A Unified Model Specification for Sparse and Dense Functional/Longitudinal Data

Supplementary Material

S1 Proof of Main Results

Lemma 1. *Under the conditions of Theorem 1, we have*

$$\sqrt{n}(\tilde{\boldsymbol{\alpha}} - \boldsymbol{\alpha}) = O_p(1).$$

Proof. Similar to the procedure of Lemma 2 in the supplement materials of Liu et al. (2022), we can show that

$$\frac{1}{n}\sum_{i=1}^{n} \hat{S}_i \hat{S}_i^{\tau} = \frac{1}{n}\sum_{i=1}^{n} S_i S_i^{\tau} + O_p\Big(\sup_{1 \le k \le p} \sup_{x \in [a_k, b_k]} |\hat{\beta}_k - \beta_k| \Big),$$

which implies that $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mathrm{E}[\hat{S}_i \hat{S}_i^{\tau}] = \Omega$. Under Assumption T2, it holds that $\frac{1}{n}\sum_{i=1}^{n} \hat{S}_i \hat{S}_i^{\tau}$ is invertible with probability approaching to one. According to the formula of $\tilde{\boldsymbol{\alpha}}$ given by (3) in the main body, we obtain that

$$\tilde{\boldsymbol{\alpha}} = \Big(\sum_{i=1}^{n} \hat{S}_i \hat{S}_i^{\tau} \Big)^{-1} \sum_{i=1}^{n} \hat{S}_i \boldsymbol{\delta}_i = \boldsymbol{\alpha} + \Big(\sum_{i=1}^{n} \hat{S}_i \hat{S}_i^{\tau} \Big)^{-1} \sum_{i=1}^{n} \hat{S}_i \boldsymbol{\delta}_i,$$

where $\boldsymbol{\delta}_i = (\delta_{i1}, ..., \delta_{im_i})^{\tau}$. Let $\hat{S}_{i,j}^{\otimes 2}$ be the (i,j)-element of the matrix $\sum_{i=1}^{n} \hat{S}_i \hat{S}_i^{\tau}$. Employing Theorem 3 in Liu et al. (2022), we have

$$\begin{aligned}
\mathrm{Var}\Big(\frac{1}{n}\sum_{i=1}^{n} \hat{\boldsymbol{S}}_i \boldsymbol{\delta}_i \Big) &= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{i=1}^{n} \mathrm{E}[\boldsymbol{\delta}_i^{\tau} \hat{S}_i \hat{S}_i^{\tau} \boldsymbol{\delta}_i] \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{k=1}^{m_i} \mathrm{E}\big[\delta_{ik} \hat{S}_{k,k}^{\otimes 2}\big] + \frac{1}{n^2}\sum_{i=1}^{n}\sum_{k\neq l}^{m_i} \mathrm{E}\big[\delta_{ik}\delta_{il} \hat{S}_{k,l}^{\otimes 2}\big] \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{k=1}^{m_i} \mathrm{E}\big[\mathrm{E}[\delta_{ik}^2 | T_{ij}] \hat{S}_{k,k}^{\otimes 2}\big] + \frac{1}{n^2}\sum_{i=1}^{n}\sum_{k\neq l}^{m_i} \mathrm{E}\big[\mathrm{E}[\delta_{ik}\delta_{il} | T_{ij}] \hat{S}_{k,l}^{\otimes 2}\big] \\
&\le c_1 \frac{1}{n} tr \mathrm{E}\Big[\frac{1}{n}\sum_{i=1}^{n} \hat{S}_i \hat{S}_i^{\tau} \Big] + c_2 \frac{1}{n} \Big\| \frac{1}{n}\sum_{i=1}^{n} \hat{S}_i \hat{S}_i^{\tau} \Big\|_F \\
&= O_p(1/n) + o_p(1),
\end{aligned}$$

which completes the proof. $\square$

Proof of Theorem 1.

Proof. Denote $\mathbf{Y}_i = \left\{Y_{i_1}, ..., Y_{i_{m_i}}\right\}^\tau$, $\tilde{\mathbf{e}}_i = \tilde{\mathbf{g}}_{0,i} - \mathbf{g}_{0,i}$, where

$$\begin{aligned} \mathbf{g}_{0,i} &= \{g_0(T_{i1}, \mathbf{Z}_{i1}, \mathbf{X}_{i1}; \boldsymbol{\alpha}), ..., g_0(T_{im_i}, \mathbf{Z}_{im_i}, \mathbf{X}_{im_i}; \boldsymbol{\alpha}, \boldsymbol{\beta}(\mathbf{x}))\}^\tau \quad \text{and} \\ \tilde{\mathbf{g}}_{0,i} &= \{g_0(T_{i1}, \mathbf{Z}_{i1}, \mathbf{X}_{i1}; \tilde{\boldsymbol{\alpha}}), ..., g_0(T_{im_i}, \mathbf{Z}_{im_i}, \mathbf{X}_{im_i}; \tilde{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}(\mathbf{x})\}^\tau. \end{aligned}$$

Since $\hat{\mathbf{e}}_i = \mathbf{Y}_i - \tilde{\mathbf{g}}_{0,i} = \boldsymbol{\delta}_i - (\tilde{\mathbf{g}}_{0,i} - \mathbf{g}_{0,i}) = \boldsymbol{\delta}_i - \tilde{\mathbf{e}}_i$, we can write $\hat{J}_n = \hat{J}_{n,1} - 2\hat{J}_{n,2} + \hat{J}_{n,3}$, where

$$\hat{J}_{n,1} = \frac{1}{n^2 \bar{N}_{\mathrm{H}}^2 |H|} \sum_{i=1}^n \sum_{j \neq i}^n \boldsymbol{\delta}_i^\tau W_{ij} \boldsymbol{\delta}_j, \tag{S1.2}$$

$$\hat{J}_{n,2} = \frac{1}{n^2 \bar{N}_{\mathrm{H}}^2 |H|} \sum_{i=1}^n \sum_{j \neq i}^n \boldsymbol{\delta}_i^\tau W_{ij} \tilde{\mathbf{e}}_i, \tag{S1.3}$$

$$\hat{J}_{n,3} = \frac{1}{n^2 \bar{N}_{\mathrm{H}}^2 |H|} \sum_{i=1}^n \sum_{j \neq i}^n \tilde{\mathbf{e}}_i^\tau W_{ij} \tilde{\mathbf{e}}_i. \tag{S1.4}$$

It suffices to show that

$$\frac{n^2 \bar{N}_{\mathrm{H}}^2}{\sqrt{N^2 - n\bar{N}_2}} \sqrt{|H|} \hat{J}_{n,1}/\sigma_1 \xrightarrow{D} N(0,1), \tag{S1.5}$$

$$\hat{J}_{n,2} = o_p\left(\frac{1}{n\bar{N}_{\mathrm{H}}\sqrt{|H|}}\right), \tag{S1.6}$$

$$\hat{J}_{n,3} = o_p\left(\frac{1}{n\bar{N}_{\mathrm{H}}\sqrt{|H|}}\right) \quad \text{and} \tag{S1.7}$$

$$\hat{\sigma}_1^2 \xrightarrow{p} \sigma_1^2 \tag{S1.8}$$

under $H_{0,\mathrm{C}}$.

