Supplementary material on "Identifying Cointegration by Eigenanalysis"

Rongmao Zhang[⋆] Peter Robinson[†] Qiwei Yao[‡]

[⋆]School of Mathematics, Zhejiang University, Hangzhou, 310058, China

[†]Department of Economics, London School of Economics
London, WC2A 2AE, U.K.

[‡]Department of Statistics, London School of Economics
London, WC2A 2AE, U.K.

S.1 Proofs of the lemmas

Proof of Lemma 7. For any $I(d_l)$ process x_t^l, we can write

$$\nabla^{d_l} x_t^l = \mathrm{E} z_t^l + (z_t^l - \mathrm{E} z_t^l) =: \mu_l + \zeta_t^l.$$

Let $U_t^l(0) = \zeta_t^l$, $V_t^l(0) = \mu_l$ and

$$U_t^l(j) = \sum_{s=1}^{t} U_s^l(j-1), \quad V_t^l(j) = \sum_{s=1}^{t} V_s^l(j-1).$$

Then

$$x_t^l = U_t^l(d_l) + V_t^l(d_l) = \sum_{j=1}^{t} U_j^l(d_l - 1) + \sum_{j=1}^{t} V_j^l(d_l - 1). \tag{S.1}$$

By induction, we have

$$V_t^l(d_l) = \mu_l \prod_{j=0}^{d_l - 1} (t+j)/d_l! = \mu_l G_{d_l}(t). \tag{S.2}$$

On the other hand, since $\mathrm{E}\zeta_t^l = 0$, by (i) of Condition 1 and continuous mapping theorem, it follows that

$$U_{[ns]}^l(d_l)/n^{d_l - 1/2} \overset{J_1}{\Longrightarrow} f_{d_l}^l(s), \ \text{ on } \ D[0,1]. \tag{S.3}$$

Thus, by (S.1)–(S.3),

$$(x_{[ns]}^l - \mu_l G_{d_l}([ns]))/n^{d_l - 1/2} \overset{J_1}{\Longrightarrow} f_{d_l}^l(s), \ \text{ on } \ D[0,1]. \tag{S.4}$$

Since $S_n^i(t_i)$, $1 \le i \le p$ converge to their limiting distribution jointly, (7.1) follows from (S.4) and the continuous mapping theorem.

As for conclusion (7.2), by the joint convergence condition (see (i) of Condition 1) and (7.1),

$$\left(\frac{x^i_{[nt_i]} - \bar{x}^i - \mu_i L_{d_i}([nt_i])}{n^{d_i - 1/2}}, \frac{1}{\sqrt{n}} \sum_{s=1}^{[nt_j]} (x_s^j - \mathrm{E}x_s^j),\ 1 \le i \le p-r,\ p-r+1 \le j \le p \right)$$

$$\overset{J_1}{\Longrightarrow} \left(F^i(t_i), W^j(t_j)\right)_{ij},$$

on $D[0,1] \times D[0,1]$. (ii) of Condition 1 implies that $\mathrm{E}|x_s^j|^2 < \infty$. This gives

$$\max_{1 \le s \le n} |x_s^j - \mathrm{E}x_s^j| / \sqrt{n} = o_p(1), \text{ and } \frac{1}{n} \sum_{s=1}^{n} |x_s^j - \mathrm{E}x_s^j| = O_p(1). \tag{S.5}$$

Thus, by Theorem 3.1 of Ling and Li (1998), we have

$$\frac{1}{n^{d_i + 1/2}} \sum_{t=1}^{n} (x_t^i - \bar{x}^i - \mu_i L_{d_i}(t))(x_t^j - \mathrm{E}x_1^j) \xrightarrow{p} 0.$$

Since p is fixed, we have (7.2) as desired. □

Proof of Lemma 8. For any $1 \le i \le l$, we define $\bar{\mathbf{x}}(s_i) = \frac{1}{n}\sum_{t=1}^n \mathbf{x}_t(s_i)$. When $\boldsymbol{\mu}_i = 0$, Lemma 7 gives

$$\frac{\mathbf{B}_i(\mathbf{x}_t(s_i) - \bar{\mathbf{x}}(s_i))}{n^{a_i - 1/2}} = \frac{(\mathbf{x}_t(s_i) - \bar{\mathbf{x}}(s_i))}{n^{a_i - 1/2}} \overset{J_1}{\Longrightarrow} \mathbf{F}_i(t), \quad \text{on} \quad \prod_{j=1}^{s_i} D[0,1]. \tag{S.6}$$

When $\boldsymbol{\mu}_i \neq 0$, by $\bar{\boldsymbol{\mu}}_i' \boldsymbol{\mu}_i = 1$, we have $\mathbf{B}_i \mathbf{x}_t(s_i) = ((\mathbf{P}_i' \mathbf{x}_t(s_i))', n^{-1/2} G_{d_i}(t) + n^{-1/2} \bar{\boldsymbol{\mu}}_i' \mathbf{x}_t(s_i))'$. Note that

$$\lim_{n\to\infty} \frac{1}{n^{a_i}} (G_{a_i}([nt]) + \bar{\boldsymbol{\mu}}_i' \mathbf{x}_{[nt]}(s_i)) \overset{p}{=} \lim_{n\to\infty} \frac{1}{n^{a_i}} G_{a_i}([nt]) = \lim_{n\to\infty} \frac{1}{n^{a_i}} \prod_{j=0}^{a_i - 1} ([nt] + j) = \frac{t^{a_i}}{a_i!}. \tag{S.7}$$

Thus, by $\mathbf{P}_i' \boldsymbol{\mu}_i = 0$, Lemma 7 and continuous mapping theorem, we have

$$\frac{\mathbf{B}_i(\mathbf{x}_t(s_i) - \bar{\mathbf{x}}(s_i))}{n^{a_i - 1/2}} \overset{J_1}{\Longrightarrow} ((\mathbf{P}_i' \mathbf{F}_i(t))', H^{a_i}(t))' \quad \text{on} \quad \prod_{j=1}^{s_i} D[0,1]. \tag{S.8}$$

Combining (S.6), (S.8) and the joint convergence condition ((i) of Condition 1) yields

$$\left(\left(\frac{\mathbf{B}_i(\mathbf{x}_t(s_i) - \bar{\mathbf{x}}(s_i))}{n^{a_i - 1/2}} \right)', 1 \le i \le l \right) \overset{J_1}{\Longrightarrow} (\mathbf{M}_i'(t), 1 \le i \le l) = \mathbf{M}'(t) \quad \text{on} \quad \prod_{j=1}^{p-r} D[0,1]. \tag{S.9}$$

Let $\mathbf{B} = \mathrm{diag}(\mathbf{B}_1, \mathbf{B}_2, \cdots, \mathbf{B}_l) = (b_{ij})$. By (S.9) and continuous mapping theorem, we have

$$\begin{aligned} \mathbf{D}_{n1}^{-1} \mathbf{B} \left(\frac{1}{n} \sum_{t=1}^{n} (\mathbf{x}_{t1} - \bar{\mathbf{x}}_1)(\mathbf{x}_{t1} - \bar{\mathbf{x}}_1)' \right) \mathbf{B}' \mathbf{D}_{n1}^{-1} &= \left(\frac{1}{n} \sum_{t=1}^{n} (\mathbf{D}_{n1}^{-1} \mathbf{B}(\mathbf{x}_{t1} - \bar{\mathbf{x}}_1))(\mathbf{D}_{n1}^{-1} \mathbf{B}(\mathbf{x}_{t1} - \bar{\mathbf{x}}_1))' \right) \\ &\xrightarrow{d} \int_0^1 \mathbf{M}(t) \mathbf{M}'(t)\, dt \end{aligned} \tag{S.10}$$

and complete the proof of Lemma 8. □

Proof of Lemma 9. First, we show (7.3). To this end, it is enough to show

$$\left\|\mathbf{D}_{n1}^{-1}\left(\frac{1}{n}\sum_{t=1}^{n}(\mathbf{x}_{t1}-\bar{\mathbf{x}}_1)(\mathbf{x}_{t1}-\bar{\mathbf{x}}_1)'\right)\mathbf{D}_{n1}^{-1}-\int_0^1\mathbf{F}(t)\mathbf{F}'(t)\,dt\right\|_2=o_p(1). \tag{S.11}$$

Let $\boldsymbol{\xi}_t=(\xi_t^1,\xi_t^2,\cdots,\xi_t^{p-r})'$ be an integrated process with components ξ_t^i satisfying

$$\nabla^{d_i}\xi_t^i=\sigma_{ii}v_t^i:=\widetilde{\varepsilon}_t^i.$$

For any given $1\le i,\, j\le p-r$,

$$\begin{aligned}
&\frac{1}{n^{d_i+d_j}}\sum_{t=1}^{n}[(x_t^i-\bar{x}^i)(x_t^j-\bar{x}^j)-(\xi_t^i-\bar{\xi}^i)(\xi_t^j-\bar{\xi}^j)]\\
=\;&\frac{1}{n^{d_i+d_j}}\sum_{t=1}^{n}[x_t^i-\xi_t^i-(\bar{x}^i-\bar{\xi}^i)](x_t^j-\bar{x}^j)+\frac{1}{n^{d_i+d_j}}\sum_{t=1}^{n}(\xi_t^i-\bar{\xi}^i)[x_t^j-\xi_t^i-(\bar{x}^j-\bar{\xi}^j)]\\
=:\;&r_{ij}^1+r_{ij}^2.
\end{aligned}$$

