Supplementary Material: Mathematical deductions 2

Gael Pérez-Rodríguez, Sónia Dias, Martín Pérez-Pérez, Nuno F. Azevedo and Anália Lourenço*

*Correspondence:
Author Name: Anália Lourenço
analia@uvigo.es

1 SUPPLEMENTARY DATA

The following mathematical deduction establishes the ratio of surface area of *P. aeruginosa* that releases AHL that will theoretically collide with *C. albicans*, assuming that the AHL will be released perpendicularly to the cell envelope and that *P. aeruginosa* is oriented along the y-axis (Figure S1). The deduction is expressed as a function of the distance between the two cells.

Considering the Cartesian coordinate system, the sphere, that represents *C. albicans*, is assumed to be centered at the origin of the referential, i.e $C_1 = (0, 0, 0)$, and with radius r_1. The spherocylinder is parallel to the $z - axis$ and its centroid $C_2 = (a, 0, 0)$ is aligned with the sphere center in the plan $z = 0$ (see Figure S1). The spherocylinder is composed by: a cylinder defined by the equation $(x - a)^2 + y^2 = r_2^2$ with $-\frac{h}{2} \leqslant z \leqslant \frac{h}{2}$ (i.e. the height of the cylinder is h), $r_2 < r_1$; two semi-spheres with radius r_2 and centers $(a, 0, \frac{h}{2})$ and $(a, 0, -\frac{h}{2})$.

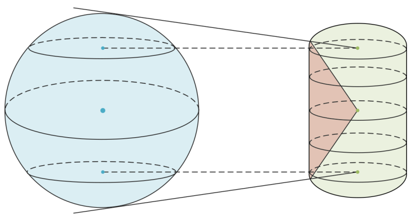

Figure S1. Schematic diagram of the sphere (*C. albicans*) and the spherocylinder (*P. aeruginosa*). In *P. aeruginosa* we can observe the surface region from where the AHL molecules that collide with *C. albicans* will emerge.

The goal of this mathematical study is to find an expression for the ratio between the surface area highlighted in the spherocylinder, represented in Figure S1, and its surface area. In this case, it is necessary to decompose the surface area of the spherocylinder in two parts. The part of the cylinder surface and the two spherical caps.

Determination of the surface area of the region of the cylinder

Half of the surface area of the part of the cylinder that is necessary to calculate is represented in Figure S2 , i.e. considering the highlight region when $0 \leqslant z \leqslant \frac{h}{2}$. This part may be planned according to the representation in Figure S3. The length of the bases of this trapezium are the length of the arcs correspondent to the circumference with center C_2 and the circumference with center $C_4 = (a, 0, \frac{h}{2})$, in Figure S2, i.e., the length of the arcs $\widehat{DF}$ and $\widehat{AG}$ represented by $L(\widehat{DF})$ and $L(\widehat{GA})$, respectively.

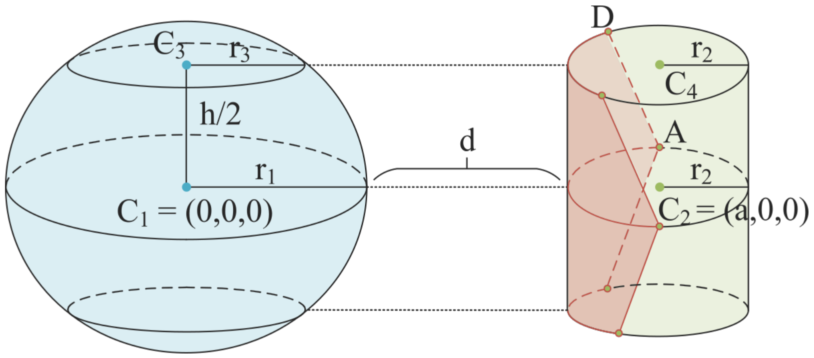

Figure S2. Schematic diagram of the sphere and the cylinder. Detail of Figure S1.

