Online Appendix to "A Simple Correction for Misspecification in Trend-Cycle Decompositions with an Application to Estimating r^*"

James Morley[*] — University of Sydney

Trung Duc Tran[†] — KPMG Australia

Benjamin Wong[‡] — Monash University

[*]Corresponding author: School of Economics, University of Sydney, Sydney, New South Wales, 2006, Australia, james.morley@sydney.edu.au

[†] KPMG Australia, Melbourne, Victoria, 3008, Australia, ductrantrung@gmail.com

[‡] Department of Econometrics and Business Statistics, Melbourne, Victoria, 3145, Australia, benjamin.wong@monash.edu

Appendices

A Details of the data

In this appendix, we provide the details of the specific variables hypothesized to drive r^* in the broad categories of productivity/demographics and safe asset supply/demand. We then report the data sources and transformations.

Productivity/demographics

Motivated by an intertemporal IS/Euler-type equation, such as in Lunsford and West (2019), we consider real consumption growth per capita. Related, we also consider TFP growth (Fernald, 2015) and S&P 500 stock returns on the basis that they might be additionally informative about expected trend growth for the economy, which Laubach and Williams (2003) highlight as the key positive determinant of r^*. By contrast, Eichengreen (2015) stresses the importance of investment-specific technological change and the subsequent decline in the price of capital goods in driving down real interest rates. Thus, we also consider real investment growth as a proxy for investment-specific technological change and expect it to have a negative relationship with r^*, at least when controlling for consumption growth and TFP growth.

Various labor-market variables reflect demographic factors and are hypothesized to influence r^* through an effect on the marginal product of capital. For example, Baker et al. (2005) note that in certain overlapping-generations models, labor-force growth is positively related to the real interest rate given that higher labor-force participation would lead, all else equal, to a lower level of capital per worker. Thus, we also consider employment growth, hours growth (to capture the intensive margin), and the change in the unemployment rate as additional possible supply-side variables, although clearly decreases in employment and hours and increases in the unemployment rate could be also be related to a decline in r^* via insufficient demand, as argued by Summers (2015). The unemployment rate also serves as a potential control for economic slack that could distort measures of trend growth and generate short-run deviations in the real interest rate from r^*.

Possible heterogeneity in marginal propensities to consume motivates consideration of income inequality and age dependency. Dynan et al. (2004) find that higher income families have lower marginal propensities to consume, suggesting that an increase in inequality will shift the savings schedule out and lower r^*. Gagnon et al. (2021), on the other hand, suggest that an increase in the dependency (older-to-working) ratio reduces aggregate savings and raises r^*. To capture these demographic factors, we consider the share of wealth held by the top 1% and the age dependency ratio, although these series are only available at an annual frequency and so are only considered in robustness analysis using a VECM with annual data in Appendix F.

Safe asset demand/supply

Caballero et al. (2017) and Del Negro et al. (2017) suggest that demand for safe assets has played a key role in lowering r^* in recent decades. To address this, we consider the change in macroeconomic uncertainty (Jurado et al., 2015), the change in the excess bond premium (Gilchrist and Zakrajšek, 2012), and growth of liquid assets held by financial and non-financial corporate businesses.

Also related to demand for safe assets, Bernanke (2005) suggests a relationship between the U.S. current account deficit and the global savings glut. Capital inflows are typically associated with a trade deficit, but the link to r^* depends on whether those capital flows are induced by a high real interest rate or reflect excess global savings. To address this, we consider the change in the U.S. current account balance (as % of GDP), the change in U.S. government debt (as % of GDP), the trade-weighted U.S. dollar exchange rate growth rate, and global central bank foreign reserves (as % of world GDP), although the global reserves variable is only available at an annual frequency and so is only considered in the robustness analysis in Appendix F. An increase in government expenditure or a decrease in tax revenues that lead to a higher level of government debt is usually thought to raise real interest rates through a crowding-out effect (see, for example, Ball and Mankiw, 1995). So the government debt measure can be thought of as reflecting the supply of safe assets, while the other measures are designed to help capture demand for safe assets that push the real interest rate in the opposite direction.

Prior to inclusion in the VECM, the data are transformed to be stationary. As discussed in Morley and Wong (2020), the BN decomposition matrix calculations require specification of the forecasting model in a stationary form. The transformations, along with the original data sources, are given in Table A1.

Table A1: Data sources and transformations

Variable Description	**Source**	**Transformation**
3-Month Treasury Bill Secondary Market Rate	FRED:TB3MS	quarterly avg., Δ
Market Yield on U.S. Treasury Securities at 10-Year Constant Maturity	FRED:GS10	quarterly avg., Δ
Personal Consumption Expenditures Excluding Food and Energy (Chain-Type Price Index)	FRED:PCEPILFE	$\%\Delta_4$
Survey of Professional Forecasters 1-Year-Ahead GDP Deflator Inflation Rate, Median Forecast	Phil.Fed:INFPGDP1YR	
Survey of Professional Forecasters 10-Year PCE Inflation Rate, Mean Response, Annual Average	Phil.Fed:PCE10,T.Clark	
Cleveland Fed 1-Month Real Rate using Model-Based Expected Inflation	clevelandfed.org	quarterly avg., Δ
Cleveland Fed 10-Year Real Rate using Model-Based Expected Inflation	clevelandfed.org	quarterly avg., Δ
Real personal consumption expenditures per capita	FRED:A794RX0Q048SBEA	ln, Δ
Business Sector TFP (annualized quarterly % growth rate)	frbsf.org	$\ln(1+\text{series}/400)$
S&P 500 Index	FRED:SP500	quarterly avg., ln, Δ
Real Gross Private Domestic Investment	FRED:GPDIC1	ln, Δ
All Employees: Total Nonfarm	FRED:PAYEMS	quarterly avg., ln, Δ
Business Sector: Hours Worked for All Employed Persons	FRED:HOABS	ln, Δ
Unemployment Rate	FRED:UNRATE	quarterly avg., Δ
Age Dependency Ratio: Older Dependents to Working-Age Population for the United States	FRED:SPPOPDPNDOLUSA	annual only, Δ^2
Top 1% Share of Pre-Tax National Income	World Inequality Database	annual only, Δ
1-Month-Ahead Economic Macro Uncertainty Index	sydneyludvigson.com	Δ
Excess Bond Premium	federalreserve.gov	quarterly avg., Δ
Nonfinancial Corporate Business and Other Financial Corporations, Money Market Funds, Insurance Companies, and Pension Funds; Liquid Assets (Broad Measure), Level	FRED:BOGZ1FL104001005Q, BOGZ1FL874001005Q	sum, ln, Δ
Balance on Current Account as a Percent of Gross Domestic Product	FRED:NETFI, GDP	ratio, Δ
Nominal Major Currencies U.S. Dollar Index (Goods Only)	FRED:TWEXMMTH	quarterly avg., ln, Δ
Federal Debt: Total Public Debt as Percent of Gross Domestic Product	FRED:GFDEGDQ188S	Δ
Total reserves comprising holdings of monetary gold, special drawing rights, reserves of members held by the IMF, and holdings of foreign exchange under the control of monetary authorities as a percent of world GDP at purchaser's prices (data are in current U.S. dollars, with gold component of reserves valued at year-end prices and GDP converted from domestic currencies using single-year official exchange rates)	IMF:FI.RES.TOTL.CD, NY.GDP.MKTP.CD	annual only, ratio, Δ

