Capture-Recapture Methods for Data on the Activations of Applications on Mobile Phones

This document gives additional details concerning the technical derivations and the proofs presented in section 4 of the manuscript. The following notation is used throughout the calculations. We distinguish (i) the notation used to describe the data themselves, (ii) important summary statistics and (iii) the robust design parameters for its analysis.

Capture-recapture data

- Subscript i denotes primary sampling period i, $i = 1, \ldots, I$;
- Subscript j denotes a secondary capture occasion within a PSP, $j = 1, ..., \ell_i$; ℓ_i is the number of secondary capture occasions within PSP i;
- $\boldsymbol{\omega}$ is then a $\sum \ell_i \times 1$ vector containing for secondary capture occasion j in PSP i the outcome $\omega_{ij} = 1$ if the unit has been captured on that occasion and 0 otherwise;
- The between PSP capture information can be summed up in a $I \times 1$ vector $\boldsymbol{\delta}$ which has entry $\delta_i = 1$ if the unit has been captured at least once during PSP i, that is if $\sum_j \omega_{ij} > 0$, and $\delta_i = 0$ otherwise; $\boldsymbol{\delta}(\boldsymbol{\omega})$ denotes the $I \times 1$ between PSP capture history derived from $\boldsymbol{\omega}$.

Statistics

- The frequency of the number of units that have capture history $\boldsymbol{\omega}$ is denoted n_ω ;
- u_i is the number of unmarked units captured during PSP i;

- m_i represents the number of marked units captured during PSP i;
- $n_i = u_i + m_i$ is the number of units captured during PSP i;
- v_i is the number of units captured for the last time at PSP i;
- $w_i = \sum_{j=1}^{i-1}(u_j - v_j)$ is the number of units captured at least once during the first $i-1$ PSPs that will be seen at least once more, either in PSP i or later;
- z_i represents the number of units captured before PSP i, not seen at PSP i, but captured subsequently;
- n is the total number of units captured at least once during the whole sampling process.

Parameters

- The survival probability, for all units in the population, between primary periods i and $i+1$ is denoted $\phi_i \in (0, 1)$;
- The probability of being captured at least once during PSP i is denoted $p_i^* = Pr(\delta_i = 1)$, this depends on the closed population model describing the captures within PSP i ;
- the probability of not being seen after PSP i, χ_i, satisfies the following recursive relationship, $\chi_i = (1 - \phi_i) + \phi_i(1 - p_{i+1}^*)\chi_{i+1}$; $\bar{\chi}_i = 1 - \chi_i$ is the probability of being seen after PSP i;
- $N_i, i = 1, \ldots, I$ is the expected population at the start of the i^{th} PSP ;

- B_i is the expected number of new units joining the population before the start of PSP $i+1$ such that $N_{i+1} = N_i\phi_i + B_i$.

- The expected number of unmarked units in the population just before PSP i, U_i, satisfies $U_i = U_{i-1}\left(1-p_i^*\right)\phi_{i-1} + B_{i-1}$ for $i = 1, ..., I-1$; $U_1 = N_1$;

- M_i is the expected number of marked units in the population just before PSP i such that $M_i = (M_{i-1} + U_{i-1}p_{i-1}^*)\phi_{i-1}$; $M_1 = 0$;

- $\eta_i = 1 - M_i/N_i$ is the proportion of unmarked units just before PSP i ; $\bar{\eta}_i = 1 - \eta_i$ is the proportion of marked units just before PSP i ;

- $D_i = 1 - (1-p_i^*)\chi_i\eta_i - \bar{\eta}_i\bar{\chi}_i$ is the probability, for an unmarked unit, to be captured at least once at PSP i or later and, for a marked unit, to be captured for the last time at PSP i.

For the asymptotic variance derivations, N_i is assumed large for every PSP i.

A. Evaluation of var $\left(\hat{N}_i\right)$ for model M_0

This section presents the main steps to evaluate the asymptotic variance of $\hat{N}_i$. The following notation is used throughout this section.

- $P_{2i} = 1 - (1-p_i)^{\ell_i} - \ell_i p_i(1-p_i)^{\ell_i - 1}$ is the probability of obtaining two captures or more at PSP i ,

- $p_{1i} = \ell_i p_i (1-p_i)^{\ell_i - 1}$ is the probability of being captured once at PSP i,

and note that $p_i^* = P_{2i} + p_{1i}$. In this section, we give further details to the proof of the asymptotic variance of $\hat{N}_i$,

$$\text{var}\left(\hat{N}_i\right) = N_i D_i (1-p_i^*)/(p_i^*\bar{\chi}_i\bar{\eta}_i + D_i P_{2i}). \tag{1}$$

In the derivations, we approximate the sufficient statistics u_i, v_i, and w_i by their expectations,

$$u_i \approx N_i \eta_i p_i^*, \quad v_i \approx N_i p_i^* \chi_i, \text{ and } w_i \approx N_i \bar{\eta}_i \bar{\chi}_i + N_i p_i^* \bar{\eta}_i \chi_i, \tag{2}$$

as, when N_i goes to ∞, the ratio of a statistic over its expectation converges to 1 in probability. Throughout the derivations, we will refer to the estimating equation that leads to the maximum likelihood estimator for N_i,

$$f_{i,N}(N_i) = N_i \left[n_i^*(w_i - n_i^* + u_i) / \{n_i^*(w_i - v_i) + u_i v_i\} - \{1 - C_i/(\ell_i N_i)\}^{\ell_i} \right], \tag{3}$$

where

$$n_i^* = N_i[1 - \{1 - C_i/(N_i \ell_i)\}^{\ell_i}]. \tag{4}$$

The limit in probability of the derivative of $f_{i,N}(N_i)$ with respect to N_i is proven in the next section to be

$$A_i = -(p_i^* \bar{\chi}_i \bar{\eta}_i + D_i \times P_{2i})/(\bar{\eta}_i \bar{\chi}_i). \tag{5}$$

A.1. Evaluation of A_i

Using (4), the derivative of n_i^* with respect to N_i, evaluated at the expectation, $N_i p_i \ell_i$, of C_i, is equal to P_{2i}. Now, differentiating (3) using the chain rule and applying (2) gives

$$\begin{aligned}
\frac{\partial f_{i,N}}{\partial N_i} &= N_i \frac{\partial n_i^*}{\partial N_i} \frac{\partial}{\partial n_i^*} [n_i^*(w_i - n_i^* + u_i) / \{n_i^*(w_i - v_i) + u_i v_i\}] - N_i \frac{\partial}{\partial N_i} \{1 - C_i/(N_i \ell_i)\}^{\ell_i} \\
&= -N_i \frac{\partial n_i^*}{\partial N_i} \left[\left\{ (n_i^*)^2 (w_i - v_i) + (2n_i^* - w_i - u_i) u_i v_i \right\} / \{n_i^*(w_i - v_i) + u_i v_i\}^2 \right] \\
&- (C_i/N_i) \{1 - C_i/(N_i \ell_i)\}^{\ell_i - 1} \\
&\approx -P_{2i} \{(1 - \eta_i \chi_i + p_i^* \eta_i \chi_i)/(\bar{\eta}_i \bar{\chi}_i)\} - p_{1i} = A_i,
\end{aligned}$$

where the last equality is obtained by setting $p_{1i} = p_i^* - P_{2i}$ and A_i is defined by (5).

A.2. Evaluation of Σ_i

The first three random variables (u_i, v_i, w_i) are, under Poisson sampling, Poisson random variables. Their variances are equal to their expectations given in (2) while the covariances are the expected number of units common to two random variables. Now C_i does not have a Poisson distribution. To find the elements of Σ_i for C_i, let $\tilde{N}_i$ be a Poisson random variable equal to the actual number of units in the population during PSP i. Given $\tilde{N}_i$, C_i has a binomial distribution with $\tilde{N}_i \times \ell_i$ trials and probability p_i. Conditioning on $\tilde{N}_i$ yields to the following expression for the variance of C_i,

$$\mathrm{var}(C_i) = E(\tilde{N}_i)\ell_i p_i(1-p_i) + \mathrm{var}(\tilde{N}_i)\ell_i^2 p_i^2 = N_i \ell_i p_i(\ell_i p_i + 1 - p_i).$$

The covariances with C_i are calculated in a similar way. For instance that with u_i involves $\tilde{U}_i$, the units that are not marked before the ith PSP. Given $(\tilde{N}_i, \tilde{U}_i)$ the conditional covariance between C_i and u_i is $\tilde{U}_i \ell_i p_i(1-p_i^*)$ while their respective expectations are $\tilde{U}_i \ell_i p_i$ and $\tilde{U}_i p_i^*$. Thus

$$\mathrm{cov}(C_i, u_i) = E\{\tilde{U}_i \ell_i p_i(1-p_i^*)\} + \mathrm{cov}(\tilde{U}_i \ell_i p_i, \tilde{U}_i p_i^*) = U_i \ell_i p_i = N_i \eta_i \ell_i p_i.$$

Similar derivations lead to the covariance matrix of the sufficient statistics u_i, v_i, w_i, C_i,

$$\Sigma_i = N_i \begin{pmatrix} \eta_i p_i^* & \eta_i p_i^* \chi_i & 0 & \eta_i \ell_i p_i \\ & p_i^* \chi_i & \bar{\eta}_i p_i^* \chi_i & \ell_i p_i \chi_i \\ & & \bar{\eta}_i(\bar{\chi}_i + p_i^* \chi_i) & \bar{\eta}_i \ell_i p_i \\ & & & \ell_i p_i(1 - p_i + \ell_i p_i) \end{pmatrix}. \qquad (6)$$

A.3. Derivation of $\nabla f_{i,N}$

We have,

$$\nabla f_{i,N} = \left(\frac{\partial f_{i,N}}{\partial u_i}, \frac{\partial f_{i,N}}{\partial v_i}, \frac{\partial f_{i,N}}{\partial w_i}, \frac{\partial f_{i,N}}{\partial C_i} \right).$$

Now, differentiating $f_{i,N}$ with respect to u_i and using (2) gives

$$\begin{aligned} \frac{\partial f_{i,N}}{\partial u_i} &= N_i n_i^* w_i (n_i - v_i) / \{ n_i^* (w_i - v_i) + u_i v_i \}^2 \\ &\approx (\bar{\chi}_i + p_i^* \chi_i)/(\bar{\eta}_i \bar{\chi}_i). \end{aligned}$$

All of the other derivatives are calculated in a similar way.

