Running head: SUPPLEMENTAL MATERIAL

Supplemental Material

Author

Affiliation

Supplemental Material

Appendix A

When y_{it}^T in the Equation (1) of the main text has an AR(1) process, $y_{it}^O - \mu_i$ has a ARMA(1,1) process. To prove this, we introduce a new variable v_{it}, and rewrite $y_{it}^O - \mu_i$ in a ARMA(1,1) form. That is, we replace the function of measurement error e_{it} and shock variable z_{it} by a MA(1) process of v_{it},

$$\begin{aligned}(y_{it}^O - \mu_i) - \phi_i(y_{i(t-1)}^O - \mu_i) &= e_{it} - \phi_i e_{i(t-1)} + z_{it} \\ &= v_{it} + \theta_i v_{i(t-1)}.\end{aligned} \tag{1}$$

To simplify the equation, we denote $m_{it} = (y_{it}^O - \mu_i) - \phi_i(y_{i(t-1)}^O - \mu_i)$. Based on the first line of Equation (1), the autocovariance of m_{it} can be calculated based on ϕ_i, e_{it}, and z_{it},

$$\begin{aligned}\gamma_{mi}(0) &= var(e_{it} - \phi_i e_{i(t-1)} + z_{it})^2 \\ &= \sigma_z^2 + (1 + \phi_i^2)\sigma_e^2,\end{aligned} \tag{2}$$

$$\begin{aligned}\gamma_{mi}(1) &= cov(e_{it} - \phi_i e_{i(t-1)} + z_{it}, e_{i(t+1)} - \phi_i e_{it} + z_{i(t+1)}) \\ &= -\phi_i \sigma_e^2.\end{aligned} \tag{3}$$

On the other hand, based on the second line of Equation (1), m_{it} is also equal to $v_{it} + \theta_i v_{i(t-1)}$, which is a MA(1) process, thus the autocovariance of m_{it} also can be calculated based on v_{it} and θ_i,

$$\gamma_{mi}(0) = \sigma_{vi}^2(1 + \theta_i^2), \tag{4}$$

$$\gamma_{mi}(1) = \theta_i \sigma_{vi}^2. \tag{5}$$

Solve the Equations (2), (3), (4), and (5), we can calculate the coefficients for the ARMA(1,1) process

$$\theta_i = \frac{\gamma_{mi}(0) - \sqrt{\gamma_{mi}^2(0) - 4\gamma_{mi}^2(1)}}{2\gamma_{mi}(1)},$$

$$\sigma_{vi}^2 = \frac{\gamma_{mi}(1)}{\theta_i}.$$

Thus the moving average coefficient of individual i (θ_i) and the variance of v_{it} are respectively

$$\theta_i = \frac{\sigma^2_{zi} + (1+\phi^2_i)\sigma^2_e - \sqrt{\sigma^4_{zi} + \sigma^4_e + \phi^4_i\sigma^4_e + 2\sigma^2_{zi}\sigma^2_e + 2\phi^2_i\sigma^2_{zi}\sigma^2_e - 2\phi^2_i\sigma^4_e}}{-2\phi_i\sigma^2_e}, \tag{6}$$

$$\sigma^2_{vi} = \frac{-\phi_i\sigma^2_e}{\theta_i}. \tag{7}$$

As a result, the population autocorrelation at lag h of y^O_{it} is $\rho_{Oyi}(h) = \phi_i^{|h|-1}\rho_{Oyi}(1)$ ($h \geq 2$). And the population variance, covariance at lag 1, and autocorrelation at lag 1 of y^O_{it} for individual i can be expressed by the coefficients for the ARMA(1,1) process (i.e., θ_i, ϕ_i, and σ^2_{vi}) respectively

$$\sigma^2_{Oyi} = \sigma^2_{vi}[1 + \frac{(\phi_i+\theta_i)^2}{1-\phi^2_i}], \tag{8}$$

$$\gamma_{Oyi}(1) = \sigma^2_{vi}[(\phi_i+\theta_i) + \frac{(\phi_i+\theta_i)^2\phi_i}{1-\phi^2_i}], \tag{9}$$

$$\rho_{Oyi}(1) = \frac{(\phi_i+\theta_i)(1+\phi_i\theta_i)}{1+2\phi_i\theta_i+\theta^2_i}. \tag{10}$$