By induction, it is easy to show that under condition (3.2),

$$\sup_{1\le i\le p-r}\sup_{1\le t\le n}\mathrm{E}[(x_t^i-\xi_t^i)/n^{d_i-1/2}]^2\le\sup_{1\le i\le p-r}\sup_{1\le t\le n}\frac{1}{n}\mathrm{E}(S_t^i-\sum_{l=1}^{t}\widetilde{\varepsilon}_l^i)^2=O(n^{2\tau-1}), \tag{S.12}$$

$$\sup_{1\le i\le p-r}\sup_{1\le t\le n}\mathrm{E}(\xi_t^i/n^{d_i-1/2})^2=O(1)\quad\text{and}\quad\sup_{1\le i\le n}\sup_{1\le t\le n}\mathrm{E}(x_t^i/n^{d_i-1/2})^2=O(1). \tag{S.13}$$

Thus, by equations (S.12), (S.13) and the independence of the components,

$$\sum_{i,j=1}^{p-r}[\mathrm{E}(r_{ij}^1)^2+\mathrm{E}(r_{ij}^2)^2]=O(p^2n^{2\tau-1}),$$

which implies

$$\left\|\mathbf{D}_{n1}^{-1}\left[\left(\frac{1}{n}\sum_{t=1}^{n}(\mathbf{x}_{t1}-\bar{\mathbf{x}}_1)(\mathbf{x}_{t1}-\bar{\mathbf{x}}_1)'\right)-\left(\frac{1}{n}\sum_{t=1}^{n}(\boldsymbol{\xi}_t-\bar{\boldsymbol{\xi}})(\boldsymbol{\xi}_t-\bar{\boldsymbol{\xi}})'\right)\right]\mathbf{D}_{n1}^{-1}\right\|_2=O_p(pn^{\tau-1/2}),$$

where $\bar{\boldsymbol{\xi}}=(\bar{\xi}^1,\bar{\xi}^2,\cdots,\bar{\xi}^{p-r})'$. Thus, for the proof of (S.11), it suffices to show

$$\sup_{1\le i,j\le p-r}\left\|\frac{1}{n^{d_i+d_j}}\sum_{t=1}^{n}(\xi_t^i-\bar{\xi}^i)(\xi_t^j-\bar{\xi}^j)-\int_0^1F^i(t)F^j(t)\,dt\right\|_2=o_{a.s.}(n^{\tau-1/2}). \tag{S.14}$$

Note that

$$\begin{aligned}
&\frac{1}{n}\sum_{t=1}^{n}\left(\frac{\xi_t^i}{n^{d_i-1/2}}\right)\left(\frac{\xi_t^j}{n^{d_j-1/2}}\right)-\sum_{t=1}^{n}\int_{(t-1)/n}^{t/n}f^i(a)f^j(a)\,da\\
=\;&\frac{1}{n}\sum_{t=1}^{n}\left(\frac{\xi_t^i}{n^{d_i-1/2}}-f^i(t/n)\right)\left(\frac{\xi_t^j}{n^{d_j-1/2}}\right)+\frac{1}{n}\sum_{t=1}^{n}f^i(t/n)\left(\frac{\xi_t^j}{n^{d_j-1/2}}-f^j(t/n)\right)\\
&-\sum_{t=1}^{n}\int_{(t-1)/n}^{t/n}(f^i(a)-f^i(t/n))f^j(a)+f^i(t/n)(f^j(a)-f^j(t/n))\,da\\
=:\;&J_{n1}(i,j)+J_{n2}(i,j)+J_{n3}(i,j).
\end{aligned}$$

From the definition of $I(d)$ process, it is easy to deduce that if $\xi_t \sim I(d)$ satisfying $\nabla^d \xi_t = \varepsilon_t$, then $\xi_t = \sum_{s=1}^t \varepsilon_s$ when $d = 1$ and when $d \geq 2$,

$$\xi_t = \sum_{s=1}^{t} \left[\prod_{i=1}^{d-1} (t - s + i)/(d-1)! \right] \varepsilon_s \tag{S.15}$$

and $f^i(t)$ can be rewritten as

$$f^i(t) = \int_0^t (t-s)^{d_i - 1} \, dW^i(s)/(d_i - 1)!. \tag{S.16}$$

By (S.16) and the continuity of $W^i(s)$, it is easy to get that

$$\sup_{1 \leq i,j \leq p-r} \sup_{(t-1)/n \leq a \leq t/n} |[f^i(a) - f^i(t/n)] f^j(a)| = O_{a.s.}(n^{-1/2} \log^2 n).$$

Thus,

$$\sup_{1 \leq i,j \leq p-r} |J_{n3}(i,j)| = O_{a.s.}(n^{-1/2} \log^2 n). \tag{S.17}$$

Set $\prod_{h=1}^0 (t-s+h)/0! = 1$. Using expressions (S.15) and (S.16), we have

$$\begin{aligned} \xi_t^i / n^{d_i - 1/2} - f^i(t/n) &= \sum_{s=1}^t \frac{\left[\prod_{h=1}^{d_i - 1} (t-s+h) - (t-s)^{d_i - 1} \right]}{n^{d_i - 1/2} (d_i - 1)!} \widetilde{\varepsilon}_s^i \\ &\quad + \left(\sum_{s=1}^t \frac{(t-s)^{d_i - 1} \widetilde{\varepsilon}_s^i}{n^{d_i - 1/2} (d_i - 1)!} - \int_0^{t/n} \frac{(t/n - s)^{d_i - 1}}{(d_i - 1)!} \, dW^i(s) \right) \\ &=: H_{n1}^i(t) + H_{n2}^i(t). \end{aligned}$$

It is easy to get that

$$\sup_{1 \leq i \leq p-r} \sup_{1 \leq t \leq n} |H_{n1}^i(t)| = O_{a.s.}(n^{-1/2} \log n). \tag{S.18}$$

On the other hand, we have for any $1 \leq t \leq n$,

$$\begin{aligned} H_{n2}^i(t) &= \frac{1}{(d_i - 1)!} \sum_{s=1}^t \left(\left(\frac{t}{n} - \frac{s}{n} \right)^{d_i - 1} \frac{\widetilde{\varepsilon}_s^i}{\sqrt{n}} - \int_{(s-1)/n}^{s/n} \left(\frac{t}{n} - a \right)^{d_i - 1} dW^i(a) \right) \\ &=^d \frac{1}{(d_i - 1)!} \sum_{s=1}^t \int_{(s-1)/n}^{s/n} \left[\left(\frac{t}{n} - \frac{s}{n} \right)^{d_i - 1} - \left(\frac{t}{n} - a \right)^{d_i - 1} \right] dW^i(a). \end{aligned} \tag{S.19}$$

This gives

$$\sup_{1 \leq i \leq p-r} \sup_{1 \leq t \leq n} |H_{n2}^i(t)| = n O_{a.s}(n^{-3/2} \log n) = O_{a.s.}(n^{-1/2} \log n). \tag{S.20}$$

Since the normal sequences $\{\nu_t^i\}$ are independent with respect to i, it follows that

$$\sup_{1 \leq j \leq p-r} \sup_{1 \leq t \leq n} |\xi_t^j| = O_{a.s.}(n^{d_j - 1/2} \log n). \tag{S.21}$$

Thus, by (S.18), (S.20) and (S.21), we have

$$\sup_{1\le i,j\le p-r} |J_{n1}(i,j)| = O_{a.s.}(n^{-1/2}\log^2 n). \tag{S.22}$$

Similarly, we have

$$\sup_{1\le i,j\le p-r} |J_{n2}(i,j)| = O_{a.s.}(n^{-1/2}\log^2 n). \tag{S.23}$$

Using the same argument, we can show

$$\sup_{1\le i,j\le p-r} \left|(\bar{\xi}^i/n^{d_i-1/2})(\bar{\xi}^j/n^{d_j-1/2}) - \int_0^1 f^i(t)\,dt \int_0^1 f^i(t)\,dt\right| = O_{a.s.}(n^{-1/2}\log^2 n). \tag{S.24}$$

Combining equations (S.17), (S.22)–(S.24), we have (S.14) and conclude (7.3). The positive definiteness of $\int_0^1 \mathbf{F}(t)\mathbf{F}'(t)\,dt$ can be shown similarly to that of Lemma 3.1.1 in Chan and Wei (1988). □

Proof of Lemma 10. We first consider the case with fixed p. To this end, we split the matrix into three parts: the nonstationary block, the cross block with elements being the product of stationary component with nonstationary component and the stationary block.

(I) As for the nonstationary block, we have for $1 \le i, h \le p-r$,

$$\begin{aligned}
&\sum_{t=1}^{n-j}(x_{t+j}^i - \bar{x}^i)(x_t^h - \bar{x}^h) - \sum_{t=1}^{n}(x_t^i - \bar{x}^i)(x_t^h - \bar{x}^h)\\
= \quad & -\sum_{t=1}^{j}(x_t^i - \bar{x}^i)(x_t^h - \bar{x}^h) - \sum_{t=1}^{n-j}(x_{t+j}^i - \bar{x}^i)(x_{t+j}^h - x_t^h)\\
= \quad & -\sum_{t=1}^{j}(x_t^i - \bar{x}^i - \mu_i L_{d_i}(t))(x_t^h - \bar{x}^h - \mu_h L_{d_h}(t)) - \sum_{t=1}^{n-j}(x_{t+j}^i - \bar{x}^i - \mu_i L_{d_i}(t))(x_{t+j}^h - x_t^h)\\
& -\mu_h \sum_{t=1}^{j} L_{d_h}(t)(x_t^i - \bar{x}^i - \mu_i L_{d_i}(t)) - \mu_i \sum_{t=1}^{j} L_{d_i}(t)(x_t^h - \bar{x}^h - \mu_h L_{d_h}(t))\\
& -\mu_i\mu_h \sum_{t=1}^{j} L_{d_h}(t)L_{d_i}(t) - \mu_i \sum_{t=1}^{n-j} L_{d_i}(t)(x_{t+j}^h - x_t^h)\\
=: \quad & \sum_{m=1}^{6} \delta_{nm}(j,i,h).
\end{aligned}$$

