Supplementary Material for "Extreme Quantile Estimation for Autoregressive Models" by Deyuan Li and Huixia Judy Wang

In this supplementary file, we provide the proofs of Propositions 2.1-2.2, Theorems 2.1-2.3 and Corollary 2.1 in the main paper.

Proof of Proposition 2.1.

We first prove that $U_t(\cdot)$ satisfies the first-order condition, i.e. as $s \to \infty$,

$$\frac{U_t(su) - U_t(s)}{a_t(s)} \to \frac{u^\gamma - 1}{\gamma}. \tag{S.1}$$

We distinguish two different cases.

Case 1: there exists a unique $i \in \{0, 1, \ldots, p\}$ such that $\gamma_i = \gamma$. Define $a_t(s) = a_0(s)$ if $i = 0$ and $y_{t-i}a_i(s)$ if $i \neq 0$. Without loss of generality, we assume that $\gamma_1 = \gamma$. Then $a_t(s) = y_{t-1}a_1(s)$ and by (2.1) we get

$$\begin{aligned}\frac{U_t(su) - U_t(s)}{a_t(s)} = &\frac{\theta_0(su) - \theta_0(s)}{a_0(s)} \times \frac{a_0(s)}{y_{t-1}a_1(s)} \\ &+ \frac{\theta_1(su) - \theta_1(s)}{a_1(s)} + \sum_{j=2}^{p} \frac{\theta_j(su) - \theta_j(s)}{a_j(s)} \times \frac{y_{t-j}}{y_{t-1}} \times \frac{a_j(s)}{a_1(s)}.\end{aligned}$$

Since $a_j(\cdot) \in RV(\gamma_j)$ for $j = 0, 1, \ldots, p$, and $\gamma_1 > \max\{\gamma_0, \gamma_2, \ldots, \gamma_p\}$, we have $a_j(s) = o(a_1(s))$ as $s \to \infty$ for $j = 0, 2, \ldots, p$, and (S.1) is thus proven.

Case 2: there exist more than one i's such that $\gamma_i = \gamma$. Without loss of generality, assume that the first $(k + 1)$ γ_i's are equal to γ, that is, $\gamma_0 = \gamma_1 = \cdots = \gamma_k = \gamma$, where $1 \leq k \leq p - 1$. Note that $a_i(\cdot) \in RV(\gamma)$ and $\rho_i < 0$ for $i = 0, 1, \ldots, k$. Then $a_i(s)/a_0(s) \to c_i$, a non-zero constant, $i = 1, \ldots, k$. Now let

$$a_t(s) = (1 + c_1y_{t-1} + ... + c_ky_{t-k})a_0(s) = (1 + \sum_{j=1}^{k} c_jy_{t-j})a_0(s).$$

On the other hand $a_i(s)/a_0(s) \to 0$ for $i = k+1, \ldots, p$. So

$$
\begin{aligned}
&\frac{U_t(su) - U_t(s)}{a_t(s)} \\
&= \frac{\theta_0(su) - \theta_0(s)}{(1 + \sum_{j=1}^k c_j y_{t-j}) a_0(s)} + \sum_{j=1}^k \Big\{ \frac{\theta_j(su) - \theta_j(s)}{a_j(s)} \times \frac{y_{t-j} a_j(s)}{(1 + \sum_{j=1}^k c_j y_{t-j}) a_0(s)} \Big\} \\
&\quad + \sum_{j=k+1}^p \frac{\theta_j(su) - \theta_j(s)}{a_j(s)} \times \frac{y_{t-j} a_j(s)}{(1 + \sum_{j=1}^k c_j y_{t-j}) a_0(s)} \\
&\to \frac{1}{1 + \sum_{j=1}^k c_j y_{t-j}} \times \frac{u^{\gamma_0} - 1}{\gamma_0} + \sum_{j=1}^k \Big\{ \frac{c_j y_{t-j}}{1 + \sum_{j=1}^k c_j y_{t-j}} \times \frac{u^{\gamma_j} - 1}{\gamma_j} \Big\} = \frac{u^\gamma - 1}{\gamma}.
\end{aligned}
$$

Next, we will prove that $U_t(\cdot)$ satisfies the second-order condition (2.3).We will only provide the proof for Case 1, as the proof for Case 2 is similar. Without loss of generality, suppose $\gamma_0 = \gamma > \max\{\gamma_1, \gamma_2, \ldots, \gamma_p\}$. Recall $a_t(s) = a_0(s)$ and

$$
\frac{U_t(su) - U_t(s)}{a_t(s)} = \frac{\theta_0(su) - \theta_0(s)}{a_0(s)} + \sum_{j=1}^p \frac{\theta_j(su) - \theta_j(s)}{a_j(s)} \times \frac{a_j(s)}{a_0(s)} \times y_{t-j}.
$$

Hence, by the second-order condition of $\theta_0(\cdot)$, we have

$$
\begin{aligned}
\frac{U_t(su) - U_t(s)}{a_t(s)} - \frac{u^\gamma - 1}{\gamma} \quad &= \quad A_0(s) \frac{1}{\rho_0} \Big(\frac{u^{\gamma_0 + \rho_0} - 1}{\gamma_0 + \rho_0} - \frac{u^{\gamma_0} - 1}{\gamma_0} \Big) \{1 + o(1)\} \\
&\quad + \sum_{j=1}^p \frac{\theta_j(su) - \theta_j(s)}{a_j(s)} \times \frac{a_j(s)}{a_0(s)} \times y_{t-j}. \qquad \text{(S.2)}
\end{aligned}
$$

Note that $A_0(\cdot) \in RV(\rho_0)$ and $a_j(\cdot)/a_0(\cdot) \in RV(\gamma_j - \gamma_0)$ for $j = 1, 2, \ldots, p$. Define $\rho = \max\{\rho_0, \gamma_1 - \gamma_0, \gamma_2 - \gamma_0, \ldots, \gamma_p - \gamma_0\}$. There are three different cases: i) $\rho_0 > \max\{\gamma_j - \gamma_0 : j = 1, 2, \ldots, p\}$; ii) there is a unique j such that $\gamma_j - \gamma_0 = \rho$; iii) there exist several elements that are equal to the maximum ρ.

Case i). $\rho_0 > \max\{\gamma_j - \gamma_0, j = 1, 2, \ldots, p\}$. In this case, $\rho = \rho_0$. Take $A_t(s) = A_0(s)$. Note that $a_j(\cdot)/\{A_0(\cdot)a_0(\cdot)\} \in RV(\gamma_j - \gamma_0 - \rho_0)$ for $j = 1, 2, \ldots, p$ with $\gamma_j - \gamma_0 - \rho_0 < 0$, so $a_j(s)/\{A_0(s)a_0(s)\} \to 0$ as $s \to \infty$. Therefore, $U_t(\cdot)$ satisfies the second-order condition (2.3) with $A_t(s) = A_0(s)$ and $\rho = \rho_0$.

Case ii). There exists a unique j such that $\gamma_j - \gamma_0$ attains the maximum ρ. Without loss of generality, suppose $j = 1$, i.e. $\gamma_1 - \gamma_0 > \max\{\rho_0, \gamma_2 - \gamma_0, \ldots, \gamma_p - \gamma_0\}$. Then $\rho = \gamma_1 - \gamma_0 < 0$. Take $A_t(s) = a_1(s)y_{t-1}/a_0(s)$. Similar to the proof for Case i), we can show that

$$\{A_t(s)\}^{-1}\Big(\frac{U_t(su)-U_t(s)}{a_t(s)}-\frac{u^\gamma-1}{\gamma}\Big)\to\frac{u^{\gamma_1}-1}{\gamma_1}=\frac{u^{\gamma+\rho}-1}{\gamma+\rho}.$$

To change the above limit to be $\rho^{-1}\{(u^{\gamma+\rho}-1)/(\gamma+\rho)-(u^\gamma-1)/\gamma\}$ as stated in (2.3), we need modify the definitions of a_t and A_t slightly. For example, let $A_t^*(s)=\rho A_t(s)$ and $a_t^*(s)=(1+\rho^{-1}A_t^*(s))a_t(s)$. Then by Taylor expansion, we have

$$\{A_t^*(s)\}^{-1}\Big(\frac{U_t(su)-U_t(s)}{a_t^*(s)}-\frac{u^\gamma-1}{\gamma}\Big)\to\frac{1}{\rho}\Big(\frac{u^{\gamma+\rho}-1}{\gamma+\rho}-\frac{u^\gamma-1}{\gamma}\Big).$$