Appendix B

Since $ISD^2_i = S^2(y^T_{it} + e_{it})$, the reliability of ISD^2 is

$$\rho^2_{ISD^2} = \left(\frac{cov(S^2(y^T_{it}+e_{it}), \sigma^2_i)}{\sqrt{var(S^2(y^T_{it}+e_{it}))var(\sigma^2_i)}}\right)^2. \tag{11}$$

The numerator is

$$\begin{aligned}
\left(cov(S^2(y^T_{it}+e_{it}), \sigma^2_i)\right)^2 &= \left(E\left[S^2(y^T_{it}+e_{it})\sigma^2_i\right] - E\left(S^2(y^T_{it}+e_{it})\right)E(\sigma^2_i)\right)^2 \\
&= \left(E\left[E(S^2(y^T_{it}+e_{it})\sigma^2_i \mid i)\right] - E\left(S^2(y^T_{it}+e_{it}) \mid i\right)E(\sigma^2_i)\right)^2 \\
&= \left(E\left[\sigma^2(y^T_{it}+e_{it})\sigma^2_i\right] - E\left(\sigma^2(y^T_{it}+e_{it})\right)E(\sigma^2_i)\right)^2 \\
&= \left(cov\left(\sigma^2(y^T_{it}+e_{it}), \sigma^2_i\right)\right)^2 \\
&= \left(var(\sigma^2_i)\right)^2.
\end{aligned}$$

Therefore we can rewrite Equation (11) as

$$\rho^2_{ISD^2} = \frac{var(\sigma^2_i)}{var\left(S^2(y^T_{it}+e_{it})\right)}. \tag{12}$$

The denominator is complex. The step by step derivation is as follows,

$$var\left(S^2(y_{it}^T+e_{it})\right) \quad = \quad E\left(S^4(y_{it}^T+e_{it})\right)-E^2\left(S^2(y_{it}^T+e_{it})\right). \tag{13}$$

It is easy to obtain $E^2\left(S^2(y_{it}^T+e_{it})\right)=\left(E(\sigma_i^2)+\sigma_e^2\right)^2$, and then we solve $S^4(y_{it}^T+e_{it})$,

$$\begin{aligned}
S^4(y_{it}^T+e_{it}) &= \left(\frac{\sum\limits_{t=1}^{T}(y_{it}^T+e_{it}-(\bar{y}_{it}^T+\bar{e_{it}}))^2}{T-1}\right)^2 \\
&= \frac{\left[T\sum\limits_{t=1}^{T}\left(y_{it}^T+e_{it}\right)^2\right]^2-2T\left(\sum\limits_{t=1}^{T}\left(y_{it}^T+e_{it}\right)^2\right)\left(\sum\limits_{t=1}^{T}\left(y_{it}^T+e_{it}\right)\right)^2+\left(\sum\limits_{t=1}^{T}\left(y_{it}^T+e_{it}\right)\right)^4}{T^2(T-1)^2} \\
&= \frac{T^2\sum\limits_{t1=1}^{T}\sum\limits_{t2=1}^{T}\left(y_{it1}^T+e_{it1}\right)^2\left(y_{it2}^T+e_{it2}\right)^2-2T\left(\sum\limits_{t1=1}^{T}\left(y_{it1}^T+e_{it1}\right)^2\right)\sum\limits_{t2=1}^{T}\sum\limits_{t3=1}^{T}\left(y_{it2}^T+e_{it2}\right)\left(y_{it3}^T+e_{it3}\right)}{T^2(T-1)^2} \\
&\quad \frac{+\sum\limits_{t1=1}^{T}\sum\limits_{t2=1}^{T}\sum\limits_{t3=1}^{T}\sum\limits_{t4=1}^{T}\left(y_{it1}^T+e_{it1}\right)\left(y_{it2}^T+e_{it2}\right)\left(y_{it3}^T+e_{it3}\right)\left(y_{it4}^T+e_{it4}\right)}{}.
\end{aligned} \tag{14}$$