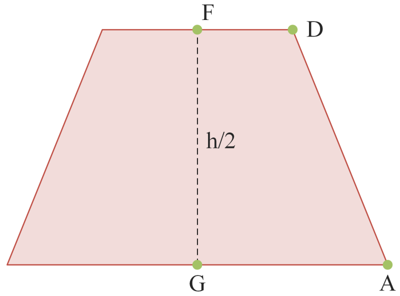

Figure S3. 2D projection of the surface area highlighted in the cylinder of Figure S2.

So, the surface area of the highlighted region in Figure S2 is obtained by

$$A_{Trapezium} = \left(L(\widehat{DF}) + L(\widehat{GA})\right) \times \frac{h}{2} \tag{S1}$$

In order to calculate the length of the arcs $\widehat{AG}$ and $\widehat{DF}$, it is necessary to calculate the coordinates of the points A and D. (see Figure S4 and S5)

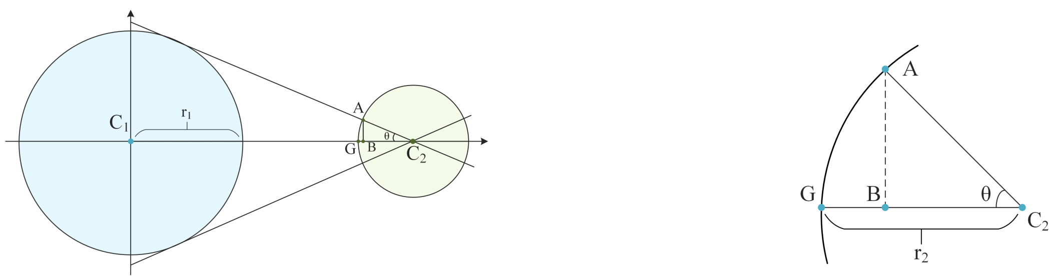

Figure S4. Detail of Figure S2, sliced in $z = 0$ (on left) and zoom of this representation in relation to the θ angle (on right).

As the mathematical deduction to calculate the length of the two arcs is similar, we will only present the calculations for the case of the arc $\widehat{AG}$.

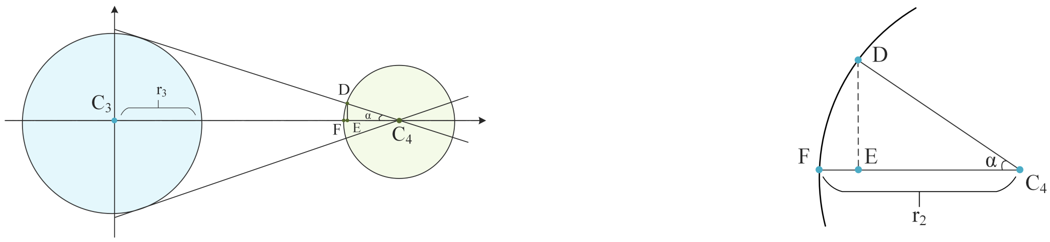

Figure S5. Detail of Figure S2, sliced in $z = \frac{h}{2}$ (on left) and zoom of this representation in relation to the α angle (on right).

From Figure S4, we may observe that the circumferences that result from the intersection of the sphere and spherocylinder with the plane $z = 0$ are the circumference with $C_1 = (0, 0, 0)$ and radius r_1 of the sphere and the circumference with $C_2 = (a, 0, 0)$ and radius r_2 of the cylinder.

The equation of the straight line that crosses the point C_2 and that is tangent to the circumference with center in C_1 (in Figure S4) is the following:

$$\begin{cases} x = a + v_1 t \\ y = v_2 t, \ t \in \mathbb{R} \\ z = 0 \end{cases}$$

where it is assumed that $\overrightarrow{v} = (v_1, v_2, 0)$ is an unitary vector, i.e. $\|\overrightarrow{v}\| = \sqrt{v_1^2 + v_2^2} = 1$.