From (S.4), it follows that

$$\begin{aligned}
\sup_{0\le j\le j_0} \frac{|\delta_{n1}(j,i,h)|}{n^{d_i+d_h}} &\le \frac{j_0}{n}\left(\sup_{1\le t\le n}\frac{|x_t^i - \bar{x}^i - \mu_i L_{d_i}(t)|}{n^{d_i-1/2}}\right)\left(\sup_{1\le t\le n}\frac{|x_t^h - \bar{x}^h - \mu_h L_{d_h}(t)|}{n^{d_h-1/2}}\right) \qquad \text{(S.25)}\\
&= O_p(1/n).
\end{aligned}$$

As for $\delta_{n2}(j,i,h)$, we have

(i) If $d_h = 1$, then $x^h_{t+j} - x^h_t = \sum_{i=t+1}^{t+j} \varepsilon^h_i$. Since $\mathrm{E}|\varepsilon^h_i| < \infty$, it follows that

$$\begin{aligned} \sup_{0\le j\le j_0} \frac{|\delta_{n2}(j,i,h)|}{n^{d_i+d_h}} &\le \left(\sup_{t\le n} \frac{|x^i_t - \bar{x}^i - \mu_i L_{d_i}(t)|}{n^{d_i-1/2}}\right)\left(\frac{1}{n^{3/2}}\sum_{t=1}^{n}\sum_{i=t+1}^{t+j_0}|\varepsilon^h_i|\right) \\ &= O_p(1/n^{1/2}). \end{aligned} \tag{S.26}$$

(ii) If $d_h \ge 2$, then $x^h_{t+j} - x^h_t = \sum_{s=t+1}^{t+j}[U^h_s(d_h-1) + V^h_s(d_h-1)]$ (see Lemma 7), it follows that

$$\sup_{1\le t\le n}\sup_{1\le j\le j_0} \frac{|x^h_{t+j} - x^h_t|}{n^{d_h-1/2}} \le \frac{j_0}{n^{1/2}} \sup_{1\le s\le n} \frac{|U^h_s(d_h-1) + V^h_s(d_h-1)|}{n^{d_h-1}} = O_p(1/n^{1/2}), \tag{S.27}$$

which implies

$$\begin{aligned} \sup_{j\le j_0} \frac{|\delta_{n2}(j,i,h)|}{n^{d_i+d_h}} \le \left(\sup_{t\le n} \frac{|x^i_t - \bar{x}^i - \mu_i L_{d_i}(t)|}{n^{d_i-1/2}}\right)\left(\sup_{t\le n}\sup_{j\le j_0} \frac{|x^h_{t+j} - x^h_t|}{n^{d_h-1/2}}\right) \\ = O_p(1/n^{1/2}). \end{aligned} \tag{S.28}$$

Let $\mathbf{\Delta}_{nm}(j) = (\delta_{nm}(j,i,h))_{(p-r)\times(p-r)}$, $m = 1, 2, \cdots, 6$. Then by equations (S.25) – (S.28),

$$\begin{aligned} &\sup_{1\le j\le j_0} ||\mathbf{D}^{-1}_{n1}\mathbf{B}n^{-1}(\mathbf{\Delta}_{n1}(j) + \mathbf{\Delta}_{n2}(j))\mathbf{B}'\mathbf{D}^{-1}_{n1}||_2 \\ \le\; &\left(\sum_{i=1}^{p}\sum_{l=1}^{p}\sum_{m=1}^{p}|b_{il}b_{im}|\right) O_p(1/n^{1/2}) \\ \le\; &\left(\sum_{i=1}^{p}\sum_{l=1}^{p}\sum_{m=1}^{p}(b^2_{il} + b^2_{im})\right) O_p(1/n^{1/2}) = O_p(p^2/n^{1/2}). \end{aligned} \tag{S.29}$$

By the definition of $\mathbf{B}$, it is easy to see that the elements of $\mathbf{B\Delta}_{n3}(j)$, $\mathbf{B\Delta}_{n4}(j)$ are zero except in rows $\sum_{i=1}^{j} s_i$, $j = 1, 2, \cdots, l$ and the non-zero elements have the following forms:

$$n^{-1/2}\sum_{t=1}^{j} L_{a_h}(t)(x^i_t - \bar{x}^i - \mu_i L_{d_i}(t)).$$

Thus, by

$$\sup_{1\le j\le j_0}\left|\frac{\sum_{t=1}^{j} L_{d_h}(t)(x^i_t - \bar{x}^i - \mu_i L_{d_i}(t))}{n^{d_h+d_i-1/2}}\right| = O_p(1),$$

we have

$$\sup_{1\le j\le j_0} ||\mathbf{D}^{-1}_{n1}\mathbf{B}n^{-1}(\mathbf{\Delta}_{n3}(j) + \mathbf{\Delta}_{n4}(j))\mathbf{B}'\mathbf{D}^{-1}_{n1}||_2 \;=\; O_p(pn^{-1}). \tag{S.30}$$

Similarly, by $\sup_{1\le j\le j_0}\left|\frac{\sum_{t=1}^{j} L_{d_h}(t)L_{d_i}(t)}{n^{d_i+d_h}}\right| = O_p(1)$, we have

$$\sup_{1\le j\le j_0} ||\mathbf{D}^{-1}_{n1}\mathbf{B}n^{-1}\mathbf{\Delta}_{n5}(j)\mathbf{B}'\mathbf{D}^{-1}_{n1}||_2 = O_p(pn^{-1}). \tag{S.31}$$

Further, using (S.7), similar to (S.26) and (S.27), we can show

$$\sup_{1\le j\le j_0} \frac{1}{n^{d_i+d_h+1/2}}\sum_{t=1}^{n-j} L_{d_i}(t)(x_{t+j}^h - x_t^h) = O_p(1/n^{1/2}).$$

Thus, for $\mathbf{\Delta}_{n6}(j)$ we have

$$\sup_{1\le j\le j_0} ||\mathbf{D}_{n1}^{-1}\mathbf{B}n^{-1}\mathbf{\Delta}_{n6}(j)\mathbf{B}'\mathbf{D}_{n1}^{-1}||_2 = O_p(p/n^{1/2}). \tag{S.32}$$

Combining equations (S.29), (S.30), (S.31) and (S.32) gives

$$\begin{aligned} & \frac{\mathbf{D}_{n1}^{-1}\mathbf{B}}{n}\left|\left|\sum_{t=1}^{n-j}(\mathbf{x}_{t+j,1}-\bar{\mathbf{x}}_1)(\mathbf{x}_{t1}-\bar{\mathbf{x}}_1)' - \sum_{t=1}^{n}(\mathbf{x}_{t1}-\bar{\mathbf{x}}_1)(\mathbf{x}_{t1}-\bar{\mathbf{x}}_1)'\right|\right|_2 \mathbf{B}'\mathbf{D}_{n1}^{-1} \\ = \;& O_p(p^2/n^{1/2}). \end{aligned} \tag{S.33}$$

(II) As for the cross block, we first show

$$\begin{aligned} & \left|\left|\mathbf{D}_{n1}^{-1}\mathbf{B}n^{-1}\sum_{t=1}^{n}(\mathbf{x}_{t1}-\bar{\mathbf{x}}_1)(\mathbf{x}_{t2}-\bar{\mathbf{x}}_2)'\right|\right|_2 + \left|\left|n^{-1}\sum_{t=1}^{n}(\mathbf{x}_{t2}-\bar{\mathbf{x}}_2)(\mathbf{x}_{t1}-\bar{\mathbf{x}}_1)'\mathbf{B}'\mathbf{D}_{n1}^{-1}\right|\right|_2 \\ = \;& o_p(1). \end{aligned} \tag{S.34}$$

Note that for $1 \le i \le p-r$ and $p-r \le h \le p$,

$$\begin{aligned} \sum_{t=1}^{n}(x_t^i - \bar{x}^i)(x_t^h - \bar{x}^h) &= \sum_{t=1}^{n}(x_t^i - \bar{x}^i - \mu_i L_{d_i}(t))(x_t^h - \bar{x}^h) + \mu_i\sum_{t=1}^{n} L_{d_i}(t)(x_t^h - \bar{x}^h), \\ &=: \omega_{ih}^1 + \omega_{ih}^2. \end{aligned} \tag{S.35}$$

Let $\mathbf{\Omega}_1 = (\omega_{ih}^1)_{(p-r)\times r}$ and $\mathbf{\Omega}_2 = (\omega_{ih}^2)_{(p-r)\times r}$. Then the elements of $\mathbf{B}\mathbf{\Omega}_1 = (e_{jh})$ have the following expression:

$$e_{jh} = \sum_{i=1}^{p-r} b_{ji}\sum_{t=1}^{n}(x_t^i - \bar{x}^i - \mu_i L_{d_i}(t))(x_t^h - \bar{x}^h) = \sum_{i=1}^{p-r} b_{ji}\omega_{ih}^1.$$

By Lemma 7, we have

$$\left|\frac{e_{jh}}{n^{d_i+1/2}}\right| \le \frac{1}{n^{d_i+1/2}}\sum_{i=1}^{p-r}|b_{ji}\omega_{ih}^1| = o_p(1). \tag{S.36}$$

On the other hand, by the definition of $\mathbf{B}$, the elements of $\mathbf{B}\mathbf{\Omega}_2 = (d_{jh})$ can be represented as

$$d_{jh} = \frac{1}{n^{1/2}}\sum_{t=1}^{n} L_{a_i}(t)(x_t^h - \bar{x}^h)I(j = s_i),\; i = 1, 2, \cdots, l.$$

It is easy to get that

$$|d_{jh}|/n^{1/2+d_i} = o_p(1). \tag{S.37}$$

Consequently, by (S.36) and (S.37), it follows that

$$\begin{aligned} \left\|\mathbf{D}_{n1}^{-1}\mathbf{B}n^{-1}\sum_{t=1}^{n}(\mathbf{x}_{t1}-\bar{\mathbf{x}}_1)(\mathbf{x}_{t2}-\bar{\mathbf{x}}_2)'\right\|_2 &\leq \left\|\mathbf{D}_{n1}^{-1}\mathbf{B}n^{-1}\mathbf{\Omega}_1\right\|_2+\left\|\mathbf{D}_{n1}^{-1}\mathbf{B}n^{-1}\mathbf{\Omega}_2\right\|_2 \\ &= o_p(1). \end{aligned} \tag{S.38}$$

Similarly,

$$\left\|n^{-1}\sum_{t=1}^{n}(\mathbf{x}_{t2}-\bar{\mathbf{x}}_2)(\mathbf{x}_{t1}-\bar{\mathbf{x}}_1)'\mathbf{B}'\mathbf{D}_{n1}^{-1}\right\|_2=o_p(1). \tag{S.39}$$

(S.34) follows from (S.38) and (S.39).

