Supplementary material

Proof of Property 2.1

Following Domowitz and White (1982), we made assumptions as follows:

Assumption 1. Sequence $g(y_i, \boldsymbol{\vartheta}), i = 1, \cdots, N$ are continuous functions of $\boldsymbol{\vartheta}$ for $\forall y_i \in \Omega$ and measurable functions of y_i for each $\boldsymbol{\vartheta} \in \Theta$, where Θ is a compact subset of a finite-dimensional Euclidean space.

Assumption 2. The random vectors $\{Y_i\}$ are either ϕ-mixing, with $\phi(m)$ of size $r_1/(2r_1 - 1), r_1 \geq 1$; or α-mixing, with $\alpha(m)$ of size $r_1/(r_1 - 1), r_1 > 1$ (Domowitz and White, 1982).

Assumption 3. Sequence $g(y_i, \boldsymbol{\vartheta})$ is dominated by uniformly $(r_1 + \varrho)$-integrable functions, $r_1 \geq 1, 0 < \varrho \leq r_1$.

Assumption 4. $\bar{G}_N(\boldsymbol{\vartheta})$ has a unique maximum $\boldsymbol{\vartheta}_0$.

Assumption 5. $g(y_i, \boldsymbol{\vartheta})$ is continuously differentiable of order 2 for $\boldsymbol{\vartheta}$.

Assumption 6. $\{g'_j(y_i, \boldsymbol{\vartheta})^2\}$ are dominated by uniformly r_2-integrable functions, where $r_2 > 1, g'_j(y_i, \boldsymbol{\vartheta}) = \partial g(y_i, \boldsymbol{\vartheta})/\partial \vartheta_j$.

Assumption 7. Define $\mathbf{Q}_{a,N} = var[N^{-1/2} \sum_{i=a+1}^{a+N} g'(y_i, \boldsymbol{\vartheta}_0)]$. Assume there exists a positive definite matrix $\mathbf{Q}$ such that $\lambda^T \mathbf{Q}_{a,N} \lambda - \lambda^T \mathbf{Q} \lambda \to 0$ as $N \to \infty$ for any real non-zero vector λ.

Assumption 8. $\{g''_{jk}(y_i, \boldsymbol{\vartheta})\}$ are dominated by uniformly $r_1 + \varrho$-integrable functions, where $0 < \varrho \leq r_1, g^{''}_{jk}(y_i, \boldsymbol{\vartheta}) = \partial^2 g(y_i, \boldsymbol{\vartheta})/\partial \vartheta_j \partial \vartheta_k$.

Assumption 9. For all N sufficiently large, the matrix $\bar{G}^{''}_N(\boldsymbol{\vartheta}) = 1/N \Sigma_{i=1}^N E[g''(y_i, \boldsymbol{\vartheta})]$ has constant rank in some open ϵ-neighborhood of $\boldsymbol{\vartheta}_0$.

We can strengthen slightly the memory requirements of Assumption 2 to allow the application of Theorem 2.6 of Domowitz and White (1982).

Assumption 2'. Assumption 2 holds, and either $\phi(m)$ is of size $r_2/(r_2 - 1)$ or $\alpha(m)$ is of size $max[r_1/(r_1 - 1), r_2/(r_2 - 1)], r_1, r_2 > 1$.

Under Assumptions $1 - 4$, $g(y_i, \boldsymbol{\vartheta})$ satisfies conditions of Theorem 2.5 of Domowitz and White (1982), then, we have

$$\left| N^{-1} \sum_{i=1}^{N} [g(y_i, \boldsymbol{\vartheta}) - E(g(y_i, \boldsymbol{\vartheta}))] \right| \to 0, a.s.$$

Furthermore, apply Theorem 2.2 of Domowitz and White (1982), Property 2.1(a) can be proved.

Under assumptions 2', 5, 8 and 9, $g^{''}_{jk}(y_i, \boldsymbol{\vartheta})$ satisfies conditions of Theorem 2.5 of Domowitz and White (1982), therefore,

$$\left| N^{-1} \sum_{i=1}^{N} [g^{''}_{jk}(y_i, \boldsymbol{\vartheta}) - E(g^{''}_{jk}(y_i, \boldsymbol{\vartheta}))] \right| \to 0, a.s.$$

Thus, $|G_N^{''}(\boldsymbol{y},\boldsymbol{\vartheta}) - \bar{G}_N^{''}(\boldsymbol{\vartheta})| \rightarrow 0, a.s.$ From $\widehat{\boldsymbol{\vartheta}}_N \rightarrow \boldsymbol{\vartheta}_0 \quad a.s.$, by the result in Theorem 2.3 in Domowitz and White (1982), we have $|G_N^{''}(\boldsymbol{y},\tilde{\boldsymbol{\vartheta}}) - \bar{G}_N^{''}(\boldsymbol{\vartheta}_0)| \rightarrow 0, a.s.$, where $\tilde{\boldsymbol{\vartheta}}$ is between $\widehat{\boldsymbol{\vartheta}}_N$ and $\boldsymbol{\vartheta}_0$. Under assumptions 2', 5, 6 and 7, according to Theorem 2.6 in Domowitz and White (1982), we get $\sqrt{N}G_N^{'}(\boldsymbol{y},\boldsymbol{\vartheta}_0) \xrightarrow{L} N(\mathbf{0},\mathbf{Q}_N)$, where $\mathbf{Q}_N = \mathbf{Q}_{0,N} = var(\sqrt{N}G_N^{'}(\boldsymbol{y},\boldsymbol{\vartheta}_0))$. Applying mean-value argument analogous (2.6) Property 2.1(b) is proved.