On the other hand, the condition $\lim_n \sup \bar{N}_2/(\bar{N}_{\mathrm{H}})^2 < \infty$ implies $\bar{N}_2 = O(\bar{N}_{\mathrm{H}}^2)$. By the convex inequality, it holds that

$$\left(\frac{1}{n}\sum_{i=1}^n m_i\right)^2 \leq \frac{1}{n}\sum_{i=1}^n m_i^2 = O(\bar{N}_{\mathrm{H}}^2),$$

which means $N = O(n\bar{N}_{\mathrm{H}})$, and $N^2 - n\bar{N}_2 = O\big(n(n-1)\bar{N}_{\mathrm{H}}^2\big)$. Therefore,

$$\frac{n^2\bar{N}_{\mathrm{H}}^2}{\sqrt{N^2 - n\bar{N}_2}}\sqrt{|H|}\hat{J}_n/\hat{\sigma}_1 = \frac{\sigma_1}{\hat{\sigma}_1}\frac{n^2\bar{N}_{\mathrm{H}}^2}{\sqrt{N^2 - n\bar{N}_2}}\sqrt{|H|}\hat{J}_{n,1}/\sigma_1 + o_p(1) \xrightarrow{D} N(0,1),$$

which completes the proof. □

Proof of (S1.5)

Proof. We write $\hat{J}_{n,1} = \frac{1}{n^2\bar{N}_{\mathrm{H}}^2|H|}\sum_{i=1}^n \eta_{in}$, where $\eta_{in} = \sum_{j\neq i}^n \boldsymbol{\delta}_i^\tau W_{ij}\boldsymbol{\delta}_j = 2\sum_{j<i}\boldsymbol{\delta}_i^\tau W_{ij}\boldsymbol{\delta}_j$. It is easy to see that η_{in} is a martingale difference (MD) process w.r.t. the filtration $\mathcal{F}_i^n = \sigma((T_i, X_i)_{i=1}^n, \boldsymbol{\delta}_1, ..., \boldsymbol{\delta}_n)$ and

$$\begin{aligned} s_n^2 := \sum_{i=1}^n \mathrm{E}[\eta_{in}^2] &= 4\sum_{i=1}^n\sum_{j<i}^n \mathrm{E}[\mathrm{E}(\boldsymbol{\delta}_i^\tau W_{ij}\boldsymbol{\delta}_j)^2|\mathcal{J}_i,\mathcal{J}_j]] \\ &= 4\sum_{i=1}^n\sum_{j<i}^n(\hat{J}_{n,11} + \hat{J}_{n,12} + \hat{J}_{n,13} + \hat{J}_{n,14}), \end{aligned}$$

where

$$\begin{aligned} \hat{J}_{n,11} &= \sum_{l=1}^{m_i}\sum_{v=1}^{m_j}\mathrm{E}\big[\mathrm{E}[\delta_{il}^2|\mathcal{J}_i]\mathrm{E}[\delta_{jv}^2|\mathcal{J}_j]\big(w_{ij}^{(l,v)}\big)^2\big], \\ \hat{J}_{n,12} &= \sum_{l=1}^{m_i}\sum_{v\neq v'}^{m_j}\mathrm{E}\big[\mathrm{E}[\delta_{il}|\mathcal{J}_i]\mathrm{E}[\delta_{jv}\delta_{jv'}|\mathcal{J}_j]w_{ij}^{(l,v)}w_{ij}^{(l,v')}\big] \\ \hat{J}_{n,13} &= \sum_{l\neq l'}^{m_i}\sum_{v=1}^{m_j}\mathrm{E}\big[\mathrm{E}[\delta_{il}\delta_{il'}|\mathcal{J}_i]\mathrm{E}[\delta_{jv}^2|\mathcal{J}_j]w_{ij}^{(l,v)}w_{ij}^{(l',v)}\big] \\ \hat{J}_{n,14} &= \sum_{l\neq l'}^{m_i}\sum_{v\neq v'}^{m_j}\mathrm{E}\big[\mathrm{E}[\delta_{il}\delta_{il'}|\mathcal{J}_i]\mathrm{E}[\delta_{jv}\delta_{jv'}|\mathcal{J}_j]w_{ij}^{(l,v)}w_{ij}^{(l',v')}\big] \end{aligned}$$

According to (C.8) and (C.9) in the supplement materials of Liu et al. (2022), after a straightforward computation, we obtain that

$$
\begin{aligned}
\hat{J}_{n,11} &= m_i m_j \kappa^{p+1} |H| \mathrm{E}\left[\{\gamma(T,T)+\sigma^2(T)\}^2 f_{\mathrm{T}}(T)\right] \prod_{k=1}^{p} \mathrm{E}\left[f_k(X_k(T);T)\right] \\
&\overset{\triangle}{=} m_i m_j \kappa^{p+1} |H| \Delta_1, \\
\hat{J}_{n,12} &= m_i m_j (m_j-1) |H|^2 \mathrm{E}[\{\gamma(T,T)+\sigma^2(T)\}\gamma(T,T) f_{\mathrm{T}}^2(T)] \prod_{k=1}^{p} \mathrm{E}\left[f_k^2(X_k(T);T)\right] \\
&\overset{\triangle}{=} m_i m_j (m_j-1) |H|^2 \Delta_2, \\
\hat{J}_{n,13} &= m_i m_j (m_i-1) |H|^2 \mathrm{E}[\{\gamma(T,T)+\sigma^2(T)\}\gamma(T,T) f_{\mathrm{T}}^2(T)] \prod_{k=1}^{p} \mathrm{E}\left[f_k^2(X_k(T);T)\right] \\
&\overset{\triangle}{=} m_i m_j (m_i-1) |H|^2 \Delta_3, \\
\hat{J}_{n,14} &= m_i m_j (m_i-1)(m_j-1) |H|^2 \mathrm{E}[\gamma^2(T,T) f_{\mathrm{T}}(T)] \prod_{k=1}^{p} \left(\mathrm{E}[f_k(X_k(T);T)]\right)^2 \\
&\overset{\triangle}{=} m_i m_j (m_i-1)(m_j-1) |H|^2 \Delta_4.
\end{aligned}
$$

Note that $2\sum_{i=1}^{n}\sum_{j\neq i}^{n} m_i m_j = N^2 - n\bar{N}_2$, where $\bar{N}_2 = \frac{1}{n}\sum_{i=1}^{n} m_i^2$. Furthermore,

$$
\begin{aligned}
&\sum_{i=1}^{n}\sum_{j\neq i}^{n} m_i m_j (m_i-1) \leq \sum_{i=1}^{n} m_i^2 \sum_{j=1}^{n} m_j = nN\bar{N}_2 = O(n^2\bar{N}_{\mathrm{H}}^3), \\
&\sum_{i=1}^{n}\sum_{j\neq i}^{n} m_i m_j (m_i-1)(m_j-1) \leq \left(\sum_{i=1}^{n} m_i^2\right)^2 = (n\bar{N}_2)^2 = O(n^2\bar{N}_{\mathrm{H}}^4).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
s_n^2 &= 2\sum_{i=1}^{n}\sum_{j\neq i}^{n} m_i m_j \kappa^{p+1}|H|\Delta_1 + 2\sum_{i=1}^{n}\sum_{j\neq i}^{n} m_i m_j (m_i-1)|H|^2\Delta_2 \\
&\quad + 2\sum_{i=1}^{n}\sum_{j\neq i}^{n} m_i m_j (m_j-1)|H|^2\Delta_3 + 2\sum_{i=1}^{n}\sum_{j\neq i}^{n} m_i m_j (m_i-1)(m_j-1)|H|^2\Delta_4 \\
&= (N^2 - n\bar{N}_2)\kappa^{p+1}|H|\Delta_1 + O(n^2\bar{N}_{\mathrm{H}}^3|H|^2) + O(n^2\bar{N}_{\mathrm{H}}^4|H|^2) \\
&= (N^2 - n\bar{N}_2)|H|\sigma_0^2 + s.o.
\end{aligned}
$$

since $\bar{N}_{\mathrm{H}}^2|H| = o(1)$. Hence,

$$\mathrm{Var}(\hat{J}_{n,1}) = \frac{N^2 - n\bar{N}_2}{n^4\bar{N}_{\mathrm{H}}^4|H|}\sigma_0^2$$

with $\sigma_0^2 = \kappa^{p+1}\mathrm{E}\left[\{\gamma(T,T) + \sigma^2(T)\}^2 f_{\mathrm{T}}(T)\right]\Pi_{k=1}^p \mathrm{E}[f_k(X_k(T);T)]$.