Next, we show

$$\sup_{j\leq j_0}\left\|\frac{\mathbf{D}_{n1}^{-1}\mathbf{B}}{n}\left(\sum_{t=1}^{n-j}(\mathbf{x}_{t+j,1}-\bar{\mathbf{x}}_1)(\mathbf{x}_{t,2}-\bar{\mathbf{x}}_2)'-\sum_{t=1}^{n}(\mathbf{x}_{t1}-\bar{\mathbf{x}}_1)(\mathbf{x}_{t2}-\bar{\mathbf{x}}_2)'\right)\right\|_2=o_p\left(1\right) \tag{S.40}$$

and

$$\begin{aligned} &\sup_{j\leq j_0}\left\|\frac{1}{n}\left(\sum_{t=1}^{n-j}(\mathbf{x}_{t,2}-\bar{\mathbf{x}}_2)(\mathbf{x}_{t+j,1}-\bar{\mathbf{x}}_1)'-\sum_{t=1}^{n}(\mathbf{x}_{t2}-\bar{\mathbf{x}}_2)(\mathbf{x}_{t1}-\bar{\mathbf{x}}_1)'\right)\mathbf{B}'\mathbf{D}_{n1}^{-1}\right\|_2 \\ = &\ o_p\left(1\right). \end{aligned} \tag{S.41}$$

As for (S.40), note that for any $1\leq i\leq p-r$, $p-r+1\leq h\leq p$,

$$\begin{aligned} &\frac{1}{n^{d_i+1/2}}\left(\sum_{t=1}^{n-j}(x_{t+j}^i-\bar{x}^i)(x_t^h-\bar{x}^h)-\sum_{t=1}^{n}(x_t^i-\bar{x}^i)(x_t^h-\bar{x}^h)\right) \\ =&\ \frac{1}{n}\sum_{t=1}^{n-j}\left(\frac{x_{t+j}^i-x_t^i}{n^{d_i-1/2}}\right)(x_t^h-\mathrm{E}x_1^h)-\frac{(\bar{x}^h-\mathrm{E}x_1^h)}{n}\sum_{t=1}^{n-j}\left(\frac{x_{t+j}^i-x_t^i}{n^{d_i-1/2}}\right) \\ &-\frac{1}{n^{d_i+1/2}}\sum_{t=n-j+1}^{n}(x_t^i-\bar{x}^i)(x_t^h-\bar{x}^h) \\ =&\ L_{1n}(j,i,h)+L_{2n}(j,i,h)+L_{3n}(j,i,h). \end{aligned} \tag{S.42}$$

By (S.27) and $\frac{1}{n}\sum_{t=1}^{n}\mathrm{E}|x_t^h|=O(1)$, it follows that when $d_i\geq 2$,

$$\sup_{0\leq j\leq j_0}|L_{1n}(j,i,h)|=O_p(1/n^{1/2}). \tag{S.43}$$

When $d_i=1$, by $x_{t+j}^i-x_t^i=\sum_{s=t+1}^{t+j}\varepsilon_s^i$, we have

$$\mathrm{E}\sup_{0\leq j\leq j_0}|L_{1n}(j,i,h)|\leq\max_{1\leq t\leq n}\frac{1}{n^{1/2}}\sum_{s=t+1}^{t+j_0}\mathrm{E}|\varepsilon_s^i(x_t^h-\mathrm{E}x_1^h)|=O(1/n^{1/2}).$$

Thus, (S.43) also holds for $d_i=1$. Similar to $L_{1n}(j,i,h)$, we have

$$\sup_{1\leq j\leq j_0}\left|\frac{1}{n}\sum_{t=1}^{n-j}\frac{x_{t+j}^i-x_t^i}{n^{d_i-1/2}}\right|=O_p(1/n^{1/2}).$$

This combining with Condition 1 show

$$\sup_{j\le j_0}|L_{2n}(j,i,h)| = n^{-1/2}\left(n^{1/2}|\bar{x}^h - \mathrm{E}x_1^h|\right)\left|\frac{1}{n}\sum_{t=1}^{n-j}\frac{x_{t+j}^i - x_t^i}{n^{d_i-1/2}}\right| = O_p(1/n). \tag{S.44}$$

For $L_{3n}(j,i,h)$, by Lemma 7 and (S.7), we have

$$\sup_{1\le t\le n}|x_t^i - \bar{x}^i|/n^{d_i} = O_p(1),$$

thus by $\sum_{t=n-j_0+1}^{n}\mathrm{E}|x_t^h|/n^{1/2} = O(1/n^{1/2})$, we have

$$\sup_{j\le j_0}|L_{3n}(j,i,h)| = O_p(1/n^{1/2}). \tag{S.45}$$

Therefore, by (S.42)–(S.45),

$$\sup_{j\le j_0}\frac{1}{n^{d_i+1/2}}\left|\sum_{t=1}^{n-j}(x_{t+j}^i - \bar{x}^i)(x_t^h - \bar{x}^h) - \sum_{t=1}^{n}(x_t^i - \bar{x}^i)(x_t^h - \bar{x}^h)\right| = O_p(1/n^{1/2}), \tag{S.46}$$

which shows (S.40).

For (S.41), note that for any $1\le i\le p-r$, $p-r+1\le h\le p$,

$$\begin{aligned}
&\frac{1}{n^{d_i+1/2}}\left(\sum_{t=1}^{n-j}(x_t^i - \bar{x}^i)(x_{t+j}^h - \bar{x}^h) - \sum_{t=1}^{n}(x_t^i - \bar{x}^i)(x_t^h - \bar{x}^h)\right)\\
=\;&\frac{1}{n^{d_i+1/2}}\sum_{t=1}^{n-j}(x_{t+j}^h - x_t^h)(x_t^i - \bar{x}^i) + L_{3n}(j,i,h) =: L_{4n}(j,i,h) + L_{3n}(j,i,h).
\end{aligned}$$

Let $\mathbf{L}(j) = (L_{4n}(j,i,h))'_{(p-r)\times r}$ and decompose $L_{4n}(j,i,h)$ into two terms as in (S.35). Using the same arguments as in (S.36) and (S.37), we can show

$$\sup_{j\le j_0}||n^{-1}\mathbf{L}(j)\mathbf{B}'\mathbf{D}_{n1}^{-1}||_2 = o_p(1), \tag{S.47}$$

thus, by (S.45), we have (S.41). Combining equations (S.34) with (S.40) and (S.41) shows that the cross blocks tend to 0 in probability.

(III) As for the stationary block, let $\mathbf{\Upsilon}_j^x$ and $\widehat{\mathbf{\Upsilon}}_j^x$ be the matrixes obtained by replacing the stationary block $\frac{1}{n}\sum_{j=1}^{n-j}(\mathbf{x}_{t+j,2} - \bar{\mathbf{x}}_2)(\mathbf{x}_{t2} - \bar{\mathbf{x}}_2)'$ in $\mathbf{\Sigma}_j^x$ and $\widehat{\mathbf{\Sigma}}_j^x$ with $\mathrm{Cov}(\mathbf{x}_{1+j,2}, \mathbf{x}'_{1,2})$. By (ii) of Condition 1, we have

$$||\mathbf{\Sigma}_j^x - \mathbf{\Upsilon}_j^x||_2 = o_p(1) \text{ and } ||\widehat{\mathbf{\Sigma}}_j^x - \widehat{\mathbf{\Upsilon}}_j^x||_2 = o_p(1). \tag{S.48}$$

Thus, by (S.33) and the fact that the cross blocks tend to 0 in probability (see (II)), we have

$$\begin{aligned}
||\mathbf{D}_n^{-1}\mathbf{\Theta}_n(\mathbf{\Sigma}_j^x - \mathbf{\Gamma}_j^x)\mathbf{\Theta}_n'\mathbf{D}_n^{-1}||_2 &\le ||\mathbf{D}_n^{-1}\mathbf{\Theta}_n(\mathbf{\Sigma}_j^x - \mathbf{\Upsilon}_j^x)\mathbf{\Theta}_n'\mathbf{D}_n^{-1}||_2 + ||\mathbf{D}_n^{-1}\mathbf{\Theta}_n(\mathbf{\Upsilon}_j^x - \mathbf{\Gamma}_j^x)\mathbf{\Theta}_n'\mathbf{D}_n^{-1}||_2\\
&= o_p(1)
\end{aligned}$$

and

$$
\begin{aligned}
||\mathbf{D}_n^{-1}\boldsymbol{\Theta}_n(\widehat{\boldsymbol{\Sigma}}_j^x - \boldsymbol{\Gamma}_j^x)\boldsymbol{\Theta}_n'\mathbf{D}_n^{-1}||_2 &\leq ||\mathbf{D}_n^{-1}\boldsymbol{\Theta}_n(\widehat{\boldsymbol{\Sigma}}_j^x - \widehat{\boldsymbol{\Upsilon}}_j^x)\boldsymbol{\Theta}_n'\mathbf{D}_n^{-1}||_2 + ||\mathbf{D}_n^{-1}\boldsymbol{\Theta}_n(\widehat{\boldsymbol{\Upsilon}}_j^x - \boldsymbol{\Gamma}_j^x)\boldsymbol{\Theta}_n'\mathbf{D}_n^{-1}||_2 \\
&= o_p(1).
\end{aligned}
$$

Hence, Lemma 10 holds for finite p.

