Supplementary Material: Mathematical deductions

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1 SUPPLEMENTARY DATA

The following mathematical deduction establishes the ratio of surface area of *P. aeruginosa* that releases AHL that will theoretically collide with *C. albicans*, assuming that the AHL will be released perpendicularly to the cell envelope and that *P. aeruginosa* is oriented along the $x - axis$ (Figure S1). The deduction is expressed as a function of the distance between the two cells.

The sphere S_1 represents *C. albicans*, that is assumed to be a sphere with center at the origin, $C_1 = (0, 0, 0)$, and radius r_1. The *P. aeruginosa* is represented by a spherocylinder but in Figure S1 only the semi-sphere S_2 is represented, because the rest of the spherocylinder will not contribute with AHL that will collide with *C. albicans*. We consider S_2 with center at $C_2 = (a, b, c)$, and radius r_2, having $r_2 < r_1$. In Figure S1 the centers of the sphere S_1 and the semi-sphere S_2 are aligned to a distance a in relation to the x coordinate, so that $C_2 = (a, 0, 0)$.

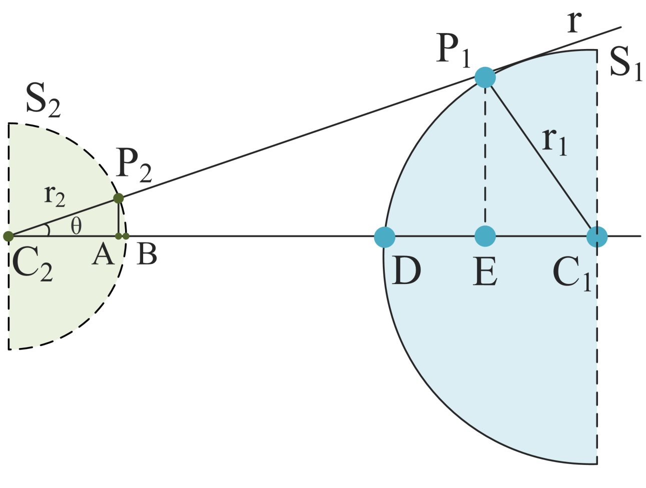

Figure S1. Schematic diagram of the sphere S_1 (*C. albicans*) and the spherocylinder (*P. aeruginosa*).

The goal of this mathematical study is to find an expression for the ratio between the surface area of the spherical cap with height $\|\overrightarrow{AB}\|$ and radius $\|\overrightarrow{AP_2}\|$, that is part of the semi-sphere S_2 that composes the spherocylinder, and the surface area of the spherocylinder. The surface area of the spherical cap that we need to calculate is represented in detail in Figure S2.

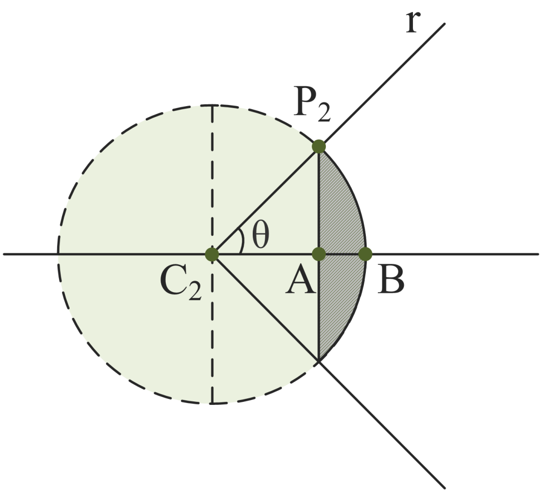

Figure S2. Spherical diagram showing the surface area of *P. aeruginosa* (darker green shade) that will release AHL that will collide with *C. albicans*

The surface area of the spherical cap, the highlighted region in Figure S2, is calculated from:

$$A_{sup} = 2\pi\|\overrightarrow{AB}\|\|\overrightarrow{AP_2}\| \tag{S1}$$

where $\|\overrightarrow{AB}\|$ and $\|\overrightarrow{AP_2}\|$ represent the distance between the points A and B; and between the points A and P_2, respectively.

Observing Figure S2 we may verify that the distance $\|\overrightarrow{AP_2}\|$ is obtained by the equation:

$$\|\overrightarrow{AP_2}\|^2 = \|\overrightarrow{C_2P_2}\|^2 - \|\overrightarrow{C_2A}\|^2. \tag{S2}$$

As $||\overrightarrow{C_2P_2}|| = r_2$, it is only necessary to calculate $||\overrightarrow{AC_2}||$. However, to calculate this distance some mathematical deductions are necessary.

