1 Appendix: Estimation

It is beyond the scope of this review of identification to also exhaustively cover estimation, therefore we only outline the broad approaches to inference that have been proposed in partial identification settings. Readers interested in more depth may consider beginning with the review by Tamer (2010), or the references cited in this section.

We may consider estimating confidence regions around the identified lower and upper bounds separately (e.g., as done early on by Manski et al. (1992)), or estimating confidence regions for the contained set jointly (e.g., as done by Horowitz and Manski (2000)). The first approach does not take into account the correlation between the lower and upper bounds, but may be all that is desired in settings where only one of the boundaries is of interest. However, many of the bounds described in this review involve minimum and maximum operations. These functionals are irregular, and therefore common techniques like the non-parametric bootstrap will generally not be valid (Andrews, 2000; Andrews and Guggenberger, 2009; Romano and Shaikh, 2008; Chernozhukov et al., 2007, 2013). Subsampling or related resampling techniques have been proposed to overcome this issue (Romano and Shaikh, 2008; Chernozhukov et al., 2007; Ramsahai and Lauritzen, 2011). In particular, Chernozhukov et al. (2013) has developed methods when a scalar is bounded above (below) by the minimum (maximum) of several quantities; these methods can be applied to many of the bounds given in Tables 2 and 3. For more general discussion of estimating the confidence region for the contained set under an optimization framework, see Kaido (2016). For a discussion of these estimation procedures in the context of testing the model itself (as in Section 3), see Bugni et al. (2015).

We may also consider estimating confidence regions for the ATE, i.e., the partially identified parameter itself, even though we do not identify it. Imbens and Manski (2004) introduced the concept and construction of confidence intervals for the true value of the parameter itself; see Stoye (2009) for important extensions and clarifications and Kaido et al. (2016) for recent developments. The length of the confidence interval for the ATE will be no greater than that of the confidence region for the identified set.

In general, more work is needed to translate the valid approaches into practice. To date, there are relatively few applications of these estimation procedures – even fewer beyond the methodological literature, e.g., Blanco et al. (2013); Mealli and Pacini (2013); Huber et al. (2015) – and no software packages that are readily applied to all bounds given in Tables 2 and 3.

2 Appendix: Lower and upper bounds on $\mathrm{E}[Y^x]$

Here we provide expressions for the bounds for the mean counterfactuals under the assumption sets discussed in Section 2. Note that bounds for the relative risk, $\mathrm{E}[Y^{x=1}]/\mathrm{E}[Y^{x=0}]$, follow from these expressions by variation independence of the numerator and the denominator (Dawid, 2000).

Supplemental Table 1: Bounds for identification of $\mathrm{E}[Y^x]$

Assumption Set	Bound*
Lower Bound for $\mathrm{E}[Y^0]$	
Data only	$p_{y_1\|x_0}p_{x_0}$
A1+A2	$\max\left\{\begin{array}{c} p_{y_1\|x_0,z_0}p_{x_0\|z_0} \\ p_{y_1\|x_0,z_1}p_{x_0\|z_1} \end{array}\right\}$
A3+A4**	$\max\left\{\begin{array}{c} p_{y_1,x_0\|z_0}+p_{y_1,x_1\|z_0}-p_{y_0,x_0\|z_1}-p_{y_1,x_1\|z_1} \\ p_{y_1,x_0\|z_1} \\ p_{y_1,x_0\|z_0} \\ p_{y_0,x_1\|z_0}+p_{y_1,x_0\|z_0}-p_{y_0,x_0\|z_1}-p_{y_0,x_1\|z_1} \end{array}\right\}$
Upper Bound for $\mathrm{E}[Y^0]$	
Data only	$p_{y_1\|x_0}p_{x_0}+p_{x_1}$
A1+A2	$\min\left\{\begin{array}{c} p_{y_1\|x_0,z_0}p_{x_0\|z_0}+p_{x_1\|z_0} \\ p_{y_1\|x_0,z_1}p_{x_0\|z_1}+p_{x_1\|z_1} \end{array}\right\}$
A3+A4**	$\min\left\{\begin{array}{c} p_{y_0,x_1\|z_0}+p_{y_1,x_0\|z_0}+p_{y_1,x_0\|z_1}+p_{y_1,x_1\|z_1} \\ 1-p_{y_0,x_0\|z_1} \\ 1-p_{y_0,x_0\|z_0} \\ p_{y_1,x_0\|z_0}+p_{y_1,x_1\|z_0}+p_{y_0,x_1\|z_1}+p_{y_1,x_0\|z_1} \end{array}\right\}$
Lower Bound for $\mathrm{E}[Y^1]$	
Data only	$p_{y_1\|x_1}p_{x_1}$
A1+A2	$\max\left\{\begin{array}{c} p_{y_1,x_1,z_0}p_{x_1\|z_0} \\ p_{y_1,x_1,z_1}p_{x_1\|z_1} \end{array}\right\}$
A3+A4**	$\max\left\{\begin{array}{c} p_{y_1,x_1\|z_0} \\ p_{y_1,x_1\|z_1} \\ -p_{y_0,x_0\|z_0}-p_{y_0,x_1\|z_0}+p_{y_0,x_0\|z_1}+p_{y_1,x_1\|z_1} \\ -p_{y_0,x_1\|z_0}-p_{y_1,x_0\|z_0}+p_{y_1,x_0\|z_1}+p_{y_1,x_1\|z_1} \end{array}\right\}$
Upper Bound for $\mathrm{E}[Y^1]$	
Data only	$p_{y_1\|x_1}p_{x_1}+p_{x_0}$
A1+A2	$\min\left\{\begin{array}{c} p_{y_1,x_1,z_0}p_{x_1\|z_0}+p_{x_0\|z_0} \\ p_{y_1,x_1,z_1}p_{x_1\|z_1}+p_{x_0\|z_1} \end{array}\right\}$
A3+A4**	$\min\left\{\begin{array}{c} 1-p_{y_0,x_1\|z_0} \\ 1-p_{y_0,x_1\|z_1} \\ p_{y_0,x_0\|z_0}+p_{y_1,x_1\|z_0}+p_{y_1,x_0\|z_1}+p_{y_1,x_1\|z_1} \\ p_{y_1,x_0\|z_0}+p_{y_1,x_1\|z_0}+p_{y_0,x_0\|z_1}+p_{y_1,x_1\|z_1} \end{array}\right\}$

