_{\mathbf{A}}\boldsymbol{\Sigma}_{\mathbf{A}} - \boldsymbol{\Sigma}_{\mathbf{P}}\mathbf{I}_{p,q}\mathbf{U}_{\mathbf{P}}^{\top}\mathbf{U}_{\mathbf{A}}\mathbf{I}_{p,q}) + \boldsymbol{\Sigma}_{\mathbf{P}}(\mathbf{I}_{p,q}\mathbf{U}_{\mathbf{P}}^{\top}\mathbf{U}_{\mathbf{A}}\mathbf{I}_{p,q} - \mathbf{W}).\end{aligned}$$

Proposition 4 shows that the terms $\|(\mathbf{W}-\mathbf{U}_{\mathbf{P}}^{\top}\mathbf{U}_{\mathbf{A}})\boldsymbol{\Sigma}_{\mathbf{A}}\|_F$ and $\|\boldsymbol{\Sigma}_{\mathbf{P}}(\mathbf{I}_{p,q}\mathbf{U}_{\mathbf{P}}^{\top}\mathbf{U}_{\mathbf{A}}\mathbf{I}_{p,q}-\mathbf{W})\|_F$ are both $\mathrm{O}\left(\log^{2\alpha+1}(n)\right)$, while $\|\mathbf{U}_{\mathbf{P}}^{\top}\mathbf{U}_{\mathbf{A}}\boldsymbol{\Sigma}_{\mathbf{A}} - \boldsymbol{\Sigma}_{\mathbf{P}}\mathbf{I}_{p,q}\mathbf{U}_{\mathbf{P}}^{\top}\mathbf{U}_{\mathbf{A}}\mathbf{I}_{p,q}\|_F$ is $\mathrm{O}\left(\log^{2\alpha+1}(n)\right)$, and so $\|\mathbf{W}\boldsymbol{\Sigma}_{\mathbf{A}} - \boldsymbol{\Sigma}_{\mathbf{P}}\mathbf{W}\|_F = \mathrm{O}\left(\log^{2\alpha+1}(n)\right)$.

ii. We will bound the absolute value of the terms $\left(\mathbf{W}\boldsymbol{\Sigma}_{\mathbf{A}}^{1/2} - \boldsymbol{\Sigma}_{\mathbf{P}}^{1/2}\mathbf{W}\right)_{ij}$. Note that

$$\begin{aligned}\left|\left(\mathbf{W}\boldsymbol{\Sigma}_{\mathbf{A}}^{1/2} - \boldsymbol{\Sigma}_{\mathbf{P}}^{1/2}\mathbf{W}\right)_{ij}\right| &= \left|\mathbf{W}_{ij}\left(\sigma_j(\mathbf{A})^{1/2} - \sigma_i(\mathbf{P})^{1/2}\right)\right| = \left|\frac{\mathbf{W}_{ij}(\sigma_j(\mathbf{A}) - \sigma_i(\mathbf{P}))}{\sigma_j(\mathbf{A})^{1/2} + \sigma_i(\mathbf{P})^{1/2}}\right| \\ &= \frac{\left|(\mathbf{W}\boldsymbol{\Sigma}_{\mathbf{A}} - \boldsymbol{\Sigma}_{\mathbf{P}}\mathbf{W})_{ij}\right|}{\sigma_j(\mathbf{A})^{1/2} + \sigma_i(\mathbf{P})^{1/2}} \leq \frac{\|\mathbf{W}\boldsymbol{\Sigma}_{\mathbf{A}} - \boldsymbol{\Sigma}_{\mathbf{P}}\mathbf{W}\|_F}{\sigma_d(\mathbf{P})^{1/2}},\end{aligned}$$

and consequently we find that $\|\mathbf{W}\boldsymbol{\Sigma}_{\mathbf{A}}^{1/2} - \boldsymbol{\Sigma}_{\mathbf{P}}^{1/2}\mathbf{W}\|_F = \mathrm{O}\left(n^{-1/2}\log^{2\alpha+1}(n)\right)$ by summing over all $i, j \in \{1, \ldots, d\}$ and applying part ***i.***

iii. We will bound the absolute value of the terms $\left(\mathbf{W}\mathbf{\Sigma}_{\mathbf{A}}^{-1/2} - \mathbf{\Sigma}_{\mathbf{P}}^{-1/2}\mathbf{W}\right)_{ij}$. Note that

$$\begin{aligned}\left|\left(\mathbf{W}\mathbf{\Sigma}_{\mathbf{A}}^{-1/2} - \mathbf{\Sigma}_{\mathbf{P}}^{-1/2}\mathbf{W}\right)_{ij}\right| &= \left|\frac{\mathbf{W}_{ij}(\sigma_i(\mathbf{P})^{1/2} - \sigma_j(\mathbf{A})^{1/2})}{\sigma_i(\mathbf{P})^{1/2}\sigma_j(\mathbf{P})^{1/2}}\right| \\ &= \frac{\left|\left(\mathbf{W}\mathbf{\Sigma}_{\mathbf{A}}^{1/2} - \mathbf{\Sigma}_{\mathbf{P}}^{1/2}\mathbf{W}\right)_{ij}\right|}{\sigma_i(\mathbf{P})^{1/2}\sigma_j(\mathbf{A})^{1/2}} = \mathrm{O}\left(n^{-3/2}\log^{2\alpha+1}(n)\right)\end{aligned}$$

by part ***ii.*** The result follows by summing over all $i, j \in \{1, \ldots, d\}$.

□

Proposition 6. Let

$$\begin{aligned}\mathbf{R}_1 &= \mathbf{U}_{\mathbf{P}}(\mathbf{U}_{\mathbf{P}}^\top\mathbf{U}_{\mathbf{A}}\mathbf{\Sigma}_{\mathbf{A}}^{1/2} - \mathbf{\Sigma}_{\mathbf{P}}^{1/2}\mathbf{W}) \\ \mathbf{R}_2 &= (\mathbf{I} - \mathbf{U}_{\mathbf{P}}\mathbf{U}_{\mathbf{P}}^\top)(\mathbf{A} - \mathbf{P})(\mathbf{U}_{\mathbf{A}}\mathbf{I}_{p,q} - \mathbf{U}_{\mathbf{P}}\mathbf{I}_{p,q}\mathbf{W})\mathbf{\Sigma}_{\mathbf{A}}^{-1/2} \\ \mathbf{R}_3 &= -\mathbf{U}_{\mathbf{P}}\mathbf{U}_{\mathbf{P}}^\top(\mathbf{A} - \mathbf{P})\mathbf{U}_{\mathbf{P}}\mathbf{I}_{p,q}\mathbf{W}\mathbf{\Sigma}_{\mathbf{A}}^{-1/2} \\ \mathbf{R}_4 &= (\mathbf{A} - \mathbf{P})\mathbf{U}_{\mathbf{P}}\mathbf{I}_{p,q}(\mathbf{W}\mathbf{\Sigma}_{\mathbf{A}}^{-1/2} - \mathbf{\Sigma}_{\mathbf{P}}^{-1/2}\mathbf{W}).\end{aligned}$$

