Second Online Supplement:

Proof of Propositions 3.1, 3.2 and 3.3

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- For Proposition 3.1:

Straight from

$$\begin{aligned} f_x(X_t) &= f_x(I_t) + \sum_{h_q=1}^{q} \sum_{\substack{n_1,\ldots,n_{h_q}=1 \\ n_1<\ldots<n_{h_q}}}^{q} \sum_{j_{n_1},\ldots,j_{n_{h_q}}=v_1,\ldots,v_k} d^{h_q} f_x(I_t^{(\mathbf{0}_p|(\mathbf{0}_{n_1-1},j_{n_1},\ldots,j_{n_{h_q}},\mathbf{0}_{q-n_{h_q}}))}) \cdot \\ &\qquad \left(\prod_{r=1}^{h_q} f_{j_{n_r}}(I_{t-n_r})\right) + \\ &\quad \sum_{h_p=1}^{p}\sum_{h_q=0}^{q} \sum_{\substack{l_1,\ldots,l_{h_p}=1 \\ l_1<\ldots<l_{h_p}}}^{p} \sum_{\substack{n_1,\ldots,n_{h_q}=1 \\ n_1<\ldots<n_{h_q}}}^{q} \sum_{i_{l_1},\ldots,i_{l_{h_p}},\, j_{n_1},\ldots,j_{n_{h_q}}=v_1,\ldots,v_k} \\ &\quad d^{h_p+h_q} f_x(I_t^{((\mathbf{0}_{l_1-1},i_{l_1},\ldots,i_{l_{h_p}},\mathbf{0}_{p-l_{h_p}})|(\mathbf{0}_{n_1-1},j_{n_1},\ldots,j_{n_{h_q}},\mathbf{0}_{q-n_{h_q}}))}) \cdot \\ &\qquad \left(\prod_{r=1}^{h_p} f_{i_{l_r}}(X_{t-l_r})\right) \left(\prod_{r=1}^{h_q} f_{j_{n_r}}(I_{t-n_r})\right), \end{aligned} \tag{1}$$

it can be seen that it holds that $f_x(X_t) =$

$$\begin{aligned} &[f_x(I_t) + \mathrm{sf}_{x,t}^{(0)}] + \sum_{j=1}^{p} \sum_{i_j=v_1,\ldots,v_k} [d f_x(I_t^{((\mathbf{0}\ i_j\ \mathbf{0})|\mathbf{0}_q)}) + \mathrm{sf}_{x,t}^{(1)}(j,i_j))]\ f_{i_j}(X_{t-j}) + \sum_{\substack{j_1,j_2=1 \\ j_1<j_2}}^{p} \\ &\sum_{i_{j_1},i_{j_2}=v_1,\ldots,v_k} [d^2 f_x(I_t^{((\mathbf{0}\ i_{j_1}\ \mathbf{0}\ i_{j_2}\ \mathbf{0})|\mathbf{0}_q)}) + \mathrm{sf}_{x,t}^{(2)}((j_1,j_2),(i_{j_1},i_{j_2}))]\ f_{i_{j_1}}(X_{t-j_1})\ f_{i_{j_2}}(X_{t-j_2}) + \\ &\qquad \vdots \\ &\sum_{i_1,\ldots,i_p=v_1,\ldots,v_k} [d^p f_x(I_t^{((i_1\ldots i_p)|\mathbf{0}_q)}) + \mathrm{sf}_{x,t}^{(p)}((1,\ldots,p),(i_1,\ldots,i_p))]\ f_{i_1}(X_{t-1})\ldots f_{i_p}(X_{t-p}) \end{aligned}$$

which using

$$\begin{aligned} \mathrm{Coef}_{x,t}^{(0)} &:= f_x(I_t) + \mathrm{sf}_{x,t}^{(0)}, \quad \text{and} \\ \mathrm{Coef}_{x,t}^{(\nu)}((j_1,\ldots j_\nu),(i_{j_1},\ldots,i_{j_\nu})) &:= d^\nu f_x(I_t^{((\mathbf{0}_{j_1-1}\ i_{j_1}\ldots i_{j_\nu}\ \mathbf{0}_{p-j_\nu})|\mathbf{0}_q)}) + \\ &\quad\ \mathrm{sf}_{x,t}^{(\nu)}((j_1,\ldots,j_\nu),(i_{j_1},\ldots,i_{j_\nu})), \end{aligned}$$

(as in the statement of the proposition), it becomes

$$\begin{aligned} &f_x(X_t) = \mathrm{Coef}_{x,t}^{(0)} + \sum_{j=1}^{p} \sum_{i_j=v_1,\ldots,v_k} \mathrm{Coef}_{x,t}^{(1)}(j,i_j)\ f_{i_j}(X_{t-j}) + \\ &\sum_{\substack{j_1,j_2=1 \\ j_1<j_2}}^{p} \sum_{i_{j_1},i_{j_2}=v_1,\ldots,v_k} \mathrm{Coef}_{x,t}^{(2)}((j_1,j_2),(i_{j_1},i_{j_2}))\ f_{i_{j_1}}(X_{t-j_1})\ f_{i_{j_2}}(X_{t-j_2}) + \\ &\qquad \vdots \\ &\sum_{i_1,\ldots,i_p=v_1,\ldots,v_k} \mathrm{Coef}_{x,t}^{(p)}((1,\ldots,p),(i_1,\ldots,i_p))\ f_{i_1}(X_{t-1})\ldots f_{i_p}(X_{t-p}). \end{aligned} \tag{2}$$

It is accepted that after the substitution of $f_{i_1}(X_{t-1})$ first, then $f_{i_2}(X_{t-2})$,... then $f_{i_w}(X_{t-w})$ for fixed $w \in \mathbb{N}$, it holds that

$$
\begin{aligned}
f_x(X_t) \quad = \quad & \mathrm{Coef}^{(0)}_{x,t} + \sum_{n=1}^{w} \sum_{i_n=v_1,\ldots,v_k} \mathrm{Cr}^{n-1}\mathrm{oef}^{(1)}_{x,t}(n,i_n) \cdot \mathrm{Coef}^{(0)}_{i_n,t-n} + \\
& \sum_{j=1+w}^{p+w} \sum_{i_j=v_1,\ldots,v_k} \mathrm{Cr}^{w}\mathrm{oef}^{(1)}_{x,t}(j,i_j)\ f_{i_j}(X_{t-j}) + \\
& \sum_{\substack{j_1,j_2=1+w \\ j_1<j_2}}^{p+w} \sum_{i_{j_1},i_{j_2}=v_1,\ldots,v_k} \mathrm{Cr}^{w}\mathrm{oef}^{(2)}_{x,t}((j_1,j_2),(i_{j_1},i_{j_2}))\ f_{i_{j_1}}(X_{t-j_1})\ f_{i_{j_2}}(X_{t-j_2}) \\
& \vdots \\
& \sum_{i_{1+w},\ldots,i_{p+w}=v_1,\ldots,v_k} \mathrm{Cr}^{w}\mathrm{oef}^{(p)}_{x,t}((1+w,\ldots,p+w),(i_{1+w},\ldots,i_{p+w}))\ f_{i_{1+w}}(X_{t-1-w}) \\
& \qquad \ldots f_{i_{p+w}}(X_{t-p-w}),
\end{aligned} \tag{3}
$$

where formula

$$
\begin{aligned}
& \mathrm{Cr}^{n}\mathrm{oef}^{(\nu)}_{x,t}((j_1,\ldots,j_\nu),(i_{j_1},\ldots,i_{j_\nu})) := \mathrm{Cr}^{n-1}\mathrm{oef}^{(\nu)}_{x,t}((j_1,\ldots,j_\nu),(i_{j_1},\ldots,i_{j_\nu})) + \\
& \sum_{m=0}^{\min\{\nu/\nu-1,p-1\}} \sum_{\substack{j_{(1)},\ldots,j_{(m)} = \\ j_1,j_2\ldots,\min\{j_\nu,n+p-1\} \\ j_{(1)}<\ldots<j_{(m)}}} \sum_{m_1=0}^{m} \sum_{\substack{j^*_{(1)},\ldots,j^*_{(m_1)} = \\ j_{(1)},\ldots,j_{(m)} \\ j^*_{(1)}<\ldots<j^*_{(m_1)}}} \sum_{i_n=v_1,\ldots,v_k} \\
& \qquad \mathrm{Cr}^{n-1}\mathrm{oef}^{(1+m)}_{x,t}((n,j_{(1)},\ldots,j_{(m)}),(i_n,i_{j_{(1)}},\ldots,i_{j_{(m)}})) \cdot \\
& \mathrm{Coef}^{(\nu-m+m_1)}_{i_n,t-n}((j-n\ (j\neq j_{()}),\ j^*_{(1)}-n,\ldots,j^*_{(m_1)}-n),(i_j\ (j\neq j_{()}),i_{j^*_{(1)}},\ldots,i_{j^*_{(m_1)}})),
\end{aligned} \tag{4}
$$

(as well as

$$
\mathrm{Cr}^{n-1}\mathrm{oef}^{(\nu)}_{x,t}((j_1,\ldots,j_{\nu-1},n+p),(i_{j_1},\ldots,i_{j_{\nu-1}},i_{n+p})) \equiv 0) \tag{5}
$$

from the proposition statement, is accepted to hold *up to* $n=w$.