*$p_{y_k,x_j|z_i} \equiv \Pr[Y = k, X = j|Z = i]; p_{y_k|x_j,z_i} \equiv \Pr[Y = k|X = j, Z = i]; p_{y_k|x_j} \equiv \Pr[Y = k|X = j]; p_{y_k|z_i} \equiv \Pr[Y = k|Z = i]; p_{x_j|z_i} \equiv \Pr[X = j|Z = i]; p_{x_j} \equiv \Pr[X = j]; p_{z_i} \equiv \Pr[Z = i]$

**See Section 2 for additional assumption sets that likewise lead to the Balke-Pearl bounds.

3 Appendix: Lower and upper bounds within compliance types with known proportion of defiers

We will denote the marginal probability of an always-taker, never-taker, complier, and defier as π_{AT}, π_{NT}, π_{CO}, and π_{DE}, respectively. We will further make use of the same compact notation as appears in Tables 2 and 3:

$$p_{y_k,x_j|z_i} \equiv \Pr[Y = k, X = j|Z = i]$$

$$p_{y_k|x_j,z_i} \equiv \Pr[Y = k|X = j, Z = i]$$

$$p_{y_k|x_j} \equiv \Pr[Y = k|X = j]$$

$$p_{y_k|z_i} \equiv \Pr[Y = k|Z = i]$$

$$p_{x_j|z_i} \equiv \Pr[X = j|Z = i]$$

$$p_{x_j} \equiv \Pr[X = j]$$

$$p_{z_i} \equiv \Pr[Z = i]$$

Under (A5) and (A12), the proportion of defiers can be bounded as follows:

$$\max\left\{\begin{array}{c} 0 \\ p_{x_1|z_0} - p_{x_1|z_1} \\ p_{x_1|z_0} - 1 - \sum_j p_{y_j,x_0|z_j} \\ p_{x_1|z_0} - 1 - \sum_k p_{y_k,x_0|z_{1-k}} \\ p_{x_1|z_0} - \sum_j p_{y_j,x_1|z_j} \\ p_{x_1|z_0} - \sum_k p_{y_k,x_1|z_{1-k}} \end{array}\right\} \leq \pi_{DE} \leq \min\left\{\begin{array}{c} p_{x_1|z_0} \\ p_{x_0|z_1} \end{array}\right\}$$

See Richardson and Robins (2010) for further details. As discussed in the text, for any given value of π_{DE}, the proportion of the other three compliance types is determined by the observed distribution of (X, Z). For the remainder of Appendix 3, we assume that the distribution of compliance types is known. Under assumptions (A5), (A12), and (A13), Richardson and Robins (2010) identified bounds for counterfactual risks within each compliance type.