Then the following bounds hold almost surely:

i. $\|\mathbf{R}_1\|_{2\to\infty} = \mathrm{O}\left(n^{-1}\log^{2\alpha+1}(n)\right)$;

ii. $\|\mathbf{R}_2\|_{2\to\infty} = \mathrm{O}\left(n^{-\frac{3}{4}}\log^{3\alpha+3/2}(n)\right)$;

iii. $\|\mathbf{R}_3\|_{2\to\infty} = \mathrm{O}\left(n^{-1}\log^{\alpha+1/2}(n)\right)$;

iv. $\|\mathbf{R}_4\|_{2\to\infty} = \mathrm{O}\left(n^{-1}\log^{3\alpha+3/2}(n)\right)$

In particular, we have $\|n^{1/2}\mathbf{R}_i\|_{2\to\infty} \to 0$ for all i.

Proof. ***i.*** Recall that $\mathbf{U_P}\mathbf{\Sigma}_\mathbf{P}^{1/2} = \mathbf{X}\mathbf{Q_X}$ for some $\mathbf{Q_X} \in \mathbb{O}(p,q)$ of bounded spectral norm. Using the relation $\|\mathbf{AP}\|_{2\to\infty} \leq \|\mathbf{A}\|_{2\to\infty}\|\mathbf{P}\|$ (see, for example, Cape et al. (2019b), Proposition 6.5) we find that $\|\mathbf{U_P}\|_{2\to\infty} \leq \|\mathbf{X}\|_{2\to\infty}\|\mathbf{Q_X}\|\|\mathbf{\Sigma}_\mathbf{P}^{-1}\|$, and thus $\|\mathbf{U_P}\|_{2\to\infty} = \mathrm{O}\left(n^{-1/2}\right)$ as the rows of $\mathbf{X}$ are by definition bounded in Euclidean norm.

Thus

$$\begin{aligned}\|\mathbf{R}_1\|_{2\to\infty} &\leq \|\mathbf{U_P}\|_{2\to\infty}\|\mathbf{U_P}^\top\mathbf{U_A}\mathbf{\Sigma}_\mathbf{A}^{1/2} - \mathbf{\Sigma}_\mathbf{P}^{1/2}\mathbf{W}\| \\ &\leq \|\mathbf{U_P}\|_{2\to\infty}\left(\|(\mathbf{U_P}^\top\mathbf{U_A} - \mathbf{W})\mathbf{\Sigma}_\mathbf{A}^{1/2}\|_F + \|\mathbf{W}\mathbf{\Sigma}_\mathbf{A}^{1/2} - \mathbf{\Sigma}_\mathbf{P}^{1/2}\mathbf{W}\|_F\right)\end{aligned}$$

The first summand is $\mathrm{O}\left(n^{-1/2}\log^{2\alpha+1}(n)\right)$ by Proposition 4, while Proposition 5 shows that the second is $\mathrm{O}\left(n^{-1/2}\log^{2\alpha+1}(n)\right)$, and so

$$\|\mathbf{R}_1\|_{2\to\infty} = \mathrm{O}\left(n^{-1}\log^{2\alpha+1}(n)\right).$$

ii. We first observe that

$$\begin{aligned}&\|\mathbf{U_P}\mathbf{U_P}^\top(\mathbf{A}-\mathbf{P})(\mathbf{U_A}\mathbf{I}_{p,q} - \mathbf{U_P}\mathbf{I}_{p,q}\mathbf{W})\mathbf{\Sigma}_\mathbf{A}^{-1/2}\|_{2\to\infty} \\ &\leq \|\mathbf{U_P}\|_{2\to\infty}\|\mathbf{U_P}^\top\|\|\mathbf{A}-\mathbf{P}\|\|\mathbf{U_A}\mathbf{I}_{p,q} - \mathbf{U_P}\mathbf{I}_{p,q}\mathbf{W}\|\|\mathbf{\Sigma}_\mathbf{A}^{-1/2}\| \\ &= \mathrm{O}\left(n^{-1/2}\cdot n^{1/2}\log^{\alpha+1/2}(n)\cdot n^{-1/2}\log^{2\alpha+1}(n)\cdot n^{-1/2}\right) \\ &= \mathrm{O}\left(n^{-1}\log^{3\alpha+3/2}(n)\right),\end{aligned}$$

where we have bounded $\|\mathbf{U_A}\mathbf{I}_{p,q} - \mathbf{U_P}\mathbf{I}_{p,q}\mathbf{W}\|$ by noting that

$$\begin{aligned}\|\mathbf{U_A}\mathbf{I}_{p,q} - \mathbf{U_P}\mathbf{I}_{p,q}\mathbf{W}\| &\leq \|\mathbf{U_A}\mathbf{I}_{p,q} - \mathbf{U_P}\mathbf{U_P}^\top\mathbf{U_A}\mathbf{I}_{p,q}\| + \|\mathbf{U_P}\mathbf{I}_{p,q}(\mathbf{I}_{p,q}\mathbf{U_P}^\top\mathbf{U_A}\mathbf{I}_{p,q} - \mathbf{W})\| \\ &= \mathrm{O}\left(n^{-1/2}\log^{2\alpha+1}(n)\right),\end{aligned}$$

by Propositions 3 and 4.