Next, $f_{i_{1+w}}(X_{t-1-w})$ is substituted from (2) (it is noted for the following formula that i_{1+w} is not $i_1,\ldots,i_p$), i.e.

$$
\begin{aligned}
& f_{i_{1+w}}(X_{t-1-w}) = \mathrm{Coef}^{(0)}_{i_{1+w},t-1-w} + \sum_{j=1}^{p} \sum_{i_j=v_1,\ldots,v_k} \mathrm{Coef}^{(1)}_{i_{1+w},t-1-w}(j,i_j)\ f_{i_j}(X_{t-1-w-j}) + \\
& \sum_{\substack{j_1,j_2=1 \\ j_1<j_2}}^{p} \sum_{i_{j_1},i_{j_2}=v_1,\ldots,v_k} \mathrm{Coef}^{(2)}_{i_{1+w},t-1-w}((j_1,j_2),(i_{j_1},i_{j_2}))\ f_{i_{j_1}}(X_{t-1-w-j_1})\ f_{i_{j_2}}(X_{t-1-w-j_2}) +
\end{aligned}
$$

$$\vdots$$

$$\sum_{i_1,\ldots,i_p=v_1,\ldots,v_k} \mathrm{Coef}^{(p)}_{i_{1+w},t-1-w}((1,\ldots,p),(i_1,\ldots,i_p))\ f_{i_1}(X_{t-1-w-1})\ldots f_{i_p}(X_{t-1-w-p})$$

or, it can be re-written as $f_{i_{1+w}}(X_{t-1-w}) =$

$$\mathrm{Coef}^{(0)}_{i_{1+w},t-1-w} + \sum_{j=(1+w)+1}^{(1+w)+p} \sum_{i_j=v_1,\ldots,v_k} \mathrm{Coef}^{(1)}_{i_{1+w},t-1-w}(j-(1+w),i_j)\ f_{i_j}(X_{t-j})$$

$$+ \sum_{\substack{j_1,\,j_2\,=\,(1\,+\,w)\,+\,1\\ j_1\,<\,j_2}}^{(1+w)+p} \sum_{i_{j_1},i_{j_2}=v_1,\ldots,v_k} \mathrm{Coef}^{(2)}_{i_{1+w},t-1-w}((j_1-(1+w),j_2-(1+w)),(i_{j_1},i_{j_2}))$$

$$f_{i_{j_1}}(X_{t-j_1})\ f_{i_{j_2}}(X_{t-j_2}) +$$

$$\vdots$$

$$\sum_{i_{(1+w)+1},\ldots,i_{(1+w)+p}=v_1,\ldots,v_k} \mathrm{Coef}^{(p)}_{i_{1+w},t-1-w}((1,\ldots,p),(i_{(1+w)+1},\ldots,i_{(1+w)+p}))$$

$$f_{i_{(1+w)+1}}(X_{t-(1+w)-1})\ldots f_{i_{(1+w)+p}}(X_{t-(1+w)-p}), \tag{6}$$

which is now ready to use in (3). Clearly, there is going to be a

$$\mathrm{Coef}^{(0)}_{x,t} + \sum_{n=1}^{w} \sum_{i_n=v_1,\ldots,v_k} \mathrm{Cr}^{n-1}\mathrm{oef}^{(1)}_{x,t}(n,i_n)\cdot \mathrm{Coef}^{(0)}_{i_n,t-n}$$

(from the first line in (3)), as well as a $\mathrm{Cr}^{w}\mathrm{oef}^{(1)}_{x,t}(1+w,i_{1+w})\ \mathrm{Coef}^{(0)}_{i_{1+w},t-1-w}$, $i_{1+w} = v_1,\ldots,v_k$ (second line of (3)) to be the only terms that are not multiplied by at least one $f(X)$; the remaining terms will either be multiplied by $f_{i_j}(X_{t-j})$, $j=(1+w)+1$,...$(1+w)+p$ ($i_j = v_1,\ldots,v_k$), or $f_{i_{j_1}}(X_{t-j_1})\ f_{i_{j_2}}(X_{t-j_2})$, $j_1,j_2 = (1+w)+1$,...$(1+w)+p$ ($j_1 < j_2$) ($i_{j_1},i_{j_2} = v_1,\ldots,v_k$),... or $f_{i_{(1+w)+1}}(X_{t-(1+w)-1})$...$f_{i_{(1+w)+p}}(X_{t-(1+w)-p})$. So the question that remains is to determine their coefficients and verify (4).

The coefficient multiplying $f_{i_{j_1}}(X_{t-j_1})\ldots f_{i_{j_\nu}}(X_{t-j_\nu})$, $\nu = 1,\ldots,p$, $j_1,\ldots,j_\nu = (1+w)+1$,...$(1+w)+p$ ($j_1<\ldots<j_\nu$), $i_{j_1},\ldots,i_{j_\nu} = v_1,\ldots,v_k$ is a sum of the coefficient from before, plus the sum of relevant coefficients after substitution of $f_{i_{1+w}}(X_{t-1-w})$, say $S_{x,t}((j_1,\ldots,j_\nu),(i_{j_1},\ldots,i_{j_\nu}))$: if $j_v = (1+w)+p$ then there is no coefficient from the previous step, i.e. the coefficient from before is

$$\begin{cases} \mathrm{Cr}^{w}\mathrm{oef}^{(\nu)}_{x,t}((j_1,\ldots,j_\nu),(i_{j_1},\ldots,i_{j_\nu})), & \text{if } (1+w)+1 \le j_\nu \le (1+w)+(p-1) \\ 0, & \text{if } j_\nu = (1+w)+p \end{cases},$$

which verifies (5).

Next, $S_{x,t}((j_1,\ldots,j_\nu),(i_{j_1},\ldots,i_{j_\nu}))$ is determined. Again there is the slight distinction whether $j_\nu = (1+w)+p$, or not. Let us work the way through the more general case $(1+w)+1 \le j_\nu \le (1+w)+(p-1)$ first, for which $1\le \nu \le p-1$. Since $S_{x,t}$ refers to the coefficients that happen via the substitution of $f(X_{t-1-w})$, there is always going to be a $\mathrm{Cr}^w\mathrm{oef}_{x,t}^{(1+...)}((1+w,...),(i_{1+w},...))$, which is there before the substitution as in (3). Since Cr^w can take up to p entries j, with $1+w$ being the smallest entry and $\nu+1 \leq p$, all the different $\mathrm{Cr}^w\mathrm{oef}^{(1+m)}(1+w,j)$ for $m=0,1,\ldots,\nu$ and all possible selections $\begin{pmatrix} \nu \\ m \end{pmatrix}$ of j will be considered: to make sure that these are the coefficients of $f_{i_{j_1}}(X_{t-j_1})\ldots f_{i_{j_\nu}}(X_{t-j_\nu})$ though, the correct versions of Coef have to be taken in, which will be from (6).

Once a specific $\mathrm{Cr}^w\mathrm{oef}_{x,t}^{(1+m)}((1+w,j_{(1)},\ldots,j_{(m)}),(i_{1+w},i_{j_{(1)}},\ldots,i_{j_{(m)}}))$, $m=0,1,\ldots,\nu$, $j_{(1)},\ldots,j_{(m)} = j_1,\ldots,j_\nu$ $(j_{(1)} < \ldots < j_{(m)})$ is being used, write $jj_1 < ... < jj_{\nu-m}$ for the remaining $j \neq j_{()}$, and see why it should be multiplied by

$$\sum_{m_1=0}^{m} \sum_{\substack{j^*_{(1)},\ldots,j^*_{(m_1)} = \\ j_{(1)},\ldots,j_{(m)} \\ j^*_{(1)} < \ldots < j^*_{(m_1)}}} \mathrm{Coef}_{i_{1+w},t-(1+w)}^{(\nu-m+m_1)}((jj_1-(1+w),\ldots,jj_{\nu-m}-(1+w),$$
$$j^*_{(1)}-(1+w),\ldots,j^*_{(m_1)}-(1+w)),(i_{jj_1},\ldots,i_{jj_{\nu-m}},i_{j^*_{(1)}},\ldots,i_{j^*_{(m_1)}}));$$

this is because there have to be the remaining $(\nu-m)$ js that were not used already, in order to complete the ν desired js, and there can be $m_1=0,\ldots,m$ repetitions j^* of the ones used and for the same i_{j^*}, of course.

As for the case when $j_\nu = (1+w)+p$, it cannot be that j_ν is a choice in $\mathrm{Cr}^w\mathrm{oef}$ as the maximum j allowed is $p+w$, hence j_ν *has* to take one of the positions in Coef. So the first part of $\mathrm{Cr}^w\mathrm{oef}_{x,t}^{(1+m)}(1+w,j_{(1)},\ldots,j_{(m)})$ will use $m=0,1,\ldots,\nu-1$ with $j_{(1)},\ldots,j_{(m)} = j_1,\ldots,j_{\nu-1} \leq p+w < j_\nu$ $(j_{(1)} < \ldots < j_{(m)})$. The second part

$$\mathrm{Coef}(jj_1-(1+w),\ldots,jj_{\nu-m-1}-(1+w),j^*_{(1)}-(1+w),\ldots,j^*_{(m_1)}-(1+w),j_\nu-(1+w))$$

will use $\nu-m$ js at least (the 'jj'), to secure the ones that have not been used and j_ν (the one in the end) is one of them. The repetitions $m_1=0,\ldots,m$ for all possible $j^*_{(1)},\ldots,j^*_{(m_1)} = j_{(1)},\ldots,j_{(m)}$ $(j^*_{(1)} < \ldots < j^*_{(m_1)})$ remain in the same spirit.

Consequently, (4) (and (5)) holds up to $n = w + 1$ and

$$
\begin{aligned}
f_x(X_t) = \mathrm{Coef}^{(0)}_{x,t} + \sum_{n=1}^{w+1} \sum_{i_n = v_1,\dots,v_k} \mathrm{Cr}^{n-1}\mathrm{oef}^{(1)}_{x,t}(n, i_n) \cdot \mathrm{Coef}^{(0)}_{i_n, t-n} + \\
\sum_{j=1+(w+1)}^{p+(w+1)} \sum_{i_j = v_1,\dots,v_k} \mathrm{Cr}^{w+1}\mathrm{oef}^{(1)}_{x,t}(j, i_j)\ f_{i_j}(X_{t-j}) + \\
\sum_{\substack{j_1, j_2 = 1+(w+1) \\ j_1 < j_2}}^{p+(w+1)} \sum_{i_{j_1}, i_{j_2} = v_1,\dots,v_k} \mathrm{Cr}^{w+1}\mathrm{oef}^{(2)}_{x,t}((j_1, j_2), (i_{j_1}, i_{j_2}))\ f_{i_{j_1}}(X_{t-j_1})\ f_{i_{j_2}}(X_{t-j_2}) + \\
\vdots \\
\sum_{i_{1+(w+1)},\dots,i_{p+(w+1)} = v_1,\dots,v_k} \mathrm{Cr}^{w+1}\mathrm{oef}^{(p)}_{x,t}((1+(w+1),\dots,p+(w+1)), (i_{1+(w+1)},\dots,i_{p+(w+1)})) \\
f_{i_{1+(w+1)}}(X_{t-1-(w+1)}) \dots f_{i_{p+(w+1)}}(X_{t-p-(w+1)})
\end{aligned}
$$

holds as well. An induction argument is used to generalize for any $n \in \mathbb{N}$, which results in

$$
f_x(X_t) = \mathrm{Coef}^{(0)}_{x,t} + \sum_{n \in \mathbb{N}} \sum_{i_n = v_1,\dots,v_k} \mathrm{Cr}^{n-1}\mathrm{oef}^{(1)}_{x,t}(n, i_n) \cdot \mathrm{Coef}^{(0)}_{i_n, t-n}, \tag{7}
$$

i.e. the desired representation.