A.4. Evaluation of $\nabla f_{i,N}^\top \Sigma_i \nabla f_{i,N}$

The denominator $(\bar{\eta}_i \bar{\chi}_i)$ of $\nabla f_{i,N}$ simplifies with that of A_i; only the numerator is considered here. To evaluate it we calculate separately the quadratic form Q_1 involving the covariance matrix of (u_i, v_i, w_i) and Q_2, involving the variance and the covariances with C_i. As $D_i = -2\chi_i \eta_i + \chi_i + \eta_i + p_i^* \chi_i \eta_i$, patient calculations lead to

$$\begin{aligned} Q_1 &= p_i^* \{ \eta_i + \chi_i - 2\eta_i \chi_i - \eta_i^2 \chi_i - \eta_i \chi_i^2 + 2\eta_i^2 \chi_i^2 \\ &\quad + p_i^* (1 - \eta_i - \chi_i + \eta_i \chi_i + 2\eta_i^2 \chi_i + 2\eta_i \chi_i^2 - 4\eta_i^2 \chi_i^2) \\ &\quad + (p_i^*)^2 (\eta_i \chi_i - \eta_i^2 \chi_i - \eta_i \chi_i^2 + 2\eta_i^2 \chi_i^2) \} \\ &= p_i^* \{ D_i^2 + \eta_i + \chi_i - 4\eta_i \chi_i - \eta_i^2 - \chi_i^2 + 3\eta_i \chi_i^2 + 3\eta_i^2 \chi_i - 2\eta_i^2 \chi_i^2 \\ &\quad + p_i^* (1 - \eta_i)(1 - \chi_i) + (p_i^*)^2 \eta_i \chi_i (1 - \eta_i)(1 - \chi_i) \}. \end{aligned}$$

Since $\eta_i + \chi_i - 4\eta_i \chi_i - \eta_i^2 - \chi_i^2 + 3\eta_i^2 \chi_i + 3\eta_i \chi_i^2 - 2\eta_i^2 \chi_i^2 = (1 - \eta_i)(1 - \chi_i)(\eta_i + \chi_i - 2\eta_i \chi_i)$, we have

$$Q_1 = p_i^* D_i^2 + p_i^* \bar{\eta}_i \bar{\chi}_i \{ \chi_i + \eta_i - 2\chi_i \eta_i + p_i^* + (p_i^*)^2 \eta_i \chi_i \}.$$

To evaluate Q_2 note that $p_{1i}(1-p_i+\ell_i p_i) = \ell_i p_i(1-P_{2i})$. Thus

$$\begin{aligned}
Q_2 &= -2p_{1i}D_i\{\eta_i(1-\chi_i+\chi_i p_i^*)+\chi_i(1-\eta_i)(1-p_i^*)+p_i^*(1-\eta_i)\} \\
&+ D_i^2 p_{1i}^2(1-p_i+\ell_i p_i)/(\ell_i p_i) \\
&= -2p_{1i}D_i\{p_i^*(1-\eta_i)(1-\chi_i)+D_i\}+D_i^2 p_{1i}(1-P_{2i}) \\
&= -2p_{1i}D_i p_i^*\bar{\eta}_i\bar{\chi}_i - D_i^2 p_{1i}(1+P_{2i}).
\end{aligned}$$

Since $p_i^* - p_{1i} - p_{1i}P_{2i} = P_{2i}(1-p_{1i})$, the quadratic form is

$$Q_1+Q_2 = p_i^*\bar{\eta}_i\bar{\chi}_i\{\chi_i+\eta_i-2\chi_i\eta_i+p_i^*+(p_i^*)^2\eta_i\chi_i-2p_{1i}D_i\}+D_i^2P_{2i}(1-p_{1i}).$$

As $D_i^2P_{2i}(1-p_{1i}-P_{2i}) = D_i^2P_{2i}(1-p_i^*)$, subtracting (5) squared, that is $\{p_i^*(1-\eta_i)(1-\chi_i)\}^2+2D_iP_{2i}p_i^*(1-\eta_i)(1-\chi_i)+D_i^2P_{2i}^2$, leads to

$$Q_3 = -\{p_i^*\bar{\eta}_i\bar{\chi}_i\}^2 + D_i^2P_{2i}(1-p_i^*) + p_i^*\bar{\eta}_i\bar{\chi}_i \times Q_4,$$

where

$$\begin{aligned}
Q_4 &= \eta_i+\chi_i-2\eta_i\chi_i+p_i^*+(p_i^*)^2\eta_i\chi_i-2(p_{1i}+P_{2i})D_i \\
&= \eta_i+\chi_i-2\eta_i\chi_i+p_i^*+(p_i^*)^2\eta_i\chi_i+2p_i*(2\eta_i\chi_i-\eta_i-\chi_i-p_i^*\eta_i\chi_i) \\
&= p_i^*\bar{\eta}_i\bar{\chi}_i+(1-p_i^*)D_i.
\end{aligned}$$

Thus $Q_3 = (1-p_i^*)D_i\{D_iP_{2i}+p_i^*\bar{\eta}_i\bar{\chi}_i\}$. It gives (1) when divided by the numerator of A_i^2 and multiplied by N_i ; A_i is defined by (5).

B. Derivation of var $\left(\hat{\phi}_i\right)$ for model M_0

In this section, we show the calculations that lead to the variance of $\hat{\phi}_i$,

$$\begin{aligned}\mathrm{var}(\hat{\phi}_i) &= \phi_i^2\left\{\frac{(1-p_{i+1}^*)\chi_{i+1}\left\{\bar{\eta}_{i+1}+p_{i+1}^*\eta_{i+1}\right\}}{N_{i+1}p_{i+1}^*\bar{\eta}_{i+1}\bar{\chi}_{i+1}}+\frac{(1-p_i^*)\bar{\eta}_i\chi_i}{N_ip_i^*\bar{\chi}_i(\bar{\eta}_i+\eta_ip_i^*)}+\frac{1-\phi_i}{N_{i+1}\bar{\eta}_{i+1}}\right\} \\ &- \phi_i^2\left\{\frac{(1-p_{i+1}^*)(\bar{\eta}_{i+1}+p_{i+1}^*\eta_{i+1})^2\chi_{i+1}^2P_{2,i+1}}{N_{i+1}\bar{\eta}_{i+1}\bar{\chi}_{i+1}p_{i+1}^*(\bar{\eta}_{i+1}\bar{\chi}_{i+1}p_{i+1}^*+D_{i+1}P_{2,i+1})}+\frac{(1-p_i^*)\bar{\eta}_i\chi_i^2P_{2,i}}{N_i\bar{\chi}_ip_i^*(\bar{\eta}_i\bar{\chi}_ip_i^*+D_iP_{2,i})}\right\}.\end{aligned} \tag{7}$$

It is useful to define the conditional expectation of the units that are marked before session i and available for capture at session i or later,

$$\hat{M}_i = n_i^* - u_i + n_i^*(w_i - n_i^* + u_i)/(n_i^* - v_i). \tag{8}$$

There are two intermediate steps, the evaluation of the variance of $\hat{n}_i$ and of $\hat{M}_i$,

$$\mathrm{var}\left(\hat{M}_i\right) = N_i(1-p_i^*)\bar{\eta}_i\chi_i(\bar{\eta}_i+p_i^*\eta_i)(p_i^*\bar{\eta}_i+P_{2i}\eta_i)/\left\{p_i^*(p_i^*\bar{\chi}_i\bar{\eta}_i+D_i\times P_{2i})\right\}. \tag{9}$$

B.1. Derivation of var $(\hat{n}_i)$

Since $\hat{n}_i = \hat{N}_i\left\{1-\left(1-C_i/(\ell_i\hat{N}_i)\right)^{\ell_i}\right\}$, a standard linearization argument leads to the following approximation, $\hat{n}_i \approx N_ip_i^* + P_{2i}(\hat{N}_i - N_i) + (1-p_i)^{\ell_i-1}(C_i - N_i\ell_ip_i)$. The covariance between C_i and $\hat{N}_i$ is evaluated as N_i times the fourth entry of the vector $-\Sigma_i\nabla f_{i,N}/A_i$. It is found to be equal to $N_i\ell_ip_i$. Thus the linearization approximation to the variance of $\hat{n}_i$

is

$$
\begin{aligned}
\operatorname{var}(\hat{n}_i) &= P_{2i}^2\{\operatorname{var}(\hat{N}_i) + N_i\} + 2P_{2i}(1-p_i)^{\ell_i - 1}N_i\ell_i p_i \\
&\quad +(1-p_i)^{2\ell_i-2}N_i\ell_i p_i(1-p_i+\ell_i p_i) \\
&= P_{2i}^2\operatorname{var}(\hat{N}_i) + N_i\{(p_i^*)^2 + (p_i^* - P_{2i})(1-p_i^*)\} \\
&= N_i\left[P_{2i}(1-p_i^*)\left\{P_{2i}D_i/(p_i^*\bar{\eta}_i\bar{\chi}_i + P_{2i}D_i) - 1\right\} + p_i^*\right] \\
&= N_i p_i^*\left\{1 - (1-p_i^*)\bar{\eta}_i\bar{\chi}_i P_{2i}/(p_i^*\bar{\eta}_i\bar{\chi}_i + D_i \times P_{2i})\right\}.
\end{aligned}
$$

Thus the variance of $\hat{n}_i$ is equal to $N_i p_i^*$, the variance under an open population model, minus a term representing the variance reduction under a robust design.