B Bayesian estimation of the VECM

In this appendix, we present the details of the Bayesian estimation of the VECM.

Recall that $\boldsymbol{\Delta}\mathbf{x}_t$ consists of the first differences of the interest rates Δr_t^s and Δr_t^l (which we now denote as Δx_{1t} and Δx_{2t} for convenience) and the correlates $\boldsymbol{\Delta}\mathbf{x}_{3:n,t}$. Because of the error correction term, the regressors differ between the interest-rates and correlates blocks of the VECM. Thus, we specify the i^{th} equation of the VECM as

$$\Delta x_{it} = \mu_i + \mathbf{w}_{it}'\mathbf{b}_i + e_{it}, \tag{B1}$$

where $\mathbf{w}_{it} = [(\boldsymbol{\Delta}\mathbf{x}_{t-1} - \boldsymbol{\mu})', \dots, (\boldsymbol{\Delta}\mathbf{x}_{t-p} - \boldsymbol{\mu})', r_{t-1}^l - r_{t-1}^s - \alpha]'$ if $i = 1, 2$ and $\mathbf{w}_{it} = [(\boldsymbol{\Delta}\mathbf{x}_{t-1} - \boldsymbol{\mu})', \dots, (\boldsymbol{\Delta}\mathbf{x}_{t-p} - \boldsymbol{\mu})']'$ if $i > 2$, with $\mathbf{b}_i$ corresponding to all of the parameters associated with equation i. Following Morley and Wong (2020), unconditional means μ_i are based on sample averages, equivalent to setting a flat prior on these parameters and concentrating them out of the likelihood, except for $i = 1, 2$ where the means of the changes in real interest rates are set to zero, implying no deterministic drift in levels.

Defining

$$\boldsymbol{y}_t \equiv \boldsymbol{\Delta}\mathbf{x}_t - \boldsymbol{\mu}, \;\; \boldsymbol{\beta} \equiv \begin{bmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_n \end{bmatrix} \text{ and } \boldsymbol{Z}_t \equiv \begin{bmatrix} \mathbf{w}_{1t}' & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{w}_{nt}' \end{bmatrix}, \;\; \boldsymbol{e}_t \equiv \begin{bmatrix} e_{1t} \\ \vdots \\ e_{nt} \end{bmatrix},$$

we can stack all the equations and regressors in (B1) and rewrite the system as

$$\boldsymbol{y}_t = \boldsymbol{Z}_t\boldsymbol{\beta} + \boldsymbol{e}_t,$$

or

$$\boldsymbol{y} = \boldsymbol{Z}\boldsymbol{\beta} + \boldsymbol{E},$$

where

$$\boldsymbol{y_i} = \begin{bmatrix} y_{i1} \\ \vdots \\ y_{iT} \end{bmatrix}, \quad \boldsymbol{e_i} = \begin{bmatrix} e_{i1} \\ \vdots \\ e_{iT} \end{bmatrix},$$

and

$$\boldsymbol{y} = \begin{bmatrix} \boldsymbol{y}_1 \\ \vdots \\ \boldsymbol{y}_n \end{bmatrix}, \quad \boldsymbol{Z} = \begin{bmatrix} \boldsymbol{Z}_1 \\ \vdots \\ \boldsymbol{Z}_T \end{bmatrix}, \quad \boldsymbol{E} = \begin{bmatrix} \boldsymbol{e}_1 \\ \vdots \\ \boldsymbol{e}_n \end{bmatrix}.$$

Let $\boldsymbol{\Sigma}$ be an $n \times n$ covariance matrix for the VECM residuals. If one sets a Normal-Wishart prior on $\boldsymbol{\beta}$ and $\boldsymbol{\Sigma}$ (Koop and Korobilis (2010)), where

$$\boldsymbol{\beta} \sim N(\boldsymbol{\beta}_0, \boldsymbol{V_\beta}), \tag{B2}$$

$$\boldsymbol{\Sigma}^{-1} \sim W(\boldsymbol{S}_0^{-1}, \nu_0), \tag{B3}$$

this implies conditional distributions

$$p(\boldsymbol{\beta} \mid \boldsymbol{y}, \boldsymbol{\Sigma}^{-1}) \sim N(\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{V_\beta}}), \tag{B4}$$

$$p(\boldsymbol{\Sigma}^{-1} \mid \boldsymbol{y}, \boldsymbol{\beta}) \sim W(\hat{\boldsymbol{S}}^{-1}, \hat{\boldsymbol{\nu}}), \tag{B5}$$

where

$$\begin{aligned} \hat{\boldsymbol{V_\beta}} &= \left(\boldsymbol{V_\beta}^{-1} + \sum_{t=1}^{T} \boldsymbol{Z}_t' \boldsymbol{\Sigma}^{-1} \boldsymbol{Z}_t \right), \\ \hat{\boldsymbol{\beta}} &= \hat{\boldsymbol{V_\beta}} \left[\boldsymbol{V_\beta}^{-1} \boldsymbol{\beta}_0 + \sum_{t=1}^{T} \boldsymbol{Z}_t' \boldsymbol{\Sigma}^{-1} \boldsymbol{Z}_t \right], \end{aligned}$$

and

$$\begin{aligned} \hat{\boldsymbol{S}} &= \boldsymbol{S}_0 + \sum_{t=1}^{T} (\boldsymbol{y}_t - \boldsymbol{Z}_t \boldsymbol{\beta})(\boldsymbol{y}_t - \boldsymbol{Z}_t \boldsymbol{\beta})', \\ \hat{\boldsymbol{\nu}} &= T + \nu_0. \end{aligned}$$

We elaborate how priors $\boldsymbol{\beta}_0$, $\boldsymbol{V_\beta}$, $\boldsymbol{S}_0^{-1}$ and ν_0 are elicited below. Given the priors, (B4) and (B5) define a Gibbs-sampling scheme, where one can sequentially take draws from these conditional distributions, conditioning on the previous draw in the chain. We take 12,000 draws with the sampling scheme, discarding the first 2,000 draws and use the remaining 10,000 draws to make inferences about the posterior distribution.