This leaves us to bound the term $\|(\mathbf{A}-\mathbf{P})(\mathbf{U_A}\mathbf{I}_{p,q}-\mathbf{U_P}\mathbf{I}_{p,q}\mathbf{W})\mathbf{\Sigma}_\mathbf{A}^{-1/2}\|_{2\to\infty}$. Now,

$$\begin{aligned}(\mathbf{A}-\mathbf{P})(\mathbf{U_A}\mathbf{I}_{p,q}-\mathbf{U_P}\mathbf{I}_{p,q}\mathbf{W})\mathbf{\Sigma}_\mathbf{A}^{-1/2} &= (\mathbf{A}-\mathbf{P})(\mathbf{I}-\mathbf{U_P}\mathbf{U_P}^\top)\mathbf{U_A}\mathbf{I}_{p,q}\mathbf{\Sigma}_\mathbf{A}^{-1/2} \\ &\quad+(\mathbf{A}-\mathbf{P})\mathbf{U_P}\mathbf{I}_{p,q}(\mathbf{I}_{p,q}\mathbf{U_P}^\top\mathbf{U_A}\mathbf{I}_{p,q}-\mathbf{W})\mathbf{\Sigma}_\mathbf{A}^{-1/2},\end{aligned}$$

and

$$\begin{aligned}\|(\mathbf{A}-\mathbf{P})\mathbf{U_P}\mathbf{I}_{p,q}(\mathbf{I}_{p,q}\mathbf{U_P}^\top\mathbf{U_A}\mathbf{I}_{p,q}-\mathbf{W})\mathbf{\Sigma}_\mathbf{A}^{-1/2}\|_{2\to\infty} &\le \|\mathbf{A}-\mathbf{P}\|\|\mathbf{U_P}\|\|\mathbf{I}_{p,q}\mathbf{U_P}^\top\mathbf{U_A}\mathbf{I}_{p,q}-\mathbf{W}\|\|\mathbf{\Sigma}_\mathbf{A}^{-1/2}\| \\ &= \mathrm{O}\left(n^{1/2}\log^{\alpha+1/2}(n)\cdot n^{-1}\log^{2\alpha+1}(n)\cdot n^{-1/2}\right) \\ &= \mathrm{O}\left(n^{-1}\log^{3\alpha+3/2}(n)\right)\end{aligned}$$

by Propositions 1 and 4.

To bound the remaining term, observe that we can rewrite

$$(\mathbf{A}-\mathbf{P})(\mathbf{I}-\mathbf{U_P}\mathbf{U_P}^\top)\mathbf{U_A}\mathbf{I}_{p,q}\mathbf{\Sigma}_\mathbf{A}^{-1/2} = (\mathbf{A}-\mathbf{P})(\mathbf{I}-\mathbf{U_P}\mathbf{U_P}^\top)\mathbf{U_A}\mathbf{U}\mathbf{A}^\top\mathbf{U_A}\mathbf{I}_{p,q}\mathbf{\Sigma}_\mathbf{A}^{-1/2}$$

and so

$$\|(\mathbf{A}-\mathbf{P})(\mathbf{I}-\mathbf{U_P}\mathbf{U_P}^\top)\mathbf{U_A}\mathbf{I}_{p,q}\mathbf{\Sigma}_\mathbf{A}^{-1/2}\|_{2\to\infty} \le \|\mathbf{R}\|_{2\to\infty}\|\mathbf{U_A}\mathbf{I}_{p,q}\mathbf{\Sigma}_\mathbf{A}^{-1/2}\|,$$

where $\mathbf{R}=(\mathbf{A}-\mathbf{P})(\mathbf{I}-\mathbf{U_P}\mathbf{U_P}^\top)\mathbf{U_A}\mathbf{U_A}^\top$.

The latter term is $\mathrm{O}\left(n^{-1/2}\right)$, so it suffices to bound $\|\mathbf{R}\|_{2\to\infty}$. To do this, we claim that the Frobenius norms of the rows of the matrix $\mathbf{R}$ are exchangeable, and thus have the same expectation, which implies that $\mathbb{E}\left(\|\mathbf{R}\|_F^2\right)=n\mathbb{E}\left(\|\mathbf{R}_i\|^2\right)$ for any $i\in\{1,\dots,n\}$. Applying Markov's inequality, we therefore see that

$$\mathbb{P}\left(\|\mathbf{R}_i\|>t\right)\le\frac{\mathbb{E}\left(\|\mathbf{R}_i\|^2\right)}{t^2}=\frac{\mathbb{E}\left(\|\mathbf{R}\|_F^2\right)}{nt^2}.$$

Now,

$$\begin{aligned}\|\mathbf{R}\|_F &\le \|\mathbf{A}-\mathbf{P}\|\|\mathbf{A}-\mathbf{U_P}\mathbf{U_P}^\top\mathbf{A}\|_F\|\mathbf{U_A}^\top\|_F\\ &= \mathrm{O}\left(n^{1/2}\log^{\alpha+1/2}(n)\cdot n^{-1/2}\log^{\alpha+1/2}(n)\right)\\ &= \mathrm{O}\left(\log^{2\alpha+1}(n)\right)\end{aligned}$$

by Propositions 1 and 3.

It follows that

$$\mathbb{P}\left(\|\mathbf{R}_i\| > n^{-\frac{1}{4}}\log^{2\alpha+1}(n)\right) \le cn^{-1/2}$$

and thus

$$\|\mathbf{R}\|_{2\to\infty} = \mathrm{O}\left(n^{-\frac{1}{4}}\log^{2\alpha+1}(n)\right)$$

almost surely.

We must therefore show that the Frobenius norms of the rows of $\mathbf{R}$ are exchangeable. Let $\mathbf{Q} \in \mathbb{O}(n)$ be a permutation matrix, and observe that for any matrix $\mathbf{G} \in \mathbb{R}^{n\times n}$, right multiplication by $\mathbf{Q}^\top$ simply permutes the columns of $\mathbf{G}$, and thus does not alter the Frobenius norms of its rows. In particular, the Frobenius norms of the rows of $\mathbf{QGQ}^\top$ are the same as the Frobenius norms of the rows of $\mathbf{QG}$. For any symmetric matrix $\mathbf{G} \in \mathbb{R}^{n\times n}$, let $\mathcal{P}_d(\mathbf{G})$ denote the projection onto the subspace spanned by the eigenvectors corresponding to the top d singular values of $\mathbf{G}$, and let $\mathcal{P}_d^\perp(\mathbf{G})$ denote the projection onto the orthogonal complements of this subspace.

