Space-filling fractional factorial designs
Supplementary Materials

1. Proof of Theorem 3

From equation (18), when $s = 3$, we have

$$\bar{\phi}(D) = 9^{-n} \sum_{i=0}^{n} (3 - \rho^2 - 2\rho)^i (3 + 2\rho^2 + 4\rho)^{n-i} A_i(D).$$

Using the formula

$$(a + b + c)^n = \sum_{k=0}^{n} \sum_{l=0}^{k} \binom{n}{k}\binom{k}{l} a^{n-k} b^{k-l} c^l,$$

we have

$$\begin{aligned}
&\sum_{i=0}^{n} (3 - \rho^2 - 2\rho)^i (3 + 2\rho^2 + 4\rho)^{n-i} A_i(D) \\
&= \sum_{i=0}^{n} \left(\sum_{k_1=0}^{i} \sum_{l_1=0}^{k_1} \binom{i}{k_1}\binom{k_1}{l_1} 3^{i-k_1} (-\rho^2)^{k_1-l_1} (-2\rho)^{l_1} \right) \\
&\qquad \left(\sum_{k_2=0}^{n-i} \sum_{l_2=0}^{k_2} \binom{n-i}{k_2}\binom{k_2}{l_2} 3^{n-i-k_2} (2\rho^2)^{k_2-l_2} (4\rho)^{l_2} \right) A_i(D) \\
&= \sum_{i=0}^{n} \left(\sum_{k_1=0}^{i} \sum_{k_2=0}^{n-i} \sum_{l_1=0}^{k_1} \sum_{l_2=0}^{k_2} \binom{i}{k_1}\binom{k_1}{l_1}\binom{n-i}{k_2}\binom{k_2}{l_2} (-1)^{k_1} 3^{n-k_1-k_2} 2^{k_2+l_1+l_2} \rho^{2k_1+2k_2-l_1-l_2} \right) A_i(D).
\end{aligned}$$

Let $2k_1 + 2k_2 - l_1 - l_2 = k$. Comparing coefficient of ρ^k in equation (19), we have

$$\bar{B}_k(D) = \frac{N}{9^n} \sum_{i=0}^{n} \left(\sum_{\substack{2k_1+2k_2-l_1-l_2=k \\ 0 \le l_1 \le k_1 \le i, \quad 0 \le l_2 \le k_2 \le n-i}} \binom{i}{k_1}\binom{n-i}{k_2}\binom{k_1}{l_1}\binom{k_2}{l_2} (-1)^{k_1} 3^{n-k_1-k_2} 2^{k_2+l_1+l_2} \right) A_i(D).$$

Using the convention $\binom{n}{x} = 0$ for $x \le 0$ or $x > n$, we can ignore the constraints $0 \le l_1 \le k_1 \le i$ and $0 \le l_2 \le k_2 \le n - i$ to simplify the notation.

Further let $k_1 + k_2 - l_1 - l_2 = j$, so $k_1 + k_2 = k - j$ and $l_1 + l_2 = k - 2j$. Then we have

$$\begin{aligned}
\bar{B}_k(D) &= \frac{N}{9^n} \sum_{i=0}^{n} \left(\sum_{j=0}^{n} \sum_{k_1+k_2=k-j} \sum_{l_1+l_2=k-2j} 3^{n+j-k} 2^{k-2j} (-1)^{k_1} 2^{k_2} \binom{i}{k_1}\binom{n-i}{k_2}\binom{k_1}{l_1}\binom{k_2}{l_2} \right) A_i(D) \\
&= \frac{2^k N}{3^{n+k}} \sum_{i=0}^{n} \left(\sum_{j=0}^{n} \sum_{k_1+k_2=k-j} \sum_{l_1+l_2=k-2j} \left(\frac{3}{4}\right)^j (-1)^{k_1} 2^{k_2} \binom{i}{k_1}\binom{n-i}{k_2}\binom{k_1}{l_1}\binom{k_2}{l_2} \right) A_i(D).
\end{aligned}$$