Because y_{it}^T is a Gaussian process, when $|\phi_i|<1$, it is a strictly stationary Gaussian process. All of the multivariate functions for y_{it}^T's agree with their counterparts for any number of time lags shifted (e.g., $E\left(\left(y_{i1}^T\right)^2\left(y_{i2}^T\right)^2\right)=E\left(\left(y_{i2}^T\right)^2\left(y_{i3}^T\right)^2\right)$). And y_{it}^T has the same moments as a random variable in a normal distribution, but y_{it}^T is not independently distributed (i.e., $E\left(\left(y_{it}^T\right)^2\mid i\right)=\sigma_i^2=\frac{\sigma_{zi}^2}{1-\phi_i^2}$, $E(\left(y_{it}^T\right)^3\mid i)=0$, $E\left(\left(y_{it}^T\right)^4\mid i\right)=3\sigma_i^4=3\left(\frac{\sigma_{zi}^2}{1-\phi_i^2}\right)^2$). The expectation of the first numerator component of $S^4(y_{it}^T+e_{it})$ is

$$\begin{aligned}
T^2E\left[\sum_{t1=1}^{T}\sum_{t2=1}^{T}\left(y_{it1}^T+e_{it1}\right)^2\left(y_{it2}^T+e_{it2}\right)^2\right] &= T^2E\left[\sum_{t1=1}^{T}\sum_{t2=1}^{T}\left(y_{it1}^T\right)^2\left(y_{it2}^T\right)^2+\sum_{t1=1}^{T}\sum_{t2=1}^{T}e_{it1}^2e_{it2}^2\right. \\
&\quad \left.+4\sum_{t1=1}^{T}\sum_{t2=1}^{T}y_{it1}^Ty_{it2}^Te_{it1}e_{it2}+2\sum_{t1=1}^{T}\sum_{t2=1}^{T}\left(y_{it1}^T\right)^2e_{it2}^2\right] \\
&= T^2E\left(\sum_{t1=1}^{T}\sum_{t2=1}^{T}\left(y_{it1}^T\right)^2\left(y_{it2}^T\right)^2\right)+\left(T^4+2T^3\right)\sigma_e^4 \\
&\quad +\left(4T^3+2T^4\right)\sigma_e^2E\left(\sigma_i^2\right),
\end{aligned} \tag{15}$$

where

$$
\begin{aligned}
E\left(\sum_{t1=1}^{T}\sum_{t2=1}^{T}\left(y_{it1}^{T}\right)^{2}\left(y_{it2}^{T}\right)^{2}\right) &= TE\left(\left(y_{it}^{T}\right)^{4}\right)+2(T-1)E\left(\left(y_{i1}^{T}\right)^{2}\left(y_{i2}^{T}\right)^{2}\right)+2(T-2)E\left(\left(y_{i1}^{T}\right)^{2}\left(y_{i3}^{T}\right)^{2}\right) \\
&\quad +\cdots+2E\left(\left(y_{i1}^{T}\right)^{2}\left(y_{iT}^{T}\right)^{2}\right) \\
&= 3TE\left(\sigma_i^4\right)+6E\left(\sigma_i^4\left(\frac{(T-1)\phi_i^2-\phi_i^{2T}}{1-\phi_i^2}-\frac{\phi_i^4(1-\phi_i^{2(T-2)})}{\left(1-\phi_i^2\right)^2}\right)\right) \\
&\quad +2E\left(\sigma_i^4\left(\frac{\left(1-\phi_i^2\right)T(T-1)}{2(1-\phi_i)^2}+\frac{(T-1)\phi_i^2-\phi_i^{2T}}{(1-\phi_i)^2}-\frac{\phi_i^4\left(1-\phi_i^{2(T-2)}\right)}{\left(1-\phi_i^2\right)^2}\right.\right. \\
&\quad \left.\left. -\frac{\left(2(T-1)\phi_i-\phi_i^T\right)}{1-\phi_i}-\frac{2\phi_i^2\left(1-\phi_i^{T-2}\right)}{(1-\phi_i)^2}\right)\right) \\
\text{when } \sigma_i^2 \text{ is independent of } \phi_i &= 3TE\left(\sigma_i^4\right)+6E\left(\sigma_i^4\right)E\left(\frac{(T-1)\phi_i^2-\phi_i^{2T}}{1-\phi_i^2}-\frac{\phi_i^4(1-\phi_i^{2(T-2)})}{\left(1-\phi_i^2\right)^2}\right) \\
&\quad +2E\left(\sigma_i^4\right)E\left(\frac{\left(1-\phi_i^2\right)T(T-1)}{2(1-\phi_i)^2}+\frac{(T-1)\phi_i^2-\phi_i^{2T}}{(1-\phi_i)^2}-\frac{\phi_i^4\left(1-\phi_i^{2(T-2)}\right)}{\left(1-\phi_i^2\right)(1-\phi_i)^2}\right. \\
&\quad \left. -\frac{\left(1-\phi_i^2\right)\left(2(T-1)\phi_i-\phi_i^T\right)}{(1-\phi_i)^3}-\frac{2\phi_i^2\left(1-\phi_i^2\right)\left(1-\phi_i^{T-2}\right)}{(1-\phi_i)^4}\right).
\end{aligned}
\tag{16}
$$