Now, we consider $\frac{n^2\bar{N}_{\mathrm{H}}^2}{\sqrt{N^2 - n\bar{N}_2}}\sqrt{|H|}\hat{J}_n/\sigma_0 = \sum_{i=1}^n \zeta_{in}$, where $\zeta_{in} = \eta_{in}/s_n$ and $s_n^2 = (N^2 - n\bar{N}_2)|H|\sigma_0^2$. To derive the asymptotic normality of $\sum_{i=1}^n \zeta_{in}$, it remains to show the following three conditions in Lemma D1 of Jun and Pinkse (2012):

(i) $\sup_n \mathrm{E}\left[\max_i \zeta_{in}^2\right] < \infty$,

(ii) $\sum_{i=1}^n \zeta_{in}^2 \xrightarrow{p} 1$, and

(iii) $\max_i |\zeta_{in}| \xrightarrow{p} 0$.

It is obvious that (i) holds since $\max_i \zeta_{in}^2 \le \sum_{i=1}^n \eta_{in}^2/s_n^2$ and $s_n^2 = \sum_{i=1}^n \mathrm{E}[\eta_{in}^2] + o(1)$. By Markov's inequality, it follows that

$$\mathrm{P}(\max_i |\zeta_{in}| > \varepsilon) \le \sum_{i=1}^n \mathrm{P}(|\zeta_{in}| > \varepsilon) \le \frac{1}{\varepsilon^r}\sum_{i=1}^n \mathrm{E}[|\zeta_{in}|^r]. \tag{S1.9}$$

Note that

$$\mathrm{E}\left[|\zeta_{in}|^r\right] = \frac{1}{s_n^r}\mathrm{E}\left[|\eta_{in}|^r\right] = \frac{1}{s_n^r}\mathrm{E}\Big[\Big|\sum_{j<i}^n \boldsymbol{\delta}_i^\tau W_{ij}\boldsymbol{\delta}_j\Big|^r\Big].$$

Obviously, $s_n^r = O(\{N^2 - n\bar{N}_2)|H|\}^{r/2}) = O(\{n(n-1)\bar{N}_{\mathrm{H}}^2|H|\}^{r/2})$. On the other hand, using C_r inequality and Assumption E3, we obtain that

$$\begin{aligned}
\sum_{j<i}\mathrm{E}\left[|\boldsymbol{\delta}_i^\tau W_{ij}\boldsymbol{\delta}_j|^r\right] &= \sum_{j<i}^n \mathrm{E}\Big|\sum_{l=1}^{m_i}\sum_{v=1}^{m_j}\delta_{il}w_{ij}^{(l,v)}\delta_{jv}\Big|^r \\
&\le \sum_{j<i}^n\sum_{l=1}^{m_i}\sum_{v=1}^{m_j}\mathrm{E}\big|\delta_{il}w_{ij}^{(l,v)}\delta_{jv}\big|^r \\
&\le C\sum_{j<i}^n\sum_{l=1}^{m_i}\sum_{v=1}^{m_j}\mathrm{E}\big[k_{h_{\mathrm{C}}}^r(T_{il},T_{jv})K_{h_{\mathrm{A}}}^r(\mathbf{X}_{il},\mathbf{X}_{jv})\big] \\
&= O\Big(\sum_{j<i}^n m_i m_j|H|\Big).
\end{aligned}$$

Therefore,

$$\sum_{i=1}^{n} \mathrm{E}[|\zeta_{in}|^r] = O(\{n^2 \bar{N}_{\mathrm{H}}^2 |H|\}^{-\frac{r-2}{2}}) = o(1) \qquad \text{(S1.10)}$$

for $r > 2$, which shows that (iii) from (S1.9). Furthermore, by Burkholder inequality and C_r inequality, we have that

$$\begin{aligned} \mathrm{P}\Big(\Big|\sum_{i=1}^{n} \zeta_{in}^2 - 1\Big| > \varepsilon\Big) &= \mathbf{P}\Big(\Big|\sum_{i=1}^{n} \zeta_{in}^2 - \mathrm{E}\Big[\zeta_{in}^2|\mathcal{J}_i\Big]\Big| > \varepsilon\Big) \\ &\le \frac{1}{\varepsilon^{r/2}} \mathrm{E}\Big[\Big|\sum_{i=1}^{n} \zeta_{in}^2 - \mathrm{E}\Big[\zeta_{in}^2|\mathcal{J}_i\Big]\Big|^{r/2}\Big] \\ &\le \frac{1}{\varepsilon^{r/2}} \mathrm{E}\Big[\Big\{\sum_{i=1}^{n} \zeta_{in}^2\Big\}^{r/2}\Big] \le \frac{1}{\varepsilon^{r/2}} \mathrm{E}\Big[\Big|\sum_{i=1}^{n} \zeta_{in}\Big|^{r}\Big] \\ &\le \frac{1}{\varepsilon^{r/2}} \sum_{i=1}^{n} \mathrm{E}[|\zeta_{in}|^r] \to 0 \end{aligned}$$

which shows condition (ii). Hence, we complete the proof of (S1.5). □

Proof of (S1.6)

Proof. Let $\tilde{\mathbf{e}}_{i,j}$ be the j-th element of $\tilde{\mathbf{e}}_i$, which is given by

$$\begin{aligned} \tilde{\mathbf{e}}_{i,j} &= \mathbf{Z}_{ij}^{\tau}(\tilde{\boldsymbol{\alpha}}_0 - \boldsymbol{\alpha}_0) + \sum_{k=1}^{p} (\tilde{\alpha}_k - \alpha_k)\beta_k(X_{ijk}) \\ &\quad + \sum_{k=1}^{p} \alpha_k[\hat{\beta}_k(X_{ijk}) - \beta_k(X_{ijk})] + \sum_{k=1}^{p} (\tilde{\alpha}_k - \alpha_k)[\hat{\beta}_k(X_{ijk}) - \beta_k(X_{ijk})]. \end{aligned} \qquad \text{(S1.11)}$$

Represent $\hat{\boldsymbol{\beta}}_{i,k} = \{\hat{\beta}_k(X_{i1k}), ..., \hat{\beta}_k(X_{im_ik})\}^{\tau}$ and $\boldsymbol{\beta}_{i,k} = \{\beta_k(X_{i1k}), ..., \beta_k(X_{im_ik})\}^{\tau}$, then we can write $\hat{J}_{n,2} = \hat{J}_{n,21} + \hat{J}_{n,22} + \hat{J}_{n,23} + \hat{J}_{n,24}$, where

$$\hat{J}_{n,21} = \frac{1}{n^2\bar{N}_{\mathrm{H}}^2|H|}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\boldsymbol{\delta}_i^\tau W_{ij}\mathbf{Z}_{ij}^\tau(\tilde{\boldsymbol{\alpha}}_0 - \boldsymbol{\alpha}_0),$$

$$\hat{J}_{n,22} = \frac{1}{n^2\bar{N}_{\mathrm{H}}^2|H|}\sum_{k=1}^{p}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\boldsymbol{\delta}_i^\tau W_{ij}(\tilde{\alpha}_k - \alpha_k)\boldsymbol{\beta}_{j,k},$$

$$\hat{J}_{n,23} = \frac{1}{n^2\bar{N}_{\mathrm{H}}^2|H|}\sum_{k=1}^{p}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\boldsymbol{\delta}_i^\tau W_{ij}\alpha_k(\hat{\boldsymbol{\beta}}_{j,k} - \boldsymbol{\beta}_{j,k}), \quad \text{and}$$

$$\hat{J}_{n,24} = \frac{1}{n^2\bar{N}_{\mathrm{H}}^2|H|}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\boldsymbol{\delta}_i^\tau W_{ij}(\tilde{\alpha}_k - \alpha_k)(\hat{\boldsymbol{\beta}}_{j,k} - \boldsymbol{\beta}_{j,k}).$$