Case iii). There exist several elements that are equal to the maximum ρ. Without loss of generality, suppose that $\rho_0=\gamma_1-\gamma_0=...=\gamma_k-\gamma_0>\max\{\gamma_j-\gamma_0,j=k+1,\dots,p\}$, where $1<k<p$. In this case, $\rho=\rho_0$. Take $A_t(s)=A_0(s)$. Without loss of generality, we assume $A_0(s)\sim c_0s^{\rho_0}$ for some $c_0\neq 0$ as $s\to\infty$. Then $a_j(\cdot)/a_0(\cdot)\in RV(\gamma_j-\gamma_0)$ for $j=1,\dots,p$, and $A_t(\cdot)\in RV(\rho_0)$. Hence $a_j(s)/\{a_0(s)A_0(s)\}\to c_j/c_0$ for $j=1,\dots,k$ and $a_j(s)/(a_0(s)A_0(s))\to 0$ for $j=k+1,\dots,p$. Consequently we have

$$\begin{aligned}
&\{A_t(s)\}^{-1}\Big\{\frac{U_t(su)-U_t(s)}{a_t(s)}-\frac{u^\gamma-1}{\gamma}\Big\}\\
&=\frac{1}{\rho_0}\Big(\frac{u^{\gamma_0+\rho_0}-1}{\gamma_0+\rho_0}-\frac{u^{\gamma_0}-1}{\gamma_0}\Big)\{1+o(1)\}+\sum_{j=1}^{p}\frac{\theta_j(su)-\theta_j(s)}{a_j(s)}\times\frac{a_j(s)y_{t-j}}{a_0(s)A_t(s)}\\
&\to\frac{1}{\rho}\Big(\frac{u^{\gamma+\rho}-1}{\gamma+\rho}-\frac{u^\gamma-1}{\gamma}\Big)+\sum_{j=1}^{k}\frac{c_jy_{t-j}}{c_0}\times\frac{u^{\gamma+\rho}-1}{\gamma+\rho}\\
&=\Big(\frac{1}{\rho}+\sum_{j=1}^{k}\frac{c_jy_{t-j}}{c_0}\Big)\frac{u^{\gamma+\rho}-1}{\gamma+\rho}-\frac{1}{\rho}\left(\frac{u^\gamma-1}{\gamma}\right).
\end{aligned}$$

Let $A_t^*(s)=\Big(1/\rho+\sum_{j=1}^k c_jy_{t-j}/c_0\Big)\rho A_t(s)$ and $a_t^*(s)=\{1+A_t^*(s)/\rho\}\{1-A_t(s)/\rho\}a_t(s)$. It is easy to show that $U_t(\cdot)$ satisfies (2.3) with $A_t(\cdot)$ replaced by $A_t^*(\cdot)$, and $a_t(\cdot)$ replaced by $a_t^*(\cdot)$.

□

Proof of Proposition 2.2.

By the second-order condition of θ_j and its differentiability at the tail, it follows that, as $s\to\infty$,

$$\{A_j(s)\}^{-1}\Bigg(\frac{\frac{\partial\theta_j(su)}{\partial u}}{a_j(s)}-u^{\gamma_j-1}\Bigg)\to\frac{1}{\rho_j}\big(u^{\gamma_j+\rho_j-1}-u^{\gamma_j-1}\big)\tag{S.3}$$

Similar to the proof of Proposition 2.1, we can show that

$$\{A_t(s)\}^{-1}\Big(\frac{U_t'(su)}{s^{-1}a_t(s)} - u^{\gamma-1}\Big) \to \frac{1}{\rho}\big(u^{\gamma+\rho-1} - u^{\gamma-1}\big), \tag{S.4}$$

where $U_t'(s) = \partial U_t(s)/\partial s$. Therefore, we have

$$U_t'(su) = u^{\gamma-1}\{s^{-1}a_t(s)\}[1 + \rho^{-1}A_t(s)\{u^\rho - 1 + o(1)\}]$$

and

$$U_t'(s) = \{s^{-1}a_t(s)\}\{1 + o(A_t(s))\}.$$

Recall that

$$f_t\{U_t(s)\} = s^{-2}\Big(\frac{\partial U_t(s)}{\partial s}\Big)^{-1}.$$

Hence

$$\frac{f_t\{U_t(su)\}}{f_t\{U_t(s)\}} = \frac{(su)^{-2}\{U_t'(su)\}^{-1}}{(s)^{-2}\{U_t'(s)\}^{-1}} \to u^{-1-\gamma}.$$

□

The following Lemma 1 is needed in the proof of Theorem 2.1.

LEMMA **1.** *Let $\bar{U}_t(s) = Q_t(1 - 1/s|uu_x)$, where $uu_x = E(\mathbf{x}_t) = (1, \mu_y, \ldots, \mu_y)^T$. Assume the conditions in Proposition 2.1 hold, then $\bar{U}_t$ satisfies the second-order condition (2.2) with parameters γ and ρ. That is, there exist functions $\bar{a}_t(\cdot) > 0$ and $\bar{A}_t(\cdot) \in RV(\rho)$ such that as $s \to \infty$*

$$\bar{A}_t(s)^{-1}\Big(\frac{\bar{U}_t(su) - \bar{U}_t(s)}{\bar{a}_t(s)} - \frac{u^\gamma - 1}{\gamma}\Big) \to \frac{1}{\rho}\Big(\frac{u^{\gamma+\rho} - 1}{\gamma+\rho} - \frac{u^\gamma - 1}{\gamma}\Big). \tag{S.5}$$

Further, functions $\bar{a}_t(\cdot)$ and $\bar{A}_t(\cdot)$ can be written as

$$\bar{a}_t(s) \sim c(\mu_y, \ldots, \mu_y)\tilde{a}(s), \text{ and } \bar{A}_t(s) \sim d(\mu_y, \ldots, \mu_y)\tilde{A}(s),$$

where functions $\tilde{a}(\cdot) > 0$ and $\tilde{A}(\cdot) \in RV(\rho)$ are defined as in Proposition 2.1.

Proof. Lemma 1 follows directly from Proposition 2.1 as a special case. □

Proof of Theorem 2.1.

Let $s = 1/(1-\tau)$. Recall $\theta_j(s) = \beta_j(1 - 1/s)$ for $j = 0, \ldots, p$, and $U_t(s) = Q_t(1 - 1/s|\mathbf{x}_t) = \mathbf{x}_t^T\boldsymbol{\beta}(\tau)$. Define $\bar{U}_t(s) = Q_t(1 - 1/s|\boldsymbol{\mu}_x) = \boldsymbol{\mu}_x^T\boldsymbol{\beta}(\tau)$, where $\boldsymbol{\mu}_x = E(\mathbf{x}_t) =$

$(1, \mu_y, \ldots, \mu_y)^T$. Then $\tilde{a}_n(\tau) = (n/s)^{1/2}\{\bar{U}_t(s) - \bar{U}_t(\hbar s)\}^{-1}$. For $\gamma > 0$, $\bar{U}_t \in RV(\gamma)$, so that $\tilde{a}_n(\tau) \sim \sqrt{n(1-\tau)}\{(1-\hbar^\gamma)\bar{U}_t(1/(1-\tau))\}^{-1}$. In the case of $\gamma < 0$, $\bar{U}_t(\infty) - \bar{U}_t(\cdot) \in RV(\gamma)$ and thus $\bar{U}_t(s) - \bar{U}_t(\hbar s) \sim (\hbar^\gamma - 1)\{\bar{U}_t(\infty) - \bar{U}_t(s)\}$ and hence $\tilde{a}_n(\tau) \sim \sqrt{n(1-\tau)}[(\hbar^\gamma - 1)\{\bar{U}_t(\infty) - \bar{U}_t(1/(1-\tau))\}]^{-1}$. So $\tilde{a}_n(\tau) \sim \sqrt{n(1-\tau)}(1-\tau)^\gamma$ and hence by assumption A4, $\tilde{a}_n(\tau) \to \infty$.

In addition, define $\epsilon_t(\tau) = y_t - \mathbf{x}_t^T\boldsymbol{\beta}(\tau)$ and

$$\mathcal{M}_n(\mathbf{z}, \tau) = \frac{\tilde{a}_n(\tau)}{\sqrt{n(1-\tau)}} \sum_{t=1}^n \left[\rho_\tau\left\{\epsilon_t(\tau) - \mathbf{x}_t^T\mathbf{z}/\tilde{a}_n(\tau)\right\} - \rho_\tau\{\epsilon_t(\tau)\}\right].$$

First, we would like to show that for a fixed $\tau \in \mathcal{T}$ and $\mathbf{z} \in \mathbb{R}^p$ such that $\|\mathbf{z}\| = O(\sqrt{\log n})$,

$$\mathcal{M}_n(\mathbf{z}, \tau) + W_n(\tau) - \frac{1}{2}\left(\frac{1-\hbar^\gamma}{\gamma}\right)\mathbf{z}^T\mathcal{Q}_H\mathbf{z} \xrightarrow{p} 0. \tag{S.6}$$

By the identity $\rho_\tau(u-v) - \rho_\tau(u) = -v\psi_\tau(u) + \int_0^v\{I(u \le s) - I(u \le 0)\}ds$ (Knight, 1998), we can write

$$\mathcal{M}_n(\mathbf{z}, \tau) = -W_n(\tau)^T\mathbf{z} + G_n(\mathbf{z}, \tau), \tag{S.7}$$

where

$$G_n(\mathbf{z}, \tau) = \frac{\tilde{a}_n(\tau)}{\sqrt{n(1-\tau)}}\sum_{t=1}^n \int_0^{\mathbf{x}_t^T\mathbf{z}/\tilde{a}_n(\tau)} \left[I\{\epsilon_t(\tau) \le v\} - I\{\epsilon_t(\tau) \le 0\}\right] dv.$$