- It is easy to work out that if $j_\nu \geq p + 1$ (i.e. it has to be that $n \geq 1$), such that it can be written $j_\nu \equiv n^* + p + 1$ (for some $n^* \geq 0$), then (4) may be replaced by

$$\begin{aligned}
&\mathrm{Cr}^n\mathrm{oef}^{(\nu)}_{x,t}((j_1,\ldots,j_\nu),(i_{j_1},\ldots,i_{j_\nu})) = \\
&\qquad \sum_{m=0}^{\nu} \sum_{\substack{j_{(1)},\ldots,j_{(m)} = \\ j_1,\ldots,j_\nu \\ j_{(1)}<\ldots<j_{(m)}}} \sum_{m_1=0}^{m} \sum_{\substack{j^*_{(1)},\ldots,j^*_{(m_1)} = \\ j_{(1)},\ldots,j_{(m)} \\ j^*_{(1)}<\ldots<j^*_{(m_1)}}} \sum_{i_n=v_1,\ldots,v_k} \\
&\qquad \mathrm{Cr}^{n-1}\mathrm{oef}^{(1+m)}_{x,t}((n,j_{(1)},\ldots,j_{(m)}),(i_n,i_{j_{(1)}},\ldots,i_{j_{(m)}}))\cdot \\
&\mathrm{Coef}^{(\nu-m+m_1)}_{i_n,t-n}((j-n\ (j\neq j_{()}),\ j^*_{(1)}-n,\ldots,j^*_{(m_1)}-n),(i_j\ (j\neq j_{()}),i_{j^*_{(1)}},\ldots,i_{j^*_{(m_1)}}))+ \\
&\qquad\qquad \vdots \\
&\qquad \sum_{m=0}^{\nu} \sum_{\substack{j_{(1)},\ldots,j_{(m)} = \\ j_1\ldots,j_\nu \\ j_{(1)}<\ldots<j_{(m)}}} \sum_{m_1=0}^{m} \sum_{\substack{j^*_{(1)},\ldots,j^*_{(m_1)} = \\ j_{(1)},\ldots,j_{(m)} \\ j^*_{(1)}<\ldots<j^*_{(m_1)}}} \sum_{i_{n^*+2}=v_1,\ldots,v_k} \\
&\quad \mathrm{Cr}^{n^*+1}\mathrm{oef}^{(1+m)}_{x,t}((n^*+2,j_{(1)},\ldots,j_{(m)}),(i_{n^*+2},i_{j_{(1)}},\ldots,i_{j_{(m)}}))\cdot\mathrm{Coef}^{(\nu-m+m_1)}_{i_{n^*+2},t-n^*-2}(\\
&(j-n^*-2\ (j\neq j_{()}),\ j^*_{(1)}-n^*-2,\ldots,j^*_{(m_1)}-n^*-2),(i_j\ (j\neq j_{()}),i_{j^*_{(1)}},\ldots,i_{j^*_{(m_1)}}))+ \\
&\qquad \sum_{m=0}^{\nu-1} \sum_{\substack{j_{(1)},\ldots,j_{(m)} = \\ j_1,\ldots,j_{\nu-1} \\ j_{(1)}<\ldots<j_{(m)}}} \sum_{m_1=0}^{m} \sum_{\substack{j^*_{(1)},\ldots,j^*_{(m_1)} = \\ j_{(1)},\ldots,j_{(m)} \\ j^*_{(1)}<\ldots<j^*_{(m_1)}}} \sum_{i_{n^*+1}=v_1,\ldots,v_k} \\
&\quad \mathrm{Cr}^{n^*}\mathrm{oef}^{(1+m)}_{x,t}((n^*+1,j_{(1)},\ldots,j_{(m)}),(i_{n^*+1},i_{j_{(1)}},\ldots,i_{j_{(m)}}))\cdot\mathrm{Coef}^{(\nu-m+m_1)}_{i_{n^*+1},t-n^*-1}(\\
&(j-n^*-1\ (j\neq j_{()}),\ j^*_{(1)}-n^*-1,\ldots,j^*_{(m_1)}-n^*-1),(i_j\ (j\neq j_{()}),i_{j^*_{(1)}},\ldots,i_{j^*_{(m_1)}})),
\end{aligned} \tag{8}$$

where it is implied in the representations above that $n^*+1 < n^*+2 < \ldots < n$ (i.e. it can be $n = n^*+1$, or $n = n^*+2$, or ...; in general, it is $n = n^* + (n-n^*)$) and the number of (combined) terms in (8) is $n - n^*$: since $n+1 \leq j_1 < \ldots < j_\nu \equiv n^*+p+1 \leq n+p$, it holds that

$$1 \leq (n-n^*) \leq p.$$

In a similar spirit to deriving (8), the coefficients $\mathrm{Cr}^{n-1}\mathrm{oef}^{(1)}_{x,t}(n,i_n)$ in (7) can be presented again below

$$\mathrm{Cr}^{n-1}\mathrm{oef}^{(1)}_{x,t}(n,i_n) = \begin{cases} [df_x(I_t^{((\mathbf{0}_{n-1},i_n,\mathbf{0}_{p-n})|\mathbf{0}_q)}) + \mathrm{sf}^{(1)}_{x,t}(n,i_n)] + \quad \sum_{w=1}^{n-1} [\\ \sum_{i_w=v_1,\ldots,v_k} \mathrm{Cr}^{w-1}\mathrm{oef}^{(1)}_{x,t}(w,i_w)\ \mathrm{Coef}^{(1)}_{i_w,t-w}(n-w,i_n) + \\ \sum_{i_w=v_1,\ldots,v_k} \mathrm{Cr}^{w-1}\mathrm{oef}^{(2)}_{x,t}((w,n),(i_w,i_n)) \\ (\mathrm{Coef}^{(0)}_{i_w,t-w} + \mathrm{Coef}^{(1)}_{i_w,t-w}(n-w,i_n))\ ], \\ \text{if } n=1,2,\ldots,p \\ \\ \sum_{w=n-p}^{n-1} [\\ \sum_{i_w=v_1,\ldots,v_k} \mathrm{Cr}^{w-1}\mathrm{oef}^{(1)}_{x,t}(w,i_w)\ \mathrm{Coef}^{(1)}_{i_w,t-w}(n-w,i_n) + \\ \sum_{i_w=v_1,\ldots,v_k} \mathrm{Cr}^{w-1}\mathrm{oef}^{(2)}_{x,t}((w,n),(i_w,i_n)) \\ (\mathrm{Coef}^{(0)}_{i_w,t-w} + \mathrm{Coef}^{(1)}_{i_w,t-w}(n-w,i_n))\ ], \\ \text{if } n=p+1,p+2,\ldots \end{cases} .$$

The 'up to p' number of terms in (8) is very important for the proof of Proposition 3.3 (following in this online supplement), which achieves a condition for a series of coefficients to converge.

- For Proposition 3.2:

(i) The proof for

$$|\sum_{i_1,\ldots,i_r=v_1,\ldots,v_k} \mathbb{E}\{\mathrm{Coef}^{(\nu)}_{x,t}()\ \prod_{n=1}^{r} \mathrm{Coef}^{(\nu_n)}_{i_n,t-n}(...,\neq i_1,\ldots,i_n)\}| \leq \beta_x\ (\beta^*)^r, \tag{9}$$

as in the first statement of the proposition, takes a step-by-step substitution of Coef from time t and going backwards. So it holds that

$$\sum_{i_1,\ldots,i_r=v_1,\ldots,v_k} \mathbb{E}\{\mathrm{Coef}^{(\nu)}_{x,t}()\ \prod_{n=1}^{r} \mathrm{Coef}^{(\nu_n)}_{i_n,t-n}(...,\neq i_1,\ldots,i_n)\} =$$
$$\sum_{i_1,\ldots,i_r=v_1,\ldots,v_k} \mathbb{E}\{d^{\nu} f_x(I_t^{((\ldots)|\mathbf{0}_q)})\ \prod_{n=1}^{r} \mathrm{Coef}^{(\nu_n)}_{i_n,t-n}(...,\neq i_1,\ldots,i_n)\} +$$
$$\sum_{h_q=1}^{q} \sum_{\substack{j_{q,1},\ldots,j_{q,h_q}=1 \\ j_{q,1}<\ldots<j_{q,h_q}}}^{q} \sum_{i_{q,j_{q,1}},\ldots i_{q,j_{q,h_q}}=v_1,\ldots,v_k} \sum_{i_1,\ldots,i_r=v_1,\ldots,v_k} \mathbb{E}\{d^{\nu+h_q} f_x(I_t^{((\ldots)|(\mathbf{0}\ i_{q,j_{q,1}}\ldots i_{q,j_{q,h_q}}\ \mathbf{0}))}) \cdot$$
$$f_{i_{q,j_{q,1}}}(I_{t-j_{q,1}}) \ldots f_{i_{q,j_{q,h_q}}}(I_{t-j_{q,h_q}})\ \prod_{n=1}^{r} \mathrm{Coef}^{(\nu_n)}_{i_n,t-n}(...,\neq i_1,\ldots,i_n)\}$$

and it is easy to see, using the definitions so far and the assumption of independence in time, that