The intersection between the circumference with center in C_1, defined by $x^2 + y^2 = r_1^2 \wedge z = 0$ with the tangent straight line defined above, is as follows:

$$\begin{aligned} & x^2 + y^2 = r_1^2 \\ \Leftrightarrow \ & (a + v_1 t)^2 + v_2^2 t^2 = r_1^2 \\ \Leftrightarrow \ & a^2 + 2av_1 t + v_1 t^2 + v_2^2 t^2 = r_1^2 \\ \Leftrightarrow \ & \underbrace{(v_1^2 + v_2^2)}_{1} t^2 + 2av_1 t - r_1^2 + a^2 = 0 \\ \Leftrightarrow \ & t^2 + 2av_1 t + (a^2 - r_1^2) = 0 \\ \Leftrightarrow \ & t = \frac{-2av_1 \pm \sqrt{(2av_1)^2 - 4(a^2 - r_1^2)}}{2} \end{aligned} \tag{S2}$$

As the straight line is tangent to the circumference with center C_1, the equation S2 only has one solution if:

$$\begin{aligned} & 4a^2 v_1^2 - 4a^2 + 4r_1^2 = 0 \\ \Leftrightarrow \ & a^2 v_1^2 = a^2 - r_1^2 \\ \Leftrightarrow \ & av_1 = \pm \underbrace{\sqrt{a^2 - r_1^2}}_{k} \end{aligned}$$

The last condition, $av_1 = k$, indicates that the scalar product between the vectors $\overrightarrow{C_2C_1} = (a, 0, 0)$ and $\overrightarrow{v}$ is constant and equal to k.

From the definition of the scalar product and considering θ as the angle between $\overrightarrow{C_2C_1}$ and $\overrightarrow{v}$, we have:

$$\overrightarrow{C_2C_1}|\overrightarrow{v} = \|\overrightarrow{C_2C_1}\|\|\overrightarrow{v}\|\cos\theta = |a|\cos\theta = k \Leftrightarrow \cos\theta = \frac{k}{|a|} = \frac{\sqrt{a^2 - r_1^2}}{|a|}. \tag{S3}$$

On the other hand, from Figure S4 we may observe that $\cos\theta = \frac{\|\overrightarrow{C_2B}\|}{r_2}$. Consequently, applying the expression S3 we may conclude that

$$\|\overrightarrow{C_2B}\| = \frac{r_2\sqrt{a^2 - r_1^2}}{|a|} \tag{S4}$$

Also from Figure S4 we may observe the classical relation between the sides of a right-angled triangle that allow us to determine $\|\overrightarrow{AB}\|$,

$$\|\overrightarrow{C_2A}\|^2 = \|\overrightarrow{AB}\|^2 + \|\overrightarrow{BC_2}\|^2. \tag{S5}$$

From expressions S4 and S5, we may calculate the distance $\|\overrightarrow{AB}\|$ as follows:

$$\begin{aligned}
&\|\overrightarrow{C_2A}\|^2 = \|\overrightarrow{AB}\|^2 + \|\overrightarrow{BC_2}\|^2 \\
&\Leftrightarrow\ r_2^2 = \|\overrightarrow{AB}\|^2 + \left(\frac{r_2\sqrt{a^2 - r_1^2}}{a}\right)^2 \\
&\Leftrightarrow\ \|\overrightarrow{AB}\|^2 = r_2^2 - \frac{(a^2 - r_1^2)r_2^2}{a^2} \\
&\Leftrightarrow\ \|\overrightarrow{AB}\| = \frac{r_1r_2}{|a|}
\end{aligned} \tag{S6}$$

Knowing the value $\|\overrightarrow{AB}\|$ it is possible to calculate the coordinates of points A and B, in Figure S4 . Assuming $A = (x, y, 0)$ and $B = (x, 0, 0)$, we have $\|\overrightarrow{AB}\| = y$. From the expression S6, we may conclude that

$$y = \|\overrightarrow{AB}\| = \frac{r_1r_2}{|a|}.$$

On the other hand, $\|\overrightarrow{BC_2}\| = |x - a|$ so, from expression S4 we may conclude that

$$\|\overrightarrow{BC_2}\| = |x-a| = r_2\sqrt{1-\frac{r_1^2}{a^2}}$$
$$\Leftrightarrow \; x = a \pm \frac{r_2}{|a|}\sqrt{a^2-r_1^2}$$