Note that

$$\begin{aligned} E\{G_n(\mathbf{z}, \tau)\} &= n\tilde{a}_n(\tau)E\left[\int_0^{\mathbf{x}_t^T\mathbf{z}/\tilde{a}_n(\tau)} \frac{I\{\epsilon_t(\tau) \le v\} - I\{\epsilon_t(\tau) \le 0\}}{\sqrt{n(1-\tau)}}\right] dv \\ &= n\tilde{a}_n(\tau)E\left[\int_0^{\mathbf{x}_t^T\mathbf{z}} \frac{I\{\epsilon_t(\tau) \le v/\tilde{a}_n(\tau)\} - I\{\epsilon_t(\tau) \le 0\}}{\tilde{a}_n(\tau)\sqrt{n(1-\tau)}}\right] dv \\ &= nE\left[\int_0^{\mathbf{x}_t^T\mathbf{z}} \frac{F_t\{\mathbf{x}_t^T\boldsymbol{\beta}(\tau) + v/\tilde{a}_n(\tau)\} - F_t\{\mathbf{x}_t^T\boldsymbol{\beta}(\tau)\}}{\sqrt{n(1-\tau)}}\right] dv. \end{aligned} \tag{S.8}$$

By the definition of $\tilde{a}_n(\tau)$, we have

$$\frac{v}{\tilde{a}_n(\tau)} = \frac{v\{Q_t(\tau|\boldsymbol{\mu}_x) - Q_t(1-(1-\tau)/\hbar|\boldsymbol{\mu}_x)\}}{\sqrt{n(1-\tau)}} = o\left(\bar{U}_t(s) - \bar{U}_t(\hbar s)\right), \tag{S.9}$$

where

$$\begin{aligned} \bar{U}_t(s) - \bar{U}_t(\hbar s) &= \{\theta_0(s) - \theta_0(\hbar s)\} + \sum_{j=1}^p\{\theta_j(s) - \theta_j(\hbar s)\}\mu_y \\ &= \{U_t(s) - U_t(\hbar s)\} + \sum_{j=1}^p\{\theta_j(s) - \theta_j(\hbar s)\}(\mu_y - y_{t-j}). \end{aligned} \tag{S.10}$$

We consider two different cases. **Case 1**: if $\gamma_0 > \gamma_j$ for $j = 1, \ldots, p$, then the second term of (S.10) will be dominated by the first term, thus

$$\bar{U}_t(s) - \bar{U}_t(\hbar s) = \{U_t(s) - U_t(\hbar s)\} + o\left(U_t(s) - U_t(\hbar s)\right). \tag{S.11}$$

Case 2: if $\max_{j=1,\ldots,p} \gamma_j > \gamma_0$, then the second term of (S.10) has the same order with the first term, so

$$\bar{U}_t(s) - \bar{U}_t(\hbar s) = O\left(U_t(s) - U_t(\hbar s)\right). \tag{S.12}$$

By assumption A3, combining (S.12) with (S.11) and (S.9) gives

$$F_t\left\{\mathbf{x}_t^T\boldsymbol{\beta}(\tau) + \frac{v}{\tilde{a}_n(\tau)}\right\} - F_t\{\mathbf{x}_t^T\boldsymbol{\beta}(\tau)\} = \frac{v}{\tilde{a}_n(\tau)} f_t\left\{U_t(s) + o(U_t(s) - U_t(\hbar s))\right\}. \tag{S.13}$$

Let $\Delta_s = o(U_t(\hbar s) - U_t(s))$. By the fact that $U_t(s) \to \infty$ when $\gamma > 0$ and $U_t(s) \to$ a constant when $\gamma < 0$, there exist two sequences $l_{1s} \uparrow 1$ and $l_{2s} \downarrow 1$ such that $U_t(l_{1s}s) \leq U_t(s) + \Delta_s \leq U_t(l_{2s}s)$. On the other hand, by Proposition 2.2,

$$\frac{f_t\{U_t(\hbar s)\}}{f_t\{U_t(s)\}} \to \hbar^{-1-\gamma} \text{ uniformly for } m \text{ in any bounded interval.}$$

Thus, we have $f_t\{U_t(s) + \Delta_s\} \sim f_t\{U_t(s)\}$, which together with (S.13) gives

$$F_t\left\{\mathbf{x}_t^T\boldsymbol{\beta}(\tau) + \frac{v}{\tilde{a}_n(\tau)}\right\} - F_t\{\mathbf{x}_t^T\boldsymbol{\beta}(\tau)\} \sim \frac{v}{\tilde{a}_n(\tau)} f_t\{U_t(s)\}. \tag{S.14}$$

Therefore,

$$\begin{aligned}
E\{G_n(\mathbf{z},\tau)\} &\sim nE\left[\int_0^{\mathbf{x}_t^T\mathbf{z}} \frac{v}{\tilde{a}_n(\tau)\sqrt{n(1-\tau)}} f_t\{\mathbf{x}_t^T\boldsymbol{\beta}(\tau)\}dv\right] \\
&= nE\left[\frac{1}{2}(\mathbf{x}_t^T\mathbf{z})^2 \frac{f_t\{U_t(s)\}}{\tilde{a}_n(\tau)\sqrt{n(1-\tau)}}\right] \\
&= E\left[\frac{1}{2}(\mathbf{x}_t^T\mathbf{z})^2 \frac{s\{\bar{U}_t(s) - \bar{U}_t(\hbar s)\}}{\{f_t(U_t(s))\}^{-1}}\right]. \qquad (S.15)
\end{aligned}$$

Since $F_t\{U_t(s)\} = 1 - 1/s$, we get $f_t\{U_t(s)\}U_t'(s) = s^{-2}$. Similarly, $\bar{f}_t\{\bar{U}_t(s)\}\bar{U}_t'(s) = s^{-2}$, where $\bar{f}_t(\cdot) = f_t(\cdot|\boldsymbol{\mu}_x)$. Therefore,

$$
\begin{aligned}
\frac{s\{\bar{U}_t(s) - \bar{U}_t(\hbar s)\}}{\{f_t(U_t(s))\}^{-1}} &= \frac{s}{\{f_t(U_t(s))\}^{-1}} \int_{\hbar s}^{s} \frac{1}{v^2 \bar{f}_t\{\bar{U}_t(v)\}} dv \\
&= \frac{s}{\{f_t(U_t(s))\}^{-1}} \int_{\hbar}^{1} \frac{1}{w^2 s \bar{f}_t\{\bar{U}_t(ws)\}} dw \\
&= \int_{\hbar}^{1} \frac{1}{w^2} \frac{\{\bar{f}_t(\bar{U}_t(ws))\}^{-1}}{\{f_t(U_t(s))\}^{-1}} dw \\
&= \int_{\hbar}^{1} \frac{\bar{U}_t'(ws)}{U_t'(s)} dw. \qquad \text{(S.16)}
\end{aligned}
$$

By (S.4), we have $U_t'(s) = s^{-1}a_t(s)\{1+o(A_t(s))\}$. Similarly, by Lemma 1, we have $\bar{U}_t'(ws) = w^{\gamma-1}s^{-1}\bar{a}_t(s)\{1 + \rho^{-1}\bar{A}_t(s)\{w^\rho - 1 + o(1)\}\}$. Therefore,

$$
\frac{\bar{U}_t'(ws)}{U_t'(s)} = w^{\gamma-1}\frac{\bar{a}_t(s)}{a_t(s)} \times \frac{1 + \rho^{-1}\bar{A}_t(s)\{w^\rho - 1 + o(1)\}}{1 + o(A_t(s))}. \qquad \text{(S.17)}
$$

Note that $\bar{A}_t(s) \to 0$ and $A_t(s) \to 0$. Then the last term of (S.17) converges to 1. Note that $\bar{a}_t(s)/a_t(s)$ is some function of μ_y and $\mathbf{x}_t$. Define $H(\mathbf{x}_t) = \lim_{s\to\infty} \bar{a}_t(s)/a_t(s)$. Then

$$
\frac{\bar{U}_t'(ws)}{U_t'(s)} \sim \int_{\hbar}^{1} w^{\gamma-1} H(\mathbf{x}_t) dw = H(\mathbf{x}_t)\frac{1-\hbar^\gamma}{\gamma},
$$

which together with (S.15) and (S.16) gives

$$
E[G_n(\mathbf{z},\tau)] \sim E\left[\frac{1}{2}(\mathbf{x}_t^T\mathbf{z})^2 \frac{1-\hbar^\gamma}{\gamma} H(\mathbf{x}_t)\right].
$$

Similar to the proof of Lemma 9.6 (ii) in Chernozhukov (2005), we can show that $\mathrm{Var}[G_n(\mathbf{z},\tau)] \to 0$ as $n \to \infty$, $\tau \to 0$ and $n(1-\tau) \to \infty$. Therefore, we get

$$
G_n(\mathbf{z},\tau) = \frac{1}{2}\left(\frac{1-\hbar^\gamma}{\gamma}\right)\mathbf{z}^T \mathcal{Q}_H \mathbf{z} + o_p(1). \qquad \text{(S.18)}
$$

and this proves (S.6).