Next, consider the case: $p = o(n^{1/2-\tau})$. We still split the matrix into three parts as above.

(The nonstationary block.) Since $b_1 \leq \lim_{n\to\infty} \mathrm{Var}(\sum_{s=1}^n z_s^i/\sqrt{n}) \equiv \sigma_{ii}^2 \leq b_2$ for all i, it follows that as $n \to \infty$,

$$
\max_{1\leq t\leq n} \mathrm{Var}\Big(\sum_{s=1}^t z_s^i/\sqrt{n}\Big) \leq b_2 \quad \text{and} \quad \max_{1\leq t\leq n} \mathrm{Var}\Big(x_t^i/n^{d_i-1/2}\Big) \leq b_2. \tag{S.49}
$$

Let $\delta_{n1}(j,i,h), \delta_{n2}(j,i,h)$ be defined as above with $\mu_i = \mu_h = 0$. Note that the components of $\{\mathbf{z}_t\}$ are independent, by (S.49) and some elementary computation, we can show

$$
\mathrm{E}\left[\sum_{i,h=1}^{p-r}\left(\sup_{j\leq j_0}\frac{|\delta_{n1}(j,i,h)|}{n^{d_i+d_h}}\right)^2\right] = O(j_0^2 p^2/n^2)
$$

and

$$
\mathrm{E}\left[\sum_{i,h=1}^{p-r}\left(\sup_{j\leq j_0}\frac{|\delta_{n2}(j,i,h)|}{n^{d_i+d_h}}\right)^2\right] = O(j_0^2 p^2/n).
$$

Combining the above two equations yields

$$
\begin{aligned}
&\mathbf{D}_{n1}^{-1} n^{-1}\left|\left|\sum_{t=1}^{n-j}(\mathbf{x}_{t+j,1} - \bar{\mathbf{x}}_1)(\mathbf{x}_{t1} - \bar{\mathbf{x}}_1)' - \sum_{t=1}^{n}(\mathbf{x}_{t,1} - \bar{\mathbf{x}}_1)(\mathbf{x}_{t1} - \bar{\mathbf{x}}_1)'\right|\right|_2 \mathbf{D}_{n1}^{-1} \qquad \text{(S.50)} \\
&= O_p(pn^{-\frac{1}{2}}).
\end{aligned}
$$

(The cross block.) Let ω_{ih} be defined as in (S.35) with $\mu_i = 0$. Since $\mathbf{z}_t$ and $\mathbf{x}_{t2}$ are independent, it follows from (S.49) that

$$
\begin{aligned}
\mathrm{E}\Big[\sum_{i=1}^{p-r}\sum_{h=p-r+1}^{p}\omega_{ih}^2\Big] &= \sum_{i=1}^{p-r}\sum_{h=p-r+1}^{p}\mathrm{E}(\omega_{ih}^2) \\
&= \sum_{i=1}^{p-r}\sum_{h=p-r+1}^{p} n^{-2}\sum_{t,t'=1}^{n}\mathrm{E}\Big[\frac{(x_t^i - \bar{x}^i)}{n^{d_i-1/2}}\frac{(x_{t'}^i - \bar{x}^i)}{n^{d_i-1/2}}\Big]\mathrm{E}[(x_t^h - \bar{x}^h)][(x_{t'}^h - \bar{x}^h)] \\
&= O\left(\sum_{i=1}^{p-r}\sum_{h=p-r+1}^{p} n^{-2}\sum_{t,t'=1}^{n}|\mathrm{E}[(x_t^h - \bar{x}^h)][(x_{t'}^h - \bar{x}^h)]|\right) = O(p^2/n^{1-2\tau})
\end{aligned}
$$

by (iii) of Condition 2, which implies

$$
\left|\left|\mathbf{D}_{n1}^{-1} n^{-1}\sum_{t=1}^{n}(\mathbf{x}_{t1} - \bar{\mathbf{x}}_1)(\mathbf{x}_{t2} - \bar{\mathbf{x}}_2)'\right|\right|_2 = \big|\big|\mathbf{D}_{n1}^{-1} n^{-1}\boldsymbol{\Omega}_1\big|\big|_2 = O_p(pn^{-1/2+\tau}). \tag{S.51}
$$

Similarly,

$$\left|\left|n^{-1}\sum_{t=1}^{n}(\mathbf{x}_{t2}-\bar{\mathbf{x}}_2)(\mathbf{x}_{t1}-\bar{\mathbf{x}}_1)'\mathbf{D}_{n1}^{-1}\right|\right|_2 = O_p(pn^{-1/2+\tau}). \tag{S.52}$$

Further, by some elementary computation, it is easy to show

$$\mathrm{E}\Big[\sum_{i=1}^{p-r}\sum_{j=p-r+1}^{p}\sum_{l=1}^{3}(L_{ln}(j,i,h))^2\Big] = O(pr/n^{1-2\tau}),$$

which gives

$$\begin{aligned}
&\sup_{j\le j_0}\left|\left|\frac{\mathbf{D}_{n1}^{-1}}{n}\left(\sum_{t=1}^{n-j}(\mathbf{x}_{t+j,1}-\bar{\mathbf{x}}_1)(\mathbf{x}_{t+j,2}-\bar{\mathbf{x}}_2)'-\sum_{t=1}^{n}(\mathbf{x}_{t1}-\bar{\mathbf{x}}_1)(\mathbf{x}_{t2}-\bar{\mathbf{x}}_2)'\right)\right|\right|_2 \\
= \;& O_p(pn^{-\frac{1}{2}+\tau}).
\end{aligned} \tag{S.53}$$

Note that $L_{4n}(j,i,h)=\sum_{t=n-j+1}^{n}(x_t^i-\bar{x}^i)x_t^h-\sum_{t=1}^{j}(x_t^i-\bar{x}^i)x_t^h-\sum_{t=1}^{n-j}(x_{t+j}^i-x_t^i)x_{t+j}^h$, it is easy to show that $\mathrm{E}\Big[\sum_{i=1}^{p-r}\sum_{j=p-r+1}^{p}(L_{4n}(j,i,h))^2\Big]=O(pr/n^{1-2\tau})$ too, thus

$$\begin{aligned}
&\sup_{j\le j_0}\left|\left|\frac{1}{n}\left(\sum_{t=1}^{n-j}(\mathbf{x}_{t+j,2}-\bar{\mathbf{x}}_2)(\mathbf{x}_{t+j,1}-\bar{\mathbf{x}}_1)'-\sum_{t=1}^{n}(\mathbf{x}_{t2}-\bar{\mathbf{x}}_2)(\mathbf{x}_{t1}-\bar{\mathbf{x}}_1)'\right)\mathbf{D}_{n1}^{-1}\right|\right|_2 \\
= \;& O_p(pn^{-\frac{1}{2}+\tau}).
\end{aligned} \tag{S.54}$$

Consequently, by equations (S.51)–(S.54), we get that the norms of the cross blocks are $O_p(pn^{-\frac{1}{2}+\tau})$.

(The stationary block.) By (ii) of Condition 2, we also have (S.48). Thus, by (S.50) and the bound of the cross blocks (see above), we have if $p=o(n^{1/2-\tau})$,

$$\begin{aligned}
||\mathbf{D}_n^{-1}(\mathbf{\Sigma}_j^x-\mathbf{\Gamma}_j^x)\mathbf{D}_n^{-1}||_2 &\le ||\mathbf{D}_n^{-1}(\mathbf{\Sigma}_j^x-\mathbf{\Upsilon}_j^x)\mathbf{D}_n^{-1}||_2+||\mathbf{D}_n^{-1}(\mathbf{\Upsilon}_j^x-\mathbf{\Gamma}_j^x)\mathbf{D}_n^{-1}||_2 \\
&= o_p(1)+O_p(pn^{-1/2})=o_p(1)
\end{aligned}$$

and

$$||\mathbf{D}_n^{-1}(\widehat{\mathbf{\Sigma}}_j^x-\mathbf{\Gamma}_j^x)\mathbf{D}_n^{-1}||_2\le||\mathbf{D}_n^{-1}(\widehat{\mathbf{\Sigma}}_j^x-\widehat{\mathbf{\Upsilon}}_j^x)\mathbf{D}_n^{-1}||_2+||\mathbf{D}_n^{-1}(\widehat{\mathbf{\Upsilon}}_j^x-\mathbf{\Gamma}_j^x)\mathbf{D}_n^{-1}||_2=o_p(1).$$

Hence, Lemma 10 follows. And the proof of Lemma 10 is complete. □

Proof of Lemma 13. We only give the proof for $\mu_j=0$, $j=1,\cdots,p$ in details, other case can be proved similarly. By Lemma 12 and the continuous mapping theorem, it follows that (i) holds for $j=0$. Thus, it suffices to show for any $1\le i,h\le p$,

$$\sup_{1\le j\le j_0}\left|\frac{1}{n^{d_i+d_h}}\left(\sum_{t=1}^{n-j}(x_{t+j}^i-\bar{x}^i)(x_t^h-\bar{x}^h)-\sum_{t=1}^{n}(x_t^i-\bar{x}^i)(x_t^h-\bar{x}^h)\right)\right|=o_p(1). \tag{S.55}$$