Using Lemma 1, we get

$$\begin{aligned}
\mathrm{Var}(\hat{J}_{n,21}) &= \frac{1}{n^5\bar{N}_{\mathrm{H}}^4|H|^2}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\mathrm{E}\big[\boldsymbol{\delta}_i^\tau W_{ij}\mathbf{Z}_{ij}^\tau\mathbf{1}_{m_j}\big]^2 \\
&\le C\frac{1}{n^5\bar{N}_{\mathrm{H}}^4|H|^2}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{l=1}^{m_i}\sum_{v=1}^{m_j}\sum_{l'=1}^{m_i}\sum_{v'=1}^{m_j}\mathrm{E}\Big[\delta_{il}\delta_{il'}w_{ij}^{(l,v)}w_{ij}^{(l',v')}\Big] \\
&= O_p\left(\frac{n(n-1)\bar{N}_{\mathrm{H}}^2|H|}{n^5\bar{N}_{\mathrm{H}}^4|H|^2}\right) = o_p\left(\frac{1}{n^2\bar{N}_{\mathrm{H}}^2|H|}\right),
\end{aligned}$$

which implies that $\hat{J}_{n,21} = o_p\Big(\frac{1}{n\bar{N}_{\mathrm{H}}\sqrt{|H|}}\Big)$. Similarly, it holds that $\hat{J}_{n,22} = o_p\Big(\frac{1}{n\bar{N}_{\mathrm{H}}\sqrt{|H|}}\Big)$.

Next, according to (C.26) and (C.29) in the supplement materials of Liu et al. (2022), we obtain that

$$\begin{aligned}
\hat{J}_{n,23} &= \frac{1}{n^2\bar{N}_{\mathrm{H}}^2|H|}\sum_{k=1}^{p}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\boldsymbol{\delta}_i^\tau W_{ij}\alpha_k\begin{pmatrix}\mathbf{e}_1^\tau\{\hat{\mathbf{r}}_k(X_{j1}) - \mathbf{r}_k(X_{j1})\} \\ \vdots \\ \mathbf{e}_1^\tau\{\hat{\mathbf{r}}_k(X_{jm_j}) - \mathbf{r}_k(X_{jm_j})\}\end{pmatrix} \\
&= \frac{1}{n^2\bar{N}_{\mathrm{H}}^2|H|}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\boldsymbol{\delta}_i^\tau W_{ij}\alpha_k\left\{O\left(h_{\mathrm{A}}^2\right) + O_p\left(\sqrt{\frac{1}{n\bar{N}_{\mathrm{H}}h_{\mathrm{A}}} + \frac{1}{n}}\right)\right\} \\
&= o_p\left(\frac{1}{n\bar{N}_{\mathrm{H}}\sqrt{|H|}}\right).
\end{aligned}$$

Here, the last step uses the fact that $\sum_{i=1}^{n}\sum_{j\neq i}^{n}\boldsymbol{\delta}_i^{\tau}W_{ij}=O_p\Big(\sqrt{n(n-1)\bar{N}_{\mathrm{H}}^2|H|}\Big)$. Similarly, $\hat{J}_{n,24}=o_p\Big(\sqrt{n(n-1)\bar{N}_{\mathrm{H}}^2|H|}\Big)$. Hence, (S1.6) is shown. $\square$

Proof of (S1.7)

Proof. From the definition of $\tilde{\mathbf{e}}_{i,j}$ given in (S1.11), we can write $\hat{J}_{n,3}=\hat{J}_{n,31}+\hat{J}_{n,32}+\hat{J}_{n,33}+\hat{J}_{n,34}+\hat{J}_{n,35}+(s.o.)$, where

$$J_{n,31}=\frac{1}{n^2\bar{N}_{\mathrm{H}}^2|H|}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\left(\tilde{\boldsymbol{\alpha}}_0-\boldsymbol{\alpha}_0\right)^{\tau}\mathbf{Z}_{ij}\mathbf{1}_{m_i}^{\tau}W_{ij}\mathbf{Z}_{ij}^{\tau}\left(\tilde{\boldsymbol{\alpha}}_0-\boldsymbol{\alpha}_0\right)\mathbf{1}_{m_j},$$

$$J_{n,32}=\frac{1}{n^2\bar{N}_{\mathrm{H}}^2|H|}\sum_{k=1}^{p}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\left(\tilde{\alpha}_k-\alpha_k\right)^2\boldsymbol{\beta}_{i,k}^{\tau}W_{ij}\boldsymbol{\beta}_{j,k},$$

$$J_{n,33}=\frac{1}{n^2\bar{N}_{\mathrm{H}}^2|H|}\sum_{k=1}^{p}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\alpha_k^2\left(\hat{\boldsymbol{\beta}}_{i,k}-\boldsymbol{\beta}_{i,k}\right)^{\tau}W_{ij}\left(\hat{\boldsymbol{\beta}}_{j,k}-\boldsymbol{\beta}_{j,k}\right),$$

$$J_{n,34}=\frac{2}{n^2\bar{N}_{\mathrm{H}}^2|H|}\sum_{k=1}^{p}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\left(\tilde{\boldsymbol{\alpha}}_0-\boldsymbol{\alpha}_0\right)^{\tau}\mathbf{Z}_{ij}\mathbf{1}_{m_i}W_{ij}\left(\hat{\boldsymbol{\beta}}_{j,k}-\boldsymbol{\beta}_{j,k}\right)\alpha_k$$

$$J_{n,35}=\frac{2}{n^2\bar{N}_{\mathrm{H}}^2|H|}\sum_{k=1}^{p}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\left(\tilde{\alpha}_k-\alpha_k\right)\boldsymbol{\beta}_{i,k}W_{ij}\left(\hat{\boldsymbol{\beta}}_{j,k}-\boldsymbol{\beta}_{j,k}\right).$$

After direct computation, we obtain that $\hat{J}_{n,31}=O_p(1/n)$, $\hat{J}_{n,32}=O_p(1/n)$, $\hat{J}_{n,33}=O_p(h_{\mathrm{A}}^4)$, $\hat{J}_{n,34}=o_p(1/\sqrt{n})$, and $\hat{J}_{n,35}=o_p(1/n)$. By Assumption T3, we have that $\hat{J}_{n,3}=o_p\Big(\frac{1}{n\bar{N}_{\mathrm{H}}\sqrt{|H|}}\Big)$, which shows that (S1.7) holds. $\square$

Proof of (S1.8)

Proof. By the proof of (S1.5), the leading term of $\hat{\sigma}_1^2$ is given by

$$\tilde{\sigma}_1^2=\frac{1}{n^2\bar{N}_{\mathrm{H}}^2|H|}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{l=1}^{m_i}\sum_{v=1}^{m_j}\delta_{il}^2\delta_{jv}^2\Big\{w_{ij}^{(l,v)}\Big\}^2=\frac{1}{n^2\bar{N}_{\mathrm{H}}^2|H|}\sum_{i=1}^{n}\sum_{j\neq i}^{n}H_{i,j},$$

where the form of $H_{i,j}$ is obvious.

After straightforward computation, we obtain that

$$\begin{aligned}
\operatorname{Var}\left(\tilde{\sigma}_1^2\right) &\leq \mathrm{E}\left[\hat{\sigma}_1^4\right] = \frac{1}{n^4\bar{N}_{\mathrm{H}}^4|H|^2}\sum_{i=1}^{n}\sum_{j\neq i}^{n} H_{i,j}^2 = \frac{n(n-1)\bar{N}_{\mathrm{H}}^2}{n^4\bar{N}_{\mathrm{H}}^4|H|^2}|H| + o_p(1)\\
&= O_p\left(\frac{1}{n^2\bar{N}_{\mathrm{H}}^2|H|}\right) = o_p(1), \quad \text{and}\\
\mathrm{E}\left[\tilde{\sigma}_1^2\right] &= \frac{1}{n^2\bar{N}_{\mathrm{H}}^2|H|}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{l=1}^{m_i}\sum_{v=1}^{m_j}\mathrm{E}\Big[\delta_{il}^2\delta_{jv}^2\left\{w_{ij}^{(l,v)}\right\}^2\Big]\\
&= \frac{n(n-1)\bar{N}_{\mathrm{H}}^2}{n^2\bar{N}_{\mathrm{H}}^2|H|}|H|\sigma_1^2 + o_p(1).
\end{aligned}$$

Therefore, $\tilde{\sigma}_1^2 = \mathrm{E}[\tilde{\sigma}_1^2] + o_p(1) \xrightarrow{p} \sigma_1^2$. Since $\hat{\sigma}_1^2 \xrightarrow{p} \tilde{\sigma}_1^2$, we finish the proof. □

Proof of Theorem 2.