$$
\begin{aligned}
&|\sum_{i_1,\ldots,i_r=v_1,\ldots,v_k} \mathbb{E}\{\mathrm{Coef}^{(\nu)}_{x,t}() \prod_{n=1}^{r} \mathrm{Coef}^{(\nu_n)}_{i_n,t-n}(...,\neq i_1,\ldots,i_n)\}| \leq \\
&\beta^{(\nu)}_x \; |\sum_{i_2,\ldots,i_r=v_1,\ldots,v_k} \mathbb{E}\{\sum_{i_1=v_1,\ldots,v_k} \mathrm{Coef}^{(\nu_1)}_{i_1,t-1}(...,\neq i_1) \prod_{n=2}^{r} \mathrm{Coef}^{(\nu_n)}_{i_n,t-n}(...,\neq i_1,\ldots,i_n)\}| + \\
&\sum_{h_q=1}^{q} \beta^{(\nu)}_{q,x}(h_q) \sum_{\substack{j_{q,1},\ldots,j_{q,h_q}=1 \\ j_{q,1}<\ldots<j_{q,h_q}}}^{q} |\sum_{i_{q,j_{q,1}},\ldots i_{q,j_{q,h_q}}=v_1,\ldots,v_k} \sum_{i_2,\ldots,i_r=v_1,\ldots,v_k} \mathbb{E}\{f_{i_{q,j_{q,1}}}(I_{t-j_{q,1}}) \\
&\ldots f_{i_{q,j_{q,h_q}}}(I_{t-j_{q,h_q}}) \sum_{i_1=v_1,\ldots,v_k} \mathrm{Coef}^{(\nu_1)}_{i_1,t-1}(...,\neq i_1) \prod_{n=2}^{r} \mathrm{Coef}^{(\nu_n)}_{i_n,t-n}(...,\neq i_1,\ldots,i_n)\}|;
\end{aligned}
$$

straight from

$$
\sum_{x=v_1,\ldots,v_k} \mathrm{Coef}^{(0)}_{x,t} = \sum_{x=v_1,\ldots,v_k} f_x(I_t) - \mathrm{sf}^{(0)}_{0,t} \equiv 1 - \mathrm{Coef}^{(0)}_{0,t}, \tag{10}
$$

as this was explained in the main text (right before the statement of the proposition), it is written

$$
\sum_{i_1=v_1,\ldots,v_k} \mathrm{Coef}^{(\nu_1)}_{i_1,t-1}(...,\neq i_1) = \begin{cases} 1-\mathrm{Coef}^{(0)}_{0,t-1} \equiv 1 - f_0(I_{t-1}) - \mathrm{sf}^{(0)}_{0,t-1}, \\ \text{if } \; \nu_1 = 0 \\ -\mathrm{Coef}^{(\nu_1)}_{0,t-1}(...) \equiv -d^{\nu_1} f_0(I^{(...|\mathbf{0}_q)}_{t-1}) - \mathrm{sf}^{(\nu_1)}_{0,t-1}(...), \\ \text{if } \; \nu_1 = 1,\ldots,p \end{cases}, \tag{11}
$$

which will be denoted by $\mathrm{Coef}^{(*,\nu_1)}{}_{0,t-1}$ for convenience, so that it is re-written

$$
\begin{aligned}
&|\sum_{i_1,\ldots,i_r=v_1,\ldots,v_k} \mathbb{E}\{\mathrm{Coef}^{(\nu)}_{x,t}() \prod_{n=1}^{r} \mathrm{Coef}^{(\nu_n)}_{i_n,t-n}(...,\neq i_1,\ldots,i_n)\}| \leq \\
&\beta^{(\nu)}_x \; |\sum_{i_2,\ldots,i_r=v_1,\ldots,v_k} \mathbb{E}\{\mathrm{Coef}^{(*,\nu_1)}_{0,t-1} \prod_{n=2}^{r} \mathrm{Coef}^{(\nu_n)}_{i_n,t-n}(...,\neq i_2,\ldots,i_n)\}| + \\
&\sum_{h_q=1}^{q} \beta^{(\nu)}_{q,x}(h_q) \sum_{\substack{j_{q,1},\ldots,j_{q,h_q}=1 \\ j_{q,1}<\ldots<j_{q,h_q}}}^{q} |\sum_{i_2,\ldots,i_r=v_1,\ldots,v_k} \mathbb{E}\{(1-f_0(I_{t-j_{q,1}})) \\
&\ldots (1-f_0(I_{t-j_{q,h_q}}))\; \mathrm{Coef}^{(*,\nu_1)}_{0,t-1} \prod_{n=2}^{r} \mathrm{Coef}^{(\nu_n)}_{i_n,t-n}(...,\neq i_2,\ldots,i_n)\}|;
\end{aligned}
$$

both expectations can be taken as $\mathbb{E}\{\mathbb{E}\{...|I_{t-i},\; i\geq 2\}\}$, so that it is re-arranged that (note that the sum must stay inside of the absolute value to use (10))

$$
\begin{aligned}
&|\sum_{i_1,\dots,i_r=v_1,\dots,v_k} \mathbb{E}\{\mathrm{Coef}^{(\nu)}_{x,t}() \prod_{n=1}^{r} \mathrm{Coef}^{(\nu_n)}_{i_n,t-n}(...,\neq i_1,\dots,i_n)\}| \leq \\
&\beta^{(\nu)}_x \ \beta^{(\nu_1)}_* \ |\sum_{i_2,\dots,i_r=v_1,\dots,v_k} \mathbb{E}\{\prod_{n=2}^{r} \mathrm{Coef}^{(\nu_n)}_{i_n,t-n}(...,\neq i_2,\dots,i_n)\}| + \\
&\beta^{(\nu)}_x \sum_{h^{(1)}_q=1}^{q} \beta^{(\nu_1)}_{*,q}(h^{(1)}_q) \sum_{\substack{j^{(1)}_{q,1},\dots j^{(1)}_{q,h^{(1)}_q}=1 \\ j^{(1)}_{q,1}<\dots<j^{(1)}_{q,h^{(1)}_q}}}^{q} |\sum_{i^{(1)}_{q,j^{(1)}_{q,1}},\dots i^{(1)}_{q,j^{(1)}_{q,h^{(1)}_q}}=v_1,\dots,v_k} \ \sum_{i_2,\dots,i_r=v_1,\dots,v_k} \\
&\mathbb{E}\{f_{i^{(1)}_{q,j^{(1)}_{q,1}}}(I_{t-1-j^{(1)}_{q,1}})\dots f_{i^{(1)}_{q,j^{(1)}_{q,h^{(1)}_q}}}(I_{t-1-j^{(1)}_{q,h^{(1)}_q}}) \prod_{n=2}^{r} \mathrm{Coef}^{(\nu_n)}_{i_n,t-n}(...,\neq i_2,\dots,i_n)\}| +
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{h_q=1}^{q} \beta^{(\nu)}_{q,x}(h_q) \sum_{\substack{j_{q,2},\dots,j_{q,h_q}=2 \\ j_{q,2}<\dots<j_{q,h_q}}}^{q} |\sum_{i_2,\dots,i_r=v_1,\dots,v_k} \mathbb{E}\{\ \mathbb{E}\{(1-f_0(I_{t-1}))\ \mathrm{Coef}^{(*,\nu_1)}_{0,t-1} \\
&(1-f_0(I_{t-j_{q,2}}))\dots(1-f_0(I_{t-j_{q,h_q}})) \prod_{n=2}^{r} \mathrm{Coef}^{(\nu_n)}_{i_n,t-n}(...,\neq i_2,\dots,i_n)|I_{t-i},\ i\geq 2\}\}| + \\
&\sum_{h_q=1}^{q-1} \beta^{(\nu)}_{q,x}(h_q) \sum_{\substack{j_{q,1},\dots,j_{q,h_q}=2 \\ j_{q,1}<\dots<j_{q,h_q}}}^{q} |\sum_{i_2,\dots,i_r=v_1,\dots,v_k} \mathbb{E}\{\ \mathbb{E}\{\mathrm{Coef}^{(*,\nu_1)}_{0,t-1}\ (1-f_0(I_{t-j_{q,1}}))\dots \\
&(1-f_0(I_{t-j_{q,h_q}})) \prod_{n=2}^{r} \mathrm{Coef}^{(\nu_n)}_{i_n,t-n}(...,\neq i_2,\dots,i_n)|I_{t-i},\ i\geq 2\}\}|
\end{aligned}
$$

and from the last two terms ($j_q \geq 2$), there are

$$
\begin{aligned}
&\sum_{h_q=1}^{q}\begin{pmatrix} q-1 \\ h_q-1 \end{pmatrix} + \sum_{h_q=1}^{q-1}\begin{pmatrix} q-1 \\ h_q \end{pmatrix} = \sum_{h_q-1=0}^{q-1}\begin{pmatrix} q-1 \\ h_q-1 \end{pmatrix} + \sum_{h_q=0}^{q-1}\begin{pmatrix} q-1 \\ h_q \end{pmatrix} - 1 \\
&2^{q-1}+2^{q-1}-1 = 2\cdot 2^{q-1}-1 = 2^q-1 \equiv \sum_{h_q=1}^{q}\begin{pmatrix} q \\ h_q \end{pmatrix},
\end{aligned}
$$

i.e. the original sum has been separated into two parts. From the first of these parts, there is going to be a (straight from (11), it is abbreviated $\mathrm{Coef}^{(*,\nu_1)}_{0,t-1} = d^{(*,\nu_1)} f_0(I^{(\dots|\mathbf{0}_q)}_{t-1}) - \mathrm{sf}^{(\nu_1)}_{0,t-1}()$: the reader may look below to find this)

$$
\begin{aligned}
&|\mathbb{E}\{(1-f_0(I_{t-1}))\ \mathrm{Coef}^{(*,\nu_1)}_{0,t-1}\ (1-f_0(I_{t-j_{q,2}}))\dots(1-f_0(I_{t-j_{q,h_q}})) \\
&\sum_{i_2,\dots,i_r=v_1,\dots,v_k} \prod_{n=2}^{r} \mathrm{Coef}^{(\nu_n)}_{i_n,t-n}(...,\neq i_2,\dots,i_n)|I_{t-i},\ i\geq 2\}| =
\end{aligned}
$$

$$|\mathbb{E}\{(1-f_0(I_{t-1}))\ d^{(*,\nu_1)} f_0(I_{t-1}^{(\ldots|\mathbf{0}_q)})\} \sum_{i_2,\ldots,i_r=v_1,\ldots,v_k}$$