For the always-takers, we have:

$$0 \leq \Pr[Y^0 = 1 | X^1 = X^0 = 1] \leq 1$$

$$\Pr[Y^1 = 1 | X^1 = X^0 = 1] \geq \max\left\{\begin{array}{c} 0 \\ \frac{p_{y_1|x_1,z_1} - \frac{\pi_{CO}}{\pi_{CO}+\pi_{AT}}}{\frac{\pi_{AT}}{\pi_{CO}+\pi_{AT}}} \\ \frac{p_{y_1|x_1,z_0} - \frac{\pi_{DE}}{\pi_{DE}+\pi_{AT}}}{\frac{\pi_{AT}}{\pi_{DE}+\pi_{AT}}} \end{array}\right\}$$

$$\Pr[Y^1 = 1 | X^1 = X^0 = 1] \leq \min\left\{\begin{array}{c} 1 \\ \frac{p_{y_1|x_1,z_1}}{\frac{\pi_{AT}}{\pi_{CO}+\pi_{AT}}} \\ \frac{p_{y_1|x_1,z_0}}{\frac{\pi_{AT}}{\pi_{DE}+\pi_{AT}}} \end{array}\right\}$$

For the never-takers, we have:

$$\Pr[Y^0 = 1|X^1 = X^0 = 0] \geq \max \left\{ \begin{array}{c} 0 \\ \frac{p_{y_1|x_0,z_0} - \frac{\pi_{CO}}{\pi_{CO}+\pi_{NT}}}{\frac{\pi_{NT}}{\pi_{CO}+\pi_{NT}}} \\ \frac{p_{y_1|x_0,z_1} - \frac{\pi_{DE}}{\pi_{DE}+\pi_{NT}}}{\frac{\pi_{NT}}{\pi_{DE}+\pi_{NT}}} \end{array} \right\}$$

$$\Pr[Y^0 = 1|X^1 = X^0 = 0] \leq \min \left\{ \begin{array}{c} 1 \\ \frac{p_{y_1|x_0,z_0}}{\frac{\pi_{NT}}{\pi_{CO}+\pi_{NT}}} \\ \frac{p_{y_1|x_0,z_1}}{\frac{\pi_{NT}}{\pi_{DE}+\pi_{NT}}} \end{array} \right\}$$

$$0 \leq \Pr[Y^1 = 1|X^1 = X^0 = 0] \leq 1$$

To simplify notation, let $l\gamma^x_{NT}$ and $u\gamma^x_{NT}$ refer to the lower and upper bounds on $\Pr[Y^x = 1|X^1 = X^0 = 0]$ for each $x \in \{0, 1\}$, and similarly let $l\gamma^x_{AT}$, and $u\gamma^x_{AT}$ refer to the lower and upper bounds on $\Pr[Y^x = 1|X^1 = X^0 = 1]$ for each $x \in \{0, 1\}$ derived above. The bounds in the compliers and defiers are then functions of these.

For the compliers, we have:

$$\Pr[Y^0 = 1|X^1 > X^0] \geq (p_{y_1|x_0,z_0} - u\gamma^0_{NT}\frac{\pi_{NT}}{\pi_{NT} + \pi_{CO}})/\frac{\pi_{CO}}{\pi_{NT} + \pi_{CO}}$$

$$\Pr[Y^0 = 1|X^1 > X^0] \leq (p_{y_1|x_0,z_0} - l\gamma^0_{NT}\frac{\pi_{NT}}{\pi_{NT} + \pi_{CO}})/\frac{\pi_{CO}}{\pi_{NT} + \pi_{CO}}$$

$$\Pr[Y^1 = 1|X^1 > X^0] \geq (p_{y_1|x_1,z_1} - u\gamma^1_{AT}\frac{\pi_{AT}}{\pi_{AT} + \pi_{CO}})/\frac{\pi_{CO}}{\pi_{AT} + \pi_{CO}}$$

$$\Pr[Y^1 = 1|X^1 > X^0] \leq (p_{y_1|x_1,z_1} - l\gamma^1_{AT}\frac{\pi_{AT}}{\pi_{AT} + \pi_{CO}})/\frac{\pi_{CO}}{\pi_{AT} + \pi_{CO}}$$

For the defiers we have:

$$\Pr[Y^0 = 1|X^1 < X^0] \geq (p_{y_1|x_0,z_1} - u\gamma^0_{NT}\frac{\pi_{NT}}{\pi_{NT}+\pi_{DE}})/\frac{\pi_{DE}}{\pi_{NT}+\pi_{DE}}$$

$$\Pr[Y^0 = 1|X^1 < X^0] \leq (p_{y_1|x_0,z_1} - l\gamma^0_{NT}\frac{\pi_{NT}}{\pi_{NT}+\pi_{DE}})/\frac{\pi_{DE}}{\pi_{NT}+\pi_{DE}}$$

$$\Pr[Y^1 = 1|X^1 < X^0] \geq (p_{y_1|x_1,z_0} - u\gamma^1_{AT}\frac{\pi_{AT}}{\pi_{AT}+\pi_{DE}})/\frac{\pi_{DE}}{\pi_{AT}+\pi_{DE}}$$

$$\Pr[Y^1 = 1|X^1 < X^0] \leq (p_{y_1|x_1,z_0} - l\gamma^1_{AT}\frac{\pi_{AT}}{\pi_{AT}+\pi_{DE}})/\frac{\pi_{DE}}{\pi_{AT}+\pi_{DE}}$$