By going through the similar chaining arguments as in the proof of Lemma 3.1 of Gutenbrunner et al. (1993), we can strengthen the convergence in (S.6) to uniformly over $\mathbf{z}$ and $\tau \in \mathcal{T}$. Since $\hat{\boldsymbol{\beta}}(\tau)$ minimizes $\sum_{t=1}^{n} \rho_\tau(y_t - \mathbf{x}_t^T\boldsymbol{\beta})$, then $\mathbf{Z}_n(\tau) = \tilde{a}_n(\tau)\{\hat{\boldsymbol{\beta}}(\tau) - \boldsymbol{\beta}(\tau)\}$ minimizes the convex function $\mathcal{M}_n(\mathbf{z},\tau)$ with respect to $\mathbf{z} \in \mathbb{R}^p$. Therefore, uniformly in $\tau \in \mathcal{T}$,

$$
\min_{\mathbf{z}=O(\sqrt{\log n})} \mathcal{M}_n(\mathbf{z},\tau) = \min_{\mathbf{z}=O(\sqrt{\log n})} \left\{-W_n(\tau)^T\mathbf{z} + \frac{1}{2}\left(\frac{1-\hbar^\gamma}{\gamma}\right)\mathbf{z}^T\mathcal{Q}_H\mathbf{z} + o_p(1)\right\}.
$$

At the tail quantiles, equation (1.10) of Shorack (1991) implies that $W_n(\tau)$ has the following probability bound

$$\sup_{\tau \in \mathcal{T}} W_n(\tau) \leq O_p(1) + C \sup_{\tau \in \mathcal{T}} \{(1-\tau)^{-1/2} W(\tau)\},$$

where C is some constant and $W(\tau)$ is a Brownian Bridge Gaussian process. By Shorack and Wellner (1986, p. 599), we have $W_n(\tau) = O_p(\sqrt{\log n})$ uniformly in $\tau \in \mathcal{T}$. Therefore, uniformly over $\tau \in \mathcal{T}$,

$$\operatorname*{argmin}_{\mathbf{z} \in \mathbb{R}^p} \left\{ -W_n(\tau)^T \mathbf{z} + \frac{1}{2} \left(\frac{1-\hbar^\gamma}{\gamma} \right) \mathbf{z}^T \mathcal{Q}_H \mathbf{z} \right\} = \frac{\gamma}{1-\hbar^\gamma} \mathcal{Q}_H^{-1} W_n(\tau) = O_p(\sqrt{\log n}).$$

The Bahadur representation in Theorem 2.1 can then be proven by following the similar arguments as in the proof of Theorem 3.1 of Gutenbrunner et al. (1993). □

Proof of Corollary 2.1.

Following the arguments in Portnoy (1984) and Gutenbrunner and Jurečková (1992), the quantile autoregressive process is tight. In addition, note that $E[\psi_\tau\{\epsilon_t(\tau)\}|\mathbf{x}_t\} = 0$, $\mathbf{x}_t \psi_\tau\{\epsilon_t(\tau)\}$ is a martingale difference sequence. Therefore, there exists a sequence of $(p+1)$-dimensional standard Brownian bridge $\tilde{B}_n(\tau)$ such that

$$\sqrt{\tau(1-\tau)} W_n(\tau) = \Omega_0^{-1/2} \tilde{B}_n(\tau)\{1 + o_p(1)\}.$$

Note that as $\tau = 1 - ku/n \to 1$,

$$\tilde{B}_n(1 - ku/n) \stackrel{d}{=} \tilde{B}_n(ku/n) \stackrel{d}{=} B_n(ku/n) - (ku/n) B_n(1),$$

where B_n is a sequence of $(p+1)$-dimensional standard Brownian motions. Thus

$$\begin{aligned} W_n(1 - ku/n) &\stackrel{d}{=} (ku/n)^{-1/2} \Omega_0^{-1/2} \{B_n(ku/n) - (ku/n) B_n(1)\}\{1 + o_p(1)\} \\ &\stackrel{d}{=} \Omega_0^{-1/2} u^{-1/2} B_n(u) + o_p(1). \end{aligned}$$

Without loss of generality, we assume W_n and B_n are on the same probability space. Then by Theorem 2.1 the statement of Corollary 2.1 follows. □

Proof of Theorem 2.2.

Let $q_j = \mathbf{x}_t^T \boldsymbol{\beta}(\tau_j)$ denote the true conditional quantile at the quantile level $\tau_j = j/n$ for $j = n-k, \ldots, n-k'$. Note that, for $j = k', ..., k$,

$$\begin{aligned}
\log \frac{\hat{q}_{n-j}}{\hat{q}_{n-k}} &= \log \frac{q_{n-j}(1 + \frac{\hat{q}_{n-j} - q_{n-j}}{q_{n-j}})}{q_{n-k}(1 + \frac{\hat{q}_{n-k} - q_{n-k}}{q_{n-k}})} \\
&= \log \frac{q_{n-j}}{q_{n-k}} + \frac{\hat{q}_{n-j} - q_{n-j}}{q_{n-j}}\{1 + o_p(1)\} - \frac{\hat{q}_{n-k} - q_{n-k}}{q_{n-k}}\{1 + o_p(1)\} \\
&\doteq E_{1j} + E_{2j}\{1 + o_p(1)\} - E_{3k}\{1 + o_p(1)\},
\end{aligned}$$

and that

$$\begin{aligned}
\Big(\log \frac{\hat{q}_{n-j}}{\hat{q}_{n-k}}\Big)^2 &= \Big(\log \frac{q_{n-j}}{q_{n-k}}\Big)^2 + \Big\{\Big(\frac{\hat{q}_{n-j} - q_{n-j}}{q_{n-j}}\Big)^2 + \Big(\frac{\hat{q}_{n-k} - q_{n-k}}{q_{n-k}}\Big)^2 \\
&\quad + 2\Big(\log \frac{q_{n-j}}{q_{n-k}} \times \frac{\hat{q}_{n-j} - q_{n-j}}{q_{n-j}}\Big) - 2\Big(\log \frac{q_{n-j}}{q_{n-k}} \times \frac{\hat{q}_{n-k} - q_{n-k}}{q_{n-k}}\Big) \\
&\quad - 2\Big(\frac{\hat{q}_{n-j} - q_{n-j}}{q_{n-j}} \times \frac{\hat{q}_{n-k} - q_{n-k}}{q_{n-k}}\Big)\Big\}\{1 + o_p(1)\} \\
&= E_{1j}^2 + \big(E_{2j}^2 + E_{3k}^2 + 2E_{1j}E_{2j} - 2E_{1j}E_{3k} - 2E_{2j}E_{3k}\big)\{1 + o_p(1)\}.
\end{aligned}$$

Therefore, we have

$$M_n^{(1)} = \frac{1}{k-k'} \sum_{j=k'}^{k} \big(E_{1j} + E_{2j}\{1 + o_p(1)\} - E_{3k}\{1 + o_p(1)\}\big), \tag{S.19}$$

$$M_n^{(2)} = \frac{1}{k-k'} \sum_{j=k'}^{k} \big[E_{1j}^2 + \big(E_{2j}^2 + E_{3k}^2 + 2E_{1j}E_{2j} - 2E_{1j}E_{3k} - 2E_{2j}E_{3k}\big)\{1 + o_p(1)\}\big]. \tag{S.20}$$

We will approximate E_{1j}, E_{2j} and E_{3k} separately. By (3.5.11) in de Haan and Ferreira (2006, p. 103), it follows that

$$E_{1j} = \log \frac{q_{n-j}}{q_{n-k}} = \tilde{q}_t(n/k)\Big(\frac{(k/j)^{\gamma_-} - 1}{\gamma_-} + \lambda_t(n/k) H_{\gamma_-, \rho'}(k/j)\{1 + o(1)\}\Big),$$

where $\gamma_- = \min\{0, \gamma\}$, $\tilde{q}_t(s) = a_t(s)/U_t(s)$, λ_t is defined as (2.10),

$$\rho' = \begin{cases} \rho, & \gamma < \rho \leq 0, \\ \gamma, & \rho < \gamma \leq 0, \\ -\gamma, & (0 < \gamma < -\rho \text{ and } l \neq 0), \\ \rho, & (0 < \gamma < -\rho \text{ and } l = 0) \text{ or } \gamma \geq -\rho > 0, \end{cases}$$

(for more details on ρ', see page 398 in de Haan and Ferreira (2006)) and

$$H_{\gamma,\rho}(u) = \frac{1}{\rho}\left(\frac{u^{\gamma+\rho}-1}{\gamma+\rho} - \frac{u^{\gamma}-1}{\gamma}\right).$$

Note that $\tilde{q}_t(s) \to \gamma_+ \doteq \max\{0,\gamma\}$ (see (3.5.5) in de Haan and Ferreria, 2006, p. 101) and $\lambda_t(s) \to 0$ as $s \to \infty$.