$$\mathbb{E}\{(1-f_0(I_{t-j_{q,2}}))\ldots(1-f_0(I_{t-j_{q,h_q}}))\ \prod_{n=2}^{r}\mathrm{Coef}_{i_n,t-n}^{(\nu_n)}(...,\neq i_2,\ldots,i_n)|I_{t-i},\ i\geq 2\} -$$

$$\sum_{h_q^{(1)}=1}^{q} \sum_{\substack{j_{q,1}^{(1)},\ldots,j_{q,h_q^{(1)}}^{(1)}=1 \\ j_{q,1}^{(1)}<\ldots<j_{q,h_q^{(1)}}^{(1)}}}^{q} \sum_{i_{q,j_{q,1}^{(1)}}^{(1)},\ldots i_{q,j_{q,h_q^{(1)}}^{(1)}}^{(1)}=v_1,\ldots,v_k} \mathbb{E}\{(1-f_0(I_{t-1}))\ d^{\nu_1+h_q^{(1)}} f_0(I_{t-1}^{(\ldots|(\mathbf{0}\ i^{(1)}\ \mathbf{0}))})\}$$

$$\sum_{i_2,\ldots,i_r=v_1,\ldots,v_k} \mathbb{E}\{f_{i^{(1)}_{j_{q,1}^{(1)}}}(I_{t-1-j_{q,1}^{(1)}})\ldots f_{i^{(1)}_{j^{(1)}_{q,h_q^{(1)}}}}(I_{t-1-j^{(1)}_{q,h_q^{(1)}}})\ (1-f_0(I_{t-j_{q,2}}))\ldots(1-f_0(I_{t-j_{q,h_q}}))$$

$$\prod_{n=2}^{r}\mathrm{Coef}_{i_n,t-n}^{(\nu_n)}(...,\neq i_2,\ldots,i_n)|I_{t-i},\ i\geq 2\}|;$$

since

$$\begin{aligned}
|\mathbb{E}\{(1-f_0(I_{t-1}))\ d^{(*,\nu_1)} f_0(I_{t-1}^{(\ldots|\mathbf{0}_q)})\}| &\leq \mathbb{E}|d^{(*,\nu_1)} f_0(I_{t-1}^{(\ldots|\mathbf{0}_q)})| \leq \beta_*^{(\nu_1)} \\
|\mathbb{E}\{(1-f_0(I_{t-1}))\ d^{\nu_1+h_q^{(1)}} f_0(I_{t-1}^{(\ldots|(\mathbf{0}\ i^{(1)}\ \mathbf{0}))})\}| &\leq \mathbb{E}|d^{\nu_1+h_q^{(1)}} f_0(I_{t-1}^{(\ldots|(\mathbf{0}\ i^{(1)}\ \mathbf{0}))})| \leq \beta_{*,q}^{(\nu_1)}(h_q^{(1)}),
\end{aligned}$$

that part (consider the expectation of the conditional expectation now) can be bounded by

$$\beta_*^{(\nu_1)}\ |\mathbb{E}\{(1-f_0(I_{t-j_{q,2}}))\ldots(1-f_0(I_{t-j_{q,h_q}}))\ \sum_{i_2,\ldots,i_r=v_1,\ldots,v_k}\prod_{n=2}^{r}\mathrm{Coef}_{i_n,t-n}^{(\nu_n)}(...,\neq i_2,\ldots,i_n)\}| +$$

$$\sum_{h_q^{(1)}=1}^{q} \beta_{*,q}^{(\nu_1)}(h_q^{(1)}) \sum_{\substack{j_{q,1}^{(1)},\ldots,j_{q,h_q^{(1)}}^{(1)}=1 \\ j_{q,1}^{(1)}<\ldots<j_{q,h_q^{(1)}}^{(1)}}}^{q} |\mathbb{E}\{(1-f_0(I_{t-1-j_{q,1}^{(1)}}))\ldots(1-f_0(I_{t-1-j^{(1)}_{q,h_q^{(1)}}}))$$

$$\sum_{i_2,\ldots,i_r=v_1,\ldots,v_k}(1-f_0(I_{t-j_{q,2}}))\ldots(1-f_0(I_{t-j_{q,h_q}}))\ \prod_{n=2}^{r}\mathrm{Coef}_{i_n,t-n}^{(\nu_n)}(...,\neq i_2,\ldots,i_n)\}|;$$

the second part will be even easier to bound without the index function at time $t-1$, i.e. for $j_q \geq 2$ it holds that

$$\begin{aligned}
&|\mathbb{E}\{\ \mathbb{E}\{\mathrm{Coef}_{0,t-1}^{(*,\nu_1)}\ (1-f_0(I_{t-j_{q,1}}))\ldots\ \ (1-f_0(I_{t-j_{q,h_q}})) \\
&\sum_{i_2,\ldots,i_r=v_1,\ldots,v_k}\prod_{n=2}^{r}\mathrm{Coef}_{i_n,t-n}^{(\nu_n)}(...,\neq i_2,\ldots,i_n)|I_{t-i},\ i\geq 2\}\}| \leq \\
&\beta_*^{(\nu_1)}\ |\mathbb{E}\{(1-f_0(I_{t-j_{q,1}}))\ldots(1-f_0(I_{t-j_{q,h_q}})) \\
&\qquad \sum_{i_2,\ldots,i_r=v_1,\ldots,v_k}\prod_{n=2}^{r}\mathrm{Coef}_{i_n,t-n}^{(\nu_n)}(...,\neq i_2,\ldots,i_n)\}| +
\end{aligned}$$

$$\sum_{h_q^{(1)}=1}^{q} \beta_{*,q}^{(\nu_1)}(h_q^{(1)}) \sum_{\substack{j_{q,1}^{(1)},\ldots,j_{q,h_q^{(1)}}^{(1)}=1 \\ j_{q,1}^{(1)}<\ldots<j_{q,h_q^{(1)}}^{(1)}}}^{q} |\mathbb{E}\{(1-f_0(I_{t-1-j_{q,1}^{(1)}}))\ldots(1-f_0(I_{t-1-j_{q,h_q^{(1)}}^{(1)}}))$$

$$(1-f_0(I_{t-j_{q,1}}))\ldots(1-f_0(I_{t-j_{q,h_q}})) \sum_{i_2,\ldots,i_r=v_1,\ldots,v_k} \prod_{n=2}^{r} \mathrm{Coef}_{i_n,t-n}^{(\nu_n)}(...,\neq i_2,\ldots,i_n)\}|,$$

so that altogether, it is written

$$|\sum_{i_1,\ldots,i_r=v_1,\ldots,v_k} \mathbb{E}\{\mathrm{Coef}_{x,t}^{(\nu)}() \prod_{n=1}^{r} \mathrm{Coef}_{i_n,t-n}^{(\nu_n)}(...,\neq i_1,\ldots,i_n)\}| \leq$$

$$\beta_x^{(\nu)}\ \beta_*^{(\nu_1)}\ |\sum_{i_3,\ldots,i_r=v_1,\ldots,v_k} \mathbb{E}\{\mathrm{Coef}_{0,t-2}^{(*,\nu_2)} \prod_{n=3}^{r} \mathrm{Coef}_{i_n,t-n}^{(\nu_n)}(...,\neq i_3,\ldots,i_n)\}| +$$

$$\beta_x^{(\nu)} \sum_{h_q^{(1)}=1}^{q} \beta_{*,q}^{(\nu_1)}(h_q^{(1)}) \sum_{\substack{j_{q,1}^{(1)},\ldots j_{q,h_q^{(1)}}^{(1)}=1 \\ j_{q,1}^{(1)}<\ldots<j_{q,h_q^{(1)}}^{(1)}}}^{q} |\sum_{i_3,\ldots,i_r=v_1,\ldots,v_k} \mathbb{E}\{(1-f_0(I_{t-1-j_{q,1}^{(1)}}))\ldots$$

$$(1-f_0(I_{t-1-j_{q,h_q^{(1)}}^{(1)}}))\ \mathrm{Coef}_{0,t-2}^{(*,\nu_2)} \prod_{n=3}^{r} \mathrm{Coef}_{i_n,t-n}^{(\nu_n)}(...,\neq i_3,\ldots,i_n)\}| +$$

$$\sum_{h_q=1}^{q} \beta_{q,x}^{(\nu)}(h_q)\ \beta_*^{(\nu_1)} \sum_{\substack{j_{q,2},\ldots,j_{q,h_q}=2 \\ j_{q,2}<\ldots<j_{q,h_q}}}^{q} |\sum_{i_3,\ldots,i_r=v_1,\ldots,v_k} \mathbb{E}\{(1-f_0(I_{t-j_{q,2}}))\ldots$$

$$(1-f_0(I_{t-j_{q,h_q}}))\ \mathrm{Coef}_{0,t-2}^{(*,\nu_2)} \prod_{n=3}^{r} \mathrm{Coef}_{i_n,t-n}^{(\nu_n)}(...,\neq i_3,\ldots,i_n)\}| +$$

$$\sum_{h_q=1}^{q} \beta_{q,x}^{(\nu)}(h_q) \sum_{\substack{j_{q,2},\ldots,j_{q,h_q}=2 \\ j_{q,2}<\ldots<j_{q,h_q}}}^{q} \sum_{h_q^{(1)}=1}^{q} \beta_{*,q}^{(\nu_1)}(h_q^{(1)}) \sum_{\substack{j_{q,1}^{(1)},\ldots,j_{q,h_q^{(1)}}^{(1)}=1 \\ j_{q,1}^{(1)}<\ldots<j_{q,h_q^{(1)}}^{(1)}}}^{q} |\sum_{i_3,\ldots,i_r=v_1,\ldots,v_k}$$

$$\mathbb{E}\{(1-f_0(I_{t-1-j_{q,1}^{(1)}}))\ldots(1-f_0(I_{t-1-j_{q,h_q^{(1)}}^{(1)}}))\ (1-f_0(I_{t-j_{q,2}}))\ldots(1-f_0(I_{t-j_{q,h_q}}))$$

$$\mathrm{Coef}_{0,t-2}^{(*,\nu_2)} \prod_{n=3}^{r} \mathrm{Coef}_{i_n,t-n}^{(\nu_n)}(...,\neq i_3,\ldots,i_n)\}| +$$