Priors

Our goal in setting the prior is to apply shrinkage to mitigate possible overfitting. To keep the application of shrinkage as standard as possible, we use a "Minnesota Prior" (e.g., see Litterman, 1986). The idea behind this type of prior is to shrink parameters for persistent variables towards a random walk.

Accordingly, given that the variables in the VECM are included in first differences, we set the prior mean $\boldsymbol{\beta}_0$ in (B2) to a vector of zeros, except for the element associated with the error correction term in the short-rate equation. In that case, we set the prior mean to 0.5, consistent with the expectation hypothesis for the term structure of interest rates (see, for example, Modigliani and Shiller, 1973) that motivates our assumption of cointegration between the interest rates. Setting the prior mean for this parameter to zero is somewhat contrary to the assumption of cointegration. However, we note that our posterior inferences are robust to setting this prior mean to zero, as demonstrated in Appendix C.

In specifying the prior variance, which dictates how tightly we shrink the coefficients towards zero, we follow the Minnesota prior approach and treat shorter lags as more important than longer lags when applying shrinkage. Let $V_{i,j}^{k}$ be the prior variance on the parameter in the i^{th} equation for the j^{th} variable on the k^{th} lag. Accordingly, we set

$$V_{i,j}^{k} = \frac{\lambda^2}{k^2}\frac{\sigma_i^2}{\sigma_j^2}. \tag{B6}$$

where σ_i^2 is the sample variance of the residuals from a univariate AR(4) regression fitted using least squares on the i^{th} variable and σ_i^2/σ_j^2 acts as a scaling factor to account for different units of the variables (note that we set $\sigma_j^2 = \sigma_i^2$ in the case of the error correction coefficients). The overall tightness of the prior is then governed by one hyperparameter, λ. We set $\lambda = 0.2$ in our empirical analysis, which is a fairly common choice within the BVAR literature (e.g., Sims and Zha, 1998) and corroborated as a reasonable choice in forecasting settings by Carriero et al. (2015). We stress, however, that our main results are robust to departures from this particular prior, including optimizing λ to minimize the one-step-ahead out-of-sample RMSFE for the short-run interest rate equation along the lines of Morley and Wong (2020), again as demonstrated in Appendix C.

Once $V_{i,j}^k$ in (B6) is specified, $\boldsymbol{V}_\beta$ is constructed as

$$\boldsymbol{V_\beta} = \begin{bmatrix} \boldsymbol{V}_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & \boldsymbol{V}_n \end{bmatrix},$$

where

$$\boldsymbol{V}_i = \begin{bmatrix} \boldsymbol{V}_i^1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & \boldsymbol{V}_i^p \end{bmatrix} \text{ and } \boldsymbol{V}_i^k = \begin{bmatrix} V_{i,1}^k & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & V_{i,n}^k \end{bmatrix},$$

except for $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$, which each have an additional row and column of zeros and $V_{i,n+1}^1 = \lambda^2$ for $i = 1, 2$ on the respective last diagonal as the prior variance for the corresponding error correction coefficient.

For the remaining quantities in (B3), we set

$$\boldsymbol{S} = \begin{bmatrix} \nu_0\sigma_1^2 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & \nu_0\sigma_n^2 \end{bmatrix},$$

where ν_0 is set to $n + 1$ (i.e., one greater than the total number of variables), σ_i^2 is obtained from the same AR(4) regression on the i^{th} variable as what used in (B6), and the variance is thus scaled up by a factor of ν_0 so that the prior on the sum of squared residuals is consistent with the prior on the degrees of freedom.