Proof. Under $H_{1,\mathrm{C}}$,

$$\hat{e}_{ij} = Y_{ij} - g_0(T_{ij}, \mathbf{Z}_{ij}, \mathbf{X}_{ij}; \tilde{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}(\mathbf{X}_{ij})) = \Delta(T_{ij}, \mathbf{Z}_{ij}, \mathbf{X}_{ij}) - \tilde{e}_{ij} + \delta_{ij},$$

where $\Delta(t, \mathbf{z}, \mathbf{x}) = g(t, \mathbf{z}, \mathbf{x}; \boldsymbol{\alpha}(t), \boldsymbol{\beta}(\mathbf{x})) - g_0(t, \mathbf{z}, \mathbf{x}; \boldsymbol{\alpha}, \boldsymbol{\beta}(\mathbf{x}))$. Let $\boldsymbol{\Delta}_i = (\Delta_{i1}, ..., \Delta_{im_i})^\tau$ with $\Delta_{ij} := \Delta(T_{ij}, \mathbf{Z}_{ij}, \mathbf{X}_{ij})$, then

$$\begin{aligned}
\hat{J}_n &= \frac{1}{n^2\bar{N}_{\mathrm{H}}^2|H|}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\hat{\mathbf{e}}_i^\tau W_{ij}\hat{\mathbf{e}}_j = \frac{1}{n^2\bar{N}_{\mathrm{H}}^2|H|}\sum_{i=1}^{n}\sum_{j\neq i}^{n}(\boldsymbol{\Delta}_i - \tilde{\mathbf{e}}_i + \boldsymbol{\delta}_i)^\tau W_{ij}(\boldsymbol{\Delta}_j - \tilde{\mathbf{e}}_j + \boldsymbol{\delta}_j)\\
&= \hat{J}_{n,1} - 2\hat{J}_{n,2} + \hat{J}_{n,3} + I_{n,1} - 2I_{n,2} + 2I_{n,3},
\end{aligned}$$

where $\hat{J}_{n,i}, i = 1, 2, 3$ are given by (S1.2)–(S1.4), and

$$\begin{aligned}
I_{n,1} &= \frac{1}{n^2\bar{N}_{\mathrm{H}}^2|H|}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\boldsymbol{\Delta}_i^\tau W_{ij}\boldsymbol{\Delta}_j,\\
I_{n,2} &= \frac{1}{n^2\bar{N}_{\mathrm{H}}^2|H|}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\boldsymbol{\Delta}_i^\tau W_{ij}\boldsymbol{\delta}_j, \quad \text{and}\\
I_{n,3} &= \frac{1}{n^2\bar{N}_{\mathrm{H}}^2|H|}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\boldsymbol{\Delta}_i^\tau W_{ij}\tilde{\mathbf{e}}_j.
\end{aligned}$$

In combination with the obtained results given in (S1.5)-(S1.7), it remains to show that

$$I_{n,1} \xrightarrow{p} C_\Delta > 0, \tag{S1.12}$$

$$I_{n,2} = O_p\left(\frac{1}{n\bar{N}_{\mathrm{H}}\sqrt{|H|}}\right), \tag{S1.13}$$

$$I_{n,3} = o_p\left(\frac{1}{n\bar{N}_{\mathrm{H}}\sqrt{|H|}}\right), \quad \text{and} \tag{S1.14}$$

$$\hat{\sigma}_1^2 \xrightarrow{p} C_\Delta > 0 \tag{S1.15}$$

under $H_{1,\mathrm{C}}$. Therefore,

$$n\bar{N}_{\mathrm{H}}\sqrt{|H|}\hat{J}_n/\hat{\sigma}_1 = n\bar{N}_{\mathrm{H}}\sqrt{|H|}\left(I_{n,1} + O_p\left(\frac{1}{n\bar{N}_{\mathrm{H}}\sqrt{|H|}}\right)\right) \to \infty$$

at the rate of $n\bar{N}_{\mathrm{H}}\sqrt{|H|}$. Hence, the proof is finished. □

Proof of (S1.12)

Proof. It is easy to see that

$$\begin{aligned}
\mathrm{Var}(I_{n,1}) &\le \frac{1}{n^4\bar{N}_{\mathrm{H}}^4|H|^2}\mathrm{E}\Big[\sum_{i=1}^n\sum_{j\neq i}^n \boldsymbol{\Delta}_i^\tau W_{ij}\boldsymbol{\Delta}_j\Big]^2 \\
&= \frac{1}{n^4\bar{N}_{\mathrm{H}}^4|H|^2}\sum_{i=1}^n\sum_{j\neq i}^n\sum_{i'=1}^n\sum_{j'\neq i}^n \mathrm{E}\left[\boldsymbol{\Delta}_i^\tau W_{ij}\boldsymbol{\Delta}_j\boldsymbol{\Delta}_{i'}^\tau W_{i'j'}\boldsymbol{\Delta}_{j'}\right] \\
&= V_1 + V_2 + V_3 + V_4,
\end{aligned}$$

where

$$\begin{aligned}
V_1 &= \frac{1}{n^4\bar{N}_{\mathrm{H}}^4|H|^2}\sum_{i=1}^n\sum_{j\neq i}^n \mathrm{E}\left[\boldsymbol{\Delta}_i^\tau W_{ij}\boldsymbol{\Delta}_j\right]^2, \\
V_2 &= \frac{1}{n^4\bar{N}_{\mathrm{H}}^4|H|^2}\sum_{i=1}^n\sum_{j\neq i}^n\sum_{j'\neq i}^n \mathrm{E}\left[\boldsymbol{\Delta}_i^\tau W_{ij}\boldsymbol{\Delta}_j\boldsymbol{\Delta}_i^\tau W_{ij'}\boldsymbol{\Delta}_{j'}\right], \\
V_3 &= \frac{1}{n^4\bar{N}_{\mathrm{H}}^4|H|^2}\sum_{i=1}^n\sum_{i'=1}^n\sum_{\substack{j\neq i\\ j\neq i'}}^n \mathrm{E}\left[\boldsymbol{\Delta}_i^\tau W_{ij}\boldsymbol{\Delta}_j\boldsymbol{\Delta}_{i'}^\tau W_{i'j}\boldsymbol{\Delta}_j\right], \quad \text{and}
\end{aligned}$$

$$V_4 = \frac{1}{n^4 \bar{N}_{\mathrm{H}}^4 |H|^2} \sum_{i=1}^{n} \sum_{i'=1}^{n} \sum_{j \neq i}^{n} \sum_{j' \neq i'}^{n} \mathrm{E}\left[\boldsymbol{\Delta}_i^{\tau} W_{ij} \boldsymbol{\Delta}_j \boldsymbol{\Delta}_{i'}^{\tau} W_{i'j'} \boldsymbol{\Delta}_{j'}\right].$$