$$\sum_{h_q=1}^{q-1} \beta_{q,x}^{(\nu)}(h_q)\ \beta_*^{(\nu_1)} \sum_{\substack{j_{q,1},\dots,j_{q,h_q}=2 \\ j_{q,1}<\dots<j_{q,h_q}}}^{q} |\sum_{i_3,\dots,i_r=v_1,\dots,v_k} \mathbb{E}\{(1-f_0(I_{t-j_{q,1}}))\dots$$

$$(1-f_0(I_{t-j_{q,h_q}}))\ \mathrm{Coef}_{0,t-2}^{(*,\nu_2)} \prod_{n=3}^{r} \mathrm{Coef}_{i_n,t-n}^{(\nu_n)}(\dots,\neq i_3,\dots,i_n)\}| +$$

$$\sum_{h_q=1}^{q-1} \beta_{q,x}^{(\nu)}(h_q) \sum_{\substack{j_{q,1},\dots,j_{q,h_q}=2 \\ j_{q,1}<\dots<j_{q,h_q}}}^{q} \sum_{h_q^{(1)}=1}^{q} \beta_{*,q}^{(\nu_1)}(h_q^{(1)}) \sum_{\substack{j_{q,1}^{(1)},\dots,j_{q,h_q^{(1)}}^{(1)}=1 \\ j_{q,1}^{(1)}<\dots<j_{q,h_q^{(1)}}^{(1)}}}^{q} |\sum_{i_3,\dots,i_r=v_1,\dots,v_k}$$

$$\mathbb{E}\{(1-f_0(I_{t-1-j_{q,1}^{(1)}}))\dots(1-f_0(I_{t-1-j_{q,h_q^{(1)}}^{(1)}}))\ (1-f_0(I_{t-j_{q,1}}))\dots(1-f_0(I_{t-j_{q,h_q}}))$$

$$\mathrm{Coef}_{0,t-2}^{(*,\nu_2)} \prod_{n=3}^{r} \mathrm{Coef}_{i_n,t-n}^{(\nu_n)}(\dots,\neq i_3,\dots,i_n)\}|.$$

As this process will continue for all $n \leq r$ there will be more and more terms generated due to the fact that the index functions $(1-f_0(I_{t-n-j_{q,\dots}^{(n)}}))$, $n=0,\dots,r$ might be at the same timing as the Coef under study (this has two parts). Nevertheless, in the same way that it has been done already, all index functions will be eventually and at the right time taken away providing the upper bound unity, so that it will be possible to put together again the parts that were separated due to the index function timing, and it is re-written

$$|\sum_{i_1,\dots,i_r=v_1,\dots,v_k} \mathbb{E}\{\mathrm{Coef}_{x,t}^{(\nu)}() \prod_{n=1}^{r} \mathrm{Coef}_{i_n,t-n}^{(\nu_n)}(\dots,\neq i_1,\dots,i_n)\}| \leq$$

$$\left[\beta_x^{(\nu)} + \sum_{h_q=1}^{q} \binom{q}{h_q} \beta_{q,x}^{(\nu)}(h_q)\right] \prod_{n=1}^{r} \left[\beta_*^{(\nu_n)} + \sum_{h_q^{(n)}=1}^{q} \binom{q}{h_q^{(n)}} \beta_{*,q}^{(\nu_n)}(h_q^{(n)})\right] \leq$$

$$\beta_x\ (\beta^*)^r.$$

If it is still unclear, an induction argument can be used.

For the second part in (i) of the proposition, i.e.

$$|\sum_{i_{nn+1},\dots,i_{nn+r}=v_1,\dots,v_k} \mathbb{E}\{\mathrm{Cr}^{nn}\mathrm{oef}_{x,t}^{(\nu)}() \prod_{n=1}^{r} \mathrm{Coef}_{i_{nn+n},t-nn-n}^{(\nu_{nn+n})}(\dots,\neq i_{nn+1},\dots,i_{nn+n})\}|$$
$$\leq \beta_x^{[nn]}\ (\beta^*)^r, \qquad (12)$$

there is certainly going to be an induction argument. The case (9) will be used as the first proof. It is then accepted that for fixed $nn \in \mathbb{N}$ and any $r \in \mathbb{N}$, (12) holds. Next, (4) is used to write

$$|\sum_{i_{nn+2},\ldots,i_{nn+1+r}=v_1,\ldots,v_k} \mathbb{E}\{\mathrm{Cr}^{nn+1}\mathrm{oef}^{(\nu)}_{x,t}()\prod_{n=1}^{r}\mathrm{Coef}^{(\nu_{nn+1+n})}_{i_{nn+1+n},t-nn-1-n}(...,\neq i_{nn+2},\ldots,i_{nn+1+n})\}| \leq$$

$$|\sum_{i_{nn+2},\ldots,i_{nn+1+r}=v_1,\ldots,v_k} \mathbb{E}\{\mathrm{Cr}^{nn}\mathrm{oef}^{(\nu)}_{x,t}()\prod_{n=1}^{r}\mathrm{Coef}^{(\nu_{nn+1+n})}_{i_{nn+1+n},t-nn-1-n}(...,\neq i_{nn+2},\ldots,i_{nn+1+n})\}|+$$

$$\sum_{m=0}^{\min\{nn+1/nn,p-1\}} \sum_{\substack{j_{(1)},\ldots,j_{(m)}=\\ \ldots\min\{...,nn+p\}\\ j_{(1)}<\ldots<j_{(m)}}} \sum_{m_1=0}^{m} \sum_{\substack{j^*_{(1)},\ldots,j^*_{(m_1)}=\\ j_{(1)},\ldots,j_{(m)}\\ j^*_{(1)}<\ldots<j^*_{(m_1)}}} |\sum_{i_{nn+1},\ldots,i_{nn+1+r}=v_1,\ldots,v_k}$$

$$\mathbb{E}\{\mathrm{Cr}^{nn}\mathrm{oef}^{(1+m)}_{x,t}((nn+1,j_{(1)},\ldots,j_{(m)}),(i_{nn+1},i_{j_{(1)}},\ldots,i_{j_{(m)}}))\cdot$$

$$\mathrm{Coef}^{(\nu-m+m_1)}_{i_{nn+1},t-nn-1}((...,\ j^*_{(1)}-nn-1,\ldots,j^*_{(m_1)}-nn-1),(...,i_{j^*_{(1)}},\ldots,i_{j^*_{(m_1)}}))$$

$$\prod_{n=1}^{r}\mathrm{Coef}^{(\nu_{nn+1+n})}_{i_{nn+1+n},t-nn-1-n}(...,\neq i_{nn+2},\ldots,i_{nn+1+n})\}| \qquad .$$

It is easy to figure out for the first part, that

$$\sum_{i_{nn+2},\ldots,i_{nn+1+r}=v_1,\ldots,v_k} \mathbb{E}\{\mathrm{Cr}^{nn}\mathrm{oef}^{(\nu)}_{x,t}()\prod_{n=1}^{r}\mathrm{Coef}^{(\nu_{nn+1+n})}_{i_{nn+1+n},t-nn-1-n}(...,\neq i_{nn+2},\ldots,i_{nn+1+n})\} =$$

$$\sum_{i_{nn+2},\ldots,i_{nn+1+r}=v_1,\ldots,v_k} \mathbb{E}\{\mathrm{Cr}^{nn}\mathrm{oef}^{(\nu)}_{x,t-1}()\prod_{n=1}^{r}\mathrm{Coef}^{(\nu_{nn+1+n})}_{i_{nn+1+n},t-nn-1-n}(...,\neq i_{nn+2},\ldots,i_{nn+1+n})\} \equiv$$

$$\sum_{i_{nn+1},\ldots,i_{nn+r}=v_1,\ldots,v_k} \mathbb{E}\{\mathrm{Cr}^{nn}\mathrm{oef}^{(\nu)}_{x,t}()\prod_{n=1}^{r}\mathrm{Coef}^{(\nu_{nn+n})}_{i_{nn+n},t-nn-n}(...,\neq i_{nn+1},\ldots,i_{nn+n})\}$$

because in the first line, left-hand side, $\mathrm{Cr}^{nn}\mathrm{oef}^{(\nu)}_{x,t}$ is a function of products of nn Coef taking place at timings $t-1$,...$t-nn$, while in the product following next there are the timings $t-nn-2$,...$t-nn-r-1$ and the assumption of independence in time of the variables can be used in the same way as if there was the $\mathrm{Cr}^{nn}\mathrm{oef}^{(\nu)}_{x,t-1}$ using Coef at timings $t-2,\ldots,t-nn-1$ instead, followed by the ones in the product starting from $t-nn-2$ and going backwards: the random variables are not equal but the expectations can be shown to be the same quantity and it should be obvious to the reader how. So it will be written that

$$|\sum_{i_{nn+2},\ldots,i_{nn+1+r}=v_1,\ldots,v_k} \mathbb{E}\{\mathrm{Cr}^{nn}\mathrm{oef}^{(\nu)}_{x,t}()\prod_{n=1}^{r}\mathrm{Coef}^{(\nu_{nn+1+n})}_{i_{nn+1+n},t-nn-1-n}(...,\neq i_{nn+2},\ldots,i_{nn+1+n})\}|$$

$$\leq \beta^{[nn]}_{x}\ (\beta^*)^r$$

as it has been accepted.