Observe that

$$
\begin{aligned}
&\sum_{t=1}^{n-j}(x^i_{t+j}-\bar{x}^i)(x^h_t-\bar{x}^h)-\sum_{t=1}^{n}(x^i_t-\bar{x}^i)(x^h_t-\bar{x}^h)\\
=\;&\sum_{t=1}^{n-j}(x^i_{t+j}-x^i_t)(x^h_t-\bar{x}^h)-\sum_{t=n-j+1}^{n}(x^i_t-\bar{x}^i)(x^h_t-\bar{x}^h)=:\alpha_{n1}(j,i,h)+\alpha_{n2}(j,i,h).
\end{aligned}
$$

By Lemma 11, it follows that for any $1\le i,h\le p$,

$$\Big((x^i_{[nt]}-\bar{x}^i)/n^{d_i-1/2},\,(x^h_{[ns]}-\bar{x}^h)/n^{d_h-1/2}\Big)\xRightarrow{J_1}(U^i(t),U^h(s))\text{ on }D[0,1]^2. \tag{S.56}$$

This gives

$$\sup_{0\le j\le j_0}|\alpha_{n2}(j,i,h)|/n^{d_i+d_h}=O_p(1/n). \tag{S.57}$$

Further, for any $\varepsilon>0$, then

$$\lim_{n\to\infty}P\{\sup_{1\le j\le j_0}\sup_{1\le t\le n-j}|x^i_{t+j}-x^i_t|/n^{d_i-1/2}>\varepsilon\}=0. \tag{S.58}$$

Thus,

$$\sup_{0\le j\le j_0}|\alpha_{n1}(j,i,h)|/n^{d_i+d_h}=o_p(1). \tag{S.59}$$

Combining (S.57) and (S.59) gives (S.55) as desired.

By (i) of Condition 3, it follows that

$$\frac{1}{n}\sum_{t=1}^{n}(\mathbf{x}_{t+j,I_1}-\bar{\mathbf{x}}_{I_1})(\mathbf{x}_{t,I_1}-\bar{\mathbf{x}}_{I_1})\xrightarrow{p}\mathrm{Cov}(\mathbf{x}_{t+j,I_1}\mathbf{x}_{t,I_1}).$$

Thus, by (i) of Lemma 12, we have (7.28).

As for (7.27), it is enough to show for any $i\in I_1^c$ and $h\in I_1$,

$$
\begin{aligned}
\frac{1}{n^{d_i+1/2}}\sum_{i=1}^{n-j}(x^i_{t+j}-\bar{x}^i)(x^h_t-\bar{x}^h)\;&=\;\frac{\alpha_{n1}(j,i,h)}{n^{d_i+1/2}}+\frac{\alpha_{n2}(j,i,h)}{n^{d_i+1/2}}+\frac{1}{n^{d_i+1/2}}\sum_{i=1}^{n}(x^i_t-\bar{x}^i)(x^h_t-\bar{x}^h)\\
&\xrightarrow{p}\;0,
\end{aligned} \tag{S.60}
$$

holds for all $0\le j\le j_0$. Similar to Lemma 7, we can show

$$\frac{1}{n^{d_i+1/2}}\sum_{i=1}^{n}(x^i_t-\bar{x}^i)(x^h_t-\bar{x}^h)\xrightarrow{p}0. \tag{S.61}$$

By (S.56) and $n^{-1}\sum_{t=n-j_0}^{n}\mathrm{E}|x^h_t-\bar{x}^h|=O(j_0n^{-1})$, we have

$$\sup_{j\le j_0}\alpha_{n2}(j,i,h)/n^{d_i+1/2}=O_p(1/n). \tag{S.62}$$

By (S.58) and $\frac{1}{n}\sum_{t=1}^{n}\mathrm{E}|x^h_t-\bar{x}^h|=O(1)$, we have

$$\sup_{j\le j_0}\alpha_{n1}(j,i,h)/n^{d_i+1/2}=o_p(1). \tag{S.63}$$

(S.60) follows by equations (S.61)–(S.63). □

S.2 Proofs of Remarks 5 and 6

Proof of Remark 5. (i) By the martingale version of the Skorokhod representation theorem (Strassen 1967, Hall and Heyde 1980, and Wu 2007), we have for all i, on a richer probability space, there exists a standard Brownian motion $\{W(t)\}$ and a non-negative stopping times $\{\tau_j^i\}$ such that for $t \geq 1$,

$$S_t^i = W(\sum_{j=1}^{t} \tau_j^i) \quad \text{and} \quad \mathrm{E}[\tau_t^i | \mathcal{F}_{t-1}(i)] = \mathrm{E}[(\varepsilon_t^i)^2 | \mathcal{F}_{t-1}(i)], \tag{S.64}$$

where $\mathcal{F}_t(i)$ is the σ-algebra generated by $\{\varepsilon_s^i,\, s \leq t\}$. This implies that

$$\begin{aligned}
&\mathrm{E}|S_t^i - W(\sigma_{ii} t)|^2 = \mathrm{E}|\sum_{j=1}^{t} \tau_j^i - \sigma_{ii} t| \\
\leq \;& \mathrm{E}\left|\sum_{j=1}^{t}(\tau_j^i - \mathrm{E}(\tau_j^i|\mathcal{F}_{t-1}(i))\right| + \mathrm{E}\left|\sum_{j=1}^{t}[(\varepsilon_j^i)^2 - \mathrm{E}((\varepsilon_j^i)^2|\mathcal{F}_{t-1}(i))]\right| + \mathrm{E}\left|\sum_{j=1}^{t}(\varepsilon_j^i)^2 - \sigma_{ii} t\right|.
\end{aligned}$$

Since both $\{\tau_j^i - \mathrm{E}(\tau_j^i|\mathcal{F}_{t-1}(i))\}$ and $\{(\varepsilon_j^i)^2 - \mathrm{E}((\varepsilon_j^i)^2|\mathcal{F}_{t-1}(i))\}$ are martingale difference and $\mathrm{E}|\varepsilon_t^i|^q < \infty$, it follows that

$$\sup_{1 \leq t \leq n} \mathrm{E}\left|\sum_{j=1}^{t}(\tau_j^i - \mathrm{E}(\tau_j^i|\mathcal{F}_{t-1}(i))\right| = O\left(\mathrm{E}\left|\sum_{j=1}^{n}[\tau_j^i - \mathrm{E}(\tau_j^i|\mathcal{F}_{t-1}(i))]\right|\right) = O(n^{2/q^*}).$$

Similarly, $\sup_{1 \leq t \leq n} \mathrm{E}\left|\sum_{j=1}^{t}[(\varepsilon_j^i)^2 - \mathrm{E}((\varepsilon_j^i)^2|\mathcal{F}_{t-1}(i))]\right| = O(n^{2/q^*})$. Further, condition $\mathrm{E}\Big|\sum_{k=1}^{n}[(\varepsilon_t^i)^2$ $-\sigma_{ii}^2]\Big| = O(n^{2/q^*})$ implies that $\sup_{1 \leq t \leq n} \mathrm{E}\left|\sum_{k=1}^{t}[(\varepsilon_t^i)^2 - \sigma_{ii}^2]\right| = O(n^{2/q^*})$. Thus, Condition 2(i) holds for any $\tau > 1/q^*$. If $p = o(n^{1/2})$, Condition 2(ii) holds. Since the components of $\boldsymbol{\varepsilon}_t$ are independent, Condition 2(iii) follows with $\sup_j \sum_{s,t=1}^{n} \mathrm{E}|\varepsilon_s^j \varepsilon_t^j| = O(n)$.

(ii) By the proof of Theorem 9.3.1 of Lin and Lu (1996), we know that there exists a martingale difference sequence $\{m_t^i\}$ such that $R_t = S_t^i - M_t^i$ satisfying $\mathrm{E}|R_t|^q = O(1)$, where $M_t^i = \sum_{j=1}^{t} m_j^i$. Further,

$$\mathrm{E}\left|\sum_{j=1}^{n}[(m_j^i)^2 - \mathrm{E}(m_j^i)^2]\right|^{q/2} \leq C n \log n. \tag{S.65}$$

As a result, Condition 2(i) holds for any $\tau > 1/q$. Similarly, Condition 2(iii) can be easily obtained by basic inequality for mixing processes, see Lemma 1.2.2 of Lin and Lu (1996). Note that for any given j,

$$\begin{aligned}
&\mathrm{E}\Big\|\frac{1}{n}\sum_{t=1}^{n-j}(\mathbf{x}_{t+j,2} - \bar{\mathbf{x}}_2)(\mathbf{x}_{t2} - \bar{\mathbf{x}}_2)' - \mathrm{Cov}(\mathbf{x}_{1+j,2}, \mathbf{x}_{1,2})\Big\|_2 \\
\leq \;& \sum_{i,j=p-r+1}^{p} \mathrm{E}\left(\frac{1}{n}\sum_{t=1}^{n}[(x_t^i - \bar{x}^i)(x_t^j - \bar{x}^j) - \mathrm{Cov}(x_t^i, x_t^j)]\right)^2 = O(p^2/n) \to 0
\end{aligned}$$

as $p = o(n^{1/2})$. Condition 2(ii) holds too.