Since $\sum_{l_1+l_2=k-2j}\binom{k_1}{l_1}\binom{k_2}{l_2} = \binom{k_1+k_2}{k-2j} = \binom{k-j}{j}$, we have

$$\bar{B}_k(D) \;=\; \frac{2^k N}{3^{n+k}} \sum_{i=0}^{n} \left(\sum_{j=0}^{n} \left(\frac{3}{4}\right)^j \binom{k-j}{j} \sum_{k_1+k_2=k-j} (-1)^{k_1} 2^{k_2} \binom{i}{k_1}\binom{n-i}{k_2} \right) A_i(D).$$

The definition of Krawtchouk polynomials implies $\sum_{k_1+k_2=k-j}(-1)^{k_1}2^{k_2}\binom{i}{k_1}\binom{n-i}{k_2} = P_{k-j}(i;n,3)$. Then

$$\bar{B}_k(D) \;=\; \frac{2^k N}{3^{n+k}} \sum_{i=0}^{n} \left(\sum_{j=0}^{n} \left(\frac{3}{4}\right)^j \binom{k-j}{j} P_{k-j}(i;n,3) \right) A_i(D).$$

This completes the proof.

2. Some space-filling fractional factorial designs

16-run four-level designs

$3-factors$			$4-factors$				$5-factors$				
1	1	1	1	1	1	0	1	1	1	1	1
1	3	2	1	3	2	1	1	3	2	3	3
1	2	0	1	2	0	2	1	2	0	2	2
1	0	3	1	0	3	3	1	0	3	0	0
3	3	3	3	3	3	2	3	3	3	2	1
3	1	0	3	1	0	3	3	1	0	0	3
3	0	2	3	0	2	0	3	0	2	1	2
3	2	1	3	2	1	1	3	2	1	3	0
2	2	2	2	2	2	3	2	2	2	0	1
2	0	1	2	0	1	2	2	0	1	2	3
2	1	3	2	1	3	1	2	1	3	3	2
2	3	0	2	3	0	0	2	3	0	1	0
0	0	0	0	0	0	1	0	0	0	3	1
0	2	3	0	2	3	0	0	2	3	1	3
0	3	1	0	3	1	3	0	3	1	0	2
0	1	2	0	1	2	2	0	1	2	2	0

25-run five-level designs

$3-factors$			$4-factors$				$5-factors$					$6-factors$					
0	0	0	0	0	1	2	0	0	0	0	0	0	0	0	1	0	2
0	1	3	0	1	4	3	0	1	3	4	2	0	1	1	4	2	3
0	2	1	0	2	2	4	0	2	1	3	4	0	2	3	3	3	0
0	3	4	0	3	0	0	0	3	4	2	1	0	3	4	2	1	4
0	4	2	0	4	3	1	0	4	2	1	3	0	4	2	0	4	1
1	0	3	1	0	4	0	1	0	3	2	4	1	0	1	2	3	1
1	1	1	1	1	2	1	1	1	1	1	1	1	1	3	0	1	2
1	2	4	1	2	0	2	1	2	4	0	3	1	2	4	1	4	3
1	3	2	1	3	3	3	1	3	2	4	0	1	3	2	4	0	0
1	4	0	1	4	1	4	1	4	0	3	2	1	4	0	3	2	4
2	0	1	2	0	2	3	2	0	1	4	3	2	0	3	4	4	4
2	1	4	2	1	0	4	2	1	4	3	0	2	1	4	3	0	1
2	2	2	2	2	3	0	2	2	2	2	2	2	2	2	2	2	2
2	3	0	2	3	1	1	2	3	0	1	4	2	3	0	0	3	3
2	4	3	2	4	4	2	2	4	3	0	1	2	4	1	1	1	0
3	0	4	3	0	0	1	3	0	4	1	2	3	0	4	0	2	0
3	1	2	3	1	3	2	3	1	2	0	4	3	1	2	1	3	4
3	2	0	3	2	1	3	3	2	0	4	1	3	2	0	4	1	1
3	3	3	3	3	4	4	3	3	3	3	3	3	3	1	3	4	2
3	4	1	3	4	2	0	3	4	1	2	0	3	4	3	2	0	3
4	0	2	4	0	3	4	4	0	2	3	1	4	0	2	3	1	3
4	1	0	4	1	1	0	4	1	0	2	3	4	1	0	2	4	0
4	2	3	4	2	4	1	4	2	3	1	0	4	2	1	0	0	4
4	3	1	4	3	2	2	4	3	1	0	2	4	3	3	1	2	1
4	4	4	4	4	0	3	4	4	4	4	4	4	4	4	4	3	2