By Corollary 1 in the supplementary file (i.e. $W_n(\tau_{n-j}) = \Omega_0^{-1/2}(j/k)^{-1/2}B_n(j/k)\{1+o_p(1)\}$), we have

$$\begin{aligned}
E_{2j} = \frac{\hat{q}_{n-j}-q_{n-j}}{q_{n-j}} &= \frac{\tilde{a}_n(\tau_{n-j})(\hat{q}_{n-j}-q_{n-j})}{\tilde{a}_n(\tau_{n-j})q_{n-j}} \\
&= \left\{\tilde{a}_n(\tau_{n-j})q_{n-j}\right\}^{-1}\Big(\frac{\gamma}{1-\hbar^{\gamma}}\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}(j/k)^{-1/2}B_n(j/k)\Big)\{1+o_p(1)\}
\end{aligned}$$

and

$$E_{3k} = \frac{\hat{q}_{n-k}-q_{n-k}}{q_{n-k}} = \big(\tilde{a}_n(\tau_{n-k})q_{n-k}\big)^{-1}\Big(\frac{\gamma}{1-\hbar^{\gamma}}\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}B_n(1)\Big)\{1+o_p(1)\}.$$

Hence

$$E_{1j}^2 = \tilde{q}_t(n/k)^2\left(\left\{\frac{(k/j)^{\gamma_-}-1}{\gamma_-}\right\}^2 + 2\left\{\frac{(k/j)^{\gamma_-}-1}{\gamma_-}\right\}\lambda_t(n/k)H_{\gamma_-,\rho'}(k/j)\{1+o(1)\}\right),$$

$$\begin{aligned}
E_{2j}^2 &= \left\{\tilde{a}_n(\tau_{n-j})q_{n-j}^0\right\}^{-2}\Big(\frac{\gamma}{1-\hbar^{\gamma}}\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}(j/k)^{-1/2}B_n(j/k)\Big)^2\{1+o_p(1)\}, \\
E_{3k}^2 &= \left\{\tilde{a}_n(\tau_{n-k})q_{n-k}^0\right\}^{-2}\Big(\frac{\gamma}{1-\hbar^{\gamma}}\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}B_n(1)\Big)^2\{1+o_p(1)\}, \\
E_{1j}E_{2j} &= \tilde{q}_t(n/k)\Big(\frac{(k/j)^{\gamma_-}-1}{\gamma_-} + \lambda_t(n/k)H_{\gamma_-,\rho'}(k/j)\{1+o(1)\}\Big) \\
&\quad \times \left\{\tilde{a}_n(\tau_{n-j})q_{n-j}\right\}^{-1}\Big(\frac{\gamma}{1-\hbar^{\gamma}}\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}(j/k)^{-1/2}B_n(j/k)\Big)\{1+o_p(1)\}, \\
E_{1j}E_{3k} &= \tilde{q}_t(n/k)\Big(\frac{(k/j)^{\gamma_-}-1}{\gamma_-} + \lambda_t(n/k)H_{\gamma_-,\rho'}(k/j)\{1+o(1)\}\Big) \\
&\quad \times \left\{\tilde{a}_n(\tau_{n-k})q_{n-k}\right\}^{-1}\Big(\frac{\gamma}{1-\hbar^{\gamma}}\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}B_n(1)\Big)\{1+o_p(1)\}, \\
E_{2j}E_{3k} &= \left\{\tilde{a}_n(\tau_{n-j})q_{n-j}\right\}^{-1}\Big(\frac{\gamma}{1-\hbar^{\gamma}}\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}(j/k)^{-1/2}B_n(j/k)\Big) \\
&\quad \times \left\{\tilde{a}_n(\tau_{n-k})q_{n-k}\right\}^{-1}\Big(\frac{\gamma}{1-\hbar^{\gamma}}\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}B_n(1)\Big)\{1+o_p(1)\}.
\end{aligned}$$

Recall that $\tilde{a}_n(\tau) = \sqrt{n(1-\tau)}\{\bar{U}_t(1/(1-\tau)) - \bar{U}_t(\hbar/(1-\tau))\}^{-1}$. For $\gamma > 0$, $\bar{U}_t \in RV(\gamma)$, so that $\tilde{a}_n(\tau) \sim \sqrt{n(1-\tau)}\{(1-\hbar^{\gamma})\bar{U}_t(1/(1-\tau))\}^{-1}$. In the case of $\gamma < 0$, $\bar{U}_t(\infty) -$

$\bar{U}_t(\cdot) \in RV(\gamma)$ and thus $\bar{U}_t(s) - \bar{U}_t(\hbar s) \sim (\hbar^\gamma - 1)\{\bar{U}_t(\infty) - \bar{U}_t(s)\}$ and hence $\tilde{a}_n(\tau) \sim \sqrt{n(1-\tau)}[(\hbar^\gamma - 1)\{\bar{U}_t(\infty) - \bar{U}_t(1/(1-\tau))\}]^{-1}$.

Next we establish the asymptotic representations of $\hat{\gamma}, \hat{a}_t(n/k)$ and $\hat{b}_t(n/k)$ separately by considering two different cases: $\gamma > 0$ and $\gamma < 0$.

(I) The asymptotic representation of $\hat{\gamma}$.

Case 1: $\gamma > 0$. In this case, $\gamma_- = 0$, $\gamma_+ = \gamma$, $\tilde{q}_t(s) \to \gamma$ as $s \to \infty$ and $\tilde{a}_n(\tau) \sim \sqrt{n(1-\tau)}\{(1-\hbar^\gamma)\bar{U}_t(1/(1-\tau))\}^{-1}$. Then,

$$E_{1j} = \gamma\Big(- \log(j/k) + \lambda_t(n/k) H_{0,\rho'}\{(j/k)^{-1}\}\Big)\{1 + o(1)\},$$

$$\begin{aligned} E_{2j} &= \Big(\frac{\sqrt{n(1-\tau_j)}q_{n-j}}{(1-\hbar^\gamma)\bar{U}_t\{1/(1-\tau_j)\}}\Big)^{-1}\Big(\frac{\gamma}{1-\hbar^\gamma}\mathbf{x}_t^T \mathcal{Q}_H^{-1}\Omega_0^{-1/2}(j/k)^{-1/2}B_n(j/k)\Big)(1+o_p(1)) \\ &= \gamma\sqrt{\frac{1}{k}}\Big(\frac{j}{k}\Big)^{-1/2} H(\mathbf{x}_t)\mathbf{x}_t^T \mathcal{Q}_H^{-1}\Omega_0^{-1/2}(j/k)^{-1/2}B_n(j/k)\{1+o_p(1)\}, \end{aligned}$$

since $\bar{U}_t(s)/U_t(s) \sim \bar{a}_t(s)/a_t(s) \to H(\mathbf{x}_t)$ as $s \to \infty$, and

$$E_{3k} = \gamma\sqrt{\frac{1}{k}} H(\mathbf{x}_t)\mathbf{x}_t^T \mathcal{Q}_H^{-1}\Omega_0^{-1/2}B_n(1)\{1+o_p(1)\}. \tag{S.21}$$

Hence, by (S.19), we have

$$\begin{aligned} M_n^{(1)} &= \gamma\left[\int_0^1 \Big(- \log u + \lambda_t(n/k)H_{0,\rho'}(u^{-1})\Big)du\right]\{1+o(1)\} \\ &\quad + \gamma k^{-1/2}H(\mathbf{x}_t)\mathbf{x}_t^T \mathcal{Q}_H^{-1}\Omega_0^{-1/2}\left[\int_0^1 \{u^{-1}B_n(u) - B_n(1)\}du\right]\{1+o_p(1)\} \\ &= \gamma\left[1 + \lambda_t(n/k)\int_0^1 H_{0,\rho'}(u^{-1})du + k^{-1/2}H(\mathbf{x}_t)\mathbf{x}_t^T \mathcal{Q}_H^{-1}\Omega_0^{-1/2}I_1(B_n)\right] + \Delta_n, \end{aligned} \tag{S.22}$$

where $I_1(B_n) = \int_0^1\{u^{-1}B_n(u) - B_n(1)\}du$ and $\Delta_n = o_p(k^{-1/2} \vee |\lambda_t(n/k)|)$.

Similarly, we have

$$E_{1j}^2 = \gamma^2\Big(\{-\log(j/k)\}^2 - 2\log(j/k)\lambda_t(n/k)H_{0,\rho'}\{(j/k)^{-1}\}\Big)\{1+o(1)\},$$

$$E_{2j}^2 = \gamma^2\frac{1}{k}\Big(\frac{j}{k}\Big)^{-2}\{H(\mathbf{x}_t)\mathbf{x}_t^T \mathcal{Q}_H^{-1}\Omega_0^{-1/2}B_n(j/k)\}^2\{1+o_p(1)\},$$

$$E_{3k}^2 = \gamma^2\frac{1}{k}\{H(\mathbf{x}_t)\mathbf{x}_t^T \mathcal{Q}_H^{-1}\Omega_0^{-1/2}B_n(1)\}^2\{1+o_p(1)\},$$

$$\begin{aligned} E_{1j}E_{2j} &= \gamma^2 k^{-1/2}\Big(- \log(j/k) + \lambda_t(n/k)H_{0,\rho'}\{(j/k)^{-1}\}\Big) \\ &\quad \times \{H(\mathbf{x}_t)\mathbf{x}_t^T \mathcal{Q}_H^{-1}\Omega_0^{-1/2}(j/k)^{-1}B_n(j/k)\}\{1+o_p(1)\}, \end{aligned}$$

$$
\begin{aligned}
E_{1j}E_{3k} &= \gamma^2 k^{-1/2}\Big(-\log(j/k)+\lambda_t(n/k)H_{0,\rho'}\{(j/k)^{-1}\}\Big)\\
&\quad\times\{H(\mathbf{x}_t)\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}B_n(1)\}\{1+o_p(1)\},\\
E_{2j}E_{3k} &= \gamma^2 k^{-1}\{H(\mathbf{x}_t)\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}(j/k)^{-1}B_n(j/k)\}\\
&\quad\times\{H(\mathbf{x}_t)\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}B_n(1)\}\{1+o_p(1)\}.
\end{aligned}
$$