The expectations of the second and third numerator components of $S^4(y_{it}^T+e_{it})$ can be obtained in the same way. They have much more tedious forms but still functions of σ_i^2, σ_e^2, T, and ϕ_i. Especially, when σ_i^2 is independent of ϕ_i, $E\left(S^4(y_{it}^T+e_{it})\right)$ is a function of $E(\sigma_i^4)$ (implies $var(\sigma_i^2)$), $E(\sigma_i^2)$, σ_e^2, T, and distribution of ϕ_i. And σ_e^2 is a function of $E(\sigma_i^2)$, σ_μ^2, and ρ_y^2. Because of Equations (12), (13), (14), (15), and (16), overall the reliability of ISD^2 depends on the joint distribution of σ_i^2 and ϕ_i, and values of σ_μ^2, T, and ρ_y^2.

Appendix C

The asymptotic distribution for $\hat{\rho}_i(1)$ is given by Shumway & Stoffer (2010),

$$\hat{\rho}_i(1) \sim AN(\rho_{Oyi}(1), \frac{1}{n}w_i).$$

For the time series model in the current study,

$$
\begin{aligned}
w_i &= \sum_{u=-\infty}^{\infty}\left[\rho_{Oyi}(u+1)^2+\rho_{Oyi}(u-1)\rho_{Oyi}(u+1)+2\rho_{Oyi}(1)^2\rho_{Oyi}(u)^2-4\rho_{Oyi}(1)\rho_{Oyi}(u)\rho_{Oyi}(u+1)\right] \\
&= 1+2\phi_i\rho_{Oyi}(1)+\frac{7\phi_i^2-3}{1-\phi_i^2}\rho_{Oyi}^2(1)-\frac{8\phi_i}{1-\phi_i^2}\rho_{Oyi}^3(1)+\frac{4}{1-\phi_i^2}\rho_{Oyi}^4(1).
\end{aligned}
\tag{17}
$$