Let us consider the straight line r, represented in Figure S1, that crosses the point $C_2 = (a, b, c)$ and that is tangent to the sphere S_1. This straight line r is defined by the equation:

$$\begin{cases} x = a + v_1 t \\ y = b + v_2 t, \ t \in \mathbb{R} \\ z = c + v_3 t \end{cases} \tag{S3}$$

where it is assumed that $\overrightarrow{v} = (v_1, v_2, v_3)$ is an unitary vector, that is, $\|\overrightarrow{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} = 1$.

The intersection between the sphere S_1, defined by the equation $x^2 + y^2 + z^2 = r_1^2$, with the straight line r, is as follows:

$$
\begin{aligned}
&(a+v_1t)^2+(b+v_2t)^2+(c+v_3t)^2=r_1^2\\
&\Leftrightarrow\ a^2+2av_1t+v_1^2t^2+b^2+2bv_2t+v_2^2t^2+c^2+2cv_3t+v_3^2t^2=r_1^2\\
&\Leftrightarrow\ (v_1^2+v_2^2+v_3^2)t^2+(2av_1+2bv_2+2cv_3)t+(a^2+b^2+c^2-r_1^2)=0\\
&\Leftrightarrow\ \underbrace{1}_{\alpha}t^2+\underbrace{2(av_1+bv_2+cv_3)}_{\beta}t+\underbrace{(a^2+b^2+c^2-r_1^2)}_{\gamma}=0
\end{aligned}
\tag{S4}
$$

As the line r is tangent to the sphere S_1, the quadratic equation (S4) has only one solution, if:

$$\beta^2-4\alpha\gamma=0$$

That means that:

$$
\begin{aligned}
&4(av_1+bv_2+cv_3)^2-4(a^2+b^2+c^2-r_1^2)=0\\
&\Leftrightarrow\ av_1+bv_2+cv_3=\pm\underbrace{\sqrt{a^2+b^2+c^2-r_1^2}}_{k}\\
&\Leftrightarrow\ \alpha_1v_1+bv_2+cv_3=k\vee-(\alpha_1v_1+\alpha_2v_2+\alpha_3v_3)=k
\end{aligned}
\tag{S5}
$$

Note that, if $\overrightarrow{v}$ is a vector of the straight line r, the vector $\overrightarrow{-v}$ is also a vector of this straight line.

The condition $av_1+bv_2+cv_3=k$ indicates that the scalar product between the director vector of the straight line r, $\overrightarrow{v}=(v_1,v_2,v_3)$ and the vector $\overrightarrow{C_1C_2}=C_2-C_1=(a,b,c)$ is constant and equal to k.

From the definition of the scalar product and considering θ as the angle between $\overrightarrow{v}$ and $\overrightarrow{C_1C_2}$, we have:

$$
\begin{aligned}
\overrightarrow{v}|\overrightarrow{C_1C_2}&=\|\overrightarrow{v}\|\|\overrightarrow{C_1C_2}\|\cos\theta=k\\
\Leftrightarrow\cos\theta&=\frac{k}{\|\overrightarrow{v}\|\|\overrightarrow{C_1C_2}\|}=\frac{k}{\sqrt{a^2+b^2+c^2}}
\end{aligned}
\tag{S6}
$$

As such, the angles between $\overrightarrow{C_1C_2}$ and the vectors $\overrightarrow{v}$, that are defining the straight tangent lines to sphere S_1 and go through C_2, are constants.

As we may observe in Figure S2,

$$\cos\theta=\frac{\|\overrightarrow{C_2A}\|}{\|\overrightarrow{C_2P_2}\|}=\frac{\|\overrightarrow{C_2A}\|}{r_2}. \tag{S7}$$

So, from the expressions (S6) and (S7), we may calculate the distance $\|\overrightarrow{C_2A}\|$, as follows:

$$
\begin{aligned}
\frac{k}{\sqrt{a^2+b^2+c^2}} &= \frac{\|\overrightarrow{C_2A}\|}{r_2} \\
\Leftrightarrow\ \|\overrightarrow{C_2A}\| &= \frac{kr_2}{\sqrt{a^2+b^2+c^2}} \\
\Leftrightarrow\ \|\overrightarrow{C_2A}\| &= \frac{r_2\sqrt{a^2+b^2+c^2-r_1^2}}{\sqrt{a^2+b^2+c^2}}
\end{aligned}
\tag{S8}
$$