B.2. Derivation of var $\left(\hat{M}_i\right)$

From equation (8),

$$\hat{M}_i = \hat{n}_i - u_i + \hat{n}_i(w_i - \hat{n}_i + u_i)/(\hat{n}_i - v_i).$$

The linearization approximation to $\hat{M}_i$ is

$$\hat{M}_i = M_i + \begin{pmatrix} u_i - N_i\eta_i p_i^* \\ v_i - N_i p_i^*\chi_i \\ w_i - N_i\bar{\eta}_i\left\{1-(1-p_i^*)\chi_i\right\} \\ \hat{n}_i - N_i p_i^* \end{pmatrix}^{\top} \nabla M_i$$

and ∇M_i is limit of the vector of partial derivatives of $\hat{M}_i$ with respect to $(u_i, v_i, w_i, \hat{n}_i)$. Using (2), it is given by

$$\nabla M_i = (1/\bar{\chi}_i)\begin{pmatrix} \chi_i \\ \bar{\eta}_i(1-p_i^*)/p_i^* \\ 1 \\ -\chi_i(\bar{\eta}_i + \eta_i p_i^*)/p_i^* \end{pmatrix}.$$

Following Jolly (1965), as $M_i = N_i \bar{\eta}_i$, the quadratic form for the variance is $\mathrm{var}\left(\hat{M}_i\right) = \nabla M_i^\top \Sigma_i^* \nabla M_i - N_i \bar{\eta}_i$, where Σ_i^* is the covariance matrix of $u_i, v_i, w_i, \hat{n}_i$. The covariance matrix for u_i, v_i, w_i is already discussed in sections (A.1) and (A.2). Using the linearization for $\hat{n}_i$, one has

$$\mathrm{cov}(\hat{n}_i, u_i) = P_{2i} \mathrm{cov}(\hat{N}_i, u_i) + (1 - p_i)^{\ell_i - 1} \mathrm{cov}(C_i, u_i).$$

The covariance between u_i and $\hat{N}_i$ is evaluated as N_i times the first entry of the vector $-\Sigma_i \nabla f_{i,N}/A_i$. It is equal to $N_i \eta_i p_i^*$. The covariance between C_i and u_i is $N_i \eta_i \ell_i p_i$, see (6). This gives $\mathrm{cov}(\hat{n}_i, u_i) = N_i \eta_i p_i^*$. In the same fashion, we derive $\mathrm{cov}(\hat{n}_i, v_i) = N_i p_i^* \chi_i$ and $\mathrm{cov}(\hat{n}_i, w_i) = N_i \bar{\eta}_i p_i^*$. Thus, the covariance matrix between $u_i, v_i, w_i, \hat{n}_i$ is,

$$\Sigma_i^* = N_i \begin{pmatrix} \eta_i p_i^* & \eta_i p_i^* \chi_i & 0 & \eta_i p_i^* \\ & p_i^* \chi_i & \bar{\eta}_i p_i^* \chi_i & p_i^* \chi_i \\ & & \bar{\eta}_i \left\{1 - (1 - p_i^*) \chi_i\right\} & \bar{\eta}_i p_i^* \\ & & & \mathrm{var}\left(\hat{n}_i\right) \end{pmatrix}.$$

To evaluate the quadratic form $\nabla M_i^\top \Sigma_i^* \nabla M_i$, first observe that the contribution of the covariance matrix of (u_i, v_i, w_i) is

$$Q_1 = (N_i / \bar{\chi}_i^2) \left[\eta_i p_i^* \chi_i^2 + \bar{\eta}_i + (1 - p_i^*) \bar{\eta}_i \chi_i \left\{\bar{\eta}_i / p_i^* - \eta_i + 2 \eta_i \chi_i\right\}\right].$$

The sum of the the three terms involving a covariance with $\hat{n}_i$ is

$$Q_2 = \left\{-2 N_i p_i^* \chi_i (\bar{\eta}_i + \eta_i p_i^*) / (p_i^* \bar{\chi}_i)\right\} \left[\bar{\eta}_i + \chi_i (\bar{\eta}_i + \eta_i p_i^*) / (p_i^* \bar{\chi}_i)\right].$$

The last term, involving the variance of $\hat{n}_i$, is given by

$$
\begin{aligned}
Q_3 &= N_i p_i^* \left\{(\bar{\eta}_i + p_i^* \eta_i)\chi_i / p_i^* \bar{\chi}_i\right\}^2 \left\{1 - (1-p_i^*)\bar{\eta}_i \bar{\chi}_i P_{2i} / (p_i^* \bar{\chi}_i \bar{\eta}_i + D_i \times P_{2i})\right\} \\
&= \underbrace{N_i p_i^* \left\{(\bar{\eta}_i + p_i^* \eta_i)\chi_i / (p_i^* \bar{\chi}_i)\right\}^2}_{Q_{3,1}} \\
&- \underbrace{N_i p_i^* \left\{(\bar{\eta}_i + p_i^* \eta_i)\chi_i / (p_i^* \bar{\chi}_i)\right\}^2 \left\{(1-p_i^*)\bar{\eta}_i \bar{\chi}_i P_{2i} / (p_i^* \bar{\chi}_i \bar{\eta}_i + D_i \times P_{2i})\right\}}_{Q_{3,2}}.
\end{aligned}
$$

First we calculate the variance under an open population model, corresponding to the case $Q_{3,2} = P_{2i} = 0$. This gives

$$
Q_1 + Q_2 + Q_{3,1} = N_i \chi_i \bar{\eta}_i (1 - p_i^*) \left\{\bar{\eta}_i + p_i^* \eta_i\right\} / (p_i^* \bar{\chi}_i) + N_i \bar{\eta}_i.
$$

Now adding $Q_{3,2}$ yields the expression for $\nabla M_i^\top \Sigma_i^* \nabla M_i - N_i \bar{\eta}_i$ given in (9) .

In the next sections, we give some details on the last calculations that lead to the variance of $\hat{\phi}_i$.

B.3. Derivation of $\nabla \hat{\phi}_i$

We have,

$$
\nabla \hat{\phi}_i = \left(\frac{\partial \hat{\phi}_i}{\partial \hat{M}_{i+1}}, \frac{\partial \hat{\phi}_i}{\partial \hat{M}_i}, \frac{\partial \hat{\phi}_i}{\partial u_i} \right).
$$

Differentiating $\hat{\phi}_i$ with respect to $\hat{M}_{i+1}$ and using (2) leads to

$$
\begin{aligned}
\frac{\partial \hat{\phi}_i}{\partial \hat{M}_{i+1}} &= 1/(\hat{M}_i + u_i) \\
&\approx 1/\left\{N_i(\bar{\eta}_i + \eta_i p_i^*)\right\}.
\end{aligned}
$$

The derivations with respect to $\hat{M}_i$ and u_i are calculated in a similar way.

B.4. Evaluation of Γ_i

The variance of $\hat{M}_{i+1}$ and $\hat{M}_i$ have been derived in section (B.2) ; the expectation of u_i is calculated in equation (2). The covariance between $\hat{M}_{i+1}$ and $\hat{M}_i$ is evaluated as

$$\mathrm{cov}(\hat{M}_{i+1}, \hat{M}_i) = \nabla M_{i+1}^{\top} \Gamma_i^* \nabla M_i,$$

where Γ_i^* is the 4×4 matrix of the covariances between the vectors $(u_i, v_i, w_i, \hat{n}_i)$ and $(u_{i+1}, v_{i+1}, w_{i+1}, \hat{n}_{i+1})$. The covariance between u_i and u_{i+1} is 0. The covariance between u_i and v_{i+1} is $N_i \eta_i p_i^* \phi_i p_{i+1}^* \chi_{i+1}$. The other covariances involving the random variables u_i, v_i, w_i and $u_{i+1}, v_{i+1} w_{i+1}$ are derived in a similar way. The covariances involving $\hat{n}_i$ and $\hat{n}_{i+1}$ are derived using the approach discussed in section (B.2). For example, the covariance between $\hat{n}_i$ and $\hat{n}_{i+1}$ gives $N_i p_i^* \phi_i p_{i+1}^*$. Finally, the covariance matrix Γ_i^* is,

$$\Gamma_i^* = N_i \begin{pmatrix} 0 & \eta_i p_i^* \phi_i p_{i+1}^* \chi_{i+1} & \eta_i p_i^* \phi_i (\bar{\chi}_{i+1} + p_{i+1}^* \chi_{i+1}) & \eta_i p_i^* \phi_i p_{i+1}^* \\ 0 & 0 & 0 & 0 \\ 0 & \bar{\eta}_i \phi_i p_{i+1}^* \chi_{i+1} & \bar{\eta}_i (1 - p_i^*) \phi_i (1 - p_{i+1}^*) \bar{\chi}_{i+1} & \bar{\eta}_i (1 - p_i^*) \phi_i p_{i+1}^* \\ 0 & p_i^* \phi_i p_{i+1}^* \chi_{i+1} & p_i^* \phi_i (\bar{\chi}_{i+1} + p_{i+1}^* \chi_{i+1}) & p_i^* \phi_i p_{i+1}^* \end{pmatrix}^{\top}.$$