(iii) By Beveridge-Nelson decomposition, ε_t^i can be represented as

$$\varepsilon_t^i = \Big(\sum_{j=0}^{\infty} c_{ij}\Big)\eta_t^i - (\epsilon_t - \epsilon_{t-1}),\ \epsilon_t = \sum_{j=0}^{\infty} \widetilde{c}_{ij}\eta_{t-j}^i,\ \widetilde{c}_{ij} = \sum_{h=j+1}^{\infty} c_h.$$

Let $R_t^i = S_t^i - \Big(\sum_{j=0}^{\infty} c_{ij}\Big)\sum_{j=1}^{t}\eta_j^i = \epsilon_t - \epsilon_0$. Then

$$\sup_{1\le t\le n}\left|S_t^i - \Big(\sum_{j=0}^{\infty} c_{ij}\Big)\sum_{j=1}^{t}\eta_j^i\right|^2 = O(1) \tag{S.66}$$

Since $\{\eta_t^i\}$ is an i.i.d sequence with $\mathrm{E}|\eta_j^i|^q < \infty$, by Theorem 4.3 of Strassen (1967), the stopping $\{\tau_t^i\}$ defined as in (S.64) is an independent sequence with $\mathrm{E}\tau_t^i = \mathrm{E}(\eta_j^i)^2$ and $\mathrm{E}|\tau_t^i|^{q/2} < \infty$. Thus, $\sup_{1\le t\le n}\mathrm{E}|\sum_{j=1}^{t}[\tau_j^i - \mathrm{E}(\eta_j^i)^2]|^{q/2} = O(n + n^{q/4})$. As a result, we have for $q_0 = \min\{q, 4\}$,

$$\sup_{1\le t\le n}\mathrm{E}|\sum_{j=1}^{t}[\tau_j^i - \mathrm{E}(\eta_j^i)^2]| = O(n^{2/q_0}).$$

Let $a_i = \mathrm{E}(\eta_j^i)^2$, then on a richer space there exist a standard Brownian motion $W(t)$ such that

$$\mathrm{E}(\sum_{j=1}^{t}\eta_j^i - W(a_i t))^2 = O(n^{2/q_0}). \tag{S.67}$$

Thus, by (S.66), (S.67), Condition 2(i) holds for $\tau = 1/q_0$. It is easy to show that $\sup_{p-r<j\le p}\sum_{s,t=1}^{n}|\mathrm{E}(\varepsilon_t^j\varepsilon_s^j)|$ $= O(n)$. Thus, Condition 2(iii) follows by the independence of the components. Condition 2(ii) can be shown similarly to Remark 5(ii). □

Proof of Remark 6. It is easy to get $\lambda^* = O_e(1)$ when $p - r$ is fixed. We only show the case when $m := p - r \to \infty$ as $n \to \infty$. Let $d = d_{\min}$, ξ_t^i be $I(1)$ process defined as in Lemma 9, $\bar{\boldsymbol{\xi}}^i = (\bar{\xi}^i, \cdots, \bar{\xi}^i)'$ and $\mathbf{e} = (1, \cdots, 1)'$ be two n dimensional vectors. Let $\mathbf{E}_n$ and $\mathbf{\Pi}_n$ be $n\times n$ matrices given by

$$\mathbf{E}_n = \mathbf{I}_n - \frac{1}{n}\mathbf{e}\mathbf{e}' \text{ and } \mathbf{\Pi}_n = \begin{pmatrix} 1 & 0 & 0 & & 0\cdots & 0 \\ -1 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix},$$

then for any $1 \le i, j \le m$,

$$\begin{aligned} \boldsymbol{\xi}^i &= (\xi_1^i, \cdots, \xi_n^i)' = \sigma_{ii}\mathbf{\Pi}_n^{-d}(v_1^i, v_2^i, \cdots, v_n^i)' =: \sigma_{ii}\mathbf{\Pi}_n^{-d}\mathbf{V}^i, \text{ and} \\ (\boldsymbol{\xi}^i - \bar{\boldsymbol{\xi}}^i)'(\boldsymbol{\xi}^j - \bar{\boldsymbol{\xi}}^j) &= \sigma_{ii}\sigma_{jj}(\mathbf{V}^i)'(\mathbf{\Pi}_n^{-d})'\mathbf{E}_n\mathbf{E}_n'\mathbf{\Pi}_n^{-d}\mathbf{V}^j. \end{aligned}$$

Let $\delta_1 \leq \cdots \leq \delta_n$ and $\gamma_1 \geq \cdots \geq \gamma_n$ be the eigenvalues of $\mathbf{\Pi}_n\mathbf{\Pi}_n'$ and $(\mathbf{\Pi}_n^{-d})'\mathbf{E}_n\mathbf{E}_n'\mathbf{\Pi}_n^{-d}$ respectively. Since $\lambda_1(\mathbf{E}_n\mathbf{E}_n') = \cdots = \lambda_{n-1}(\mathbf{E}_n\mathbf{E}_n') = 1$, by Theorem 9 of Merikoski and Kumar (2004), it follows that

$$\delta_{i+1}^{-d} = \lambda_{i+1}((\mathbf{\Pi}_n^{-d})'\mathbf{\Pi}_n^{-d})\lambda_{n-1}(\mathbf{E}_n\mathbf{E}_n') \leq \gamma_i \leq \lambda_i((\mathbf{\Pi}_n^{-d})'\mathbf{\Pi}_n^{-d})\lambda_1(\mathbf{E}_n\mathbf{E}_n') = \delta_i^{-d}. \tag{S.68}$$

Further, $\delta_k = 2 - 2\cos(2k\pi/(2n+1))$, $k = 1, 2, \cdots, n$ (see Yueh (2005)), which implies

$$\delta_k \sim 4(k\pi/(2n+1))^2, \text{ as } k/n \to 0. \tag{S.69}$$

Let $\mathbf{U}$ be an orthogonal matrix with row vectors $\mathbf{u}_1, \cdots, \mathbf{u}_n$ such that $(\mathbf{\Pi}_n^{-d})'\mathbf{E}_n\mathbf{E}_n'\mathbf{\Pi}_n^{-d} = \mathbf{U}\text{diag}(\gamma_1, \cdots, \gamma_n)\mathbf{U}'$ and let $\mathbf{\Omega} = (\mathbf{V}^1, \cdots, \mathbf{V}^m)$. For $\mathbf{x} \in \mathcal{R}^m$ with $\mathbf{x}'\mathbf{x} = 1$, define $\mathbf{U\Omega x} = (b_{1\mathbf{x}}, \cdots, b_{n\mathbf{x}})' = \mathbf{b}_\mathbf{x} \in \mathcal{R}^n$. By (S.68), we have

$$\begin{aligned}
& \lambda_{\min}\left(\frac{1}{n^{2d}}\sum_{t=1}^n(\boldsymbol{\xi}_t - \bar{\boldsymbol{\xi}})(\boldsymbol{\xi}_t' - \bar{\boldsymbol{\xi}}')\right) \\
= \; & \lambda_{\min}\left(\frac{1}{n^{2d}}(\boldsymbol{\xi}^1 - \bar{\boldsymbol{\xi}}^1, \cdots, \boldsymbol{\xi}^m - \bar{\boldsymbol{\xi}}^m)'(\boldsymbol{\xi}^1 - \bar{\boldsymbol{\xi}}^1, \cdots, \boldsymbol{\xi}^m - \bar{\boldsymbol{\xi}}^m)\right) \\
\geq \; & \{\min_i(\sigma_{ii})\}^2 \min_\mathbf{x} \frac{1}{n^{2d}}\mathbf{x}'(\mathbf{U\Omega})'\text{diag}(\gamma_1, \cdots, \gamma_n)(\mathbf{U\Omega})\mathbf{x} \\
= \; & \{\min_i(\sigma_{ii})\}^2 \min_\mathbf{x} \frac{(\mathbf{b}_\mathbf{x}'\mathbf{b}_\mathbf{x})}{n^{2d}} \frac{\mathbf{b}_\mathbf{x}'(\gamma_1, \cdots, \gamma_n)\mathbf{b}_\mathbf{x}}{\mathbf{b}_\mathbf{x}'\mathbf{b}_\mathbf{x}} \\
\geq \; & \{\min_i(\sigma_{ii})\}^2 n^{1-2d}\lambda_{\min}(\mathbf{\Omega}'\mathbf{\Omega}/n) \min_\mathbf{x}\left(\sum_{l=1}^k \delta_{k+1}^{-d} b_{l\mathbf{x}}^2 / \sum_{l=1}^n b_{l\mathbf{x}}^2\right) \\
\geq \; & \{\min_i(\sigma_{ii})\}^2 (k/n^{2d})[\lambda_{\min}(\mathbf{\Omega}'\mathbf{\Omega}/n)/\lambda_{\max}(\mathbf{\Omega}'\mathbf{\Omega}/n)]\delta_{k+1}^{-d} \min_\mathbf{x} \frac{1}{k}\sum_{l=1}^k b_{l\mathbf{x}}^2 \\
= \; & O_e(k^{1-2d})\lambda_{\min}\{[\mathbf{\Omega}'(\mathbf{u}_1', \cdots, \mathbf{u}_k')][(\mathbf{u}_1', \cdots, \mathbf{u}_k')'\mathbf{\Omega}]\},
\end{aligned} \tag{S.70}$$

where $\lambda_{\min}$, $\lambda_{\max}$ denote the smallest and largest eigenvalues of a matrix respectively, the last equation follows by (S.69) and the fact that there exist two positive constants C_1, C_2 such that $C_1 < \lambda_{\min}(\mathbf{\Omega}'\mathbf{\Omega}/n) \leq \lambda_{\max}(\mathbf{\Omega}'\mathbf{\Omega}/n) < C_2$ in probability when $p/n^{1/2} \to 0$. Since U is orthogonal and the elements of $\mathbf{\Omega}$ are independent standard normal variables, it follows that the elements of $(\mathbf{u}_1', \cdots, \mathbf{u}_k')'\mathbf{\Omega}$ are independent and standard normal variables, thus by Theorem 2 of Bai and Yin (1993), we have if $m/k \in (0, 1)$,

$$\lambda_{\min}\{[\mathbf{\Omega}'(\mathbf{u}_1', \cdots, \mathbf{u}_k')][(\mathbf{u}_1', \cdots, \mathbf{u}_k')'\mathbf{\Omega}]\} = (1 - \sqrt{m/k})^2, \text{ a.s.} \tag{S.71}$$