After having calculated the distance $\|\overrightarrow{C_2A}\|$, it is then possible to calculate $\|\overrightarrow{AP_2}\|$ from expression (S2),

$$
\begin{aligned}
\|\overrightarrow{C_2A}\|^2 + \|\overrightarrow{AP_2}\|^2 &= \|\overrightarrow{C_2P_2}\|^2 \\
\Leftrightarrow\ \|\overrightarrow{AP_2}\|^2 &= r_2^2 - \frac{(a^2+b^2+c^2-r_1^2)r_2^2}{a^2+b^2+c^2} \\
\Leftrightarrow\ \|\overrightarrow{AP_2}\|^2 &= \frac{r_1^2 r_2^2}{a^2+b^2+c^2} \\
\Leftrightarrow\ \|\overrightarrow{AP_2}\| &= \frac{r_1 r_2}{\sqrt{a^2+b^2+c^2}}
\end{aligned}
\tag{S9}
$$

For calculating the surface area of the spherical cap in expression (S1), it is now only needed to calculate the distance $\|\overrightarrow{AB}\|$. This value is easily obtained in accordance with the following result (see Figure S2):

$$
\|\overrightarrow{AB}\| = \|\overrightarrow{C_2B}\| - \|\overrightarrow{C_2A}\|
$$

That is, from expression (S8)

$$
\begin{aligned}
\|\overrightarrow{AB}\| &= r_2 - \frac{\sqrt{a^2+b^2+c^2-r_1^2}\,r_2}{\sqrt{a^2+b^2+c^2}} \\
\Leftrightarrow\ \|\overrightarrow{AB}\| &= \frac{r_2\sqrt{a^2+b^2+c^2} - r_2\sqrt{a^2+b^2+c^2-r_1^2}}{\sqrt{a^2+b^2+c^2}} \\
\Leftrightarrow\ \|\overrightarrow{AB}\| &= \frac{r_2(\sqrt{a^2+b^2+c^2} - \sqrt{a^2+b^2+c^2-r_1^2})}{\sqrt{a^2+b^2+c^2}}
\end{aligned}
\tag{S10}
$$

Consequently, replacing the expressions (S9) and (S10) in the expression S1, the surface area of the spherical cap is given by:

$$A_{sup} = 2\pi \|\overrightarrow{AB}\| \|\overrightarrow{AP_2}\|$$

$$A_{sup} = \frac{2\pi r_1 r_2^2 \left(\sqrt{a^2+b^2+c^2} - \sqrt{a^2+b^2+c^2-r_1^2} \right)}{a^2+b^2+c^2}$$

$$A_{sup} = \frac{2\pi r_1 r_2^2 \left(\|\overrightarrow{C_1C_2}\| - \sqrt{\|C_1C_2\|^2 - r_1^2} \right)}{\|\overrightarrow{C_1C_2}\|^2}$$

Finally, the expression from the ratio between the surface area of the spherical cap and the surface area of the spherocylinder in accordance to the conditions stated in the beginning of this deduction, i.e. $C_2 = (a, 0, 0)$, is the following:

$$\frac{A_{sup}}{A_{total}} = \frac{\frac{2\pi r_1 r_2^2 \left(a - \sqrt{a^2 - r_1^2} \right)}{a^2}}{\underbrace{4\pi r_2^2}_{sphere} + \underbrace{2\pi r_2 h}_{cylinder}}$$

$$= \frac{2\pi r_1 {r_2}^2 \left(a - \sqrt{a^2 - r_1^2} \right)}{a^2 2\pi r_2 (2r_2 + h)} = \frac{r_1 r_2 \left(a - \sqrt{a^2 - r_1^2} \right)}{a^2 (2r_2 + h)}$$

$$= \frac{r_1 r_2 \left(r_1 + r_2 + d - \sqrt{(r_2 + r_2 + d)^2 - r_1^2} \right)}{(r_1 + r_2 + d)^2 (2r_2 + h)}$$

where $a = r_1 + r_2 + d$, being d is the distance in the x coordinate between the surface of the sphere $S1$ and the the surface of the semi-sphere S_2 and h represents the height of the cylinder that composes the spherocylinder.