It is not difficult to show that the leading term of V_1 is given by

$$\begin{aligned} &\frac{1}{n^4 \bar{N}_{\mathrm{H}}^4 |H|^2} \sum_{i=1}^{n} \sum_{j \neq i}^{n} \sum_{l=1}^{m_i} \sum_{v=1}^{m_j} \mathrm{E}\Big[\Delta_{il}^2 \Delta_{jv}^2 \big(w_{ij}^{(l,v)}\big)^2\Big] \\ &= O\left(\frac{(N - n\bar{N}_2)\kappa_{22}^{p+1}|H|}{n^4 \bar{N}_{\mathrm{H}}^4 |H|^2}\right) = O\left(\frac{1}{n^2 \bar{N}_{\mathrm{H}}^2 |H|}\right) \end{aligned}$$

which yields $V_1 = O\left(\frac{1}{n^2 \bar{N}_{\mathrm{H}}^2 |H|}\right)$. In the same vein, we obtain that $V_i = o\Big(\frac{1}{n^2 \bar{N}_{\mathrm{H}}^2 |H|}\Big)$ for $i = 2, 3, 4$. Therefore, $\mathrm{Var}(I_{n,1}) = O_p\Big(\frac{1}{n^2 \bar{N}_{\mathrm{H}}^2 |H|}\Big) = o(1)$, which implies that $I_{n,1} = \mathrm{E}[I_{n,1}] + o_p(1)$. Furthermore, we can write $\mathrm{E}[I_{n,1}]$ as

$$\begin{aligned} \mathrm{E}[I_{n,1}] &= \frac{1}{n^2 \bar{N}_{\mathrm{H}}^2 |H|} \sum_{i=1}^{n} \sum_{j \neq i}^{n} \mathrm{E}\left[\boldsymbol{\Delta}_i^{\tau} W_{ij} \boldsymbol{\Delta}_{\mathbf{j}}\right] \\ &= \frac{1}{n^2 \bar{N}_{\mathrm{H}}^2 |H|} \sum_{i=1}^{n} \sum_{j \neq i}^{n} \sum_{l=1}^{m_i} \sum_{v=1}^{m_j} \mathrm{E}\big[\Delta_{il} w_{ij}^{(l,v)} \Delta_{jv}\big], \end{aligned}$$

and obtain

$$\begin{aligned} &\mathrm{E}\big[\Delta_{il} w_{ij}^{(l,v)} \Delta_{jv}\big] = \mathrm{E}\left[\Delta(T_{il})\Delta(T_{jv}) k_{h_{\mathrm{C}}}(T_{il}, T_{jv}) K_{h_{\mathrm{A}}}(X_{il}, X_{jv})\right] \\ &= \int \Delta(t_1)\Delta(t_2) k\Big(\frac{t_1 - t_2}{h_{\mathrm{C}}}\Big) \prod_{k=1}^{p} k\Big(\frac{x_{1,k} - x_{2,k}}{h_{\mathrm{A}}}\Big) f_{\mathrm{T}}(t_1) f_{\mathrm{T}}(t_2) \\ &\quad \prod_{k=1}^{p} f_k(x_{1,k}) f_k(x_{2,k}) \mathrm{d}t_1 \mathrm{d}t_2 \prod_{k=1}^{p} \mathrm{d}x_{1,k} \mathrm{d}x_{2,k} \\ &= |H| \int \Delta(t_2 + h_{\mathrm{C}} z)\Delta(t_2) k(z) \prod_{k=1}^{p} k(v_k) f_{\mathrm{T}}(t_2 + h_{\mathrm{C}} z) f_{\mathrm{T}}(t_2) \mathrm{d}z \mathrm{d}t_2 \prod_{k=1}^{p} \mathrm{d}v_k \mathrm{d}X_{2,k} \\ &= |H| \mathrm{E}\Big[\Delta^2(T, \mathbf{Z}, \mathbf{X}) f_{\mathrm{T}}(T) \frac{\prod_{k=1}^{p} f_k^2(X_k)}{f(X_1, \ldots, X_k)}\Big] := |H| C_{\Delta}. \end{aligned}$$

Since C_Δ is positive under H_1, we obtain that $\mathrm{E}[I_{n,1}] = \frac{n(n-1)\bar{N}_{\mathrm{H}}^2 |H|}{n^2 \bar{N}_{\mathrm{H}}^2 |H|} C_\Delta \to C_\Delta > 0$, which yields (S1.12). □

Proof of (S1.13)

Proof. It is easy to see that

$$\mathrm{Var}(I_{n,2}) \leq \mathrm{E}\left[I_{n,2}^2\right] = \frac{1}{n^4 \bar{N}_{\mathrm{H}}^4 |H|^2} \mathrm{E}\Big[\sum_{i=1}^{n}\sum_{j\neq i}^{n} \boldsymbol{\Delta}_i^\tau W_{ij} \boldsymbol{\delta}_j\Big]^2$$
$$= \frac{1}{n^4 \bar{N}_{\mathrm{H}}^4 |H|^2} \sum_{i=1}^{n}\sum_{i'=1}^{n}\sum_{j\neq i}^{m_i}\sum_{j'\neq i'}^{m_j} \mathrm{E}\left[\boldsymbol{\Delta}_i^\tau W_{ij} \boldsymbol{\delta}_j \boldsymbol{\Delta}_{j'}^\tau W_{i'j'} \boldsymbol{\delta}_{j'}\right].$$

Similar to the proof of (S1.12), it can be shown that $\mathrm{Var}(I_{n,2}) = O_p\big(\frac{1}{n^2 \bar{N}_{\mathrm{H}}^2 |H|}\big)$. Hence, (S1.13) holds because $\mathrm{E}\left[I_{n,2}\right] = 0$. □

Proof of (S1.14)

Proof. From (S1.11), we can write $I_{n,3} = I_{n,31} + I_{n,32} + I_{n,33} + I_{n,34}$, where

$$I_{n,31} = \frac{1}{n^2 \bar{N}_{\mathrm{H}}^2 |H|} \sum_{i=1}^{n}\sum_{j\neq i}^{n} \boldsymbol{\Delta}_i^\tau W_{ij} \mathbf{Z}_{ij}^\tau (\tilde{\boldsymbol{\alpha}}_0 - \boldsymbol{\alpha}_0),$$
$$I_{n,32} = \frac{1}{n^2 \bar{N}_{\mathrm{H}}^2 |H|} \sum_{i=1}^{n}\sum_{j\neq i}^{n} \boldsymbol{\Delta}_i^\tau W_{ij} (\tilde{\alpha}_k - \alpha_k) \boldsymbol{\beta}_{j,k},$$
$$I_{n,33} = \frac{1}{n^2 \bar{N}_{\mathrm{H}}^2 |H|} \sum_{i=1}^{n}\sum_{j\neq i}^{n} \boldsymbol{\Delta}_i^\tau W_{ij} \alpha_k (\hat{\boldsymbol{\beta}}_{j,k} - \boldsymbol{\beta}_{j,k}), \quad \text{and}$$
$$I_{n,34} = \frac{1}{n^2 \bar{N}_{\mathrm{H}}^2 |H|} \sum_{i=1}^{n}\sum_{j\neq i}^{n} \boldsymbol{\Delta}_i^\tau W_{ij} (\tilde{\alpha}_k - \alpha_k)(\hat{\boldsymbol{\beta}}_{j,k} - \boldsymbol{\beta}_{j,k}).$$

By Lemma 1, we have

$$\mathrm{Var}(I_{n,31}) \leq \frac{1}{n^5 \bar{N}_{\mathrm{H}}^4 |H|^2} \sum_{i=1}^{n}\sum_{j\neq i}^{n} \mathrm{E}\left[\boldsymbol{\Delta}_i^\tau W_{ij} \mathbf{Z}_{ij} \mathbf{1}_{m_j}\right]^2$$
$$\leq C \frac{1}{n^5 \bar{N}_{\mathrm{H}}^4 |H|^2} \sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{l=1}^{m_i}\sum_{v=1}^{m_j}\sum_{l=1}^{m_i}\sum_{v=1}^{m_j} \mathrm{E}\left[\Delta_{il}\Delta_{il'} w_{ij}^{(l,v)} w_{ij}^{(l',v')}\right]$$
$$= O_p\left(\frac{n(n-1)\bar{N}_{\mathrm{H}}^2 |H|}{n^5 \bar{N}_{\mathrm{H}}^4 |H|^2}\right) + o_p(1)$$
$$= o_p\left(\frac{1}{n^2 \bar{N}_{\mathrm{H}}^2 |H|}\right).$$