Taking $k = 2m$, then by (S.70) and (S.71), we have

$$\lambda_{\min}\left(\frac{1}{n^{2d}}\sum_{t=1}^n(\boldsymbol{\xi}_t - \bar{\boldsymbol{\xi}})(\boldsymbol{\xi}_t' - \bar{\boldsymbol{\xi}}')\right) \geq Cm^{1-2d}. \tag{S.72}$$

Since $||\frac{1}{n^{2d}}\sum_{t=1}^n(\boldsymbol{\xi}_t - \bar{\boldsymbol{\xi}})(\boldsymbol{\xi}_t' - \bar{\boldsymbol{\xi}}') - \int_0^1 \mathbf{F}(t)\mathbf{F}'(t)\,dt||_2 = o_p(1)$ (see Lemma 9), Remark 4 follows from (S.72). □

S.3 Numerical studies for j_0 and m

To evaluate the impact of the choice of j_0 and m, we report some of our simulation results in this subsection.

We first present a numerical result for the choice j_0. We let $\mathbf{x}_{t2}$ in model (2.1) consist of r stationary AR(1) processes with coefficients generated independently from $U(-0.8, 0.8)$, $\mathbf{x}_{t1}$ be $p - r$ ARIMA(1,1,1) processes with coefficients generated from $U(0, 0.8)$ and $U(-0.8, 0.8)$. Let the elements of $\mathbf{A}$ be generated independently from $U(-3, 3)$. We estimate the cointegration rank r by the ACF unit-root test defined in (2.5). For each setting, we replicate the exercise 500 times with sample size $n = 300$, 500, 1000 and j_0 ranging from 5 to 90. The relative frequencies for the occurrence of events $\{\widehat{r} = r\}$ and the average distance between the true cointegrating space and its true space are reported in Table S.1. It is shown from Table S.1 that the performance is stable with respect to the choice of j_0, especially for small p and large n.

Table S.1: Relative frequencies (RF) of the occurrence of event $\{\widehat{r} = r\}$ and average distance D_1 with j_0 ranging over $(5, 90)$ and 500 replications.

(p,r)	n		j=5	10	20	30	40	50	60	70	80	85	90
(4, 2)	300	RF	.950	.928	.938	.960	.940	.924	.928	.948	.934	.926	.926
		D1	.062	.069	.057	.047	.065	.065	.070	.053	.064	.068	.068
	500	RF	.982	.982	.986	.978	.984	.974	.978	.980	.978	.976	.988
		D1	.029	.024	.023	.029	.027	.036	.028	.030	.029	.030	.024
	1000	RF	.992	.994	.996	.996	.994	.998	1.00	1.00	.994	.998	.998
		D1	.013	.012	.013	.012	.011	.014	.008	.009	.014	.012	.009
(6, 3)	300	RF	.822	.794	.834	.810	.802	.826	.828	.794	.834	.816	.812
		D1	.128	.137	.115	.131	.136	.121	.122	.136	.120	.124	.129
	500	RF	.934	.948	.946	.938	.960	.962	.964	.970	.952	.960	.958
		D1	.061	.053	.053	.062	.052	.049	.047	.045	.050	.045	.050
	1000	RF	.988	.990	.994	.994	.976	.984	.992	.994	.988	.994	.994
		D1	.024	.018	.018	.018	.026	.022	.017	.017	.021	.020	.020
(8, 4)	300	RF	.562	.564	.578	.628	.612	.648	.620	.646	.592	.598	.610
		D1	.230	.224	.223	.204	.211	.198	.209	.194	.217	.213	.213
	500	RF	.874	.886	.858	.908	.884	.920	.910	.934	.900	.898	.914
		D1	.093	.078	.101	.078	.085	.077	.077	.067	.081	.081	.078
	1000	RF	.966	.978	.986	.980	.984	.986	.986	.986	.988	.988	.990
		D1	.046	.031	.028	.032	.030	.030	.027	.029	.031	.028	.028

Next table is reported for the choice of m. In this simulation, $\mathbf{x}_{t1}$ and $\mathbf{x}_{t2}$ are generated from model (2.1) as the previous example. We also replicate the exercise 500 times in each setting with sample size $n = 300$, 500 and 1000 and the lags number m is taken from 5 to 90. The corresponding relative frequencies for the occurrence of events $\{\widehat{r} = r\}$ and the average distance between the true cointegrating space and its true space are reported in Table S.2. It is shown from Table S.2 that a relatively small m always works well for the estimation of the cointegrating space. On the contrary, if m is selected too large, the performance is relatively poor, especially when the sample size n is relatively small. This is reasonable, because from Remark 1, we know that only when $n/m \to \infty$, $\sum_{k=1}^{m} \widehat{\rho}(k)/m - 1 \to 0$, otherwise, it is difficult to distinguish the integrated process from the stationary process, which means that m could not be selected too large, especially when n is relatively small. This simulation also confirms that $m = 20$ is usually good enough for the procedure.

In our simulation, we also use a data driven procedure in selecting m, which is given by

$$\widehat{m} = \operatorname{argmin}_m \{f(m) \vee f^{-1}(m)\}$$

where $T(m) = \left\{ \frac{n}{m(m+1)} \sum_{i=1}^{p} \sum_{k=1}^{m} (1 - \widehat{\rho}_i(k)) \right\}$ and $f(m) = T(m)/T(m+1)$. It also work reasonably, for example, when $(p, r) = (6, 3)$, the corresponding relative correct frequencies and average distance is $(0.776, 0.119)$ for $n = 300$, $(0.888, 0.062)$ for $n = 500$ and $(0.970, 0.0229)$ for $n = 1000$.

Table S.2: Relative frequencies (RF) of the occurrence of event $\{\widehat{r} = r\}$ and average distance D_1 with m ranging over $(5, 90)$ and 500 replications.

(p,r)	n		j=5	10	20	30	40	50	60	70	80	85	90
(3, 2)	300	RF	.872	.976	.980	.968	.936	.882	.816	.744	.644	.602	.586
		D1	.016	.021	.025	.031	.048	.078	.115	.156	.211	.235	.243
	500	RF	.864	.980	.998	.998	.996	.986	.968	.942	.914	.896	.866
		D1	.008	.009	.010	.010	.011	.017	.026	.041	.056	.067	.084
	1000	RF	.874	.980	.996	.996	.996	.994	.994	.992	.992	.992	.992
		D1	.006	.005	.006	.006	.006	.007	.007	.008	.008	.008	.008
(6, 4)	300	RF	.850	.942	.916	.818	.702	.560	.398	.250	.140	.102	.070
		D1	.043	.050	.064	.101	.149	.211	.283	.349	.403	.422	.441
	500	RF	.848	.966	.972	.958	.916	.854	.762	.660	.568	.518	.462
		D1	0.024	.026	.029	.034	.051	.076	.115	.160	.200	.223	.248
	1000	RF	.842	.974	.996	.996	.992	.984	.976	.968	.942	.928	.906
		D1	.011	.012	.013	.013	.014	.017	.020	.023	.034	.039	.049
(9, 6)	300	RF	.802	.848	.732	.510	.290	.118	.040	.004	0	0	0
		D1	.083	.095	.128	.203	.286	.363	.414	.452	.476	.483	.494
	500	RF	.800	.944	.930	.896	.792	.638	.438	.294	.186	.132	.104
		D1	.039	.045	.052	.063	.098	.154	.227	.286	.334	.359	.375
	1000	RF	.856	.984	.990	.980	.980	.964	.946	.914	.862	.834	.786
		D1	.018	.019	.019	.021	.021	.026	.032	.044	.062	.072	.089

References

[1] Bai, Z. D. and Yin, Y. Q. (1993). Limit of smallest eigenvalue of a large dimensional sample covariance matrix. *The Annals of Probability*, **21**, 1275–1294.

[2] Chan, N. H. and Wei, C. Z. (1988). Limiting distributions of least squares estimates of unstable autoregressive processes. *The Annals of Statistics*, **16**, 367–401.

[3] Hall, P. and Heyde, C. C. (1980). *Martingale Limit Theory and Its Application*. Academic Press, New York.

[4] Lin, Z. and Lu, C. (1997). *Limit Theory on Mixing Dependent Random Variables*. Kluwer Academic Publishers, 1997.

[5] Ling, S. and Li, W. K. (1998). Limiting distributions of maximum likelihood estimators for unstable autoregressive moving-average time series with general autoregressive heteroscedastic errors. *The Annals of Statistics*, **26**, 84–125.

[6] Merikoski, J. K. and Kumar, R. (2004). Inequalities for spreads of matrix sums and products. *Applied Mathematics E-Notes*, **4**, 150–159.

[7] Strassen, V. (1967). Almost sure behaviour of sums of independent random variables and martingales. *Proceedings of the Fifth Berkeley Symposium of Mathematical Statistics and Probability*, **2**, 315-343. University of California Press, Berkeley.

[8] Wu, W. B. (2007). Strong invariance principles for dependent random variables. *The Annals of Probability*, **35**, 2294–2320.

[9] Yueh, W. C. (2005). Eigenvalues of several tridiagonal matrices. *Applied Mathematics E-Notes*, **5**, 66–74.