To evaluate the quadratic form $\nabla M_{i+1}^{\top} \Gamma_i^* \nabla M_i$, we first calculate the component featuring the covariances between (u_i, v_i, w_i) and $(u_{i+1}, v_{i+1}, w_{i+1})$,

$$\begin{aligned} Q_1 &= \{N_i \phi_i / (\bar{\chi}_i \bar{\chi}_{i+1})\} \left[\eta_i \chi_i \left\{ \bar{\eta}_{i+1} (1 - p_{i+1}^*) \chi_{i+1} + p_i^* (\bar{\chi}_{i+1} + p_{i+1}^* \chi_{i+1}) \right\} \right] \\ &+ \{N_i \phi_i / (\bar{\chi}_i \bar{\chi}_{i+1})\} \left[\bar{\eta}_i (1 - p_{i+1}^*) \left\{ \bar{\eta}_{i+1} \chi_{i+1} + (1 - p_i^*) \bar{\chi}_{i+1} \right\} \right]. \end{aligned}$$

The sum of the covariances between (u_i, v_i, w_i) and $\hat{n}_{i+1}$ is

$$Q_2 = -N_i \chi_i \chi_{i+1} \phi_i (\bar{\chi}_{i+1} + \eta_{i+1} p_{i+1}^*)(\bar{\chi}_i + \eta_i p_i^*)/\bar{\chi}_i.$$

That involving $(u_{i+1}, v_{i+1}, w_{i+1})$ and $\hat{n}_i$ is,

$$Q_3 = -N_i p_i^* \phi_i \chi_i (\bar{\eta}_i + \eta_i p_i^*) \left\{ \bar{\eta}_{i+1}(1 - p_{i+1}^*) + \bar{\chi}_{i+1} + p_{i+1}^* \chi_{i+1} \right\} / (p_i^* \bar{\chi}_i \bar{\chi}_{i+1}).$$

The component involving the covariance between $\hat{n}_{i+1}$ and $\hat{n}_i$ is

$$Q_4 = N_i \phi_i \chi_i \chi_{i+1} (\bar{\eta}_i + \eta_i p_i^*)(\bar{\eta}_{i+1} + \eta_{i+1} p_{i+1}^*) / (\bar{\chi}_i \bar{\chi}_{i+1}).$$

Now adding up Q_1, Q_2, Q_2 and Q_4 gives the quadratic form $\nabla M_{i+1}^\top \Gamma_i^* \nabla M_i$. It leads to $\mathrm{cov}(\hat{M}_{i+1}, \hat{M}_i) = N_i \bar{\eta}_i \phi_i$. In the same fashion, we derive $\mathrm{cov}(\hat{M}_{i+1}, u_i) = N_i \eta_i p_i^* \phi_i$ and $\mathrm{cov}(\hat{M}_i, u_i) = 0$. The covariance matrix Γ_i is now completely derived.

B.5. Evaluation of the variance of $\hat{\phi}_i$

To evaluate the quadratic form $\nabla \hat{\phi}_i^\top \Gamma_i \nabla \hat{\phi}_i$, first observe that the contribution of the three covariances between $\hat{M}_{i+1}$,$\hat{M}_i$, and u_i is

$$Q_1 = -2\phi_i^2 / \left\{ N_i (\bar{\eta}_i + \eta_i p_i^*) \right\},$$

while the contribution of the the variances of $(\hat{M}_{i+1}, \hat{M}_i, u_i)$ to the quadratic form is

$$\begin{aligned} Q_2 &= \phi_i^2 \frac{(1 - p_{i+1}^*)\chi_{i+1} \left\{ \bar{\eta}_{i+1} + \eta_{i+1} p_{i+1}^* \right\} \left\{ p_{i+1}^* \bar{\eta}_{i+1} + P_{2,i+1} \eta_{i+1} \right\}}{N_{i+1} p_{i+1}^* \bar{\eta}_{i+1} \left\{ p_{i+1}^* \bar{\chi}_{i+1} \bar{\eta}_{i+1} + D_{i+1} \times P_{2,i+1} \right\}} \\ &+ \phi_i^2 \left[\frac{(1 - p_i^*) \bar{\eta}_i \chi_i \left\{ p_i^* \bar{\eta}_i + P_{2i} \eta_i \right\}}{N_i p_i^* \left\{ \bar{\eta}_i + \eta_i p_i^* \right\} \left\{ p_i^* \bar{\chi}_i \bar{\eta}_i + D_i \times P_{2i} \right\}} \right] \\ &+ \phi_i^2 \left[\frac{1}{N_{i+1} \bar{\eta}_{i+1}} + \frac{1}{N_i (\bar{\eta}_i + \eta_i p_i^*)} \right]. \end{aligned}$$

Finally, adding up Q_1 and Q_2 gives (7) .

C. Derivation of var $\left(\hat{N}_i\right)$ for model M_t

In this section, we show the calculations that lead to the variance of $\hat{N}_i$. The following notation is used throughout this section.

- $p^*_{1i} = \sum_{j=1}^{\ell_i} \left\{ p_{ij} \prod_{\substack{s=1 \\ s\neq j}}^{\ell_i} (1 - p_{is}) \right\}$ the probability of being captured once at PSP i,
- $P^*_{2i} = 1 - \prod_{j=1}^{\ell_i} (1 - p_{ij}) - \sum_{j=1}^{\ell_i} \left\{ p_{ij} \prod_{\substack{s=1 \\ s\neq j}}^{\ell_i} (1 - p_{is}) \right\}$ the probability of obtaining two captures or more at PSP i.

Furthermore, for a PSP i, let p_{ij} and n_{ij} be (resp.) the probability of being captured and the number of captures in a secondary sampling level j , $j = 1, 2, ..., \ell_i$; thus, $p^*_i = 1 - \prod_{j=1}^{\ell_i} (1 - p_{ij})$.

The estimating equation that leads to the maximum likelihood estimator for N_i is

$$f^*_{i,N}(N_i) = N_i \left[n^*_i(w_i - n^*_i + u_i)/\left\{n^*_i(w_i - v_i) + u_i v_i\right\} - \prod_{j=1}^{\ell_i} (1 - n_{ij}/N_i) \right], \tag{10}$$

where

$$n^*_i = N_i \left\{ 1 - \prod_{j=1}^{\ell_i} (1 - n_{ij}/N_i) \right\}. \tag{11}$$

This section give further details to the proof of the asymptotic variance of $\hat{N}_i$,

$$\text{var}\left(\hat{N}_i\right) = N_i D_i (1 - p^*_i)/(p^*_i \bar{\chi}_i \bar{\eta}_i + D_i P^*_{2i}). \tag{12}$$

C.1. Evaluation of A_i^*

We have,

$$\begin{aligned} n_i^* &= N_i\left\{1-\prod_{j=1}^{\ell_i}(1-n_{ij}/N_i)\right\} \\ &= N_i\left[1-\exp\left\{\sum_{j=1}^{\ell_i}\log(1-n_{ij}/N_i)\right\}\right]. \end{aligned}$$

Then, its derivative with respect to N_i, evaluated at the expectation of n_{ij}, $j=1,2,...,\ell_i$, is equal to P_{2i}^*, the probability of being captured more than once during PSP i. Now, differentiating (10) using the chain rule and applying (2) gives

$$\begin{aligned} \frac{\partial f_{i,N}^*}{\partial N_i} &\approx -N_iP_{2i}^*\left[(1-p_i^*)\chi_i/(N_i\bar{\chi}_i)+\{1-(1-p_i^*)\chi_i\}/(M_i\bar{\chi}_i)\right]-p_{1i}^* \\ &= -\{1/(\bar{\chi}_i\bar{\eta}_i)\}\left(p_i^*\bar{\chi}_i\bar{\eta}_i+D_i\times P_{2i}^*\right)=A_i^*. \end{aligned}$$

C.2. Evaluation of Σ_i^*

The variance and the covariance of the three Poisson random variables u_i, v_i, w_i are already discussed in section (A.2). Now, for $j=1,2,...,\ell_i$, the variance of n_{ij}, a Poisson random variable, is

$$\text{var}(n_{ij})=N_ip_{ij}.$$

The covariances with n_{ij} are calculated in a similar way. For instance that with u_i involves $\tilde{U}_i$, the units that are not marked before the ith PSP. Given $(\tilde{N}_i,\tilde{U}_i)$ the conditional covariance between n_{ij} and u_i is $\tilde{U}_i\ell_ip_{ij}(1-p_i^*)$ while their respective expectations are $\tilde{N}_ip_{ij}$ and $\tilde{U}_ip_i^*$. Thus,

$$\text{cov}(n_{ij},u_i)=E\{\tilde{U}_ip_{ij}(1-p_i^*)\}+\text{cov}(\tilde{N}_ip_{ij},\tilde{U}_ip_i^*)=U_ip_{ij}=N_i\eta_ip_{ij}.$$

Finally, the covariance matrix of $u_i, v_i, w_i, n_{i1}, n_{i2}, ..., n_{i\ell_i}$, $\mathbf{\Sigma_i^*}$, is

$$\mathbf{\Sigma_i^*} = N_i \begin{pmatrix} \eta_i p_i^* & \eta_i p_i^* \chi_i & 0 & \eta_i p_{i1} & \dots & \eta_i p_{i\ell_i} \\ & p_i^* \chi_i & \bar{\eta}_i p_i^* \chi_i & p_{i1}\chi_i & \dots & p_{i\ell_i}\chi_i \\ & & \bar{\eta}_i \left\{1 - (1-p_i^*)\chi_i\right\} & \bar{\eta}_i p_{i1} & \dots & \bar{\eta}_i p_{i\ell_i} \\ & & & p_{i1} & \dots & p_{i1} p_{i\ell_i} \\ & & & & \ddots & p_{i\ell_i} \end{pmatrix}. \tag{13}$$