32-run four-level designs

$3-factors$			$4-factors$				$5-factors$					$6-factors$					
1	1	1	1	1	1	0	1	1	1	1	0	1	1	1	0	1	1
1	2	0	1	0	2	2	1	2	2	3	0	1	3	2	1	1	2
1	3	1	1	2	1	3	1	3	1	2	1	1	0	1	2	0	2
1	0	0	1	3	2	1	1	0	2	0	1	1	2	2	3	0	1
1	1	3	1	1	0	2	1	1	0	3	3	1	1	0	1	2	3
1	2	2	1	0	3	0	1	2	3	1	3	1	3	3	0	2	0
1	3	3	1	2	0	1	1	3	0	0	2	1	0	0	3	3	0
1	0	2	1	3	3	3	1	0	3	2	2	1	2	3	2	3	3
3	2	2	3	0	3	1	3	2	3	0	2	3	3	3	3	3	2
3	1	3	3	1	0	3	3	1	0	2	2	3	1	0	2	3	1
3	0	2	3	3	3	2	3	0	3	3	3	3	2	3	1	2	1
3	3	3	3	2	0	0	3	3	0	1	3	3	0	0	0	2	2
3	2	0	3	0	2	3	3	2	2	2	1	3	3	2	2	0	0
3	1	1	3	1	1	1	3	1	1	0	1	3	1	1	3	0	3
3	0	0	3	3	2	0	3	0	2	1	0	3	2	2	0	1	3
3	3	1	3	2	1	2	3	3	1	3	0	3	0	1	1	1	0
2	3	0	2	2	2	3	0	3	2	2	3	2	0	2	2	2	3
2	0	1	2	3	1	1	0	0	1	0	3	2	2	1	3	2	0
2	1	0	2	1	2	0	0	1	2	1	2	2	1	2	0	3	0
2	2	1	2	0	1	2	0	2	1	3	2	2	3	1	1	3	3
2	3	2	2	2	3	1	0	3	3	0	0	2	0	3	3	1	1
2	0	3	2	3	0	3	0	0	0	2	0	2	2	0	2	1	2
2	1	2	2	1	3	2	0	1	3	3	1	2	1	3	1	0	2
2	2	3	2	0	0	0	0	2	0	1	1	2	3	0	0	0	1
0	0	3	0	3	0	2	2	0	0	3	1	0	2	0	1	0	0
0	3	2	0	2	3	0	2	3	3	1	1	0	0	3	0	0	3
0	2	3	0	0	0	1	2	2	0	0	0	0	3	0	3	1	3
0	1	2	0	1	3	3	2	1	3	2	0	0	1	3	2	1	0
0	0	1	0	3	1	0	2	0	1	1	2	0	2	1	0	3	2
0	3	0	0	2	2	2	2	3	2	3	2	0	0	2	1	3	1
0	2	1	0	0	1	3	2	2	1	2	3	0	3	1	2	2	1
0	1	0	0	1	2	1	2	1	2	0	3	0	1	2	3	2	2