It follows that $\mathrm{E}|I_{n,31}| \leq o_p\Big(\frac{1}{n\bar{N}_{\mathrm{H}}\sqrt{|H|}}\Big)$, which leads to $I_{n,31} = o_p\Big(\frac{1}{n\bar{N}_{\mathrm{H}}\sqrt{|H|}}\Big)$. Similarly, we get $I_{n,32} = o_p\Big(\frac{1}{n\bar{N}_{\mathrm{H}}\sqrt{|H|}}\Big)$. Combining with Theorem 3 in Liu et al. (2022), we obtain that

$$\begin{aligned} I_{n,33} &= \frac{1}{n^2\bar{N}_{\mathrm{H}}^2|H|}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\mathbf{\Delta}_i^{\tau}W_{ij}\alpha_k\left\{O(h_{\mathrm{A}}^2)+O_p\left(\sqrt{\frac{1}{n\bar{N}_{\mathrm{H}}h_{\mathrm{A}}}+\frac{1}{n}}\right)\right\} \\ &= o_p\left(\frac{1}{n\bar{N}_{\mathrm{H}}\sqrt{|H|}}\right), \end{aligned} \tag{S1.16}$$

which is also implies that $I_{n,34} = o_p\Big(\frac{1}{n\bar{N}_{\mathrm{H}}\sqrt{|H|}}\Big)$. Thus, we complete the proof of (S1.14). □

Proof of (S1.15)

Proof. From the proof of above, the leading term of $\hat{\sigma}_1$ under H_1 is decided by

$$\frac{1}{n^2\bar{N}_{\mathrm{H}}^2|H|}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{l=1}^{m_i}\sum_{v=1}^{m_j}\Delta_{il}^2\Delta_{jv}^2\{w_{ij}^{(l,v)}\}^2 = \frac{n(n-1)\bar{N}_{\mathrm{H}}^2|H|}{n^2\bar{N}_{\mathrm{H}}^2|H|}C + o_p(1) \xrightarrow{p} C > 0, \tag{S1.17}$$

where $C = \kappa_{22}^{p+1}\mathrm{E}\Big[\Delta^4(T,\mathbf{Z},\mathbf{X})f_{\mathrm{T}}(T)\frac{\prod_{k=1}^{p}f_k^2(X_k)}{f(X_1,\ldots,X_k)}\Big] > 0$ under $H_{1,\mathrm{C}}$. Hence the proof. □

Proof of Theorem 3

Proof. Let $\tilde{\mathbf{d}}_i = \tilde{\mathbf{h}}_{0,i} - \mathbf{h}_{0,i}$, where

$$\begin{aligned} \mathbf{h}_{0,i} =&\ \{h_0(T_{i1},\mathbf{Z}_{i1},\mathbf{X}_{i1};\boldsymbol{\alpha}),...,h_0(T_{im_i},\mathbf{Z}_{im_i},\mathbf{X}_{im_i};\boldsymbol{\alpha})\}^{\tau} \quad \text{and} \\ \tilde{\mathbf{h}}_{0,i} =&\ \{h_0(T_{i1},\mathbf{Z}_{i1},\mathbf{X}_{i1};\tilde{\boldsymbol{\alpha}}),...,h_0(T_{im_i},\mathbf{Z}_{im_i},\mathbf{X}_{im_i};\tilde{\boldsymbol{\alpha}})\}^{\tau}. \end{aligned}$$

Notice that $\hat{\boldsymbol{\varsigma}}_i = \mathbf{Y}_i - \tilde{\mathbf{h}}_{0,i} = \boldsymbol{\delta}_i - (\tilde{\mathbf{h}}_{0,i} - \mathbf{h}_{0,i}) = \boldsymbol{\delta}_i - \tilde{\mathbf{d}}_i$, then, we can write $\hat{I}_n = \hat{J}_{n,1} - 2\hat{I}_{n,2} + \hat{I}_{n,3}$, where $\hat{J}_{n,1}$ is defined in the proof of Theorem 2, and

$$\begin{aligned} \hat{I}_{n,2} &= \frac{1}{n^2\bar{N}_{\mathrm{H}}^2|H|}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\boldsymbol{\delta}_i^{\tau}W_{ij}\tilde{\mathbf{d}}_j, \quad \text{and} \\ \hat{I}_{n,3} &= \frac{1}{n^2\bar{N}_{\mathrm{H}}^2|H|}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\tilde{\mathbf{d}}_i^{\tau}W_{ij}\tilde{\mathbf{d}}_j. \end{aligned}$$

We have show that $\frac{n^2\bar{N}_{\mathrm{H}}^2}{\sqrt{N^2-n\bar{N}_2}}\sqrt{|H|}\hat{J}_{n,1}/\sigma_1 \xrightarrow{D} N(0,1)$. Since the j-th elements of $\tilde{\mathbf{d}}_i$ is given by $\mathbf{Z}_{ij}^{\tau}\left(\tilde{\boldsymbol{\alpha}}_0(T_{ij})-\boldsymbol{\alpha}_0(T_{ij})\right)+\sum_{k=1}^{p}\left(\tilde{\alpha}_k\left(T_{ij}\right)-\alpha_k\left(T_{ij}\right)\right)X_{ij,k}$, we can write $\hat{I}_{n,2}=\hat{I}_{n,21}+\hat{I}_{n,22}$ with

$$\hat{I}_{n,21}=\frac{1}{n^2\bar{N}_{\mathrm{H}}^2|H|}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{l=1}^{m_i}\sum_{j=1}^{m_j}\delta_{il}\mathbf{Z}_{jv}^{\tau}\left(\tilde{\boldsymbol{\alpha}}_k\left(T_{jv}\right)-\boldsymbol{\alpha}_0(T_{jv})\right)w_{ij}^{(l,v)} \quad \text{and}$$

$$\hat{I}_{n,22}=\frac{1}{n^2\bar{N}_{\mathrm{H}}^2|H|}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{l=1}^{m_i}\sum_{j=1}^{m_j}\delta_{il}\left(\alpha_k\left(T_{jv}\right)-\alpha_k\left(T_{jv}\right)\right)X_{jv,k}w_{ij}^{(l,v)}.$$

Similar to the procedure of Zhang and Wang (2016), it can be shown that $\tilde{\boldsymbol{\alpha}}_0(t)-\boldsymbol{\alpha}_0(t)=O_p\left(\sqrt{\frac{1}{n\bar{N}_{\mathrm{H}}h_{\mathrm{C}}}+\frac{1}{n}}\right)$ and $\hat{\alpha}_k(t)-\alpha_k(t)=O_p\left(\sqrt{\frac{1}{n\bar{N}_{\mathrm{H}}h_{\mathrm{C}}}+\frac{1}{n}}\right)$. Similarly, we obtain that $\hat{I}_{n,21}=o_p\left(\frac{1}{n\bar{N}_{\mathrm{H}}\sqrt{|H|}}\right)$ and $\hat{I}_{n,22}=o_p\left(\frac{1}{n\bar{N}_{\mathrm{H}}\sqrt{|H|}}\right)$. That is, $\hat{I}_{n,2}=o_p\left(\frac{1}{n\bar{N}_{\mathrm{H}}\sqrt{|H|}}\right)$.