C.3. Derivation of $\nabla f_{i,N}^*$

We have,

$$\nabla f_{i,N}^* = \left(\frac{\partial f_{i,N}}{\partial u_i}, \frac{\partial f_{i,N}}{\partial v_i}, \frac{\partial f_{i,N}}{\partial w_i}, \left(\frac{\partial f_{i,N}}{\partial n_{ij}} \right)_{j=1,\dots,\ell_i} \right).$$

The first three partial derivates are the same as calculated in section (A.3). Now, for a fixed j ($j = 1, 2, ..., \ell_i$), deriving $f_{i,N}^*$ with respect to n_{ij} using (2) gives

$$\begin{aligned} \frac{\partial f_{i,N}}{\partial n_{ij}} &\approx \prod_{\substack{s=1 \\ s\neq j}}^{\ell_i} (1-p_{is}) - \prod_{\substack{s=1 \\ s\neq j}}^{\ell_i} (1-p_{is}) \left\{ (\bar{\chi}_i + p_i^* \chi_i)/(\bar{\eta}_i \bar{\chi}_i) + (1-p_i^*)\chi_i/(\bar{\chi}_i) \right\} \\ &= \prod_{\substack{s=1 \\ s\neq j}}^{\ell_i} (1-p_{is}) \left\{ 1 - (\bar{\chi}_i + p_i^* \chi_i)/(\bar{\eta}_i \bar{\chi}_i) - (1-p_i^*)\chi_i/(\bar{\chi}_i) \right\} \\ &= -D_i \prod_{\substack{s=1 \\ s\neq j}}^{\ell_i} (1-p_{is}) / (\bar{\eta}_i \bar{\chi}_i). \end{aligned}$$

The limit of the vector of partial derivatives of $f_{i,N}^*(N_i)$ with respect to u_i, v_i, w_i, n_{i1},n_{i2},...,$n_{i\ell_i}$, is

$$\nabla f_{i,N}^* = \{1/(\bar{\chi}_i \bar{\eta}_i)\} \times (1 - (1-p_i^*)\chi_i, \bar{\eta}_i(1-p_i^*), p_i^*, -D_i \mathbf{\Psi_i})^\top,$$

where

$$\mathbf{\Psi_i} = \left(\prod_{\substack{s=1 \\ s\neq 1}}^{\ell_i} (1-p_{is}), \prod_{\substack{s=1 \\ s\neq 2}}^{\ell_i} (1-p_{is}), ..., \prod_{\substack{s=1 \\ s\neq \ell_i}}^{\ell_i} (1-p_{is}) \right).$$

C.4. Evaluation of $\nabla f_{i,N}^{*\top} \Sigma_i^* \nabla f_{i,N}^*$

From the previous evaluation of $\nabla f_{i,N}^{\top} \Sigma_i \nabla f_{i,N}$, the only quantity that changes is the one involving the variances and the covariances with n_{ij}. Let that be A_{ij}. We have,

$$\begin{aligned}
A_{ij} &= \sum_{j=1}^{\ell_i} \left\{ D_i \prod_{\substack{s=1 \\ s\neq j}}^{\ell_i} (1-p_{is}) \right\}^2 p_{ij} + \sum_{j=1}^{\ell_i} \sum_{\substack{j'=1 \\ j'\neq j}}^{\ell_i} \left\{ D_i^2 \prod_{\substack{s=1 \\ s\neq j}}^{\ell_i} (1-p_{is}) \prod_{\substack{t=1 \\ t\neq j'}}^{\ell_i} (1-p_{it}) p_{ij} p_{ij'} \right\} \\
&- 2D_i \sum_{j=1}^{\ell_i} \prod_{\substack{s=1 \\ s\neq j}}^{\ell_i} (1-p_{is}) \left\{ \frac{\partial f_{i,N}^*}{\partial u_i} \eta_i p_{ij} + \frac{\partial f_{i,N}^*}{\partial v_i} p_{ij} \chi_i + \frac{\partial f_{i,N}^*}{\partial w_i} \bar{\eta}_i p_{ij} \right\} \\
&= D_i^2 \sum_{j=1}^{\ell_i} p_{ij} \prod_{\substack{s=1 \\ s\neq j}}^{\ell_i} (1-p_{is}) \left[\prod_{\substack{s=1 \\ s\neq j}}^{\ell_i} (1-p_{is}) - p_{ij} \prod_{\substack{s=1 \\ s\neq j}}^{\ell_i} (1-p_{is}) + \sum_{j=1}^{\ell_i} p_{ij} \prod_{\substack{s=1 \\ s\neq j}}^{\ell_i} (1-p_{is}) \right] \\
&- 2D_i p_{1i}^* \left\{ \frac{\partial f_{i,N}^*}{\partial u_i} \eta_i + \frac{\partial f_{i,N}^*}{\partial v_i} \chi_i + \frac{\partial f_{i,N}^*}{\partial w_i} \bar{\eta}_i \right\} \\
&\approx D_i^2 p_{1i}^* (1 - p_i^* + p_{1i}^*) - 2D_i p_{1i}^* \left[\eta_i \left\{ 1 - (1-p_i^*)\chi_i \right\} + \bar{\eta}_i (1-p_i^*)\chi_i + \bar{\eta}_i p_i^* \right] \\
&= -2p_{i1}^* D_i p_i^* \bar{\eta}_i \bar{\chi}_i - D_i^2 p_{i1}^* (1 + P_{2i}^*),
\end{aligned}$$

which only depends on PSP i. Finally, A_{ij} has the same form as Q_2 defined in the evaluation of $\nabla f_{i,N}^{\top} \Sigma_i \nabla f_{i,N}$. Therefore, $\nabla f_{i,N}^{*\top} \Sigma_i^* \nabla f_{i,N}^*$ and $\nabla f_{i,N}^{\top} \Sigma_i \nabla f_{i,N}$ have the same quadratic form and the vaiance of $\hat{N}_i$ is given by (12).

D. Derivation of var $\left(\hat{\phi}_i\right)$ for model M_t

In this section, we show the calculations that lead to the variance of $\hat{\phi}_i$,

$$\begin{aligned}\mathrm{var}(\hat{\phi}_i) &= \phi_i^2\left\{\frac{(1-p_{i+1}^*)\chi_{i+1}\left\{\bar{\eta}_{i+1}+p_{i+1}^*\eta_{i+1}\right\}}{N_{i+1}p_{i+1}^*\bar{\eta}_{i+1}\bar{\chi}_{i+1}}+\frac{(1-p_i^*)\bar{\eta}_i\chi_i}{N_ip_i^*\bar{\chi}_i(\bar{\eta}_i+\eta_ip_i^*)}+\frac{1-\phi_i}{N_{i+1}\bar{\eta}_{i+1}}\right\} \\ &- \phi_i^2\left\{\frac{(1-p_{i+1}^*)(\bar{\eta}_{i+1}+p_{i+1}^*\eta_{i+1})^2\chi_{i+1}^2P_{2,i+1}^*}{N_{i+1}\bar{\eta}_{i+1}\bar{\chi}_{i+1}p_{i+1}^*(\bar{\eta}_{i+1}\bar{\chi}_{i+1}p_{i+1}^*+D_{i+1}P_{2,i+1}^*)}+\frac{(1-p_i^*)\bar{\eta}_i\chi_i^2P_{2,i}^*}{N_i\bar{\chi}_ip_i^*(\bar{\eta}_i\bar{\chi}_ip_i^*+D_iP_{2,i}^*)}\right\}.\end{aligned} \tag{14}$$

Throughout the section, we give details on the evaluation of the variance of $\hat{n}_i$ and of $\hat{M}_i$,

$$\mathrm{var}\left(\hat{M}_i\right) = N_i(1-p_i^*)\bar{\eta}_i\chi_i(\bar{\eta}_i+p_i^*\eta_i)(p_i^*\bar{\eta}_i+P_{2i}^*\eta_i)/\left\{p_i^*(p_i^*\bar{\chi}_i\bar{\eta}_i+D_i\times P_{2i}^*)\right\}. \tag{15}$$

D.1. Derivation of var $(\hat{n}_i)$

The covariance between C_i and $\hat{N}_i$ is evaluated as N_i times the fourth entry of the vector $-\Sigma_i\nabla f_{i,N}^*/A_i^*$. It is found to be equal to $N_i\sum_{j=1}^{\ell_i}p_{ij}$. Thus the linearization approximation to the variance of $\hat{n}_i$ is

$$\begin{aligned}\mathrm{var}\left(\hat{n}_i\right) &= (P_{2i}^*)^2\mathrm{var}(\hat{N}_i)+N_i\{(p_i^*)^2+(p_i^*-P_{2i}^*)(1-p_i^*)\} \\ &= N_i\left[P_{2i}^*(1-p_i^*)\left\{P_{2i}^*D_i/(p_i^*\bar{\eta}_i\bar{\chi}_i+P_{2i}^*D_i)-1\right\}+p_i^*\right] \\ &= N_ip_i^*\left\{1-(1-p_i^*)\bar{\eta}_i\bar{\chi}_iP_{2i}^*/(p_i^*\bar{\eta}_i\bar{\chi}_i+D_i\times P_{2i}^*)\right\}.\end{aligned}$$