Note that

$$\mathcal{P}_d(\mathbf{P}) = \mathbf{U_P}\mathbf{U_P}^\top \quad \text{and} \quad \mathcal{P}_d(\mathbf{A}) = \mathbf{U_A}\mathbf{U_A}^\top,$$

while for any permutation matrix $\mathbf{Q} \in \mathbb{O}(n)$ we have

$$\mathcal{P}_d(\mathbf{Q}\mathbf{P}\mathbf{Q}^\top) = \mathbf{Q}\mathbf{U}_\mathbf{P}\mathbf{U}_\mathbf{P}^\top\mathbf{Q}^\top \quad \text{and} \quad \mathcal{R}_d(\mathbf{Q}\mathbf{A}\mathbf{Q}^\top) = \mathbf{Q}\mathbf{U}_\mathbf{A}\mathbf{U}_\mathbf{A}^\top\mathbf{Q}^\top.$$

For any pair of matrices $\mathbf{G}, \mathbf{H} \in \mathbb{R}^{n \times mn}$, define an operator

$$\widehat{\mathcal{P}}_d(\mathbf{G}, \mathbf{H}) = (\mathbf{G} - \mathbf{H})\mathcal{P}_d^\perp(\mathbf{H})\mathcal{P}_d(\mathbf{G})$$

and note that $\widehat{\mathcal{P}}_d(\mathbf{A}, \mathbf{P}) = \mathbf{R}$, while

$$\begin{aligned}\widehat{\mathcal{P}}_d(\mathbf{Q}\mathbf{A}\mathbf{Q}^\top, \mathbf{Q}\mathbf{P}\mathbf{Q}^\top) &= \mathbf{Q}(\mathbf{A} - \mathbf{P})\mathbf{Q}^\top\mathbf{Q}(\mathbf{I} - \mathbf{U}_\mathbf{P}\mathbf{U}_\mathbf{P}^\top)\mathbf{Q}^\top\mathbf{Q}\mathbf{U}_\mathbf{A}\mathbf{U}_\mathbf{A}^\top\mathbf{Q}^\top \\ &= \mathbf{Q}\mathbf{R}\mathbf{Q}^\top.\end{aligned}$$

By assumption, the latent positions for our graphs are independent and identically distributed, and so the entries of the pair $(\mathbf{A}, \mathbf{P})$ have the same joint distribution as those of the pair $(\mathbf{Q}\mathbf{A}\mathbf{Q}^\top, \mathbf{Q}\mathbf{P}\mathbf{Q}^\top)$. Therefore, the entries of the matrix $\mathcal{P}_\mathcal{L}(\mathbf{A}, \mathbf{P})$ have the same joint distribution as those of the matrix $\widehat{\mathcal{P}}_d(\mathbf{Q}\mathbf{A}\mathbf{Q}^\top, \mathbf{Q}\mathbf{P}\mathbf{Q}^\top)$, which implies that $\mathbf{R}$ has the same distribution as $\mathbf{Q}\mathbf{R}\mathbf{Q}^\top$, and consequently the Frobenius norms of the rows of $\mathbf{R}$ have the same distribution as those of $\mathbf{Q}\mathbf{R}$, which proves our claim.

Combining these results, we see that

$$\|\mathbf{R}_2\|_{2\to\infty} = \mathrm{O}\left(n^{-\frac{3}{4}} \log^{3\alpha+3/2}(n)\right),$$

as required.

iii. Similarly to part ***i.***, we see that

$$\begin{aligned}\|\mathbf{R}_3\|_{2\to\infty} &\leq \|\mathbf{U}_\mathbf{P}\|_{2\to\infty}\|\mathbf{U}_\mathbf{P}^\top(\mathbf{A}-\mathbf{P})\mathbf{U}_\mathbf{P}\mathbf{I}_{p,q}\mathbf{W}\boldsymbol{\Sigma}_\mathbf{A}^{-1/2}\| \\ &\leq \|\mathbf{U}_\mathbf{P}\|_{2\to\infty}\|\mathbf{U}_\mathbf{P}^\top(\mathbf{A}-\mathbf{P})\mathbf{U}_\mathbf{P}\|_F\|\mathbf{W}\boldsymbol{\Sigma}_\mathbf{A}^{-1/2}\|_F \\ &= \mathrm{O}\left(n^{-1/2}\cdot\log^{\alpha+1/2}(n)\cdot n^{-1/2}\right) \\ &= \mathrm{O}\left(n^{-1}\log^{\alpha+1/2}(n)\right)\end{aligned}$$

by Proposition 2.

iv. Observe that

$$\begin{aligned}\|\mathbf{R}_4\|_{2\to\infty} &\leq \|\mathbf{R}_4\|_F \\ &\leq \|\mathbf{A}-\mathbf{P}\|\|\mathbf{U}_\mathbf{P}\|_F\|\mathbf{W}\boldsymbol{\Sigma}_\mathbf{A}^{-1/2}-\boldsymbol{\Sigma}_\mathbf{P}^{-1/2}\mathbf{W}\|_F \\ &= \mathrm{O}\left(n^{-1}\log^{3\alpha+3/2}(n)\right)\end{aligned}$$

by Propositions 1 and 5.

□

1.1 Proof of Theorem 1

Proof. Observe that

$$\begin{aligned}\mathbf{X}_\mathbf{A}-\mathbf{X}_\mathbf{P}\mathbf{W} &= \mathbf{U}_\mathbf{A}\boldsymbol{\Sigma}_\mathbf{A}^{1/2}-\mathbf{U}_\mathbf{P}\boldsymbol{\Sigma}_\mathbf{P}^{1/2}\mathbf{W} \\ &= \mathbf{U}_\mathbf{A}\boldsymbol{\Sigma}_\mathbf{A}^{1/2}-\mathbf{U}_\mathbf{P}\mathbf{U}_\mathbf{P}^\top\mathbf{U}_\mathbf{A}\boldsymbol{\Sigma}_\mathbf{A}^{1/2}+\mathbf{U}_\mathbf{P}(\mathbf{U}_\mathbf{P}^\top\mathbf{U}_\mathbf{A}\boldsymbol{\Sigma}_\mathbf{A}^{1/2}-\boldsymbol{\Sigma}_\mathbf{P}^{1/2}\mathbf{W}) \\ &= \mathbf{U}_\mathbf{A}\boldsymbol{\Sigma}_\mathbf{A}^{1/2}-\mathbf{U}_\mathbf{P}\mathbf{U}_\mathbf{P}^\top\mathbf{U}_\mathbf{A}\boldsymbol{\Sigma}_\mathbf{A}^{1/2}+\mathbf{R}_{1,1}.\end{aligned}$$