For the second part, it can be written directly that

$$
\begin{aligned}
&|\sum_{i_{nn+1},\ldots,i_{nn+1+r}=v_1,\ldots,v_k} \mathbb{E}\{\mathrm{Cr}^{nn}\mathrm{oef}^{(1+m)}_{x,t}((nn+1,j_{(1)},\ldots,j_{(m)}),(i_{nn+1},i_{j_{(1)}},\ldots,i_{j_{(m)}}))\cdot \\
&\qquad \mathrm{Coef}^{(\nu-m+m_1)}_{i_{nn+1},t-nn-1}((\ldots,\ j^*_{(1)}-nn-1,\ldots,j^*_{(m_1)}-nn-1),(\ldots,i_{j^*_{(1)}},\ldots,i_{j^*_{(m_1)}})) \\
&\qquad\qquad \prod_{n=1}^{r}\mathrm{Coef}^{(\nu_{nn+1+n})}_{i_{nn+1+n},t-nn-1-n}(\ldots,\neq i_{nn+2},\ldots,i_{nn+1+n})\}| \\
&\qquad\qquad\qquad \leq \beta^{[nn]}_x\ \beta^*\ (\beta^*)^r ,
\end{aligned}
$$

where '$\neq i_{nn+1},\ldots,i_{nn+1+n}$' can replace '$\neq i_{nn+2},\ldots,i_{nn+1+n}$' as the index i_{nn+1} is totally unconnected with the product: this is a direct consequence of what was accepted for nn. Hence, it can be written that

$$
\begin{aligned}
&|\sum_{i_{nn+2},\ldots,i_{nn+1+r}=v_1,\ldots,v_k} \mathbb{E}\{\mathrm{Cr}^{nn+1}\mathrm{oef}^{(\nu)}_{x,t}()\ \prod_{n=1}^{r}\mathrm{Coef}^{(\nu_{nn+1+n})}_{i_{nn+1+n},t-nn-1-n}(\ldots,\neq i_{nn+2},\ldots,i_{nn+1+n})\}| \leq \\
&\beta^{[nn]}_x\ (\beta^*)^r + \sum_{m=0}^{\min\{nn+1/nn,p-1\}} \sum_{\substack{j_{(1)},\ldots,j_{(m)}= \\ \ldots\min\{\ldots,nn+p\} \\ j_{(1)}<\ldots<j_{(m)}}} \sum_{m_1=0}^{m} \sum_{\substack{j^*_{(1)},\ldots,j^*_{(m_1)}= \\ j_{(1)},\ldots,j_{(m)} \\ j^*_{(1)}<\ldots<j^*_{(m_1)}}} \beta^{[nn]}_x\ (\beta^*)^{r+1} = \\
&\beta^{[nn]}_x\ (\beta^*)^r \left(1+\sum_{m=0}^{\min\{nn+1/nn,p-1\}} \sum_{\substack{j_{(1)},\ldots,j_{(m)}= \\ \ldots\min\{\ldots,nn+p\} \\ j_{(1)}<\ldots<j_{(m)}}} \sum_{m_1=0}^{m} \sum_{\substack{j^*_{(1)},\ldots,j^*_{(m_1)}= \\ j_{(1)},\ldots,j_{(m)} \\ j^*_{(1)}<\ldots<j^*_{(m_1)}}} \beta^*\right) \leq \\
&\qquad \beta^{[nn]}_x\ (\beta^*)^r\ \left(1+\beta^*\ \sum_{m=0}^{p-1}\binom{p-1}{m}\sum_{m_1=0}^{m}\binom{m}{m_1}\right)
\end{aligned}
$$

and the proof is completed.

(ii) This statement will be shown using induction and (i). For $n=1$, the formula (4) translates to

$$\mathrm{Croef}^{(\nu)}_{x,t}((j_1,\dots,j_\nu),(i_{j_1},\dots,i_{j_\nu})) = \mathrm{Coef}^{(\nu)}_{x,t}((j_1,\dots,j_\nu),(i_{j_1},\dots,i_{j_\nu})) +$$
$$\sum_{m=0}^{\min\{\nu/\nu-1,p-1\}} \sum_{\substack{j_{(1)},\dots,j_{(m)} = \\ j_1,j_2\dots,\min\{j_\nu,p\} \\ j_{(1)}<\dots<j_{(m)}}} \sum_{m_1=0}^{m} \sum_{\substack{j^*_{(1)},\dots,j^*_{(m_1)} = \\ j_{(1)},\dots,j_{(m)} \\ j^*_{(1)}<\dots<j^*_{(m_1)}}} \sum_{i_1=v_1,\dots,v_k}$$
$$\mathrm{Coef}^{(1+m)}_{x,t}((1,j_{(1)},\dots,j_{(m)}),(i_1,i_{j_{(1)}},\dots,i_{j_{(m)}}))\cdot$$
$$\mathrm{Coef}^{(\nu-m+m_1)}_{i_1,t-1}((j-1\ (j\neq j_{()}),\ j^*_{(1)}-1,\dots,j^*_{(m_1)}-1),(i_j\ (j\neq j_{()}),i_{j^*_{(1)}},\dots,i_{j^*_{(m_1)}})),$$

where it can be $\nu = 1,\dots,p$, $j_1,\dots,j_\nu = 2,\dots,p+1$ $(j_1<\dots<j_\nu)$ and in the case that $j_\nu = p+1$, it is considered that $\mathrm{Coef}(...,p+1) := 0$. Then

$$\mathbb{E}\{\mathrm{Croef}^{(\nu)}_{x,t}((j_1,\dots,j_\nu),(i_{j_1},\dots,i_{j_\nu}))\} = \mathbb{E}\{\mathrm{Coef}^{(\nu)}_{x,t}((j_1,\dots,j_\nu),(i_{j_1},\dots,i_{j_\nu}))\} +$$
$$\sum_{m=0}^{\min\{\nu/\nu-1,p-1\}} \sum_{\substack{j_{(1)},\dots,j_{(m)} = \\ j_1,j_2\dots,\min\{j_\nu,p\} \\ j_{(1)}<\dots<j_{(m)}}} \sum_{m_1=0}^{m} \sum_{\substack{j^*_{(1)},\dots,j^*_{(m_1)} = \\ j_{(1)},\dots,j_{(m)} \\ j^*_{(1)}<\dots<j^*_{(m_1)}}}$$
$$\sum_{i_1=v_1,\dots,v_k} \mathbb{E}\{\mathrm{Coef}^{(1+m)}_{x,t}((1,j_{(1)},\dots,j_{(m)}),(i_1,i_{j_{(1)}},\dots,i_{j_{(m)}}))\cdot$$
$$\mathrm{Coef}^{(\nu-m+m_1)}_{i_1,t-1}((j-1\ (j\neq j_{()}),\ j^*_{(1)}-1,\dots,j^*_{(m_1)}-1),(i_j\ (j\neq j_{()}),i_{j^*_{(1)}},\dots,i_{j^*_{(m_1)}}))\}$$

and using (9) from (i), it holds that

$$|\mathbb{E}\{\mathrm{Croef}_{x,t}^{(\nu)}((j_1,\ldots,j_\nu),(i_{j_1},\ldots,i_{j_\nu}))\}| \leq \beta_x + \sum_{m=0}^{\min\{\nu/\nu-1,p-1\}} \sum_{\substack{j_{(1)},\ldots,j_{(m)}= \\ j_1,j_2\ldots,\min\{j_\nu,p\} \\ j_{(1)}<\ldots<j_{(m)}}} \sum_{m_1=0}^{m} \sum_{\substack{j^*_{(1)},\ldots,j^*_{(m_1)}= \\ j_{(1)},\ldots,j_{(m)} \\ j^*_{(1)}<\ldots<j^*_{(m_1)}}} \beta_x\ \beta^* \leq$$

$$\beta_x \left(1+\sum_{m=0}^{p-1} \sum_{\substack{j_{(1)},\ldots,j_{(m)}= \\ j_1,j_2\ldots,p \\ j_{(1)}<\ldots<j_{(m)}}} \sum_{m_1=0}^{m} \sum_{\substack{j^*_{(1)},\ldots,j^*_{(m_1)}= \\ j_{(1)},\ldots,j_{(m)} \\ j^*_{(1)}<\ldots<j^*_{(m_1)}}} \beta^* \right) =$$

$$\beta_x \left(1+\beta^* \sum_{m=0}^{p-1} \binom{p-1}{m} \sum_{m_1=0}^{m} \binom{m}{m_1} \right),$$

so the desired statement

$$|\mathbb{E}\{\mathrm{Cr}^n\mathrm{oef}_{x,t}^{(\nu)}()\}| \leq \beta_x^{[n]} \tag{13}$$

holds for $n=1$ (and it is still valid if $j_\nu = n+p$).

For fixed $n_a \in \mathbb{N}$, it is accepted that

$$|\mathbb{E}\{\mathrm{Cr}^{n_a}\mathrm{oef}_{x,t}^{(\nu)}(j_1,\ldots,j_\nu),(i_{j_1},\ldots,i_{j_\nu})\}| \leq \beta_x \left(1+\beta^* \sum_{m=0}^{p-1} \binom{p-1}{m} \sum_{m_1=0}^{m} \binom{m}{m_1} \right)^{n_a} \tag{14}$$

for any $j_1,\ldots,j_\nu = n_a+1,\ldots n_a+p$ $(j_1<\ldots<j_\nu)$, $i_{j_1},\ldots,i_{j_\nu} = v_1,\ldots,v_k$.

For $n = n_a+1$, it is implied from (4) that

$$\mathbb{E}\{\mathrm{Cr}^{n_a+1}\mathrm{oef}_{x,t}^{(\nu)}((j_1,\ldots,j_\nu),(i_{j_1},\ldots,i_{j_\nu}))\} = \mathbb{E}\{\mathrm{Cr}^{n_a}\mathrm{oef}_{x,t}^{(\nu)}((j_1,\ldots,j_\nu),(i_{j_1},\ldots,i_{j_\nu}))\} + \sum_{m=0}^{\min\{\nu/\nu-1,p-1\}} \sum_{\substack{j_{(1)},\ldots,j_{(m)}= \\ j_1,j_2\ldots,\min\{j_\nu,n_a+p\} \\ j_{(1)}<\ldots<j_{(m)}}} \sum_{m_1=0}^{m} \sum_{\substack{j^*_{(1)},\ldots,j^*_{(m_1)}= \\ j_{(1)},\ldots,j_{(m)} \\ j^*_{(1)}<\ldots<j^*_{(m_1)}}} \sum_{i_{n_a+1}=v_1,\ldots,v_k}$$

$$\mathbb{E}\{\mathrm{Cr}^{n_a}\mathrm{oef}_{x,t}^{(1+m)}((n_a+1,j_{(1)},\ldots,j_{(m)}),(i_{n_a+1},i_{j_{(1)}},\ldots,i_{j_{(m)}}))\cdot \mathrm{Coef}_{i_{n_a+1},t-n_a-1}^{(\nu-m+m_1)}(
(j-n_a-1\ (j\neq j_{()}),\ j^*_{(1)}-n_a-1,\ldots,j^*_{(m_1)}-n_a-1),(i_j\ (j\neq j_{()}),i_{j^*_{(1)}},\ldots,i_{j^*_{(m_1)}}))\}$$

and the first term of this will be bounded according to (14). For the second parts, the bound

$$|\sum_{i_{n_a+1}=v_1,\dots,v_k} \mathbb{E}\{\mathrm{Cr}^{n_a}\mathrm{oef}^{(1+m)}_{x,t}((n_a+1,j_{(1)},\dots,j_{(m)}),(i_{n_a+1},i_{j_{(1)}},\dots,i_{j_{(m)}}))\cdot \mathrm{Coef}^{(\nu-m+m_1)}_{i_{n_a+1},t-n_a-1}$$
$$((j-n_a-1\ (j\neq j_{()}),\ j^*_{(1)}-n_a-1,\dots,j^*_{(m_1)}-n_a-1),(i_j\ (j\neq j_{()}),i_{j^*_{(1)}},\dots,i_{j^*_{(m_1)}}))\}|$$
$$\leq \beta_x^{[n_a]}\ \beta^*\equiv \beta_x\ \beta^*\ \left(1+\beta^*\ \sum_{m=0}^{p-1}\begin{pmatrix}p-1\\m\end{pmatrix}\sum_{m_1=0}^{m}\begin{pmatrix}m\\m_1\end{pmatrix}\right)^{n_a}$$

can be claimed from (12).