Let $\varepsilon>0$ be some small constant. By (S.20), we have

$$
\begin{aligned}
&M_n^{(2)}\\
=\;& \gamma^2\left[\int_0^1\{(-\log u)^2+2\lambda_t(n/k)(-\log u)H_{0,\rho'}(u^{-1})\}du\right]\{1+o(1)\}+o_p(k^{-1+\epsilon})+o_p(k^{-1})\\
&+2\gamma^2k^{-1/2}\Big[\int_0^1\{-\log u+\lambda_t(n/k)H_{0,\rho'}(u^{-1})\}\{H(\mathbf{x}_t)\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}u^{-1}B_n(u)\}du\\
&-\int_0^1\{-\log u+\lambda_t(n/k)H_{0,\rho'}(u^{-1})\}H(\mathbf{x}_t)\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}B_n(1)du\Big]\{1+o_p(1)\}+O_p(k^{-1})\\
=\;& 2\gamma^2\Big\{1+\lambda_t(n/k)\int_0^1(-\log u)H_{0,\rho'}(u^{-1})du+k^{-1/2}H(\mathbf{x}_t)\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}I_2(B_n)+\Delta_n\Big\},
\end{aligned}
$$

where $I_2(B_n)=\int_0^1(-\log u)\{u^{-1}B_n(u)-B_n(1)\}du$.

Then we have

$$
\begin{aligned}
\frac{\{M_n^{(1)}\}^2}{M_n^{(2)}} &= \frac{\{1+2\lambda_t(n/k)\int_0^1 H_{0,\rho'}(u^{-1})du+2k^{-1/2}H(\mathbf{x}_t)\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}I_1(B_n)+\Delta_n\}}{2\big\{1+\lambda_t(n/k)\int_0^1(-\log u)H_{0,\rho'}(u^{-1})du+k^{-1/2}H(\mathbf{x}_t)\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}I_2(B_n)+\Delta_n\big\}}\\
&=\frac{1}{2}\Big\{1+\lambda_t(n/k)\int_0^1(2+\log u)H_{0,\rho'}(u^{-1})du\\
&\qquad+k^{-1/2}H(\mathbf{x}_t)\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}\{2I_1(B_n)-I_2(B_n)\}+\Delta_n\Big\}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\Big(1-\frac{\{M_n^{(1)}\}^2}{M_n^{(2)}}\Big)^{-1} \;=\;& 2\Big\{1+\lambda_t(n/k)\int_0^1(2+\log u)H_{0,\rho'}(u^{-1})du\\
&+k^{-1/2}H(\mathbf{x}_t)\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}\{2I_1(B_n)-I_2(B_n)\}+\Delta_n\Big\}. \qquad \text{(S.23)}
\end{aligned}
$$

Plugging (S.22) and (S.23) to the definition of $\hat\gamma$ gives

$$
\begin{aligned}
\hat\gamma &= M_n^{(1)}+1-\frac{1}{2}\Big(1-\frac{\{M_n^{(1)}\}^2}{M_n^{(2)}}\Big)^{-1}\\
&=\gamma+\lambda_t(n/k)\int_0^1(\gamma-2-\log u)H_{0,\rho'}(u^{-1})du\\
&\quad+k^{-1/2}H(\mathbf{x}_t)\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}\{(\gamma-2)I_1(B_n)+I_2(B_n)\}+\Delta_n.
\end{aligned}
$$

Therefore,

$$\sqrt{k}(\hat{\gamma}-\gamma)=\Gamma+o_p(1),$$

where (for $\gamma>0$)

$$\Gamma=\lambda\int_0^1(\gamma-2-\log u)H_{0,\rho'}(u^{-1})du+H(\mathbf{x}_t)\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}\{(\gamma-2)I_1(B_n)+I_2(B_n)\}. \quad \text{(S.24)}$$

Case 2: $\gamma<0$. In this case, $\gamma_-=\gamma<0$, $\gamma_+=0$. Obviously,

$$E_{1j}=\tilde{q}_t(n/k)\Big(\frac{(j/k)^{-\gamma}-1}{\gamma}+\lambda_t(n/k)H_{\gamma,\rho'}\{(j/k)^{-1}\}\Big)\{1+o(1)\},$$

In addition, in this case $a_t(s)\sim-\gamma\{U_t(\infty)-U_t(s)\}$, $\bar{a}_t(s)\sim-\gamma\{\bar{U}_t(\infty)-\bar{U}_t(s)\}$, and $\tilde{q}_t(s)=a_t(s)/U_t(s)\in RV(\gamma)$. Thus

$$\begin{aligned}\tilde{a}_n(\tau)U_t(s)&=\frac{\sqrt{n(1-\tau)}U_t(s)}{(\hbar^\gamma-1)\{\bar{U}_t(\infty)-\bar{U}_t(s)\}}\\&\sim\frac{\sqrt{n(1-\tau)}}{\hbar^\gamma-1}\times(-\gamma)\frac{a_t(s)}{\bar{a}_t(s)}\times\frac{U_t(s)}{a_t(s)}\\&\sim\frac{\sqrt{n(1-\tau)}}{\hbar^\gamma-1}(-\gamma)\{H(\mathbf{x}_t)\}^{-1}\{\tilde{q}_t(s)\}^{-1},\ \text{where } s=1/(1-\tau).\end{aligned}$$

Then we have

$$\begin{aligned}E_{2j}&=\big\{\tilde{a}_n(\tau_{n-j})q_{n-j}\big\}^{-1}\Big(\frac{\gamma}{1-\hbar^\gamma}\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}(j/k)^{-1/2}B_n(j/k)\Big)\{1+o_p(1)\}\\&=k^{-1/2}\tilde{q}_t(n/k)H(\mathbf{x}_t)\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}(j/k)^{-\gamma-1}B_n(j/k)\{1+o_p(1)\},\end{aligned}$$

where the last step is due to $\tilde{q}_t(n/j)\sim\tilde{q}_t(n/k)(k/j)^\gamma$, and

$$E_{3k}=k^{-1/2}\tilde{q}_t(n/k)H(\mathbf{x}_t)\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}B_n(1)\{1+o_p(1)\}. \quad \text{(S.25)}$$

Denote $\Delta_n=o_p(k^{-1/2}\vee|\lambda_t(n/k)|)$. By (S.19),

$$\begin{aligned}\frac{M_n^{(1)}}{\tilde{q}_t(n/k)}&=\Big[\int_0^1\Big(\frac{u^{-\gamma}-1}{\gamma}+\lambda_t(n/k)H_{\gamma,\rho'}(u^{-1})\Big)du\Big]\{1+o(1)\}\\&\quad-k^{-1/2}H(\mathbf{x}_t)\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}\int_0^1\{u^{-\gamma-1}B_n(u)-B_n(1)\}du\{1+o_p(1)\}\\&=\frac{1}{1-\gamma}+\lambda_t(n/k)\int_0^1H_{\gamma,\rho'}(u^{-1})du\\&\quad-k^{-1/2}H(\mathbf{x}_t)\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}\int_0^1\{u^{-\gamma-1}B_n(u)-B_n(1)\}du+\Delta_n. \quad \text{(S.26)}\end{aligned}$$

Similarly, we have

$$E_{1j}^2 = \tilde{q}_t^2(n/k)\Big[\big(\frac{(j/k)^{-\gamma}-1}{\gamma}\big)^2 + 2\big(\frac{(j/k)^{-\gamma}-1}{\gamma}\big)\lambda_t(n/k)H_{\gamma,\rho'}\{(j/k)^{-1}\}\Big]\{1+o(1)\},$$

$$E_{2j}^2 = k^{-1}\tilde{q}_t^2(n/k)\big\{H(\mathbf{x}_t)\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}(j/k)^{-\gamma-1}B_n(j/k)\big\}^2\{1+o_p(1)\},$$

$$E_{3k}^2 = k^{-1}\tilde{q}_t^2(n/k)\big\{H(\mathbf{x}_t)\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}B_n(1)\big\}^2\{1+o_p(1)\},$$

$$\begin{aligned}E_{1j}E_{2j} &= k^{-1/2}\tilde{q}_t^2(n/k)\Big\{\frac{(j/k)^{-\gamma}-1}{\gamma} + \lambda_t(n/k)H_{\gamma,\rho'}((j/k)^{-1})\Big\}\\ &\quad\times\big\{H(\mathbf{x}_t)\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}(j/k)^{-\gamma-1}B_n(j/k)\big\}\{1+o_p(1)\},\end{aligned}$$

$$\begin{aligned}E_{1j}E_{3k} &= k^{-1/2}\tilde{q}_t^2(n/k)\Big\{\frac{(j/k)^{-\gamma}-1}{\gamma} + \lambda_t(n/k)H_{\gamma,\rho'}((j/k)^{-1})\Big\}\\ &\quad\times\big\{H(\mathbf{x}_t)\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}B_n(1)\big\}\{1+o_p(1)\},\end{aligned}$$

$$\begin{aligned}E_{2j}E_{3k} &= k^{-1}\tilde{q}_t^2(n/k)\big\{H(\mathbf{x}_t)\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}(j/k)^{-\gamma-1}B_n(j/k)\big\}\\ &\quad\times\big\{H(\mathbf{x}_t)\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}B_n(1)\big\}\{1+o_p(1)\}.\end{aligned}$$