Noting that

$$\mathbf{U}_\mathbf{A}\boldsymbol{\Sigma}_\mathbf{A}^{1/2}=\mathbf{A}\mathbf{U}_\mathbf{A}\mathbf{I}_{p,q}\boldsymbol{\Sigma}_\mathbf{A}^{-1/2} \quad \text{and} \quad \mathbf{U}_\mathbf{P}\mathbf{U}_\mathbf{P}^\top\mathbf{P}=\mathbf{P},$$

we see that

$$
\begin{aligned}
\mathbf{X_A} - \mathbf{X_P W} &= \mathbf{A U_A I}_{p,q} \mathbf{\Sigma_A}^{-1/2} - \mathbf{U_P U_P^\top A U_A I}_{p,q} \mathbf{\Sigma_A}^{-1/2} + \mathbf{R}_{1,1} \\
&= \mathbf{A U_A I}_{p,q} \mathbf{\Sigma_A}^{-1/2} - \mathbf{P U_A I}_{p,q} \mathbf{\Sigma_A}^{-1/2} - (\mathbf{U_P U_P^\top A U_A I}_{p,q} \mathbf{\Sigma_A}^{-1/2} - \mathbf{P U_A I}_{p,q} \mathbf{\Sigma_A}^{-1/2}) + \mathbf{R}_1 \\
&= (\mathbf{A} - \mathbf{P}) \mathbf{U_A I}_{p,q} \mathbf{\Sigma_A}^{-1/2} - (\mathbf{U_P U_P^\top A U_A I}_{p,q} \mathbf{\Sigma_A}^{-1/2} - \mathbf{U_P U_P^\top P U_A I}_{p,q} \mathbf{\Sigma_A}^{-1/2}) + \mathbf{R}_1 \\
&= (\mathbf{A} - \mathbf{P}) \mathbf{U_A I}_{p,q} \mathbf{\Sigma_A}^{-1/2} - \mathbf{U_P U_P^\top} (\mathbf{A} - \mathbf{P}) \mathbf{U_A I}_{p,q} \mathbf{\Sigma_A}^{-1/2} + \mathbf{R}_1 \\
&= (\mathbf{I} - \mathbf{U_P U_P^\top})(\mathbf{A} - \mathbf{P}) \mathbf{U_A I}_{p,q} \mathbf{\Sigma_A}^{-1/2} + \mathbf{R}_1 \\
&= (\mathbf{I} - \mathbf{U_P U_P^\top})(\mathbf{A} - \mathbf{P}) \{\mathbf{U_P I}_{p,q} \mathbf{W} + (\mathbf{U_A I}_{p,q} - \mathbf{U_P I}_{p,q} \mathbf{W})\} \mathbf{\Sigma_A}^{-1/2} + \mathbf{R}_1 \\
&= (\mathbf{A} - \mathbf{P}) \mathbf{U_P I}_{p,q} \mathbf{W} \mathbf{\Sigma_A}^{-1/2} + \mathbf{R}_3 + \mathbf{R}_2 + \mathbf{R}_1 \\
&= (\mathbf{A} - \mathbf{P}) \mathbf{U_P I}_{p,q} \{\mathbf{\Sigma_P}^{-1/2} \mathbf{W} + (\mathbf{W} \mathbf{\Sigma_A}^{-1/2} - \mathbf{\Sigma_P}^{-1/2} \mathbf{W})\} + \mathbf{R}_3 + \mathbf{R}_2 + \mathbf{R}_1 \\
&= (\mathbf{A} - \mathbf{P}) \mathbf{U_P I}_{p,q} \mathbf{\Sigma_P}^{-1/2} \mathbf{W} + \mathbf{R}_4 + \mathbf{R}_3 + \mathbf{R}_2 + \mathbf{R}_1.
\end{aligned}
$$

Applying Proposition 6, we find that

$$
\|\mathbf{X_A} - \mathbf{X_P W}\|_{2\to\infty} = \|(\mathbf{A} - \mathbf{P}) \mathbf{U_P I}_{p,q} \mathbf{\Sigma_P}^{-1/2}\|_{2\to\infty} + \mathrm{O}\left(n^{-\frac{3}{4}} \log^{3\alpha+3/2}(n)\right).
$$

Consequently,

$$
\|\mathbf{X_A} - \mathbf{X_P W}\|_{2\to\infty} \le \sigma_d(\mathbf{P})^{-1/2} \|(\mathbf{A} - \mathbf{P}) \mathbf{U_P}\|_{2\to\infty} + \mathrm{O}\left(n^{-\frac{3}{4}} \log^{3\alpha+3/2}(n)\right).
$$

Letting u denote the j^{th} column of $\mathbf{U_P}$, we note that for any $i \in \{1, \dots, n\}$ we have

$$
\begin{aligned}
\{(\mathbf{A} - \mathbf{P}) \mathbf{U_P}\}_{ij} &= \sum_{k=1}^{n} (\mathbf{A}_{ik} - \mathbf{P}_{ik}) u_k \\
&= \sum_{k \ne i} (\mathbf{A}_{ik} - \mathbf{P}_{ik}) u_k - \mathbf{P}_{ii} u_i.
\end{aligned}
$$

The latter term is O(1), while the former is a sum of independent zero-mean random variables satisfying

$$\mathbb{P}\left(\left|\sum_{k\neq i}(\mathbf{A}_{ik}-\mathbf{P}_{ik})u_k\right|\geq t\right)\leq 2\exp\left(\frac{-2t^2}{4\sum_{k\neq i}|u_k|^2}\right)$$

$$\leq 2\exp\left(\frac{-t^2}{2}\right)$$

by Hoeffding's inequality. Thus $\{(\mathbf{A}-\mathbf{P})\mathbf{U}_{\mathbf{P}}\}_{ij}=\mathrm{O}\left(\log^{\alpha+1/2}(n)\right)$ almost surely, and hence $\|\{(\mathbf{A}-\mathbf{P})\mathbf{U}_{\mathbf{P}}\}_i\|=\mathrm{O}\left(\log^{\alpha+1/2}(n)\right)$ almost surely by summing over all $j\in\{1,\ldots,d\}$. Taking the union bound over all n rows then shows that

$$\sigma_d(\mathbf{P})^{-1/2}\|(\mathbf{A}-\mathbf{P})\mathbf{U}_{\mathbf{P}}\|_{2\to\infty}=\mathrm{O}\left(n^{-1/2}\log^{\alpha+1/2}(n)\right),$$

and consequently that

$$\|\mathbf{X}_{\mathbf{A}}\mathbf{Q}_n-\mathbf{X}\|_{2\to\infty}=\mathrm{O}\left(n^{-1/2}\log^{3\alpha+3/2}(n)\right)$$

by multiplying on the right by $\mathbf{Q}_n=\mathbf{W}^\top\mathbf{Q}_{\mathbf{X}}^{-1}$. □

1.2 Proof of Theorem 2

Proof. From the proof of Theorem 1, we see that

$$n^{1/2}(\mathbf{X}_{\mathbf{A}}\mathbf{W}^\top\mathbf{Q}_{\mathbf{X}}^{-1}-\mathbf{X})=n^{1/2}(\mathbf{A}-\mathbf{P})\mathbf{U}_{\mathbf{P}}\mathbf{I}_{p,q}\mathbf{\Sigma}_{\mathbf{P}}^{-1/2}\mathbf{Q}_{\mathbf{X}}^{-1}+n^{1/2}\mathbf{R},$$

where $\|n^{1/2}\mathbf{R}\|_{2\to\infty}\to 0$ by Proposition 6.