Upper and lower bounds on the ATE within compliance types can be derived directly from the above formulas, as $\Pr[Y^1 = 1]$ and $\Pr[Y^0 = 1]$ are variation independent given compliance type (Richardson and Robins, 2010). To return to the ATE, the identities

$$p_{y_1,x_1|z_1} = \pi_{CO}\Pr[Y^1 = 1|X^1 > X^0] + \pi_{AT}\Pr[Y^1 = 1|X^1 = X^0 = 1]$$

$$p_{y_1,x_0|z_0} = \pi_{CO}\Pr[Y^0 = 1|X^1 > X^0] + \pi_{NT}\Pr[Y^0 = 1|X^1 = X^0 = 0]$$

then imply $\Pr[Y^0 = 1]$, $\Pr[Y^1 = 1]$, and ATE are bounded as follows:

$$\Pr[Y^0 = 1] \geq p_{y_1,x_0|z_0} + \pi_{DE}l\gamma^0_{DE} + \pi_{AT}l\gamma^0_{AT}$$

$$\Pr[Y^0 = 1] \leq p_{y_1,x_0|z_0} + \pi_{DE}u\gamma^0_{DE} + \pi_{AT}u\gamma^0_{AT}$$

$$\Pr[Y^1 = 1] \geq p_{y_1,x_1|z_1} + \pi_{DE}l\gamma^1_{DE} + \pi_{NT}l\gamma^1_{NT}$$

$$\Pr[Y^1 = 1] \leq p_{y_1,x_1|z_1} + \pi_{DE} u\gamma_{DE}^1 + \pi_{NT} u\gamma_{NT}^1$$

$$ATE \geq p_{y_1,x_1|z_1} - p_{y_1,x_0|z_0} + \pi_{DE} lATE_{DE} + \pi_{NT} l\gamma_{NT}^1 - \pi_{AT} u\gamma_{AT}^0$$

$$ATE \leq p_{y_1,x_1|z_1} - p_{y_1,x_0|z_0} + \pi_{DE} lATE_{DE} + \pi_{NT} u\gamma_{NT}^1 - \pi_{AT} l\gamma_{AT}^0$$

where $l\gamma_{DE}^x$ and $l\gamma_{DE}^x$ are the lower and upper bounds on $\Pr[Y^x = 1|X^1 < X^0]$ for each $x \in \{0,1\}$, and $lATE_{DE}$ is the lower bound on the average treatment effect within the defiers.

The bounds under the additional assumption (A14) are as above, except with $l\gamma_{NT}^1$, $l\gamma_{AT}^0$, $u\gamma_{NT}^1$, and $u\gamma_{AT}^0$ replaced with given values. That is, one can specify in (A14) that $l\gamma_{NT}^1$ or $l\gamma_{AT}^0$ are greater than 0, or that $u\gamma_{NT}^1$ or $u\gamma_{AT}^0$ are less than 1.

4 Appendix: Comparison of constraints implied by the IV model for a trichotomous instrument

Consider the setting of a trichotomous instrument Z, binary treatment X, and binary outcome Y. Bonet (2001) showed the IV model defined by the individual-level exclusion restriction (A5) and the full exchangeability assumption (A12) implies the following constraints:

$$\max_i \left\{p_{y_1,x_1|z_i}\right\} \leq \min_i \left\{1 - p_{y_0,x_1|z_i}\right\}$$

$$\max_i \left\{p_{y_1,x_0|z_i}\right\} \leq \min_i \left\{1 - p_{y_0,x_0|z_i}\right\}$$

$$\max_i \left\{p_{y_1,x_0|z_i} + p_{y_1,x_1|z_i}\right\} + \max_i \left\{p_{y_1,x_0|z_i} + p_{y_0,x_1|z_i}\right\} + \max_i \left\{p_{y_0,x_0|z_i}\right\} \leq 2$$

where $p_{y_k,x_j|z_i}$ makes use of the same compact notation as appears in Tables 2 and 3. The first two of these three expressions were previously given by Pearl (1995). We now provide an empirical example that satisfies the first two constraints but not the third.

Supplemental Table 2: Example with a trichotomous proposed instrument

N	**Z**	**X**	**Y**
10	0	0	0
10	0	0	1
70	0	1	0
10	0	1	1
10	1	0	0
50	1	0	1
10	1	1	0
30	1	1	1
50	2	0	0
10	2	0	1
20	2	1	0
20	2	1	1

Here, the first constraint is satisfied ($0.3 \leq 0.3$), the second constraint is satisfied ($0.5 \leq 0.5$), but the third constraint is not satisfied ($0.8+0.8+0.5 = 2.1$ is not less than or equal to 2).