32-run four-level designs

$7-factors$							$8-factors$								$9-factors$								
0	1	1	1	1	1	1	1	1	1	1	1	0	0	1	0	0	3	0	0	3	2	0	1
0	2	3	0	1	2	3	1	3	2	0	1	2	3	0	0	1	0	3	0	1	0	1	0
0	3	1	3	0	2	0	1	2	1	2	0	2	1	3	0	2	3	2	3	1	1	3	3
0	0	3	2	0	1	2	1	0	2	3	0	0	2	2	0	3	0	1	3	3	3	2	2
0	1	0	0	3	3	2	1	1	0	0	2	3	2	3	0	0	2	3	2	0	3	3	1
0	2	2	1	3	0	0	1	3	3	1	2	1	1	2	0	1	1	0	2	2	1	2	0
0	3	0	2	2	0	3	1	2	0	3	3	1	3	1	0	2	2	1	1	2	0	0	3
0	0	2	3	2	3	1	1	0	3	2	3	3	0	0	0	3	1	2	1	0	2	1	2
3	2	2	2	2	2	2	3	3	3	3	3	2	2	3	3	1	1	1	1	1	3	3	1
3	1	0	3	2	1	0	3	1	0	2	3	0	1	2	3	0	2	2	1	3	1	2	0
3	0	2	0	3	1	3	3	0	3	0	2	0	3	1	3	3	1	3	2	3	0	0	3
3	3	0	1	3	2	1	3	2	0	1	2	2	0	0	3	2	2	0	2	1	2	1	2
3	2	3	3	0	0	1	3	3	2	2	0	1	0	1	3	1	0	2	3	2	2	0	1
3	1	1	2	0	3	3	3	1	1	3	0	3	3	0	3	0	3	1	3	0	0	1	0
3	0	3	1	1	3	0	3	0	2	1	1	3	1	3	3	3	0	0	0	0	1	3	3
3	3	1	0	1	0	2	3	2	1	0	1	1	2	2	3	2	3	3	0	2	3	2	2
2	3	3	3	3	3	3	2	2	2	2	2	3	3	2	2	2	0	2	2	0	0	2	1
2	0	1	2	3	0	1	2	0	1	3	2	1	0	3	2	3	3	1	2	2	2	3	0
2	1	3	1	2	0	2	2	1	2	1	3	1	2	0	2	0	0	0	1	2	3	1	3
2	2	1	0	2	3	0	2	3	1	0	3	3	1	1	2	1	3	3	1	0	1	0	2
2	3	2	2	1	1	0	2	2	3	3	1	0	1	0	2	2	1	1	0	3	1	1	1
2	0	0	3	1	2	2	2	0	0	2	1	2	2	1	2	3	2	2	0	1	3	0	0
2	1	2	0	0	2	1	2	1	3	0	0	2	0	2	2	0	1	3	3	1	2	2	3
2	2	0	1	0	1	3	2	3	0	1	0	0	3	3	2	1	2	0	3	3	0	3	2
1	0	0	0	0	0	0	0	0	0	0	0	1	1	0	1	3	2	3	3	2	1	1	1
1	3	2	1	0	3	2	0	2	3	1	0	3	2	1	1	2	1	0	3	0	3	0	0
1	2	0	2	1	3	1	0	3	0	3	1	3	0	2	1	1	2	1	0	0	2	2	3
1	1	2	3	1	0	3	0	1	3	2	1	1	3	3	1	0	1	2	0	2	0	3	2
1	0	1	1	2	2	3	0	0	1	1	3	2	3	2	1	3	3	0	1	1	0	2	1
1	3	3	0	2	1	1	0	2	2	0	3	0	0	3	1	2	0	3	1	3	2	3	0
1	2	1	3	3	1	2	0	3	1	2	2	0	2	0	1	1	3	2	2	3	3	1	3
1	1	3	2	3	2	0	0	1	2	3	2	2	1	1	1	0	0	1	2	1	1	0	2