Appendix D

References

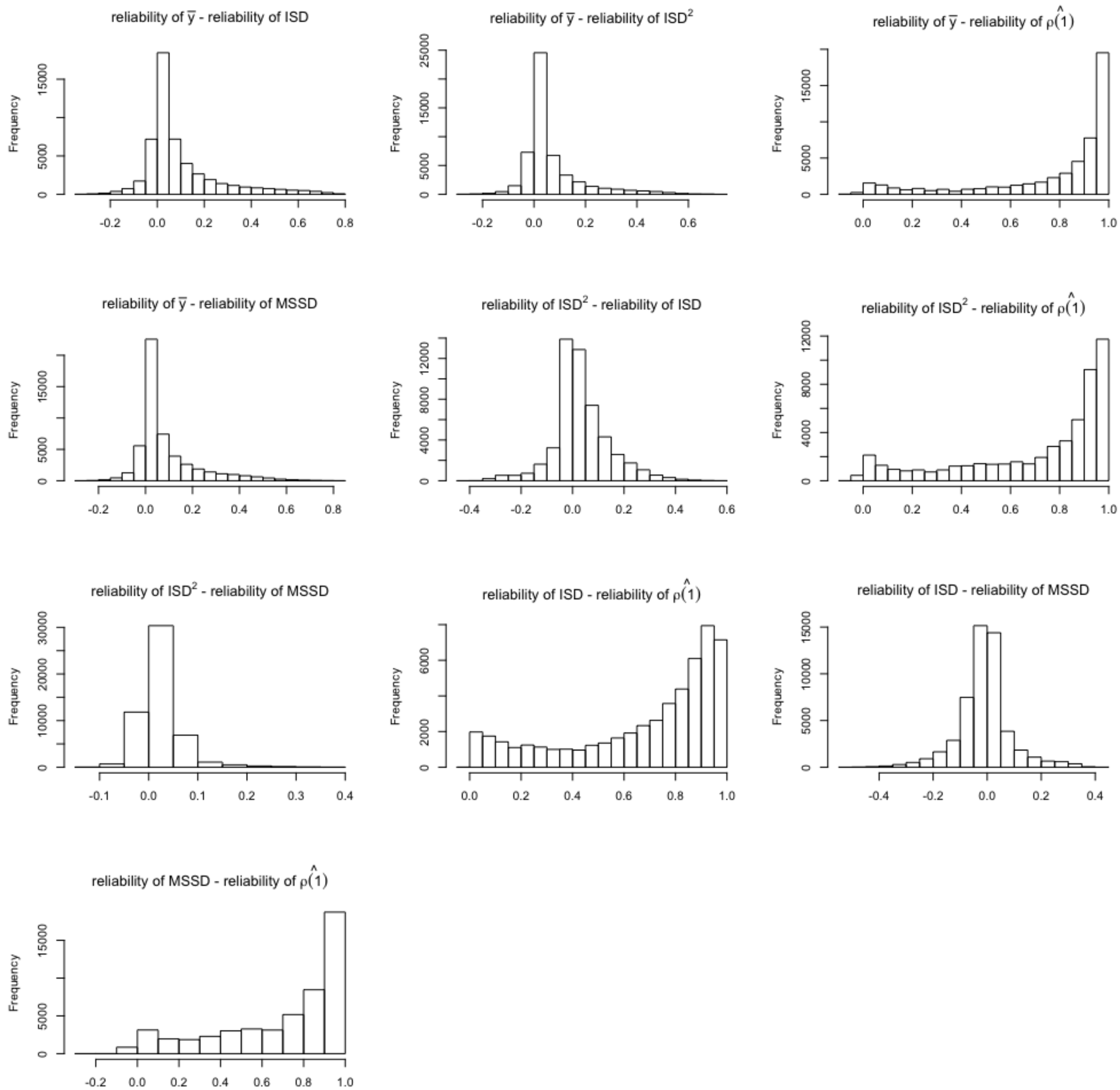
Shumway, R. H., & Stoffer, D. S. (2010). *Time series analysis and its applications: With R examples*. New York: Springer Science & Business Media.

Figure Captions

Figure 1. Comparing reliabilities of $\bar{y}$, ISD, ISD^2, $\rho(\hat{1})$, and $MSSD$ when $\sigma_\mu^2 = 0.1$

Figure 2. Comparing reliabilities of $\bar{y}$, ISD, ISD^2, $\rho(\hat{1})$, and $MSSD$ when $\sigma_\mu^2 = 5$

Figure 3. Comparing reliabilities of $\bar{y}$, ISD, ISD^2, $\rho(\hat{1})$, and $MSSD$ when $\sigma_\mu^2 = 10$