Finally, the coordinates of the points A and B are then:

$$A = \left(a - \frac{r_2}{|a|}\sqrt{a^2-r_1^2}, \frac{r_1 r_2}{|a|}, 0\right)$$
$$B = \left(a - \frac{r_2}{|a|}\sqrt{a^2-r_1^2}, 0, 0\right)$$

With an analogous procedure, it is possible to determinate the coordinates of the points C and D, in Figure S5.

$$D = \left(a - \frac{r_2}{|a|}\sqrt{a^2-r_3^2}, \frac{r_3 r_2}{|a|}, h\right)$$
$$E = \left(a - \frac{r_2}{|a|}\sqrt{a^2-r_3^2}, 0, h\right)$$

Knowing the coordinates of the points A, B, D and E, it is now possible to calculate the length of the arcs of the circumferences represented in Figures S4 and S5. Next, we will present the mathematical deductions to calculate the length of the arc in the circumference with center C_2. For the arc in the circumference with center C_4, the process is similar.

From the expression S3 we may conclude that the θ angle in Figure S4 may be defined as $\theta = \arccos\left(\frac{\sqrt{a^2-r_1^2}}{|a|}\right)$. Consequently, the arc in circumference with center C_2, i.e. the arc $\widehat{AG}$ may be parametrized, in polar coordinates, as follows:

$$r(\theta) = (r_2\cos\theta + a, r_2\sin\theta), \quad \theta \in \left[0, \arccos\left(\frac{\sqrt{a^2-r_1^2}}{|a|}\right)\right]$$

The length of the arc $\widehat{AG}$, i.e $L(\widehat{AG})$ may be calculated from the resolution of the curve integral

$$
\begin{aligned}
L(\widehat{AG}) &= \int_0^{\arccos\left(\frac{\sqrt{a^2-r_1^2}}{|a|}\right)} ||r'(\theta)||\, d\theta \\
&= \int_0^{\arccos\left(\frac{\sqrt{a^2-r_1^2}}{|a|}\right)} r_2\, d\theta \\
&\Leftrightarrow\ r_2 \arccos\left(\frac{\sqrt{a^2-r_1^2}}{|a|}\right)
\end{aligned}
\tag{S7}
$$

Analogously, the length of an arc $\widehat{DF}$, i.e. $L(\widehat{DF})$, in Figure S5, is

$$
L(\widehat{DF}) = \ r_2 \arccos\left(\frac{\sqrt{a^2-r_3^2}}{|a|}\right). \tag{S8}
$$

Knowing the length of the arc $\widehat{AG}$ and $\widehat{DF}$, the surface area of the part of the cylinder highlighted in Figure S3 may be calculated from expression S1:

$$
\begin{aligned}
A_{Trapezium} &= \left(L(\widehat{DF}) + L(\widehat{GA})\right) \times \frac{h}{2} \\
A_{Trapezium} &=\Leftrightarrow\ \frac{hr_2}{2}\left[\arccos\left(\frac{\sqrt{a^2-r_3^2}}{|a|}\right) + \arccos\left(\frac{\sqrt{a^2-r_1^2}}{|a|}\right)\right]
\end{aligned}
\tag{S9}
$$

Determination of the surface area of the region of the semi-sphere

The spherocylinder has in the upper and lower part of the cylinder two semi-spheres, the one at the top with a center at $C_4 = (a, 0, \frac{h}{2})$ and the one at the bottom with a center at $(a, 0, -\frac{h}{2})$. The area in each of the semi-spheres that contributes with AHL molecules that will collide with the sphere with a center at C_1 is an area of half of a spherical cap. In Figure S6, the surface of half of the spherical cap in the semi-sphere with center $C_4 = (a, 0, \frac{h}{2})$ is represented.