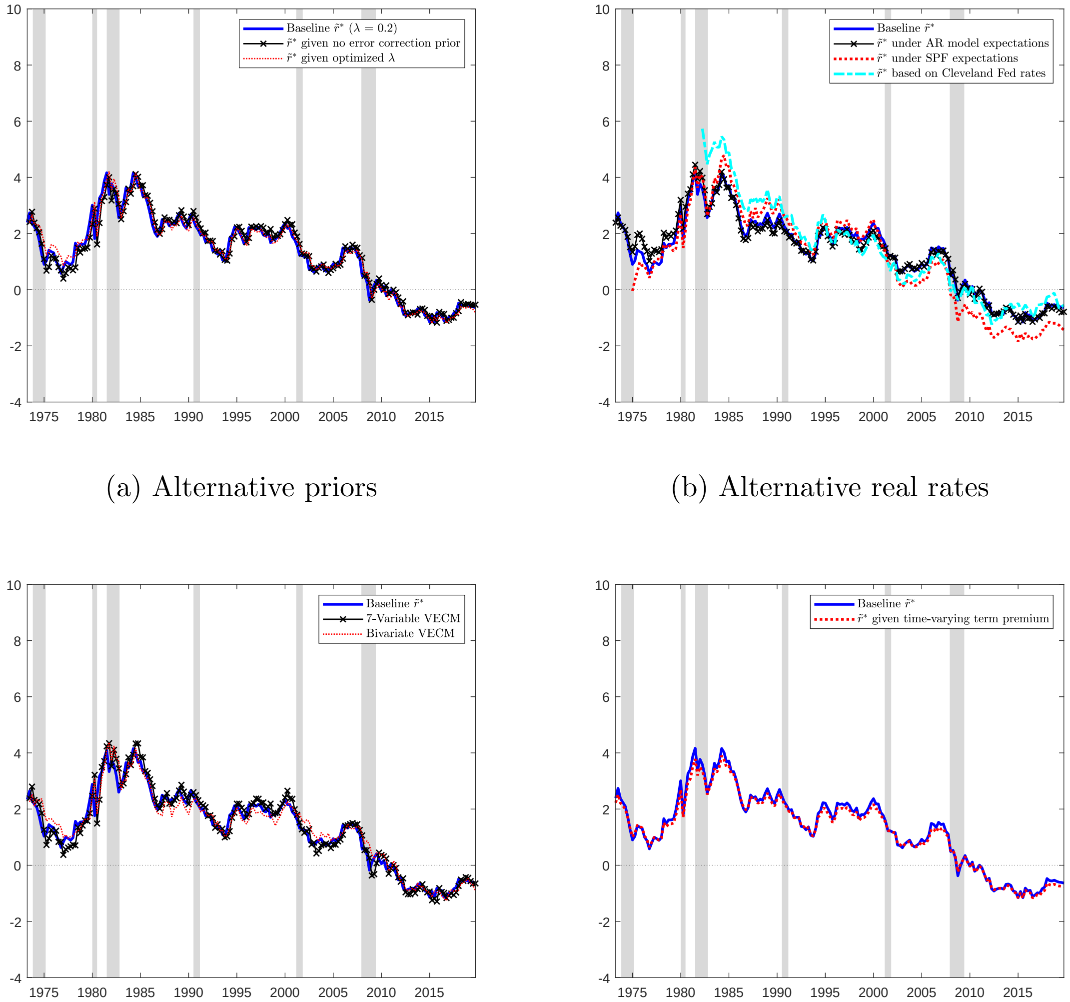
C Robustness checks

In this appendix, we consider some robustness checks to demonstrate that our main empirical findings are not particularly sensitive to choices about priors, proxies for inflation expectations, variables to include in the VECM, and possible permanent movements in the term premium.

First, we investigate the sensitivity of our r^* estimates with respect to the prior on the error-correction coefficient for the short-term interest rate and to the shrinkage hyperparameter used in the Bayesian estimation. The results are highly robust to the choice of prior, as seen in the first panel of Figure C1. Thus, while we see our baseline priors as well justified, we also note that our main findings, including the smoothness of our r^* estimates, do not hinge upon them.

Second, we investigate the sensitivity of our r^* estimates with respect to how we proxy inflation expectations when measuring *ex ante* real interest rates. As shown in the second panel of Figure C1, our estimates are generally robust to three alternative measures of inflation expectations. In the first case, following Laubach and Williams (2003), we proxy short-run inflation expectations with the forecast of the four-quarter-ahead percentage change in core PCE prices generated from a univariate AR(3) of inflation estimated over the prior 40 quarters (10 year rolling window). In the second case, we proxy the short-run (long-run) inflation expectations by the SPF short-term (long-run) inflation forecast. In the third case, we investigate the sensitivity of our r^* estimates with respect to the 1-month and 10-year real interest rates constructed by the Cleveland Fed based on their model-based expected inflation measures. The r^* estimate using the Cleveland Fed data appears to be higher in the early part of the sample for which it is available, but it soon converges to our baseline r^* estimate.

Third, to confirm the relevance of different sources of information, we consider two alternative models in terms of which variables are included in the information set. In the first case, we consider a smaller seven-variable model that, in addition to the interest-rates block, only includes the five most informationally-relevant variables for deviations of the short-term real interest rate from its trend following the variable selection procedure suggested in Morley and Wong (2020). The selected variables are the change in government debt, hours growth, employment growth, real consumption per capita growth, and stock returns. In the second case, we consider a bivariate model that only includes the interest-rates block. As shown in the third panel of Figure C1, the estimates are generally robust. Notably, by dropping the



(a) Alternative priors

(b) Alternative real rates

(c) Alternative information sets

(d) Time-varying long-run term premium

Figure C1: Robustness of r^* estimates. NBER recession dates are shaded.

less informationally-relevant variables from the model, the estimated r^* is barely affected. The estimated r^* changes a bit more, however, when we do not include any possible determinants beyond the interest-rates block. But the general similarity of the estimates even when only including interest rates has two important implications: (i) measurement error or other sources of model misspecification that generate serial correlation in the first-stage estimates of trend growth appear to be primarily related to the *ex ante* real interest rates in particular and (ii) one could obtain a reasonably robust estimate of r^* by just considering a bivariate VAR of the change in the short-term real interest rate and the spread between the long- and short-term real rates given that our correction can handle misspecification due to omitted variables. Of course, our baseline medium-scale Bayesian VECM has the advantage of allowing us to track which economic forces are most important in driving changes in r^*.

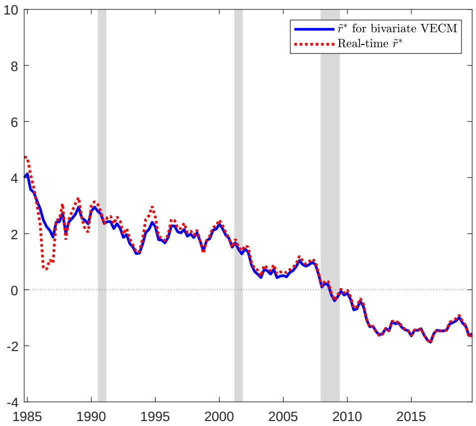
Fourth, we investigate the robustness of our result with respect to the assumption of no permanent movements in the term premium. We do so by allowing the long-run level of the term premium to be time varying, effectively by demeaning the error-correction term dynamically with a backward-looking 40-quarter average. As seen in the last panel of Figure C1, the results are robust.

D Additional exercises

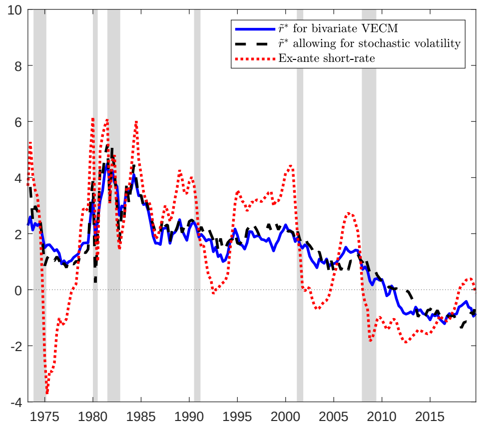
In this appendix, we conduct some additional exercises related to the real-time reliability of our estimates, allowing for stochastic volatility, and taking a frequentist approach to inference given our proposed correction.

First, we consider real-time reliability of our estimates of r^*. To abstract from the effect of data revisions, which Orphanides and van Norden (2002) argue are less important than trend-cycle decomposition method for reliability of real-time estimates, we focus on r^* estimates based on a model using only the interest-rates block and using SPF survey measures of inflation expectations so that there are no sources of data revision. The first panel of Figure D1 plots the real-time estimate using an expanding window of data for the first ten years of the sample period until the end of the period to estimate r^* and compares it with the *ex post* estimate based on the full sample of data. The real-time estimates are clearly quite reliable, although there is an upward bias earlier in the sample period compared to the revised estimates. This is likely due to some changes in the estimated long-run level of the term premium over the sample period. But the movements and general decline in r^* implied by the real-time estimates are highly robust to consideration of the full sample of data. Notably, there is clearly no end-point problem that plagues other approaches to trend-cycle decomposition such as the Hodrick-Prescott filter.