Furthermore, we can rewrite $\hat{I}_{n,3}=\hat{I}_{n,31}+\hat{I}_{n,32}+\hat{I}_{n,33}$, where

$$\hat{I}_{n,31}=\frac{1}{n^2\bar{N}_{\mathrm{H}}^2|H|}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{l=1}^{m_i}\sum_{j=1}^{m_j}\left(\tilde{\boldsymbol{\alpha}}_0(T_{il})-\boldsymbol{\alpha}_0(T_{il})\right)\mathbf{Z}_{il}^{\tau}w_{ij}^{(l,v)}\left(\tilde{\boldsymbol{\alpha}}_0(T_{jv})-\boldsymbol{\alpha}_0(T_{jv})\right)\mathbf{Z}_{jv},$$

$$\hat{I}_{n,32}=\frac{1}{n^2\bar{N}_{\mathrm{H}}^2|H|}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{l=1}^{m_i}\sum_{j=1}^{m_j}\left(\tilde{\alpha}_k(T_{il})-\alpha_k(T_{il})\right)w_{ij}^{(l,v)}\left(\tilde{\alpha}_k(T_{jv})-\alpha_k(T_{jv})\right), \quad \text{and}$$

$$\hat{I}_{n,33}=\frac{1}{n^2\bar{N}_{\mathrm{H}}^2|H|}\sum_{k=1}^{p}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{l=1}^{m_i}\sum_{j=1}^{m_j}\left(\tilde{\boldsymbol{\alpha}}_0(T_{il})-\boldsymbol{\alpha}_0(T_{il})\right)^{\tau}\mathbf{Z}_{il}w_{ij}^{(l,v)}\left(\tilde{\alpha}_k(T_{jv})-\alpha_k(T_{jv})\right).$$

It is routine to show that $\hat{I}_{n,3l}=o_p\left(\frac{1}{n\bar{N}_{\mathrm{H}}\sqrt{|H|}}\right)$ for $l=1,2,3$, which means that $\hat{I}_{n,3}=o_p\left(\frac{1}{n\bar{N}_{\mathrm{H}}\sqrt{|H|}}\right)$. Therefore, it follows that

$$\frac{n^2\bar{N}_{\mathrm{H}}^2}{\sqrt{N^2-n\bar{N}_2}}\sqrt{|H|}\hat{I}_n/\hat{\sigma}_1=\frac{\sigma_1}{\hat{\sigma}_1}\frac{n^2\bar{N}_{\mathrm{H}}^2}{\sqrt{N^2-n\bar{N}_2}}\sqrt{|H|}\hat{I}_{n,1}/\sigma_1+o_p(1)\xrightarrow{D} N(0,1),$$

which completes the proof. □

Proof of Theorem 4.

Proof. Under $H_{1,\mathrm{A}}$,

$$\hat{\varsigma}_{ij} = Y_{ij} - h_0\left(T_{ij}, \mathbf{Z}_{ij}, \mathbf{X}_{ij}; \tilde{\boldsymbol{\alpha}}(T_{ij})\right) = \tilde{\Delta}(T_{ij}, \mathbf{Z}_{ij}, \mathbf{X}_{ij}) - \tilde{d}_{ij} + \delta_{ij},$$

where $\tilde{\Delta}(t, \mathbf{z}, \mathbf{x}) = g\left(t, \mathbf{z}, \mathbf{x}; \boldsymbol{\alpha}(t), \boldsymbol{\beta}(\mathbf{x})\right) - h_0\left(t, \mathbf{z}, \mathbf{x}; \boldsymbol{\alpha}(t)\right)$. Let $\tilde{\boldsymbol{\Delta}}_i = (\tilde{\Delta}_{i1}, ..., \tilde{\Delta}_{im_i})^\tau$ with $\tilde{\Delta}_{ij} := \tilde{\Delta}(T_{ij}, \mathbf{Z}_{ij}, \mathbf{X}_{ij})$, then

$$\hat{I}_n = \hat{J}_{n,1} - 2\hat{I}_{n,2} + \hat{I}_{n,3} + \tilde{I}_{n,1} - 2\tilde{I}_{n,2} + 2\tilde{I}_{n,3},$$

where

$$\tilde{I}_{n,1} = \frac{1}{n^2 \bar{N}_{\mathrm{H}}^2 |H|} \sum_{k=1}^{p} \sum_{i=1}^{n} \sum_{j\neq i}^{n} \tilde{\boldsymbol{\Delta}}_i^\tau W_{ij} \tilde{\boldsymbol{\Delta}}_j,$$

$$\tilde{I}_{n,2} = \frac{1}{n^2 \bar{N}_{\mathrm{H}}^2 |H|} \sum_{i=1}^{n} \sum_{j\neq i}^{n} \tilde{\boldsymbol{\Delta}}_i^\tau W_{ij} \boldsymbol{\delta}_j, \quad \text{and}$$

$$\tilde{I}_{n,3} = \frac{1}{n^2 \bar{N}_{\mathrm{H}}^2 |H|} \sum_{i=1}^{n} \sum_{j\neq i}^{n} \tilde{\boldsymbol{\Delta}}_i^\tau W_{ij} \tilde{d}_i.$$

In the vein of Theorem 2, we can show that

$$\tilde{I}_{n,1} = \mathrm{E}[\tilde{I}_{n,1}] + o_p(1) \to \mathrm{E}\left[\tilde{\Delta}^2(T, \mathbf{Z}, \mathbf{X}) f_{\mathrm{T}}(T) \frac{\prod_{k=1}^{p} f_k^2(X_k)}{f(X_1, ..., X_k)}\right] > 0$$

and $\tilde{I}_{n,2} = O_p\left(\frac{1}{n\bar{N}_{\mathrm{H}}\sqrt{|H|}}\right)$. On the other hand, note that $\tilde{I}_{n,3} = \tilde{I}_{n,31} + \tilde{I}_{n,32}$ with

$$\tilde{I}_{n,31} = \frac{1}{n^2 \bar{N}_{\mathrm{H}}^2 |H|} \sum_{i=1}^{n} \sum_{j\neq i}^{n} \tilde{\boldsymbol{\Delta}}_i^\tau W_{ij} \mathbf{Z}_{ij}^\tau \left(\tilde{\alpha}_0(T_{ij}) - \alpha_0(T_{ij})\right) \quad \text{and}$$

$$\tilde{I}_{n,32} = \frac{1}{n^2 \bar{N}_{\mathrm{H}}^2 |H|} \sum_{k=1}^{p} \sum_{i=1}^{n} \sum_{j\neq i}^{n} \tilde{\boldsymbol{\Delta}}_i^\tau W_{ij} \left(\tilde{\alpha}_k(T_{ij}) - \alpha_k(T_{ij})\right) X_{ijk}.$$

Along with the line of (S1.14), we have $\tilde{I}_{n,3i} = o_p\left(\frac{1}{n\bar{N}_{\mathrm{H}}\sqrt{|H|}}\right)$ for $i = 1, 2$, which yields $\hat{I}_{n,3} = o_p\left(\frac{1}{n\bar{N}_{\mathrm{H}}\sqrt{|H|}}\right)$. In combination with (S1.5) and (S1.8), we obtain that

$$n\bar{N}_{\mathrm{H}}\sqrt{|H|}\hat{I}_n/\hat{\sigma}_1 = n\bar{N}_{\mathrm{H}}\sqrt{|H|}\left(\tilde{I}_{n,1} + O_p\left(\frac{1}{n\bar{N}_{\mathrm{H}}\sqrt{|H|}}\right)\right) \to \infty$$

at the rate of $n\bar{N}_{\mathrm{H}}\sqrt{|H|}$. $\square$

References

Jun, S. J., Pinkse, J., 2012. Testing under weak identification with conditional moment restrictions. Econometric Theory 28 (6), 1229–1282.

Liu, S., You, J., Hu, L., 2022. Unified statistical inference for a nonlinear dynamic functional/longitudinal data model. Journal of Statistical Planning and Inference 219, 175–188.

Zhang, X., Wang, J.-L., 2016. From sparse to dense functional data and beyond. The Annals of Statistics 44 (5), 2281–2321.