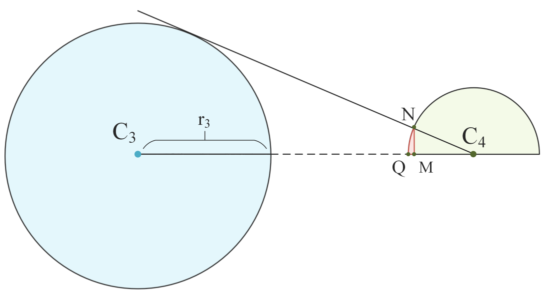

Figure S6. Schematic diagram of the sphere and the semi-sphere. Detail of Figure S1.

This situation is analogous to the mathematical deduction shown in Supplementary Material 1, where only part of the semi-sphere that represents *P. aeruginosa* contributes with AHL molecules that will theoretically collide with *C. albicans*. Whereas, in the earlier case the area of the semi-sphere that would contribute with AHL molecules was a spherical cap, in this case only half of the spherical cap contributes with molecules.

The surface area of half of the spherical cap represented in Figure S6, is given by trapezium

$$A_{\frac{calote}{2}} = \pi \|\overrightarrow{MN}\| \|\overrightarrow{MQ}\|$$

where, according to the mathematical deductions shown in Supplementary Material 1,

$$\|\overrightarrow{MN}\| = \frac{r_2 r_3}{\|\overrightarrow{C_3 C_4}\|} = \frac{r_2 r_3}{|a|} \text{ and } \|\overrightarrow{MQ}\| = \frac{r_2(a - \sqrt{a^2 - r_3^2} r_2)}{|a|} = \frac{r_2(|a| - \sqrt{a^2 - r_3^2})}{|a|}.$$

As such, the surface area of half of the spherical cap with height $\|\overrightarrow{MQ}\|$ and radius $\|\overrightarrow{MN}\|$ is given by:

$$A_{\frac{calote}{2}} = \pi \|\overrightarrow{MN}\| \|\overrightarrow{MQ}\| = \pi \frac{r_3 r_2^2 (a - \sqrt{a^2 - r_3^2})}{a^2} \tag{S10}$$

Expression of the ratio between the surface area in the spherocylinder that contributes with AHL molecules that will collide with sphere and its total surface area.

Finally, the expression from the ratio between the surface area highlighted in Figure S1 and the surface area of the spherocylinder, in the conditions defined in this supplementary material, is the following:

$$\frac{A_{sup}}{A_{total}} = \frac{2 \times A_{Trapezium} + 2 \times A_{\frac{calote}{2}}}{A_{Cylinder} + 2 \times A_{Semisphere}}$$

Considering the expressions S9 and S10, and having $a = r_1 + r_2 + d$, the ratio between the surface area of the region drawn in the spherocylinder and the total area is, in this situation, the following applies:

$$\begin{aligned}
\frac{A_{sup}}{A_{total}} &= \frac{2 \times \frac{hr_2}{2} \left[\arccos\left(\frac{\sqrt{a^2 - r_3^2}}{|a|} \right) + \arccos\left(\frac{\sqrt{a^2 - r_1^2}}{|a|} \right) \right] + \frac{2\pi r_3 r_2^2 \left(a - \sqrt{a^2 - r_3^2}\right)}{a^2}}{2\pi r_2 h + 4\pi r_2^2} \\
&= \frac{\frac{hr_2}{2} \left[\arccos\left(\frac{\sqrt{a^2 - r_3^2}}{|a|} \right) + \arccos\left(\frac{\sqrt{a^2 - r_1^2}}{|a|} \right) \right] + \frac{\pi r_3 r_2^2 \left(a - \sqrt{a^2 - r_3^2}\right)}{a^2}}{\pi r_2 h + 2\pi r_2^2} \\
&= \frac{\frac{hr_2}{2} \left[\arccos\left(\frac{\sqrt{(r_1+r_2+d)^2 - r_3^2}}{r_1+r_2+d} \right) + \arccos\left(\frac{\sqrt{(r_1+r_2+d)^2 - r_1^2}}{r_1+r_2+d} \right) \right] + \frac{\pi r_3 r_2^2 \left((r_1+r_2+d) - \sqrt{(r_1+r_2+d)^2 - r_3^2}\right)}{(r_1+r_2+d)^2}}{\pi r_2 h + 2\pi r_2^2}
\end{aligned}$$