Recall that the matrix $\mathbf{Q}_{\mathbf{X}}$ was chosen so that

$$\mathbf{X}\mathbf{Q}_{\mathbf{X}}=\mathbf{X}_{\mathbf{P}}=\mathbf{U}_{\mathbf{P}}\mathbf{\Sigma}_{\mathbf{P}}^{1/2},$$

and so

$$\mathbf{U}_{\mathbf{P}}\mathbf{I}_{p,q}\boldsymbol{\Sigma}_{\mathbf{P}}^{-1/2} = \mathbf{U}_{\mathbf{P}}\boldsymbol{\Sigma}_{\mathbf{P}}^{-1/2}\mathbf{I}_{p,q} = \mathbf{X}\mathbf{Q}_{\mathbf{X}}\boldsymbol{\Sigma}_{\mathbf{P}}^{-1}\mathbf{I}_{p,q}.$$

Thus

$$n^{1/2}(\mathbf{A}-\mathbf{P})\mathbf{U}_{\mathbf{P}}\mathbf{I}_{p,q}\boldsymbol{\Sigma}_{\mathbf{P}}^{-1/2}\mathbf{Q}_{\mathbf{X}}^{-1} = n^{1/2}(\mathbf{A}-\mathbf{P})\mathbf{X}\mathbf{Q}_{\mathbf{X}}\boldsymbol{\Sigma}_{\mathbf{P}}^{-1}\mathbf{I}_{p,q}\mathbf{Q}_{\mathbf{X}}^{-1}.$$

Consequently,

$$\begin{aligned} n^{1/2}\left\{(\mathbf{A}-\mathbf{P})\mathbf{U}_{\mathbf{P}}\mathbf{I}_{p,q}\boldsymbol{\Sigma}_{\mathbf{P}}^{-1/2}\mathbf{Q}_{\mathbf{X}}^{-1}\right\}_i^{\top} &= n^{1/2}(\mathbf{Q}_{\mathbf{X}}\boldsymbol{\Sigma}_{\mathbf{P}}^{-1}\mathbf{I}_{p,q}\mathbf{Q}_{\mathbf{X}}^{-1})^{\top}\left\{(\mathbf{A}-\mathbf{P})\mathbf{X}\right\}_i^{\top} \\ &= (n\mathbf{Q}_{\mathbf{X}}\boldsymbol{\Sigma}_{\mathbf{P}}^{-1}\mathbf{I}_{p,q}\mathbf{Q}_{\mathbf{X}}^{-1})^{\top}\left\{n^{-1/2}\sum_{j=1}^{n}(\mathbf{A}_{ij}-\mathbf{P}_{ij})\mathbf{X}_j\right\} \\ &= (n\mathbf{Q}_{\mathbf{X}}\boldsymbol{\Sigma}_{\mathbf{P}}^{-1}\mathbf{I}_{p,q}\mathbf{Q}_{\mathbf{X}}^{-1})^{\top}\left\{n^{-1/2}\sum_{j\neq i}(\mathbf{A}_{ij}-\mathbf{P}_{ij})\mathbf{X}_j\right\} \\ &\quad -(n\mathbf{Q}_{\mathbf{X}}\boldsymbol{\Sigma}_{\mathbf{P}}^{-1}\mathbf{I}_{p,q}\mathbf{Q}_{\mathbf{X}}^{-1})^{\top}(n^{-1/2}\mathbf{P}_{ii}\mathbf{X}_i). \end{aligned}$$

The latter term satisfies

$$\left\|(n\mathbf{Q}_{\mathbf{X}}\boldsymbol{\Sigma}_{\mathbf{P}}^{-1}\mathbf{I}_{p,q}\mathbf{Q}_{\mathbf{X}}^{-1})^{\top}(n^{-1/2}\mathbf{P}_{ii}\mathbf{X}_i)\right\|_{2\to\infty} \leq \left\|(n\mathbf{Q}_{\mathbf{X}}\boldsymbol{\Sigma}_{\mathbf{P}}^{-1}\mathbf{I}_{p,q}\mathbf{Q}_{\mathbf{X}}^{-1})^{\top}(n^{-1/2}\mathbf{P}_{ii}\mathbf{X}_i)\right\| = \mathrm{O}\left(n^{-1/2}\right)$$

almost surely.

Conditional on the latent variable $\mathbf{Z}_i = \mathbf{z} \in \mathcal{Z}$ from Definition 1, we have $\mathbf{P}_{ij} = \phi(\mathbf{z})^{\top}\mathbf{I}_{p,q}\mathbf{X}_j$, and so

$$n^{-1/2}\sum_{j\neq i}(\mathbf{A}_{ij}-\mathbf{P}_{ij})\mathbf{X}_j$$

is a scaled sum of $n-1$ independent, zero-mean random variables, each with covariance matrix given by

$$\widehat{\boldsymbol{\Sigma}}(\mathbf{z}) = \mathbb{E}\left\{v(\mathbf{z},\mathbf{Z})\phi(\mathbf{Z})\phi(\mathbf{Z})^{\top}\right\},$$

recalling that the function $v(\mathbf{Z}_i, \mathbf{Z}_j)$ gives the variance of $\mathbf{A}_{ij}$. Therefore, by the multivariate central limit theorem,

$$n^{-1/2}\sum_{j\neq i}(\mathbf{A}_{ij} - \mathbf{P}_{ij})\mathbf{X}_j \to \mathcal{N}(\mathbf{0}, \widehat{\mathbf{\Sigma}}(\mathbf{z})).$$

Finally, we consider the terms $(n\mathbf{Q}_\mathbf{X}\mathbf{\Sigma}_\mathbf{P}^{-1}\mathbf{I}_{p,q}\mathbf{Q}_\mathbf{X}^{-1})^\top$. Using the identities

$$\mathbf{Q}_\mathbf{X} = (\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{X}_\mathbf{P}, \qquad \text{and} \quad \mathbf{X}_\mathbf{P}^\top\mathbf{X}_\mathbf{P} = \mathbf{\Sigma}_\mathbf{P},$$

we see that

$$\begin{aligned}
\mathbf{Q}_\mathbf{X}\mathbf{\Sigma}_\mathbf{P}^{-1}\mathbf{I}_{p,q}\mathbf{Q}_\mathbf{X}^{-1} &= (\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{X}_\mathbf{P}\mathbf{\Sigma}_\mathbf{P}^{-1}\mathbf{I}_{p,q}\mathbf{Q}_\mathbf{X}^{-1} \\
&= (\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{Q}_\mathbf{X}^{-\top}\mathbf{X}_\mathbf{P}^\top\mathbf{X}_\mathbf{P}\mathbf{\Sigma}_\mathbf{P}^{-1}\mathbf{I}_{p,q}\mathbf{Q}_\mathbf{X}^{-1} \\
&= (\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{Q}_\mathbf{X}^{-\top}\mathbf{I}_{p,q}\mathbf{Q}_\mathbf{X}^{-1} \\
&= (\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{I}_{p,q}
\end{aligned}$$

and so

$$(n\mathbf{Q}_\mathbf{X}\mathbf{\Sigma}_\mathbf{P}^{-1}\mathbf{I}_{p,q}\mathbf{Q}_\mathbf{X}^{-1})^\top \to \mathbf{I}_{p,q}\mathbf{\Delta}^{-1}$$

almost surely by the law of large numbers.