By (S.20),

$$\begin{aligned}\frac{M_n^{(2)}}{\tilde{q}_t^2(n/k)} &= \int_0^1 \Big\{\big(\frac{u^{-\gamma}-1}{\gamma}\big)^2 + 2\big(\frac{u^{-\gamma}-1}{\gamma}\big)\lambda_t(n/k)H_{\gamma,\rho'}(u^{-1})\Big\}du\{1+o(1)\}\\ &\quad + 2k^{-1/2}\int_0^1\Big\{\frac{u^{-\gamma}-1}{\gamma} + \lambda_t(n/k)H_{\gamma,\rho'}((j/k)^{-1})\Big\}\\ &\qquad\times\big\{H(\mathbf{x}_t)\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}u^{-\gamma-1}B_n(u)\big\}du\{1+o_p(1)\}\\ &\quad - 2k^{-1/2}\int_0^1\Big\{\frac{u^{-\gamma}-1}{\gamma} + \lambda_t(n/k)H_{\gamma,\rho'}((j/k)^{-1})\Big\}\\ &\qquad\times\big\{H(\mathbf{x}_t)\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}B_n(1)\big\}du\{1+o_p(1)\} + O_p(k^{-1})\\ &= \frac{2}{(1-\gamma)(1-2\gamma)} + 2\lambda_t(n/k)\int_0^1\big(\frac{u^{-\gamma}-1}{\gamma}\big)H_{\gamma,\rho'}(u^{-1})du\\ &\quad + 2k^{-1/2}\{H(\mathbf{x}_t)\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}\}\int_0^1\big(\frac{u^{-\gamma}-1}{\gamma}\big)\{u^{-\gamma-1}B_n(u)-B_n(1)\}du + \Delta_n.\end{aligned}$$

Denote $I_3(B_n)=\int_0^1\left(\frac{u^{-\gamma}-1}{\gamma}\right)\{u^{-\gamma-1}B_n(u)-B_n(1)\}du$. We have

$$\begin{aligned}
&\frac{\{M_n^{(1)}\}^2}{M_n^{(2)}}\\
=&\ \frac{\left(\frac{1}{1-\gamma}\right)^2+\frac{2}{1-\gamma}[\lambda_t(n/k)\int_0^1 H_{\gamma,\rho'}(u^{-1})du-k^{-1/2}H(\mathbf{x}_t)\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}I_1(B_n)+\Delta_n]}{\frac{2}{(1-\gamma)(1-2\gamma)}+2\lambda_t(n/k)\int_0^1\left(\frac{u^{-\gamma}-1}{\gamma}\right)H_{\gamma,\rho'}(u^{-1})du+2k^{-1/2}H(\mathbf{x}_t)\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}I_3(B_n)+\Delta_n}\\
=&\ \frac{1-2\gamma}{2(1-\gamma)}\Big[1+(1-\gamma)\lambda_t(n/k)\int_0^1\{2-(1-2\gamma)\frac{u^{-\gamma}-1}{\gamma}\}H_{\gamma,\rho'}(u^{-1})du\\
&\qquad -(1-\gamma)k^{-1/2}H(\mathbf{x}_t)\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}\{2I_1(B_n)+(1-2\gamma)I_3(B_n)\}+\Delta_n\Big]
\end{aligned}$$

and

$$\begin{aligned}
&\Big(1-\frac{\{M_n^{(1)}\}^2}{M_n^{(2)}}\Big)^{-1}\\
=&\ 2(1-\gamma)\Big[1-(1-\gamma)(1-2\gamma)\lambda_t(n/k)\int_0^1\{2-(1-2\gamma)\frac{u^{-\gamma}-1}{\gamma}\}H_{\gamma,\rho'}(u^{-1})du\\
&+(1-\gamma)(1-2\gamma)k^{-1/2}H(\mathbf{x}_t)\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}\{2I_1(B_n)+(1-2\gamma)I_3(B_n)\}+\Delta_n\Big]. \qquad (S.27)
\end{aligned}$$

Denote $\tilde{\Delta}_n=o_p(k^{-1/2}\vee|\tilde{q}_t(n/k)|\vee|\lambda_t(n/k)|)$. Therefore,

$$\begin{aligned}
\hat{\gamma}\ =&\ M_n^{(1)}+1-\frac{1}{2}\Big(1-\frac{\{M_n^{(1)}\}^2}{M_n^{(2)}}\Big)^{-1}\\
=&\ \frac{\tilde{q}_t(n/k)}{1-\gamma}+\gamma+(1-\gamma)^2(1-2\gamma)\lambda_t(n/k)\int_0^1\{2-(1-2\gamma)\frac{u^{-\gamma}-1}{\gamma}\}H_{\gamma,\rho'}(u^{-1})du\\
&-(1-\gamma)^2(1-2\gamma)k^{-1/2}H(\mathbf{x}_t)\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}(2I_1(B_n)+(1-2\gamma)I_3(B_n))+\tilde{\Delta}_n.
\end{aligned}$$

Note that in the case of $\gamma<0$, $\tilde{q}_t(\cdot)\in RV(\gamma)$ and hence $\tilde{q}_t(s)\to 0$ as $s\to\infty$. For $\gamma<\rho<0$, we have $\lambda_t(\cdot)\in RV(\rho)$ and $\tilde{q}_t(s)=o(\lambda_t(s))$; for $\rho<\gamma<0$, we have $\tilde{q}_t(s)=-\lambda_t(s)$. Thus, as $\sqrt{k}\lambda_t(n/k)\to\lambda$, $\sqrt{k}(\hat{\gamma}-\gamma)=\Gamma+o_p(1)$, where (for $\gamma<0$)

$$\begin{aligned}
\Gamma\ =&-\tfrac{\lambda I\{\rho<\gamma<0\}}{1-\gamma}+\lambda(1-\gamma)^2(1-2\gamma)\int_0^1\{2-(1-2\gamma)\tfrac{u^{-\gamma}-1}{\gamma}\}H_{\gamma,\rho'}(u^{-1})du\\
&-(1-\gamma)^2(1-2\gamma)H(\mathbf{x}_t)\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}\{2I_1(B_n)+(1-2\gamma)I_3(B_n)\}. \qquad (S.28)
\end{aligned}$$

(II) The asymptotic representation of $\hat{a}_t(n/k)$. Define

$$a_{t0}(n/k)\doteq\begin{cases}\gamma U_t(n/k), & \text{if } \gamma>0,\\ -\gamma\{U_t(\infty)-U_t(n/k)\}, & \text{if } \gamma<0.\end{cases}$$

Since $a_t(n/k) \sim a_{t0}(n/k)$, without loss of generality, we assume $a_t(n/k) = a_{t0}(n/k)$. Otherwise, we approximate $\frac{\hat{a}_t(n/k)}{a_{t0}(n/k)} - 1$.

Case 1: $\gamma > 0$**.** By (S.21), (S.22) and (S.23), it follows that

$$
\begin{aligned}
&\frac{\hat{a}_t(n/k)}{a_t(n/k)} = \frac{\hat{q}_{n-k} M_n^{(1)} \frac{1}{2}\big(1 - \frac{\{M_n^{(1)}\}^2}{M_n^{(2)}}\big)^{-1}}{\gamma q_{n-k}} \\
= \;& \Big(\frac{\hat{q}_{n-k}}{q_{n-k}}\Big)\Big(\frac{M_n^{(1)}}{\gamma}\Big)\frac{1}{2}\Big(1 - \frac{\{M_n^{(1)}\}^2}{M_n^{(2)}}\Big)^{-1} \\
= \;& (1 + E_{3k})\{1 + \lambda_t(n/k)\int_0^1 H_{0,\rho'}(u^{-1})du + k^{-1/2} H(\mathbf{x}_t)\mathbf{x}_t^T \mathcal{Q}_H^{-1}\Omega_0^{-1/2} I_1(B_n) + \Delta_n\} \\
& \qquad \times\Big[1 + \lambda_t(n/k)\int_0^1 (2 + \log u) H_{0,\rho'}(u^{-1})du \\
& \qquad\qquad + k^{-1/2} H(\mathbf{x}_t)\mathbf{x}_t^T \mathcal{Q}_H^{-1}\Omega_0^{-1/2}\{2I_1(B_n) - I_2(B_n)\}\Big]\{1 + o_p(1)\} \\
= \;& 1 + \gamma k^{-1/2} H(\mathbf{x}_t)\mathbf{x}_t^T \mathcal{Q}_H^{-1}\Omega_0^{-1/2} B_n(1) \\
& + \lambda_t(n/k)\int_0^1 H_{0,\rho'}(u^{-1})du + k^{-1/2} H(\mathbf{x}_t)\mathbf{x}_t^T \mathcal{Q}_H^{-1}\Omega_0^{-1/2} I_1(B_n) \\
& + \lambda_t(n/k)\int_0^1 (2 + \log u) H_{0,\rho'}(u^{-1})du + k^{-1/2} H(\mathbf{x}_t)\mathbf{x}_t^T \mathcal{Q}_H^{-1}\Omega_0^{-1/2}\{2I_1(B_n) - I_2(B_n)\} + \Delta_n \\
= \;& 1 + \lambda_t(n/k)\int_0^1 (3 + \log u) H_{0,\rho'}(u^{-1})du \\
& + k^{-1/2} H(\mathbf{x}_t)\mathbf{x}_t^T \mathcal{Q}_H^{-1}\Omega_0^{-1/2}\{3I_1(B_n) - I_2(B_n) + \gamma B_n(1)\} + \Delta_n.
\end{aligned}
$$