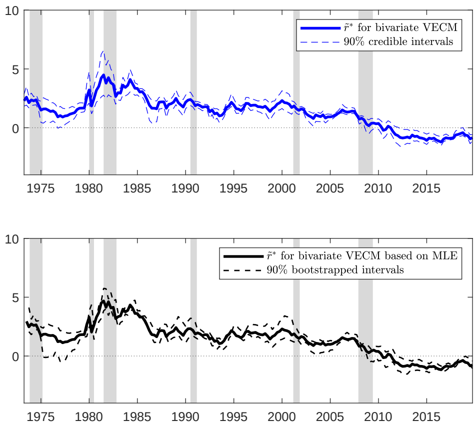
Second, we consider an additional exercise of allowing for stochastic volatility when estimating the Bayesian VECM. Again, for tractability, we consider a bivariate VECM. We set up the bivariate VECM with stochastic volatility as per Carriero et al. (2022). Specifically, the Carriero et al. (2022) approach uses the triangular factorization when it sets up the stochastic volatility component, and so we set up the model by ordering the *ex-ante* long-rate before the short-rate in the VECM. All other specifications remain identical to the baseline setting, and we also retain the prior from our baseline analysis for the VECM parameters. The MA parameters



(a) Real-time reliability



(b) Allowing for stochastic volatility



(c) Bayesian versus frequentist inference

Figure D1: Results for additional exercises. NBER recession dates are shaded.

for the correction are estimated using standardized changes in the preliminary estimated trend, i.e., $\Delta\hat{r}_t^*/\sigma^2_{\Delta\hat{r}_t^*,t}$, where $\sigma^2_{\Delta\hat{r}_t^*,t} = \mathbf{s}'_{k,1}(\mathbf{I}-\mathbf{P})^{-1}\mathbf{H}\boldsymbol{\Sigma}_t\mathbf{H}'((\mathbf{I}-\mathbf{P})^{-1})'\mathbf{s}_{k,1}$, with $\boldsymbol{\Sigma}_t$ being the residual covariance matrix for the VECM under stochastic volatility, while the correction is applied to the raw changes, i.e., $\Delta\hat{r}_t^*$. The second panel of Figure D1 plots the results when allowing for stochastic volatility, as well as the original bivariate VECM estimates. It can be seen that the estimates are largely robust.

Third, we sketch out how one might conduct frequentist inference about the corrected trend estimates if one were not inclined to do Bayesian estimation as we do. First, recall that our main motivation for conducting Bayesian inference in our baseline analysis is because we consider a medium-scale model and applying shrinkage by using Bayesian methods is reasonably standard to address in-sample overfitting given parameter proliferation in such settings (e.g., see Banbura et al., 2010). Nonetheless, if one considered the simpler bivariate model, it would be straightforward to consider OLS or MLE, as we also do in our Monte Carlo analysis with a bivariate DGP.

A bootstrap provides a natural approach to conducting frequentist inference about the corrected trend estimates given the complicated mapping from the preliminary estimates. For the bootstrap DGP, we obtain an estimate of the projection matrix $\hat{\mathbf{P}}$ and the projection errors $\hat{\boldsymbol{\eta}}_t$. Then, we propose the following four bootstrap steps:

1. Create an artificial sample of data $\boldsymbol{\Delta}\mathbf{X}_t^{(b)}$ based on $\hat{\mathbf{P}}$ and $\boldsymbol{\Delta}\check{\mathbf{X}}_t = \mathbf{P}\boldsymbol{\Delta}\check{\mathbf{X}}_{\mathrm{t}-1} + \mathbf{H}\boldsymbol{\eta}_t$ by drawing with replacement from the projection errors $\hat{\boldsymbol{\eta}}_t$ in a block bootstrap with block size of 5 to capture any small amount of serial correlation due to possible misspecification of the original forecasting model.

2. Obtain a bootstrap estimate of the projection matrix $\hat{\mathbf{P}}^{(b)}$ for the bootstrap sample $\boldsymbol{\Delta}\mathbf{X}_t^{(b)}$.

3. Calculate a preliminary bootstrap estimate of trend $\hat{\mathbf{r}}^{*(b)}$ by using the bootstrap estimate of $\mathbf{P}^{(b)}$ and the original realized data $\boldsymbol{\Delta}\mathbf{X}_t$.

4. Apply our proposed correction to $\hat{\mathbf{r}}^{*(b)}$ based on frequentist estimates of ARMA parameters for the first differences of the preliminary estimated trend. This step provides a bootstrap estimate $\tilde{\mathbf{r}}^{*(b)}$.

We repeat the above steps for 1000 bootstrap replications from $b = 1, \ldots, 1000$. We then take the $\alpha/2$ and $(100 - \alpha/2)$ quantiles across the bootstrapped replications for each $\tilde{r}_t^*$ within $\tilde{\mathbf{r}}^*$ and report these as $(100 - \alpha)\%$ bootstrapped confidence intervals.

It should be noted that our bootstrapped intervals only reflect parameter uncertainty. If we had infinite data and the population parameters, there would be no uncertainty about the corrected estimated changes in trend, as we treat the data as realized values rather than random variables when making our calculation of the corrected estimated trend. In this sense, our bootstrapped intervals are confidence intervals rather than prediction intervals even though the BN decomposition is based on a long-horizon forecast. That is, the forecast is known at time t given population parameters, even though the realized future path is not. This is related also to the idea discussed in the main text that there is no filtering uncertainty about the BN trend, even if there might be about a true underlying r^* given a UC process. Again, it would be necessary to specify the UC process to capture any such filtering uncertainty.

The third panel of Figure D1 compares the Bayesian posterior mean and 90% equal-tailed credible intervals for $\tilde{r}_t^*$ against estimates based on MLE for the bivariate VECM and an MA(8) model for the correction with 90% bootstrapped confidence intervals. Even though there are the obvious conceptual differences, we note that the Bayesian and frequentist estimates and intervals are similar, which also confirms that the degree of uncertainty with our Bayesian inferences is not really due to informative priors.

E Variable-by-variable sign probabilities and informational contributions

In this appendix, we report on variable-by-variable sign probabilities and informational contributions.