Combining all this, we find that, conditional on $\mathbf{Z}_i = \mathbf{z}$,

$$n^{-1/2}\left(\mathbf{X}_\mathbf{A}\mathbf{W}^\top\mathbf{Q}_\mathbf{X}^{-1} - \mathbf{X}\right)_i^\top \to \mathcal{N}\left(\mathbf{0}, \mathbf{\Sigma}(\mathbf{z})\right)$$

almost surely. □

2 Proofs for size-adjusted Chernoff information

Before beginning the proofs, it is convenient to write the summations in Corollary 1 as matrix products. The second moment matrix $\mathbf{\Delta}$ can expressed as

$$\mathbf{\Delta} = \sum_{k=1}^{K} \pi_k (\mathbf{X_B})_k (\mathbf{X_B})_k^\top = \mathbf{X_B^\top \Pi X_B},$$

where $\mathbf{\Pi} = \mathrm{diag}(\pi_1, \ldots, \pi_K)$. Similarly, the covariance matrix $\mathbf{\Sigma}_k$ for $k \in \{1, \ldots, K\}$, can be expressed as

$$\mathbf{\Sigma}_k = \mathbf{I}_{p,q} \mathbf{\Delta}^{-1} \left\{ \sum_{\ell=1}^{K} \pi_\ell \mathbf{C}_{k\ell} (\mathbf{X_B})_\ell (\mathbf{X_B})_\ell^\top \right\} \mathbf{\Delta}^{-1} \mathbf{I}_{p,q} = \mathbf{I}_{p,q} \mathbf{\Delta}^{-1} \mathbf{X_B^\top \Pi S}_k \mathbf{X_B} \mathbf{\Delta}^{-1} \mathbf{I}_{p,q},$$

where $\mathbf{S}_k = \mathrm{diag}(\mathbf{C}_k) \in \mathbb{R}^{K \times K}$. The expression for $\mathbf{\Sigma}_{k\ell}(t)$ has the same form as the above equation, replacing $\mathbf{S}_k$ with its corresponding counterpart $\mathbf{S}_{k\ell}(t) = (1-t)\mathbf{S}_k + t\mathbf{S}_\ell$.

2.1 Proof of Lemma 1

Proof. If $\mathbf{D} \in \mathbb{R}^{K \times K}$ is full rank, then for $\mathbf{Y} \in \mathbb{R}^{K \times d}$, the following matrix inequality holds (Marshall and Olkin, 1990),

$$\mathbf{Y} \left(\mathbf{Y}^\top \mathbf{D} \mathbf{Y} \right)^{-1} \mathbf{Y}^\top \preceq \mathbf{D}^{-1},$$

where $\mathbf{M} \succeq \mathbf{0}$ means that $\mathbf{M}$ is a positive semi-definite matrix. However, in the case where $\mathbf{D}$ and $\mathbf{X} \in \mathbb{R}^{K \times K}$ are full rank, then the two sides of this inequality are equal,

$$\mathbf{X} \left(\mathbf{X}^\top \mathbf{D} \mathbf{X} \right)^{-1} \mathbf{X}^\top = \mathbf{D}^{-1}.$$

If the block mean matrix $\mathbf{B}$ is full rank, then adjacency spectral embedding $\mathbf{X_B}$ is also full rank. Since $\mathbf{B} = \mathbf{X_B I}_{p,q} \mathbf{X_B^\top}$, this means $\mathrm{rank}(\mathbf{B}) \leq \mathrm{rank}(\mathbf{X_B})$ implying $\mathrm{rank}(\mathbf{X_B}) = K$ when $\mathrm{rank}(\mathbf{B}) = K$.

Using this matrix equality and the expression for $\mathbf{\Delta}$, we have

$$\begin{aligned}\mathbf{\Sigma}_{k\ell}(t)^{-1} &= \left(\mathbf{I}_{p,q}\mathbf{\Delta}^{-1}\mathbf{X}_{\mathbf{B}}^{\top}\mathbf{\Pi}\mathbf{S}_{k\ell}(t)\mathbf{X}_{\mathbf{B}}\mathbf{\Delta}^{-1}\mathbf{I}_{p,q}\right)^{-1} \\ &= \mathbf{I}_{p,q}\mathbf{X}_{\mathbf{B}}^{\top}\mathbf{\Pi}\mathbf{X}_{\mathbf{B}}\left(\mathbf{X}_{\mathbf{B}}^{\top}\mathbf{\Pi}\mathbf{S}_{k\ell}(t)\mathbf{X}_{\mathbf{B}}\right)^{-1}\mathbf{X}_{\mathbf{B}}^{\top}\mathbf{\Pi}\mathbf{X}_{\mathbf{B}}\mathbf{I}_{p,q} \\ &= \mathbf{I}_{p,q}\mathbf{X}_{\mathbf{B}}^{\top}\mathbf{\Pi}\mathbf{S}_{k\ell}(t)^{-1}\mathbf{X}_{\mathbf{B}}\mathbf{I}_{p,q}\end{aligned}$$

Substituting this expression into the objective function in the size-adjusted Chernoff information gives

$$\begin{aligned}&\{(\mathbf{X}_{\mathbf{B}})_k - (\mathbf{X}_{\mathbf{B}})_\ell\}^{\top}\mathbf{\Sigma}_{k\ell}(t)^{-1}\{(\mathbf{X}_{\mathbf{B}})_k - (\mathbf{X}_{\mathbf{B}})_\ell\} \\ &= (\mathbf{e}_k - \mathbf{e}_\ell)^{\top}\mathbf{X}_{\mathbf{B}}\mathbf{I}_{p,q}\mathbf{X}_{\mathbf{B}}^{\top}\mathbf{\Pi}\mathbf{S}_{k\ell}(t)^{-1}\mathbf{X}_{\mathbf{B}}\mathbf{I}_{p,q}\mathbf{X}_{\mathbf{B}}^{\top}(\mathbf{e}_k - \mathbf{e}_\ell) \\ &= (\mathbf{e}_k - \mathbf{e}_\ell)^{\top}\mathbf{B}\mathbf{\Pi}\mathbf{S}_{k\ell}(t)^{-1}\mathbf{B}(\mathbf{e}_k - \mathbf{e}_\ell),\end{aligned}$$

using the expression $\mathbf{B} = \mathbf{X}_{\mathbf{B}}\mathbf{I}_{p,q}\mathbf{X}_{\mathbf{B}}^{\top}$. □