reliability of $\bar{y}$ - reliability of ISD
Frequency
reliability of $\bar{y}$ - reliability of ISD^2
Frequency
reliability of $\bar{y}$ - reliability of $\rho\hat{(1)}$
Frequency
reliability of $\bar{y}$ - reliability of MSSD
Frequency
reliability of ISD^2 - reliability of ISD
Frequency
reliability of ISD^2 - reliability of $\rho\hat{(1)}$
Frequency
reliability of ISD^2 - reliability of MSSD
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reliability of ISD - reliability of $\rho\hat{(1)}$
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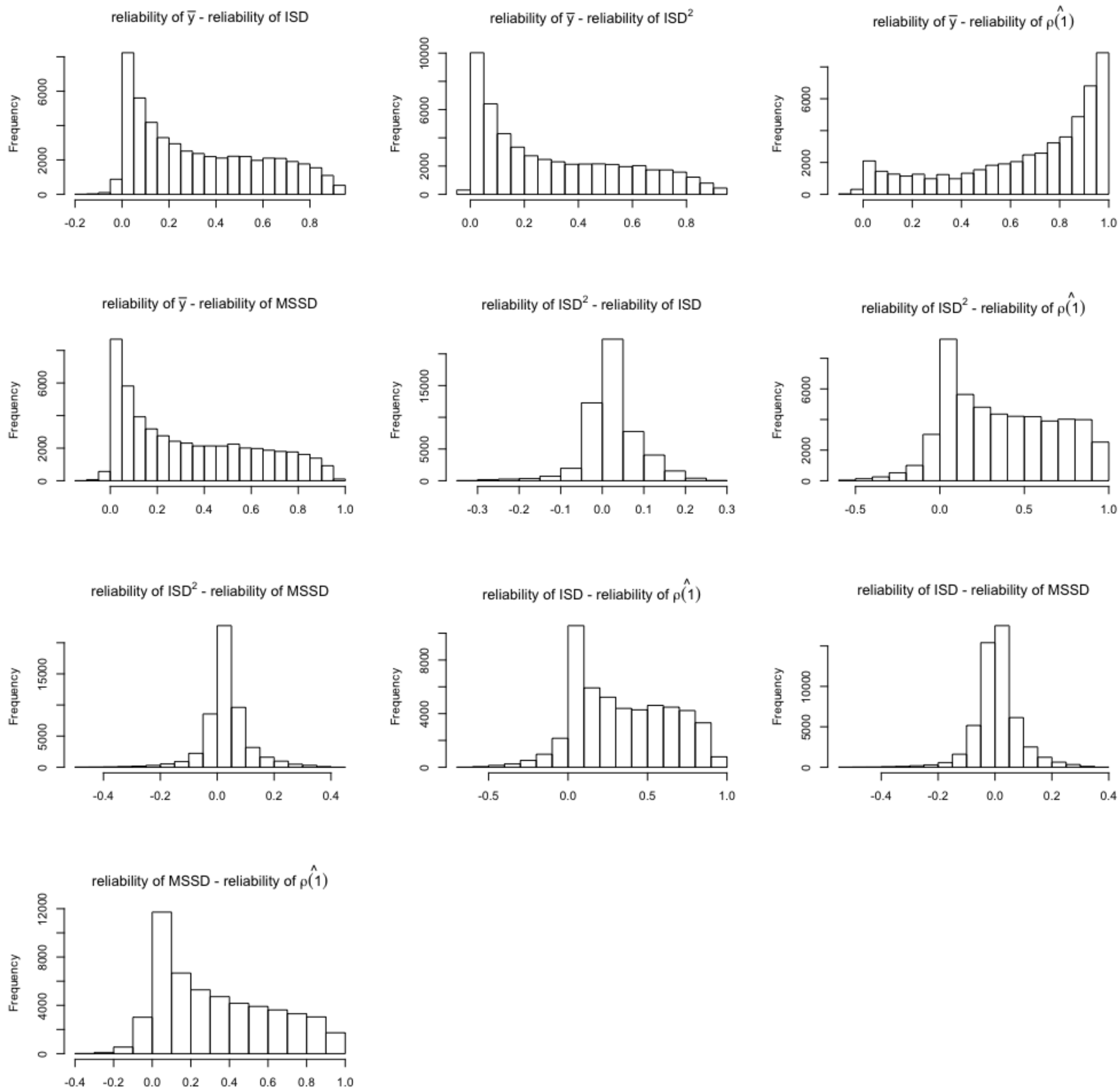
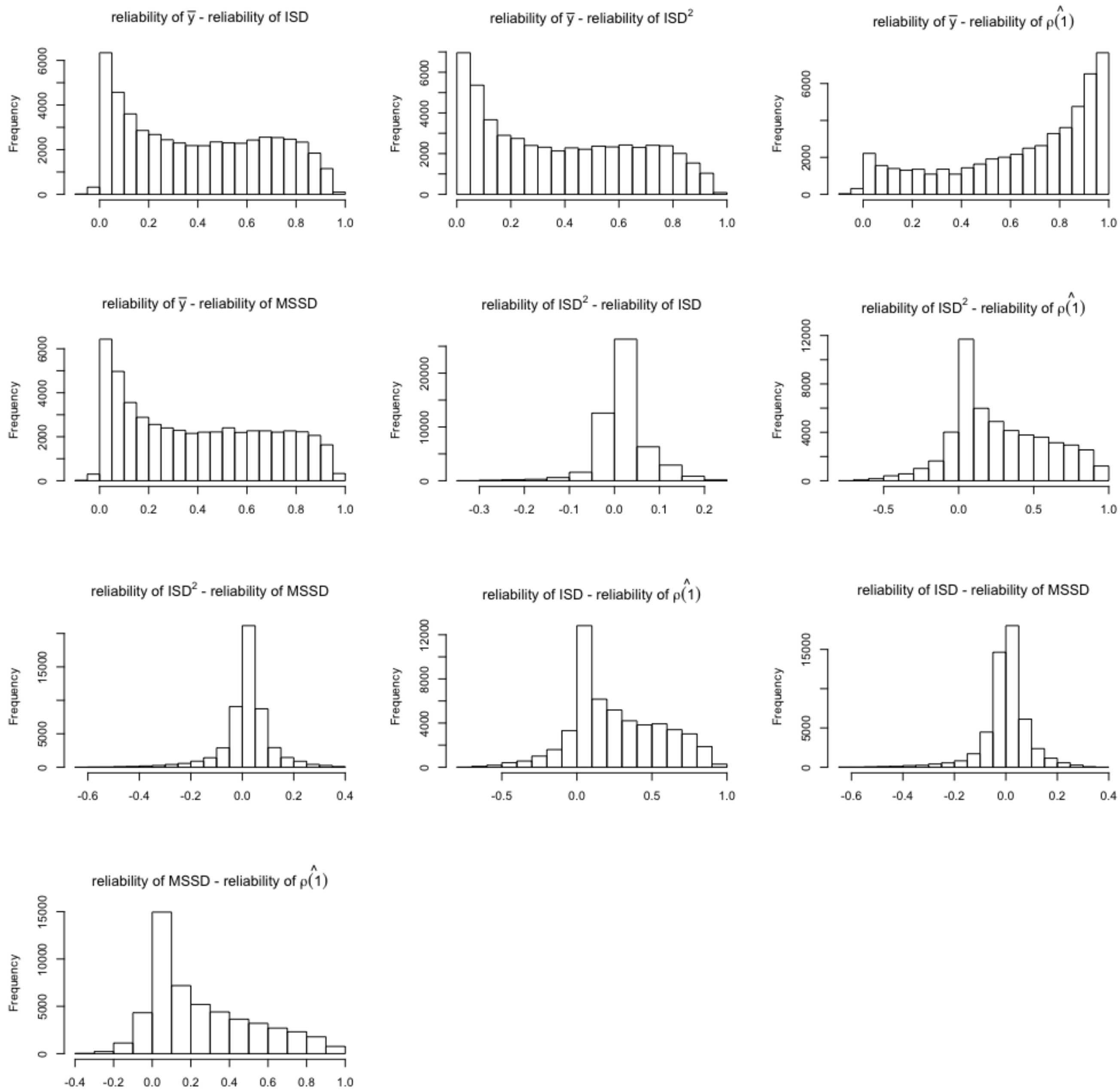
reliability of MSSD - reliability of $\rho\hat{(1)}$
Frequency


reliability of $\overline{y}$ - reliability of ISD
reliability of $\overline{y}$ - reliability of ISD^2
reliability of $\overline{y}$ - reliability of $\rho(\hat{1})$
reliability of $\overline{y}$ - reliability of MSSD
reliability of ISD^2 - reliability of ISD
reliability of ISD^2 - reliability of $\rho(\hat{1})$
reliability of ISD^2 - reliability of MSSD
reliability of ISD - reliability of $\rho(\hat{1})$
reliability of ISD - reliability of MSSD
reliability of MSSD - reliability of $\rho(\hat{1})$
Frequency



reliability of $\bar{y}$ - reliability of ISD
Frequency
reliability of $\bar{y}$ - reliability of ISD^2
Frequency
reliability of $\bar{y}$ - reliability of $\rho(\hat{1})$
Frequency
reliability of $\bar{y}$ - reliability of MSSD
Frequency
reliability of ISD^2 - reliability of ISD
Frequency
reliability of ISD^2 - reliability of $\rho(\hat{1})$
Frequency
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