Table E1 presents results for the individual variables in terms of (i) the correlation of the projection error for each variable and the implied change in the estimate of r^* based on $\Delta\tilde{r}_{it}^* \equiv \frac{\theta(1)}{\phi(1)}\tilde{\omega}_i\eta_{it}$ and (ii) their contributions to estimated r^* during the three subsample episodes considered in Figure 4 in the main text.

In terms of the signs of correlations, Table E1 reports 62% and 71% posterior probabilities that consumption and TFP growth have a positive relationship with r^*. The broad finding of a positive link between trend growth and r^* corroborates many earlier studies (e.g., Laubach and Williams, 2003; Hamilton et al., 2016; Holston et al., 2017; Berger and Kempa, 2019; Lunsford and West, 2019). Investment growth has the predicted negative relationship consistent with investment-specific technological change with a 60% posterior probability. Consistent with the theoretical prediction on the effect of the labor force on r^* (Baker et al., 2005; Lunsford and West, 2019), there are 69% and 86% posterior probabilities that employment and hours growth have a positive relationship with r^*, while the unemployment rate has a negative relationship with 67% posterior probability, which is also consistent with a labor force effect or possibly insufficient aggregate demand, as suggested in Summers (2015). Meanwhile, consistent with a safe asset demand/flight-to-safety phenomenon, there are 66% and 83% posterior probabilities that macroeconomic uncertainty and the excess bond premium have a negative relationship with r^*. On the contrary, there is only weak evidence that liquid asset growth has a positive relationship with r^*, with only a 53% posterior probability, reflecting a likely mix of supply and demand factors driving this variable. Furthermore, consistent with the global savings glut hypothesis (Bernanke, 2005), there are 88% and 83% posterior probabilities that the current account and a depreciation in the exchange rate have respective positive and negative relationships with r^*. Last, we find that there is a 84% posterior probability of a positive relationship between debt-to-GDP and r^*, consistent with a safe asset supply/crowding-out effect (Ball and Mankiw, 1995).

Table E1: Accounting for changes in r^*

	Sign	Probability	Informational Contributions (bps)		
			Onset of Great Moderation	End of Great Moderation	Secular Stagnation
I. Productivity/Demographics			68 [2, 139]	-21 [-85, 43]	-43 [-116, 24]
Real consumption growth per capita	+	0.62	4 [-14, 23]	3 [-10, 18]	-7 [-38, 20]
TFP growth	+	0.71	2 [-11, 14]	20 [-14, 56]	-18 [-51, 11]
S&P 500 stock returns	+	0.74	6 [-11, 23]	14 [-7, 38]	-8 [-28, 12]
Real investment growth	-	0.60	0 [-10, 9]	-8 [-34, 17]	7 [-17, 32]
Employment growth	+	0.69	8 [-20, 41]	-16 [-49, 14]	-8 [-42, 22]
Hours growth	+	0.86	38 [1, 78]	-27 [-55, 0]	-10 [-30, 8]
Unemployment rate (Δ)	-	0.67	11 [-9, 34]	-8 [-29, 10]	-1 [-17, 18]
II. Safe Asset Demand			35 [-58, 129]	0 [-46, 47]	-39 [-133, 53]
Macroeconomic uncertainty (Δ)	-	0.66	9 [-13, 33]	1 [-7, 10]	-10 [-38, 15]
Excess bond premium (Δ)	-	0.83	0 [-13, 12]	29 [-1, 60]	-31 [-67, 1]
Liquid assets growth	-	0.53	-1 [-89, 86]	1 [-20, 23]	0 [-75, 77]
Current account as % of GDP (Δ)	+	0.88	15 [-3, 34]	-34 [-65, -4]	16 [-4, 38]
Exchange rate return	-	0.80	12 [-6, 33]	3 [-8, 15]	-13 [-35, 7]
III. Safe Asset Supply			52 [-1, 112]	-67 [-140, 1]	29 [-5, 67]
Government debt as % of GDP (Δ)	+	0.84	52 [-1, 112]	-67 [-140, 1]	29 [-5, 67]
Changes in r^*			139 [50, 232]	-141 [-192, -90]	-213 [-244, -180]

Notes: The posterior probability of each reported sign is reported. For the informational contributions, the posterior mean is reported with 67% equal-tailed credible intervals reported in square brackets.

In terms of contributions over the subsamples considered in Figure 4 in the main text, Table E1 suggests that higher employment and hours growth helped drive the large overall contribution of productivity/demographic factors to the rise in r^* with the onset of the Great Moderation. The individual variables associated with safe asset demand had somewhat offsetting effects with the onset of the Great Moderation, while higher safe asset supply in the form of an increase in government debt-to-GDP during the Reagan years had a clear positive contribution to r^*. The effects of the key individual variables during the onset of the Great Moderation reversed by the end of the Great Moderation, especially with the debt consolidation during the Clinton years, although faster TFP growth and higher stock returns with the so-called 'New Economy' at the time meant the overall drag from productivity/demographic factors was less than otherwise, while the individual safe asset demand variables had largely offsetting effects, with a large positive effect from a lower excess bond premium and large negative effect from a current account deficit due to large capital inflows to the United States related to high savings rates in emerging market economies, especially after the Asian financial crisis and with high revenues earned by oil exporters from booming oil prices (Glick, 2020). Finally, with Secular Stagnation, lower trend growth captured by lower consumption growth, TFP growth, and weaker stock returns, as well as weaker employment and hours growth, all contributed to the fall in r^*, as did the key safe asset demand related variables of macroeconomic uncertainty and the excess bond premium, although the other safe asset demand related variables mostly had offsetting effects, as did the increase in safe asset supply with a higher debt-to-GDP ratio again.

F Estimating the model using annual data

In this appendix, we re-estimate the model using annual data as some of the possible drivers of r^* are only available at an annual frequency, specifically income inequality, age dependency and global reserves-to-GDP. The final dataset starts from 1975 after transformation and also includes all of the variables used in the baseline estimation converted to annual measures.

The r^* estimated using annual data is notably smoother than when using quarterly data as seen in Figure F1. However, this does not reflect the inclusion of the variables only available at an annual frequency, as they appear to contribute only negligibly to movements in the estimated r^*. Instead, the smoothness could reflect possible overfitting in-sample given small sample period of annual data and many parameters, although we only consider one lag for the annual VECM and two lags for the MA model used for the correction.