If the block mean matrix $\mathbf{B}$ is not full rank, then the matrix inequality effectively passes through the whole argument in the above proof,

$$\mathbf{\Sigma}_{k\ell}(t)^{-1} \preceq \mathbf{I}_{p,q}\mathbf{X}_{\mathbf{B}}^{\top}\mathbf{\Pi}\mathbf{S}_{k\ell}(t)^{-1}\mathbf{X}_{\mathbf{B}}\mathbf{I}_{p,q},$$

which becomes a regular inequality when we compute the quadratic form. We get that, for $\mathrm{rank}(\mathbf{B}) \leq K$,

$$C \leq \min_{k\neq\ell}\sup_{t\in(0,1)}\left[\frac{t(1-t)}{2}\left\{(\mathbf{e}_k - \mathbf{e}_\ell)^{\top}\mathbf{B}\mathbf{\Pi}\mathbf{S}_{k\ell}(t)^{-1}\mathbf{B}(\mathbf{e}_k - \mathbf{e}_\ell)\right\}\right].$$

with equality when $\mathbf{B}$ is full rank.

2.2 Proof of Lemma 2

Proof. By assumption $\mathbf{A}$ and $\mathbf{A}'$ have full rank block mean matrices, therefore, by Lemma 1, both stochastic block models have the same size-adjusted Chernoff information if they have the same value for

$$(\mathbf{e}_k - \mathbf{e}_\ell)^\top \mathbf{B}\mathbf{\Pi}\mathbf{S}_{k\ell}(t)^{-1}\mathbf{B}(\mathbf{e}_k - \mathbf{e}_\ell).$$

For an affine entry-wise transformation, the entries of the block mean and variance matrices are given by $\mathbf{B}'_{k\ell} = a\mathbf{B}_{k\ell} + b$ and $\mathbf{C}'_{k\ell} = a^2\mathbf{C}_{k\ell}$. Therefore,

$$\begin{aligned} \mathbf{B}'(\mathbf{e}_k - \mathbf{e}_\ell) &= a\mathbf{B}(\mathbf{e}_k - \mathbf{e}_\ell), \\ \mathbf{S}'_{k\ell}(t)^{-1} &= a^2\mathbf{S}_{k\ell}(t)^{-1}, \end{aligned}$$

where $\mathbf{S}'_{k\ell}(t)^{-1}$ is the equivalent version of $\mathbf{S}_{k\ell}(t)^{-1}$ in the entry-wise transformed stochastic block model. The contribution from a cancel and there is no contribution from b, meaning size-adjusted Chernoff information is unaffected by affine transformation. □

2.3 Changes of signature by affine transformation

We demonstrate the possible effects an affine transformation may have on the signature of a stochastic block model with dimension d and signature (p, q). We focus on affine transformations with $a > 0$ and $b > 0$ although a similar analysis for other affine transformations leads to slight variations of the following results.

Consider a weighted stochastic block model with full rank block mean matrix $\mathbf{B}$ with signature (p, q) and let $\lambda_i(\mathbf{B})$ denote the i^{th} smallest eigenvalue of $\mathbf{B}$. The signature of $\mathbf{B}$ implies that $\lambda_1(\mathbf{B}), \ldots, \lambda_q(\mathbf{B}) < 0$ and $\lambda_{q+1}(\mathbf{B}), \ldots, \lambda_{p+q}(\mathbf{B}) > 0$, since $\mathbf{B}$ is full rank, no eigenvalues are exactly zero.

The block mean matrix of the weighted stochastic block model after affine transformation is given by $\mathbf{B}' = a\mathbf{B} + b\mathbf{1}\mathbf{1}^\top$, where $\mathbf{1} \in \mathbb{R}^n$ is the all-one vector. This is a rank-one perturbation of the scaled matrix $a\mathbf{B}$, therefore, by Horn and Johnson (2012), Corollary 4.3.9, the i^{th} eigenvalue of $\mathbf{B}'$ lies between the i^{th} and $(i+1)^{\text{th}}$ eigenvalue of $\mathbf{B}$, that is, for $i = 1, \ldots, d-1$,

$$\lambda_i(\mathbf{B}) \leq \lambda_i(\mathbf{B}') \leq \lambda_{i+1}(\mathbf{B}) \quad \text{and} \quad \lambda_d(\mathbf{B}) \leq \lambda_d(\mathbf{B}').$$

Figure 5 shows an example of this eigenvalue behavior. The only eigenvalue that can change sign after affine transformation is $\lambda_q(\mathbf{B}')$. In the example shown, $\lambda_q(\mathbf{B}') > 0$, meaning the signature of the stochastic block model after affine transformation is $(p+1, q-1)$; if $\lambda_q(\mathbf{B}') < 0$, then the signature would have remained (p, q). The other remaining possibility is that $\lambda_q(\mathbf{B}') = 0$, meaning that $\mathbf{B}'$ is not full rank. In this case, the affine transformation block model would instead need to be embedded into $d-1$ dimensions and would have signature $(p, q-1)$.

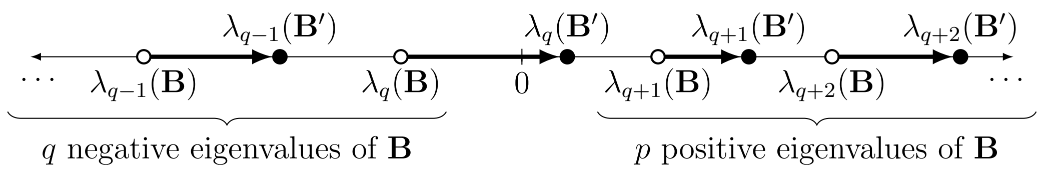

Figure 5: Number line showing the eigenvalues of mean block matrices close to the origin. White nodes represent eigenvalues of $\mathbf{B}$ with full rank d and signature (p, q), black nodes represent eigenvalues of $\mathbf{B}' = a\mathbf{B} + b\mathbf{1}\mathbf{1}^\top$ with $a > 0$, $b > 0$, full rank d and signature $(p+1, q-1)$.