D.2. Derivation of var $\left(\hat{M}_i\right)$

The only term that changes in the evaluation of var $\left(\hat{M}_i\right)$ from the previous derivations of (B.2) is the last term, involving the variance of $\hat{n}_i$,

$$\begin{aligned} Q_3 &= N_i p_i^* \left\{(\bar{\eta}_i + p_i^*\eta_i)\chi_i/p_i^*\bar{\chi}_i\right\}^2 \left\{1-(1-p_i^*)\bar{\eta}_i\bar{\chi}_i P_{2i}^*/(p_i^*\bar{\chi}_i\bar{\eta}_i + D_i \times P_{2i}^*)\right\} \\ &= \underbrace{N_i p_i^* \left\{(\bar{\eta}_i + p_i^*\eta_i)\chi_i/(p_i^*\bar{\chi}_i)\right\}^2}_{Q_{3,1}} \\ &- \underbrace{N_i p_i^* \left\{(\bar{\eta}_i + p_i^*\eta_i)\chi_i/(p_i^*\bar{\chi}_i)\right\}^2 \left\{(1-p_i^*)\bar{\eta}_i\bar{\chi}_i P_{2i}^*/(p_i^*\bar{\chi}_i\bar{\eta}_i + D_i \times P_{2i}^*)\right\}}_{Q_{3,2}}. \end{aligned}$$

First we calculate the variance under an open population model, corresponding to the case $Q_{3,2} = P_{2i}^* = 0$. This gives

$$Q_1 + Q_2 + Q_{3,1} = N_i\chi_i\bar{\eta}_i(1-p_i^*)\left\{\bar{\eta}_i + p_i^*\eta_i\right\}/(p_i^*\bar{\chi}_i) + N_i\bar{\eta}_i.$$

Now adding $Q_{3,2}$ gives (15). The derivations of $\nabla\hat{\phi}_i$ and Γ_i are the exact same obtained in (B.3) and (B.4) respectively.

D.3. Evaluation of the variance of $\hat{\phi}_i$

The only term that changes from prior calculations in (B.5) is the contribution of the variances of $(\hat{M}_{i+1}, \hat{M}_i, u_i)$ to the quadratic form,

$$\begin{aligned} Q_2 &= \phi_i^2 \frac{(1-p_{i+1}^*)\chi_{i+1}\left\{\bar{\eta}_{i+1} + \eta_{i+1}p_{i+1}^*\right\}\left\{p_{i+1}^*\bar{\eta}_{i+1} + P_{2,i+1}^*\eta_{i+1}\right\}}{N_{i+1}p_{i+1}^*\bar{\eta}_{i+1}\left\{p_{i+1}^*\bar{\chi}_{i+1}\bar{\eta}_{i+1} + D_{i+1}\times P_{2,i+1}^*\right\}} \\ &+ \phi_i^2\left[\frac{(1-p_i^*)\bar{\eta}_i\chi_i\left\{p_i^*\bar{\eta}_i + P_{2i}^*\eta_i\right\}}{N_i p_i^*\left\{\bar{\eta}_i + \eta_i p_i^*\right\}\left\{p_i^*\bar{\chi}_i\bar{\eta}_i + D_i \times P_{2i}^*\right\}}\right] \\ &+ \phi_i^2\left[\frac{1}{N_{i+1}\bar{\eta}_{i+1}} + \frac{1}{N_i(\bar{\eta}_i + \eta_i p_i^*)}\right]. \end{aligned}$$

Finally, adding up Q_1 obtained in (B.5) and Q_2 gives (14) .

E. Simulation study

This section is a complement to Section 5.2 presenting the Monte Carlo validation of the bootstrap variance estimation method. The within PSP capture probabilites were generated using Darroch model,

$$p_{ik} = \binom{7}{k} \exp\left\{\beta k + \tau k^2/2\right\} / \sum_{j=0}^{7} \binom{7}{j} \exp\left\{\beta + \tau j^2/2\right\}, \quad k = 0, \ldots, 7 \qquad (16)$$

with parameters $(\beta,\ \tau) = (-3.3,\ 0.6)\,,\ (-2.85,\ 0.6)$ corresponding to $p^* = 0.3,\ 0.5$. The daily capture probability for a unit was simulated using the method proposed in Section 2.2 of Rivest and Baillargeon (2007). We set the survival probabilities at $\phi = 0.6,\ 0.8$.

We ran 1000 replications for the Monte Carlo study; for each replication there was a burn-in period of 20 PSPs. We calculated the relative bias of $\hat{N}_5$ as $RB\left(\hat{N}_5\right) = \left(\sum_i \hat{N}_{5i}/1000 - N_5\right)/N_5$. The mean squared error, $MSE\left(\hat{N}_5\right) = \left\{\sum_i \left(\hat{N}_{5i} - N_5\right)^2/1000\right\}$, the relative root mean squared error, $RRMSE\left(\hat{N}_5\right) = MSE\left(\hat{N}_5\right)^{1/2}/N_5$. For each replication i of the Monte Carlo simulation, we ran $L = 100$ replications of the bootstrap described in Section 5.2 to calculate the bootstrap variance for $\hat{N}_5$, $v\left(\hat{N}_5\right)$. The relative bias for $E\left\{v\left(\hat{N}_5\right)\right\}$ is calculated as $RB\left[E\left\{v\left(\hat{N}_5\right)\right\}\right] = \left[E\left\{v\left(\hat{N}_5\right)\right\} - MSE\left(\hat{N}_5\right)\right]/MSE\left(\hat{N}_5\right)$; the 95% confidence interval for $\hat{N}_5$ is $\exp\left[\log\left(\hat{N}_5\right) \pm 1.96 s.e\left\{\log\left(\hat{N}_5\right)\right\}\right]$; the expected relative length of the confidence interval is calculated as $RLCI\left(\hat{N}_5\right) = (UB - LB)/N_5$, where UB and LB are respectively the expected upper and lower bounds of the confidence interval for N_5. In Table 1, the Monte Carlo standard errors, $RRMSE\left(\hat{N}_5\right)/1000^{1/2}$, of $RB\left(\hat{N}_5\right)$ are also provided in parenthesis. They show that $\hat{N}_5$ has a negative bias which is important in scenario ED2.

Table 1: Simulation results for the estimation of N_5 under the robust design model for M_{Dh}^t with $p^* = 0.3,\ 0.5$, $\phi = 0.6,\ 0.8$ and three scenarios for the entry process. All the results are presented in percentages

ϕ	p^*	Scenario	$RB(\hat{N}_5)$	$RB(E(v(\hat{N}_5)))$	$RRMSE$	95% Cov.	$RLCI(\hat{N}_5)$
0.8	0.5	RD	-0.02 (0.14)	12.79	4.54	93.3	18.34
		ED1	-0.94 (0.15)	14.92	4.73	95.9	19.58
		ED2	-6.68 (0.26)	-63.23	8.12	74.2	18.99
	0.3	RD	-2.2 (0.25)	5.8	7.8	95.5	30.7
		ED1	-3.28 (0.24)	11.41	7.54	95	30.54
		ED2	-9.17 (0.35)	-53.08	11.19	78.9	29.44
0.6	0.5	RD	0.72 (0.29)	-0.76	9.31	93.9	35.3
		ED1	-0.2 (0.30)	-2.45	9.6	93.4	36.12
		ED2	-11.36 (0.48)	-63.83	15.10	72.4	34.56
	0.3	RD	-4.34 (0.46)	8.35	14.61	93.3	58.12
		ED1	-5.3 (0.50)	-5.16	15.7	93.7	58.14
		ED2	-17.98 (0.73)	-60.07	23.17	73.8	55.59

Additional simulations, using 1000 replications, were carried out for scenario ED2 to investigate whether the relative change in population size, defined as $inc = (N_{35} - N_5)/N_5$, could be estimated accurately even if absolute population estimators were biased. The set-up of this second simulation is presented in Figure 1: after a burn-in period of 21 PSPs data are collected in weeks 1 to 9; this is used to estimate N_5. Starting on week 10 daily births are multiplied by $1 + inc$ and the process runs for 29 more weeks. Data are collected in weeks 31 to 39 to estimate N_{35}.