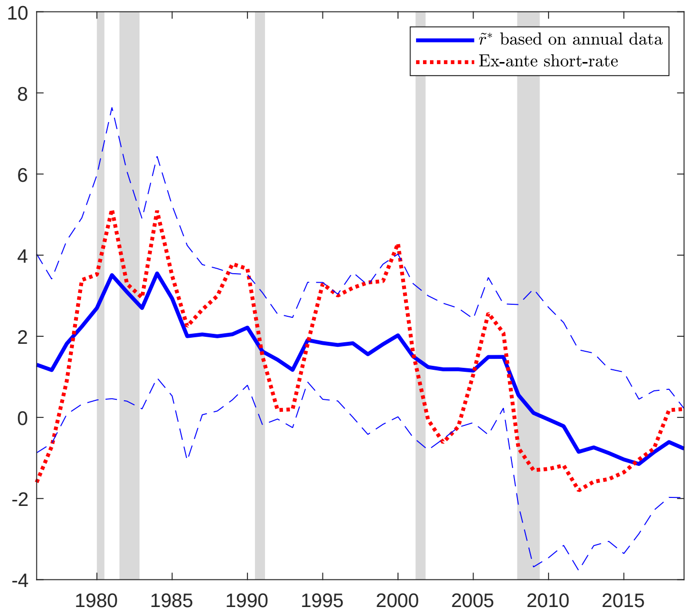


Figure F1: Corrected BN trend for the short-term real interest rate using annual data. Posterior mean and 90% equal-tailed credible intervals (dashed lines) are reported. NBER recession dates are shaded.

As seen in Table F1, we also find that the informational contributions are roughly similar, but less precise than for the baseline quarterly model. In many cases, variables that were important in the quarterly case have insignificant contributions and sometimes have higher posterior probabilities on the wrong sign in terms of predicted theoretical relationships when considering the annual model. Importantly, because the variables that are only available at an annual frequency do not appear to contribute significantly, we can infer that their omission from the baseline quarterly model does not seem to be a source of distortion of our inferences about informational contributions for our baseline model. In any event, our correction can help address misspecification due to omitted variables in the quarterly model.

Table F1: Accounting for changes in r^* based on annual data

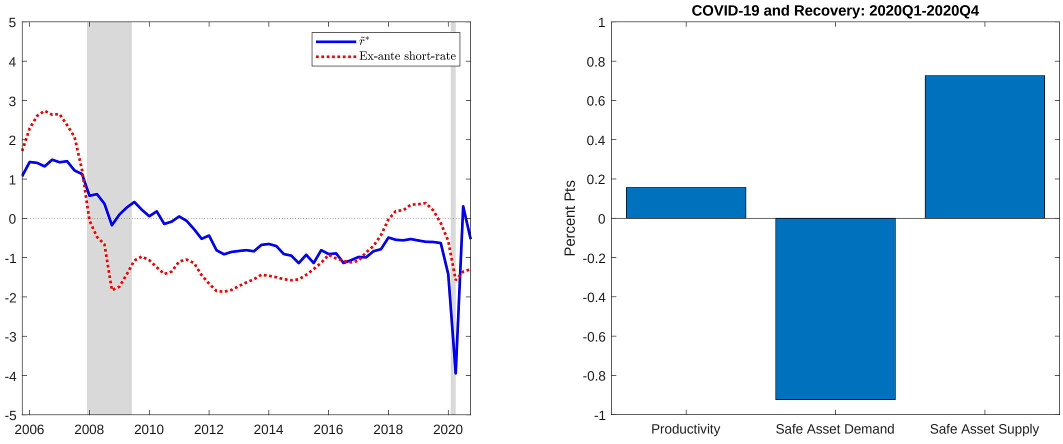
	Sign	Probability	Informational Contributions (bps)		
			Onset of Great Moderation	End of Great Moderation	Secular Stagnation
I. Productivity/Demographics			22 [-60, 122]	-10 [-109, 83]	-40 [-170, 68]
Real consumption growth per capita	+	0.59	3 [-21, 25]	2 [-16, 19]	-10 [-51, 29]
TFP growth	+	0.61	-1 [-20, 19]	8 [-18, 38]	-15 [-65, 31]
S&P 500 stock returns	-	0.52	0 [-17, 18]	-1 [-18, 16]	1 [-24, 27]
Real investment growth	+	0.62	4 [-19, 28]	1 [-16, 20]	-11 [-62, 33]
Employment growth	+	0.63	14 [-25, 54]	-10 [-38, 17]	-12 [-53, 25]
Hours growth	+	0.62	8 [-29, 52]	-7 [-43, 24]	-7 [-43, 23]
Unemployment rate (Δ)	-	0.56	5 [-22, 33]	-3 [-27, 20]	-4 [-32, 23]
Age Dependency (Δ^2)	-	0.53	2 [-23, 26]	8 [-82, 99]	-6 [-76, 65]
Inequality (Δ)	-	0.76	-8 [-30, 13]	-9 [-30, 9]	15 [-11, 45]
II. Safe Asset Demand			-29 [-156, 80]	22 [-45, 103]	14 [-101, 139]
Macroeconomic uncertainty (Δ)	+	0.67	-7 [-27, 12]	-1 [-15, 13]	16 [-14, 46]
Excess bond premium (Δ)	-	0.63	1 [-15, 19]	5 [-13, 25]	-10 [-47, 22]
Liquid assets growth	-	0.57	-17 [-127, 92]	7 [-37, 55]	16 [-77, 111]
Current account as % of GDP (Δ)	-	0.51	-1 [-17, 16]	0 [-17, 19]	1 [-28, 28]
Exchange rate return	-	0.78	12 [-14, 38]	0 [-20, 19]	-7 [-31, 19]
Reserves-to-GDP (Δ)	+	0.80	-24 [-60, 8]	16 [-8, 44]	5 [-19, 32]
III. Safe Asset Supply			0 [-25, 29]	-3 [-64, 53]	1 [-41, 47]
Government debt as % of GDP (Δ)	+	0.53	0 [-25, 29]	-3 [-64, 53]	1 [-41, 47]
Changes in r^*			89 [-20, 210]	-195 [-109, -26]	-289 [-229, -167]

Notes: The posterior probability of each reported sign is reported. For the informational contributions, the posterior mean is reported with 67% equal-tailed credible intervals reported in square brackets.

G How did r^* change during the COVID-19 pandemic?