Thus, as $\sqrt{k}\lambda_t(n/k) \to \lambda$,

$$
\sqrt{k}\Big(\frac{\hat{a}_t(n/k)}{a_t(n/k)} - 1\Big) = \Lambda + o_p(1),
$$

where (for $\gamma > 0$)

$$
\Lambda = \lambda\int_0^1 (3 + \log u) H_{0,\rho'}(u^{-1})du + H(\mathbf{x}_t)\mathbf{x}_t^T \mathcal{Q}_H^{-1}\Omega_0^{-1/2}\{3I_1(B_n) - I_2(B_n) + \gamma B_n(1)\}. \quad \text{(S.29)}
$$

Case 2: $\gamma < 0$**.** By the facts that $q_{n-k} = U_t(n/k)$ and $\tilde{q}_t(n/k) = a_t(n/k)/U_t(n/k)$, and by

(S.25), (S.26) and (S.27), it follows that, with notation $I_4(B_n)=\int_0^1(u^{-\gamma-1}B_n(u)-B_n(1))du$,

$$
\begin{aligned}
\frac{\hat{a}_t(n/k)}{a_t(n/k)} &= \frac{\hat{q}_{n-k}M_n^{(1)}\frac{1}{2}\big(1-\frac{\{M_n^{(1)}\}^2}{M_n^{(2)}}\big)^{-1}}{a_t(n/k)} \\
&= \Big(\frac{\hat{q}_{n-k}}{q_{n-k}}\Big)\Big(\frac{q_{n-k}\tilde{q}_t(n/k)}{a_t(n/k)}\Big)\Big(\frac{M_n^{(1)}}{\tilde{q}_t(n/k)}\Big)\frac{1}{2}\Big(1-\frac{\{M_n^{(1)}\}^2}{M_n^{(2)}}\Big)^{-1} \\
&= (1+E_{3k})\{\frac{1}{1-\gamma}+\lambda_t(n/k)\int_0^1 H_{\gamma,\rho'}(u^{-1})du-k^{-1/2}H(\mathbf{x}_t)\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}I_4(B_n)+\tilde{\Delta}_n\} \\
&\qquad \times(1-\gamma)\Big\{1-(1-\gamma)(1-2\gamma)\lambda_t(n/k)\int_0^1\{2-(1-2\gamma)\frac{u^{-\gamma}-1}{\gamma}\}H_{\gamma,\rho'}(u^{-1})du \\
&\qquad\qquad +(1-\gamma)(1-2\gamma)k^{-1/2}H(\mathbf{x}_t)\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}\{2I_1(B_n)+(1-2\gamma)I_3(B_n)\}+\tilde{\Delta}_n\Big\} \\
&= 1+(1-\gamma)\lambda_t(n/k)\int_0^1\{4\gamma-1+(1-2\gamma)^2\frac{u^{-\gamma}-1}{\gamma}\}H_{\gamma,\rho'}(u^{-1})du \\
&\quad +(1-\gamma)k^{-1/2}H(\mathbf{x}_t)\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}\{(2-4\gamma)I_1(B_n)+(1-2\gamma)^2I_3(B_n)-I_4(B_n)\}+\tilde{\Delta}_n,
\end{aligned}
$$

since $E_{3k}=o_p(k^{-1/2})$. Thus

$$\sqrt{k}\Big\{\frac{\hat{a}_t(n/k)}{a_t(n/k)}-1\Big\}=\Lambda+o_p(1),$$

where (for $\gamma<0$)

$$
\begin{aligned}
\Lambda &= \lambda(1-\gamma)\int_0^1\{4\gamma-1+(1-2\gamma)^2\frac{u^{-\gamma}-1}{\gamma}\}H_{\gamma,\rho'}(u^{-1})du \\
&\quad +(1-\gamma)H(\mathbf{x}_t)\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}\{(2-4\gamma)I_1(B_n)+(1-2\gamma)^2I_3(B_n)-I_4(B_n)\}. \qquad \text{(S.30)}
\end{aligned}
$$

(III) The asymptotic representation of $\hat{b}_t(n/k)$. By the definition, $b_t(n/k)=U_t(n/k)$ and $\hat{b}_t(n/k)=\hat{q}_{n-k}$.

Case 1: $\gamma>0$. Note that

$$\frac{\hat{b}_t(n/k)-b_t(n/k)}{a_t(n/k)}=\Big(\frac{\hat{q}_{n-k}}{q_{n-k}}-1\Big)\Big(\frac{a_t(n/k)}{q_{n-k}}\Big)^{-1}.$$

By $a_t(n/k)/q_{n-k}\sim\gamma$ and (S.21), we have

$$\sqrt{k}\left\{\frac{\hat{b}_t(n/k)-b_t(n/k)}{a_t(n/k)}\right\}=B+o_p(1),$$

where (for $\gamma>0$)

$$B=H(\mathbf{x}_t)\mathbf{x}_t^T\mathcal{Q}_H^{-1}\Omega_0^{-1/2}B_n(1). \qquad \text{(S.31)}$$

Case 2: $\gamma < 0$. Note that

$$\frac{\hat{b}_t(n/k) - b_t(n/k)}{a_t(n/k)} = \{\tilde{q}_t(n/k)\}^{-1}\Big(\frac{\hat{q}_{n-k}}{q_{n-k}} - 1\Big)\Big\{\frac{a_t(n/k)}{q_{n-k}\tilde{q}_t(n/k)}\Big\}^{-1}.$$

By the definition that $\tilde{q}_t(n/k) = a_t(n/k)/q_{n-k}$ and (S.25), we have

$$\sqrt{k}\{\frac{\hat{b}_t(n/k) - b_t(n/k)}{a_t(n/k)}\} = B + o_p(1)$$

where (for $\gamma < 0$)

$$B = -H(\mathbf{x}_t)\mathbf{x}_t^T \mathcal{Q}_H^{-1}\Omega_0^{-1/2}B_n(1). \tag{S.32}$$

□

Proof of Theorem 2.3.

By the definition,

$$\hat{Q}_t(\tau_n|\mathbf{x}_t) = \hat{b}_t(n/k) + \hat{a}_t(n/k)\frac{\{k/n(1-\tau_n)\}^{\hat{\gamma}} - 1}{\hat{\gamma}}.$$

Therefore, by Theorem 2.2 in this paper and Theorem 4.3.1 in de Haan and Ferreira (2006), we can easily show that

$$\sqrt{k}\left\{\frac{\hat{Q}_t(\tau_n|\mathbf{x}_t) - Q_t(\tau_n|\mathbf{x}_t)}{a_t(n/k)q_\gamma(d_n)}\right\} \xrightarrow{d} \Gamma + (\gamma_-)^2 B - \gamma_-\Lambda - \frac{\lambda\gamma_-}{\gamma_- + \rho}.$$

Since $\hat{a}_t(n/k)/a_t(n/k) \xrightarrow{p} 1$ and $q_{\hat{\gamma}}(d_n)/q_\gamma(d_n) \xrightarrow{p} 1$, by Corollary 4.3.2 in de Haan and Ferreira (2006), we have

$$\sqrt{k}\left\{\frac{\hat{Q}_t(\tau_n|\mathbf{x}_t) - Q_t(\tau_n|\mathbf{x}_t)}{\hat{a}_t(n/k)q_{\hat{\gamma}}(d_n)}\right\} \xrightarrow{d} \Gamma + (\gamma_-)^2 B - \gamma_-\Lambda - \frac{\lambda\gamma_-}{\gamma_- + \rho}.$$

□

References

Chernozhukov, V. (2005), "Extremal quantile regression," *Annals of Statistics*, 33, 806–839.

de Haan, L. and Ferreira, A. (2006), *Extreme Value Theory: An Introduction*, Springer.

Gutenbrunner, C. and Jurečková, J. (1992), "Regression rank scores and regression quantiles," *The Annals of Statistics*, 20, 305–330.

Gutenbrunner, C., Jurečková, J., Koenker, R., and Portnoy, S. (1993), "Tests of linear hypotheses based on regression rank scores," *Nonparametric Statistics*, 2, 307–331.

Knight, K. (1998), "Limiting distributions for L_1 regression estimators under general conditions," *The Annals of Statistics*, 26, 755–770.

Portnoy, S. (1984), "Tightness of the sequence of empiric cdf processes defined from regression fractiles," in *Robust and Nonlinear Time Series Analysis*, eds. Franke, J., Härdle, W., and Martin, D., Springer-Verlag, New York, pp. 231–246.

Shorack, G. (1991), "Embedding the finite sampling process at a rate," *The Annals of Probability*, 19, 826–842.

Shorack, G. and Wellner, J. (1986), *Empirical Processes with Applications to Statistics*, Wiley, New York.