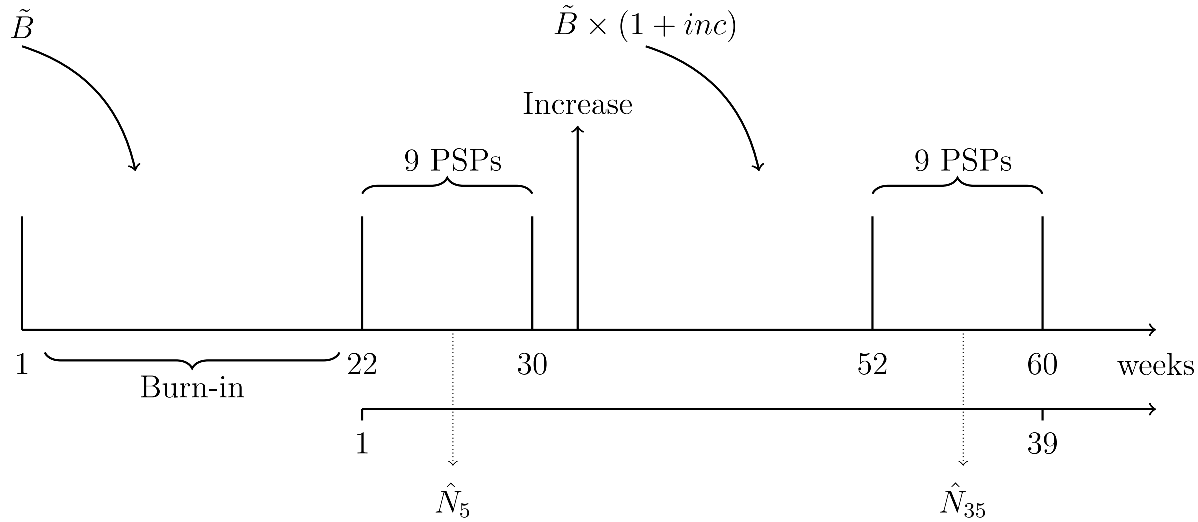

Figure 1: Data simulation process

For each replication i, the relative increase $\hat{inc}_i = \left(\hat{N}_{35,i} - \hat{N}_{5,i}\right)/\hat{N}_{5,i}$ is calculated. A bootstrap variance for $\hat{inc}$, calculated using 100 bootstrap samples, was computed using the formula $v\left(\hat{inc}\right) = \left(\hat{N}_{35}/\hat{N}_5\right)^2 \times \left[v\left\{\log\left(\hat{N}_5\right)\right\} + v\left\{\log\left(\hat{N}_{35}\right)\right\}\right]$, where $v\left\{\log\left(\hat{N}_5\right)\right\}$ and $v\left\{\log\left(\hat{N}_{35}\right)\right\}$ are the bootsrap variances for $\log\left(\hat{N}_5\right)$ and $\log\left(\hat{N}_{35}\right)$ respectively; the 95% confidence interval for $\hat{inc}$ is $\left[\hat{inc} \pm 1.96\sqrt{v\left(\hat{inc}\right)}\right]$. The bias of $\hat{inc}$, with its Monte Carlo standard error, its root mean squared error and the coverage of its 95% confidence interval are reported in Table 2. In general, the population increase is well estimated and the bootstrap confidence level is equal to the target value. In most cases, there is a small positive bias, which is always less than 10% of the true value of inc. Relatively large RMSEs are found when both p^* and ϕ are small.

Table 2: The bias, with its Monte Carlo standard error in parenthesis, the root mean squared error and the 95% coverage of the estimator of *inc*. The results are expressed in percentages.

ϕ	p^*	inc	$B(\hat{inc})$	$RMSE(\hat{inc})$	95%Cov.
0.8	0.5	20	0.45 (0.2)	8	98
		50	0.08 (0.3)	9	96
		80	0.26 (0.3)	10	98
	0.3	20	0.71 (0.4)	12	97
		50	0.8 (0.5)	14	97
		80	0.59 (0.7)	24	97
0.6	0.5	20	0.74 (0.5)	16	96
		50	1.75 (0.7)	21	94
		80	-1.40 (0.7)	22	94
	0.3	20	2.32 (1.0)	32	95
		50	1.82 (1.1)	35	94
		80	5.30 (1.3)	42	95

F. Robustness investigations

The robustness of the estimates obtained was investigated by fitting the same model to capture-recapture data from a different metropolitan area. The two sets of estimates for $\{p_i^*\}$ are provided in Figure 2. They are very similar supporting the statement that the capture mechanism is the same in the two metropolitan areas.

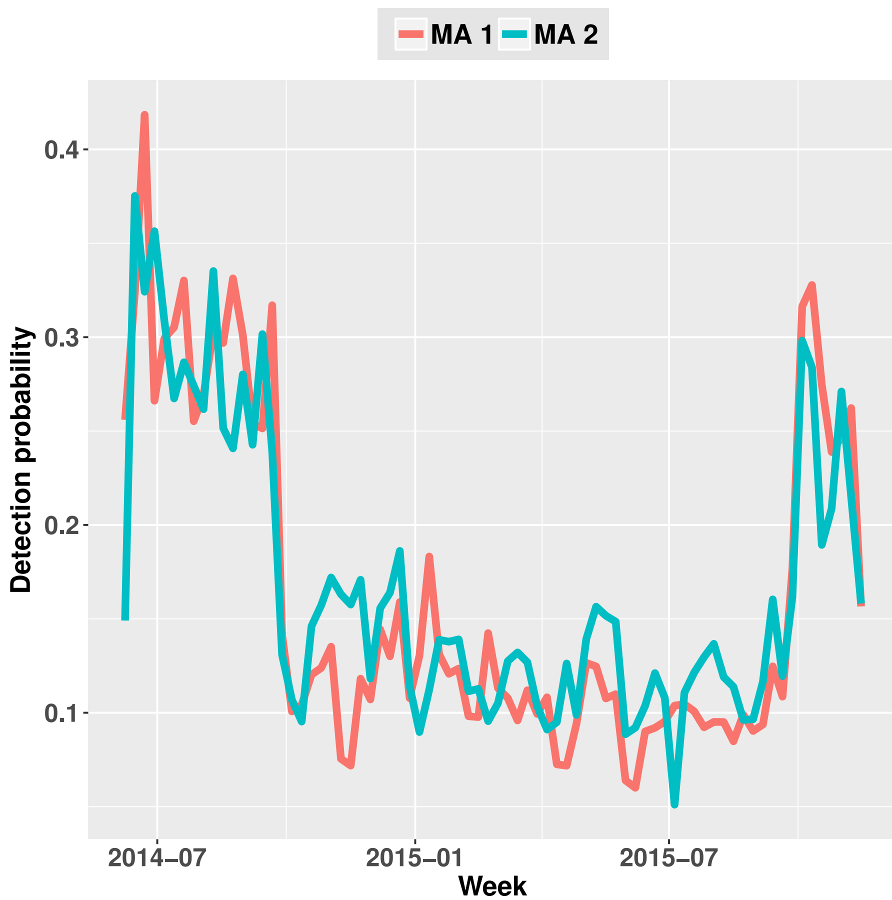

Figure 2: Evolution of the detection probability for 76 weeks and for Metropolitan Area (MA) 1 and 2.

Alternative data sets were created by removing the first few days of data. In Figure 3, the Sunday data set is the original data set. The Monday one is obtained by dropping the first day: a PSP starts one day later than in the original data. In the Tuesday data set, it starts two days later. The population size estimates appear to be invariant with respect to a redefinition of the starting day of a PSP.

G. Demographic parameter and capture probability estimates for the app data

In this section, estimates of the demographic parameters and the capture probabilities are presented along with their coefficients of variation. The results are presented for the first 20 weeks of the experiment ; results for the 76 PSPs can be obtained in .xlsx file named *Estimations* provided as a supplementary material.

Figure 4 present boxplots of the relative efficiencies computed with respect to the robust design estimators for the 76 PSPs; for the Jolly Seber estimators this efficiency is defined by $\{CV(\hat{N}_i^{JS})/CV(\hat{N}_i^{RD})\}^2$. The results show that the robust design provides estimates of N_i that are much more efficient that those of the closed population and the Jolly-Seber models. Furthermore, the gain in precision is more important for the closed population estimates.

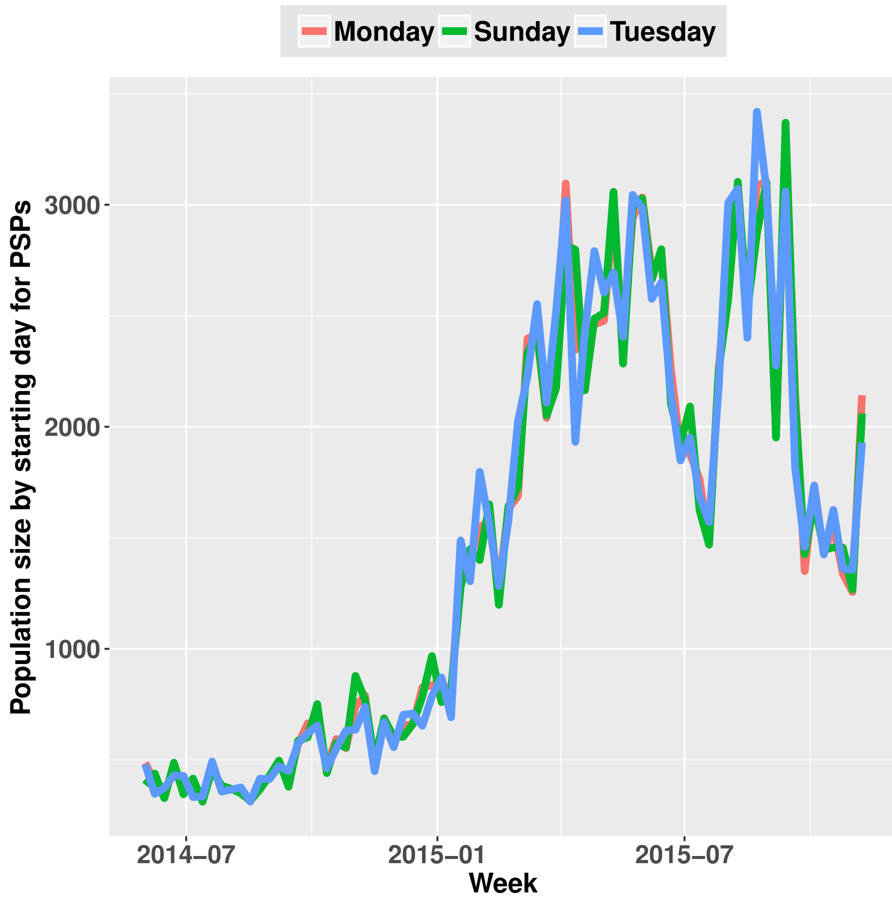

Figure 3: Evolution of the population size for 76 weeks and for three starting days (Sunday, Monday, Tuesday) for the PSPs.