In this appendix, we extend our analysis to cover the onset of the COVID-19 pandemic. In particular, we update the dataset to 2020Q4, but we use the pre-Covid parameter estimates to avoid possible distortions from large outliers in the data. Specifically, Lenza and Primiceri (2022) note that excluding data from 2020 when estimating a VAR is a simple approximation to a GLS-type approach, although their main proposal is to model a rescaling of the residual covariance matrix during 2020 when constructing density forecasts during this period. Because we are interested in point forecasts, we take the simpler approach of not including the data from 2020 in parameter estimation, which is also consistent with the findings in Schorfheide and Song (forthcoming) that forecasts based on VAR parameters estimated using only data before the pandemic appear "more stable and reasonable" than those based on updated parameter estimates.

The first panel of Figure G1 plots the posterior mean of the corrected BN trend for the short-term real interest rate over the latter part of the sample period and up to the end of 2020, noting the pre-2020 estimates of r^* are the same as in Figures 2 and 3 in the main text given the same parameter estimates. With the onset of the pandemic, the estimated r^* falls sharply to about -2.5% as various indicators related to the marginal product of capital adjusted dramatically and there was a jump up in macroeconomic uncertainty. However, the persistence of these variables was very different than normal given the unusual stop-start nature of economic activity with lockdowns, as well as the unprecedented fiscal stimulus, and the estimated r^* quickly jumped back up slightly above its pre-pandemic level. Looking at the various quarterly variables, we find that demand for safe assets was still a drag on r^* by the end of 2020, contributing an estimated 90 basis point decrease over the year, while supply of safe assets in the form of higher debt-to-GDP mostly offset this effect by contributing an estimated 75 basis point increase over the same period. These contributions are plotted in the second panel of Figure G1.



(a) Corrected BN trend including 2020 (b) Informational contributions in 2020

Figure G1: r^* during the pandemic. In panel (a), the posterior mean is reported and NBER recession dates are shaded. In panel (b), posterior means of informational contributions are reported.

References

Baker, D., J. B. De Long and P. R. Krugman, "Asset Returns and Economic Growth," *Brookings Papers on Economic Activity* 2005 (2005), 289–330.

Ball, L. and N. G. Mankiw, "What Do Budget Deficits Do?," NBER Working Paper No. 5263, National Bureau of Economic Research, 1995.

Banbura, M., D. Giannone and L. Reichlin, "Large Bayesian Vector Auto Regressions," *Journal of Applied Econometrics* 25 (2010), 71–92.

Berger, T. and B. Kempa, "Testing for Time Variation in the Natural Rate of Interest," *Journal of Applied Econometrics* 34 (2019), 836–842.

Bernanke, B. S., "The Global Saving Glut and the US Current Account Deficit, Homer Jones Lecture," *St. Louis, Missouri* (2005).

Caballero, R. J., E. Farhi and P.-O. Gourinchas, "The Safe Assets Shortage Conundrum," *Journal of Economic Perspectives* 31 (2017), 29–46.

Carriero, A., J. Chan, T. E. Clark and M. Marcellino, "Corrigendum to "Large Bayesian Vector Autoregressions with Stochastic Volatility and Non-Conjugate Priors" [J. Econometrics 212 (1) (2019) 137–154]," *Journal of Econometrics* 277 (2022), 506–512.

Carriero, A., T. E. Clark and M. Marcellino, "Bayesian VARs: Specification Choices and Forecast Accuracy," *Journal of Applied Econometrics* 30 (2015), 46–73.

Del Negro, M., D. Giannone, M. P. Giannoni and A. Tambalotti, "Safety, Liquidity, and the Natural Rate of Interest," *Brookings Papers on Economic Activity* 2017 (2017), 235–316.

Dynan, K. E., J. Skinner and S. P. Zeldes, "Do the Rich Save More?," *Journal of Political Economy* 112 (2004), 397–444.

Eichengreen, B., "Secular Stagnation: The Long View," *American Economic Review* 105 (2015), 66–70.

Fernald, J. G., "Productivity and Potential Output before, during, and after the Great Recession," *NBER Macroeconomics Annual* 29 (2015), 1–51.

Gagnon, E., B. K. Johannsen and D. López-Salido, "Understanding the New Normal: The Role of Demographics," *IMF Economic Review* 69 (2021), 357–390.

Gilchrist, S. and E. Zakrajšek, "Credit Spreads and Business Cycle Fluctuations," *American Economic Review* 102(4) (2012), 1692–1720.

Glick, R., "r* and the Global Economy," *Journal of International Money and Finance* 102 (2020), 102105.

Hamilton, J. D., E. S. Harris, J. Hatzius and K. D. West, "The Equilibrium Real Funds Rate: Past, Present, and Future," *IMF Economic Review* 64 (2016), 660–707.

Holston, K., T. Laubach and J. C. Williams, "Measuring the Natural Rate of Interest: International Trends and Determinants," *Journal of International Economics* 108 (2017), S59–S75.

Jurado, K., S. C. Ludvigson and S. Ng, "Measuring Uncertainty," *American Economic Review* 105(3) (2015), 1177–1216.

Koop, G. and D. Korobilis, *Bayesian Multivariate Time Series Methods for Empirical Macroeconomics* (Now Publishers Inc, 2010).

Laubach, T. and J. C. Williams, "Measuring the Natural Rate of Interest," *Review of Economics and Statistics* 85 (2003), 1063–1070.

Lenza, M. and G. E. Primiceri, "How to Estimate a Vector Autoregression after March 2020," *Journal of Applied Econometrics* 37 (2022), 688–699.

Litterman, R. B., "Forecasting with Bayesian Vector Autoregressions — Five Years of Experience," *Journal of Business & Economic Statistics* 4 (1986), 25–38.

Lunsford, K. G. and K. D. West, "Some Evidence on Secular Drivers of US Safe Real Rates," *American Economic Journal: Macroeconomics* 11 (2019), 113–139.

Modigliani, F. and R. J. Shiller, "Inflation, Rational Expectations and the Term Structure of Interest Rates," *Economica* 40 (1973), 12–43.

Morley, J. and B. Wong, "Estimating and Accounting for the Output Gap With Large Bayesian Vector Autoregressions," *Journal of Applied Econometrics* 35 (2020), 1–18.

Orphanides, A. and S. van Norden, "The Unreliability of Output Gap Estimates in Real Time," *Review of Economics and Statistics* 84 (2002), 569–583.

Schorfheide, F. and D. Song, "Real-Time Forecasting with a (Standard) Mixed-Frequency VAR During a Pandemic," *International Journal of Central Banking* (forthcoming).

Sims, C. and T. Zha, "Bayesian Methods for Dynamic Multivariate Models," *International Economic Review* 39(4) (1998), 949–968.

Summers, L. H., "Demand Side Secular Stagnation," *American Economic Review* 105 (2015), 60–65.