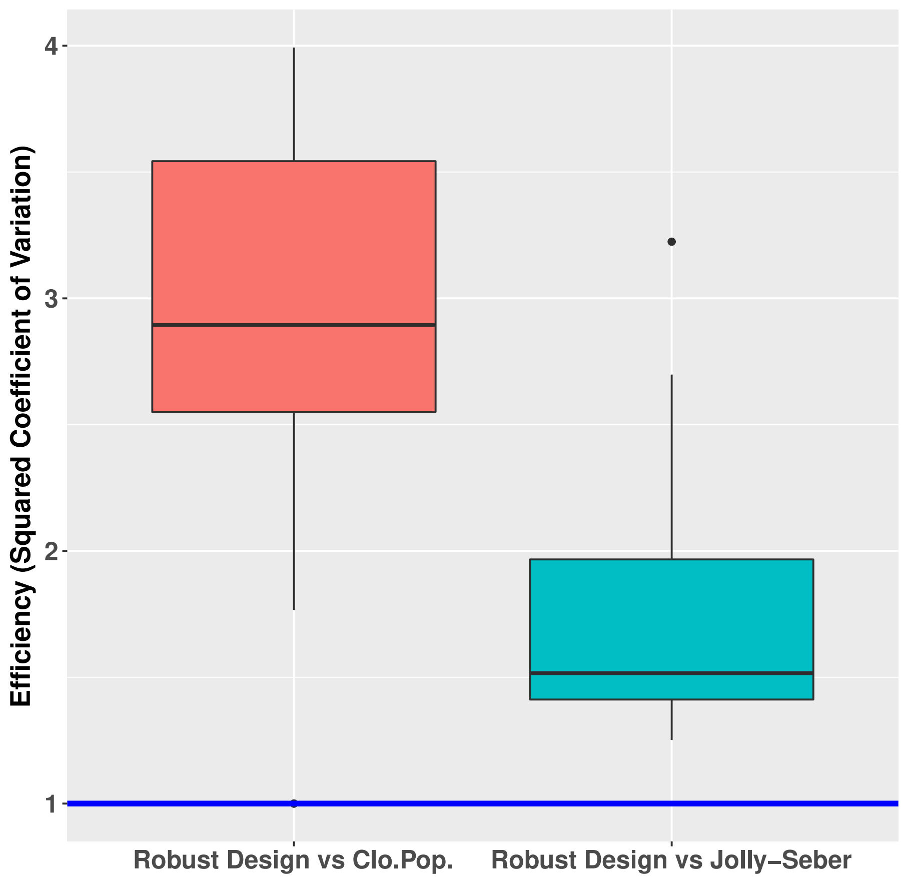

Figure 4: Efficiency comparison (Squared Coefficient of variation) between the robust design estimate of N_i and those obtained under models M_h closed population and Jolly-Seber. The relative efficiencies are calculated with 1000 bootstrap samples and their values from the 76 PSP plotted.

Table 3: Clientele size estimates and their coefficients of variation for the first 20 weeks of the experiment, under models M_h for a closed population, Jolly-Seber and M_{Dh}^t.

Parameter	Closed Pop.	Robust Design M_h	Open Pop.
$\hat{N}_1$	387 (27)	387 (27)	-
$\hat{N}_2$	672 (22)	438 (17)	315 (26)
$\hat{N}_3$	461 (21)	327 (14)	275 (18)
$\hat{N}_4$	708 (26)	487 (16)	411 (21)
$\hat{N}_5$	291 (26)	345 (14)	365 (16)
$\hat{N}_6$	446 (24)	416 (13)	405 (17)
$\hat{N}_7$	290 (27)	312 (13)	317 (15)
$\hat{N}_8$	375 (27)	457 (15)	492 (18)
$\hat{N}_9$	300 (27)	383 (14)	416 (17)
$\hat{N}_{10}$	311 (24)	370 (13)	392 (15)
$\hat{N}_{11}$	332 (25)	349 (12)	354 (15)
$\hat{N}_{12}$	280 (23)	316 (12)	327 (13)
$\hat{N}_{13}$	355 (24)	367 (12)	370 (14)
$\hat{N}_{14}$	368 (28)	428 (13)	446 (16)
$\hat{N}_{15}$	465 (25)	497 (13)	509 (16)
$\hat{N}_{16}$	248 (22)	379 (12)	464 (15)
$\hat{N}_{17}$	686 (48)	587 (21)	565 (28)
$\hat{N}_{18}$	292 (37)	601 (21)	793 (28)
$\hat{N}_{19}$	560 (49)	751 (22)	818 (29)
$\hat{N}_{20}$	212 (41)	440 (20)	533 (26)

Table 4: Survival probability estimates and their coefficients of variation for the first 20 weeks of the experiment, under Jolly-Seber and M_{Dh}^t models.

Parameter	Closed Pop.	Robust Design M_h	Open Pop.
$\hat{\phi}_1$	-	0.964 (6)	0.810 (3)
$\hat{\phi}_2$	-	0.554 (12)	0.543 (15)
$\hat{\phi}_3$	-	0.768 (11)	0.742 (12)
$\hat{\phi}_4$	-	0.632 (12)	0.689 (14)
$\hat{\phi}_5$	-	0.835 (9)	0.804 (9)
$\hat{\phi}_6$	-	0.642 (12)	0.656 (14)
$\hat{\phi}_7$	-	0.869 (8)	0.896 (8)
$\hat{\phi}_8$	-	0.766 (11)	0.784 (12)
$\hat{\phi}_9$	-	0.704 (12)	0.699 (14)
$\hat{\phi}_{10}$	-	0.705 (11)	0.694 (13)
$\hat{\phi}_{11}$	-	0.699 (10)	0.707 (11)
$\hat{\phi}_{12}$	-	0.842 (9)	0.835 (9)
$\hat{\phi}_{13}$	-	0.894 (7)	0.912 (7)
$\hat{\phi}_{14}$	-	0.829 (9)	0.826 (9)
$\hat{\phi}_{15}$	-	0.755 (10)	0.873 (10)
$\hat{\phi}_{16}$	-	0.872 (7)	0.749 (9)
$\hat{\phi}_{17}$	-	0.648 (20)	0.749 (23)
$\hat{\phi}_{18}$	-	0.988 (8)	0.932 (10)
$\hat{\phi}_{19}$	-	0.715 (17)	0.776 (20)
$\hat{\phi}_{20}$	-	0.758 (14)	0.691 (17)

Table 5: Capture probability estimates and their coefficients of variation for the first 20 weeks of the experiment, under models M_h for a closed population, Jolly-Seber and M^t_{Dh}.

Parameter	Closed Pop.	Robust Design M_h	Open Pop.
$\hat{p}^*_1$	0.256 (25)	0.256 (22)	-
$\hat{p}^*_2$	0.222 (20)	0.324 (16)	0.474 (27)
$\hat{p}^*_3$	0.310 (18)	0.418 (13)	0.520 (19)
$\hat{p}^*_4$	0.191 (23)	0.266 (16)	0.328 (22)
$\hat{p}^*_5$	0.347 (23)	0.299 (15)	0.276 (18)
$\hat{p}^*_6$	0.287 (22)	0.305 (14)	0.316 (19)
$\hat{p}^*_7$	0.352 (24)	0.330 (14)	0.321 (16)
$\hat{p}^*_8$	0.304 (25)	0.255 (16)	0.232 (20)
$\hat{p}^*_9$	0.334 (24)	0.269 (15)	0.240 (19)
$\hat{p}^*_{10}$	0.354 (22)	0.303 (13)	0.281 (17)
$\hat{p}^*_{11}$	0.310 (23)	0.297 (13)	0.291 (16)
$\hat{p}^*_{12}$	0.368 (20)	0.331 (12)	0.315 (14)
$\hat{p}^*_{13}$	0.310 (21)	0.301 (13)	0.297 (16)
$\hat{p}^*_{14}$	0.290 (26)	0.254 (15)	0.240 (18)
$\hat{p}^*_{15}$	0.267 (23)	0.251 (14)	0.244 (17)
$\hat{p}^*_{16}$	0.455 (20)	0.317 (13)	0.243 (17)
$\hat{p}^*_{17}$	0.123 (48)	0.142 (22)	0.149 (29)
$\hat{p}^*_{18}$	0.195 (34)	0.101 (22)	0.072 (30)
$\hat{p}^*_{19}$	0.138 (51)	0.104 (22)	0.094 (29)
$\hat{p}^*_{20}$	0.063 (42)	0.120 (21)	0.094 (27)

Table 6: Estimates of new arrivals and their coefficients of variation for the first 20 weeks of the experiment, under Jolly-Seber and M_{Dh}^t models.

Parameter	Closed Pop.	Robust Design M_h	Open Pop.
$\hat{B}_1$	-	65 (62)	-
$\hat{B}_2$	-	84 (90)	20 (534)
$\hat{B}_3$	-	236 (29)	262 (38)
$\hat{B}_4$	-	37 (189)	60 (346)
$\hat{B}_5$	-	128 (43)	153 (45)
$\hat{B}_6$	-	45 (134)	0 (382)
$\hat{B}_7$	-	186 (34)	284 (38)
$\hat{B}_8$	-	33 (146)	0 (198)
$\hat{B}_9$	-	100 (59)	66 (120)
$\hat{B}_{10}$	-	88 (63)	81 (106)
$\hat{B}_{11}$	-	72 (76)	81 (118)
$\hat{B}_{12}$	-	101 (43)	139 (50)
$\hat{B}_{13}$	-	100 (49)	137 (66)
$\hat{B}_{14}$	-	142 (48)	102 (72)
$\hat{B}_{15}$	-	3 (263)	44 (190)
$\hat{B}_{16}$	-	256 (40)	160 (56)
$\hat{B}_{17}$	-	221 (86)	370 (177)
$\hat{B}_{18}$	-	157 (73)	224 (74)
$\hat{B}_{19}$	-	0 (237)	0 (315)
$\hat{B}_{20}$	-	243 (70)	178 (98)