Altogether, it holds that

$$|\mathbb{E}\{\mathrm{Cr}^{n_a+1}\mathrm{oef}^{(\nu)}_{x,t}((j_1,\dots,j_\nu),(i_{j_1},\dots,i_{j_\nu}))\}|\leq \beta_x\ \left(1+\beta^*\ \sum_{m=0}^{p-1}\begin{pmatrix}p-1\\m\end{pmatrix}\sum_{m_1=0}^{m}\begin{pmatrix}m\\m_1\end{pmatrix}\right)^{n_a}$$
$$+\sum_{m=0}^{\min\{\nu/\nu-1,p-1\}}\ \sum_{\substack{j_{(1)},\dots,j_{(m)}=\\ j_1,j_2\dots,\min\{j_\nu,n_a+p\}\\ j_{(1)}<\dots<j_{(m)}}}\ \sum_{m_1=0}^{m}\ \sum_{\substack{j^*_{(1)},\dots,j^*_{(m_1)}=\\ j_{(1)},\dots,j_{(m)}\\ j^*_{(1)}<\dots<j^*_{(m_1)}}}$$
$$\beta_x\ \beta^*\ \left(1+\beta^*\ \sum_{m=0}^{p-1}\begin{pmatrix}p-1\\m\end{pmatrix}\sum_{m_1=0}^{m}\begin{pmatrix}m\\m_1\end{pmatrix}\right)^{n_a}\leq$$
$$\beta_x\ \left(1+\beta^*\ \sum_{m=0}^{p-1}\begin{pmatrix}p-1\\m\end{pmatrix}\sum_{m_1=0}^{m}\begin{pmatrix}m\\m_1\end{pmatrix}\right)^{n_a}+$$
$$\sum_{m=0}^{p-1}\begin{pmatrix}p-1\\m\end{pmatrix}\sum_{m_1=0}^{m}\begin{pmatrix}m\\m_1\end{pmatrix}\ \ \beta_x\ \beta^*\ \left(1+\beta^*\ \sum_{m=0}^{p-1}\begin{pmatrix}p-1\\m\end{pmatrix}\sum_{m_1=0}^{m}\begin{pmatrix}m\\m_1\end{pmatrix}\right)^{n_a}=$$
$$\beta_x\ \left(1+\beta^*\ \sum_{m=0}^{p-1}\begin{pmatrix}p-1\\m\end{pmatrix}\sum_{m_1=0}^{m}\begin{pmatrix}m\\m_1\end{pmatrix}\right)^{n_a}\ \left[1+\sum_{m=0}^{p-1}\begin{pmatrix}p-1\\m\end{pmatrix}\sum_{m_1=0}^{m}\begin{pmatrix}m\\m_1\end{pmatrix}\ \beta^*\right]=$$
$$\beta_x\ \left(1+\beta^*\ \sum_{m=0}^{p-1}\begin{pmatrix}p-1\\m\end{pmatrix}\sum_{m_1=0}^{m}\begin{pmatrix}m\\m_1\end{pmatrix}\right)^{n_a+1}\equiv\beta_x^{[n_a+1]}.$$

- For Proposition 3.3:

Before studying the general case, observe that for $p=0$, (1) transforms to

$$f_x(X_t)=f_x(I_t)\quad+\quad\sum_{h_q=1}^{q}\ \sum_{\substack{n_1,\dots,n_{h_q}=1\\ n_1<\dots<n_{h_q}}}^{q}\ \sum_{j_{n_1},\dots,j_{n_{h_q}}=v_1,\dots,v_k}$$
$$d^{h_q}f_x(I_t^{(\mathbf{0}_p|(\mathbf{0}_{n_1-1},j_{n_1},\dots,j_{n_{h_q}},\mathbf{0}_{q-n_{h_q}}))})\cdot\left(\prod_{r=1}^{h_q}f_{j_{n_r}}(I_{t-n_r})\right)$$

implying that there is a finite number of terms, so that the sum of expected values of absolute coefficients can be bounded with safety. Alternatively, equation

$$\left[\left(1+\beta^* \sum_{m=0}^{p-1}\binom{p-1}{m}\sum_{m_1=0}^{m}\binom{m}{m_1}\right)^p - 1\right] < 1 \tag{15}$$

can be considered (as in the statement of the proposition), i.e. it holds that $(...)^{p=0}-1 = 0 < 1$ and a finite moving-average process always satisfies $|\mathbb{E}\{\mathrm{Cr}^n\mathrm{oef}_{x,t}^{(\nu)}\}| \leq C \cdot \alpha^n$, for some $C > 0$ and $\alpha \in (0,1)$ (as in the statement of the proposition).

Straight from (13) (and (9)), it is written that

$$|\mathbb{E}\{\mathrm{Cr}^n\mathrm{oef}_{x,t}^{(\nu)}\}| \leq \beta_x \left(1+\beta^* \sum_{m=0}^{p-1}\binom{p-1}{m}\sum_{m_1=0}^{m}\binom{m}{m_1}\right)^n, \quad \text{for } n = 0, 1, \ldots, p-1,$$

while for $n = p$ and following the adjusted formula (8) ((9) and (12) are used as well for the products), it holds that

$$|\mathbb{E}\{\mathrm{Cr}^p\mathrm{oef}_{x,t}^{(\nu)}\}| \leq \left[\beta_x + \beta_x \left(1+\beta^* \sum_{m=0}^{p-1}\binom{p-1}{m}\sum_{m_1=0}^{m}\binom{m}{m_1}\right) + \ldots \right.$$
$$\left. \beta_x \left(1+\beta^* \sum_{m=0}^{p-1}\binom{p-1}{m}\sum_{m_1=0}^{m}\binom{m}{m_1}\right)^{p-1}\right] \sum_{m=0}^{p-1}\binom{p-1}{m}\sum_{m_1=0}^{m}\binom{m}{m_1}\beta^*$$

or

$$|\mathbb{E}\{\mathrm{Cr}^p\mathrm{oef}_{x,t}^{(\nu)}\}| \leq \beta_x \left[\left(1+\beta^* \sum_{m=0}^{p-1}\binom{p-1}{m}\sum_{m_1=0}^{m}\binom{m}{m_1}\right)^p - 1\right].$$

In an identical fashion (using the same recursive formula as well as Proposition 3.2), it can be demonstrated that

$$|\sum_{i_{p+1},\ldots,i_{p+r}} \mathbb{E}\{\mathrm{Cr}^p\mathrm{oef}_{x,t}^{(\nu)} \cdot \mathrm{Coef}_{i_{p+1},t-p-1} \ldots \mathrm{Coef}_{i_{p+r},t-p-r}\}| \leq$$
$$\beta_x\ (\beta^*)^r \left[\left(1+\beta^* \sum_{m=0}^{p-1}\binom{p-1}{m}\sum_{m_1=0}^{m}\binom{m}{m_1}\right)^p - 1\right].$$

The case $n = p + 1$ will also refer to (8), to come up with

$$|\mathbb{E}\{\mathrm{Cr}^{p+1}\mathrm{oef}_{x,t}^{(\nu)}\}| \leq \beta_x \left\{ \left(1+\beta^* \sum_{m=0}^{p-1}\binom{p-1}{m}\sum_{m_1=0}^{m}\binom{m}{m_1}\right) \cdot \right.$$
$$\left. \left[\left(1+\beta^* \sum_{m=0}^{p-1}\binom{p-1}{m}\sum_{m_1=0}^{m}\binom{m}{m_1}\right)^p - 1\right] \right\}$$

and this will be accompanied by

$$
\begin{aligned}
&| \sum_{i_{p+2},\ldots,i_{p+1+r}} \mathbb{E}\{\mathrm{Cr}^{p+1}\mathrm{oef}_{x,t}^{(\nu)} \cdot \mathrm{Coef}_{i_{p+2},t-p-2} \ldots \mathrm{Coef}_{i_{p+r+1},t-p-r-1}\}| \leq \\
&\beta_x \cdot (\beta^*)^r \left\{ \left(1+\beta^* \sum_{m=0}^{p-1} \binom{p-1}{m} \sum_{m_1=0}^{m} \binom{m}{m_1}\right) \cdot \right. \\
&\left. \left[\left(1+\beta^* \sum_{m=0}^{p-1} \binom{p-1}{m} \sum_{m_1=0}^{m} \binom{m}{m_1}\right)^p - 1\right] \right\},
\end{aligned}
$$

which will be needed for the other steps and so on. Following the same arguments, it will be possible to conclude (induction can be used) that

$$
\begin{aligned}
|\mathbb{E}\{\mathrm{Cr}^{n}\mathrm{oef}_{x,t}^{(\nu)}\}| \quad \leq \quad \beta_x & \left[\left(1+\beta^* \sum_{m=0}^{p-1} \binom{p-1}{m} \sum_{m_1=0}^{m} \binom{m}{m_1}\right)^p - 1\right]^{[n/p]} \\
& \left(1+\beta^* \sum_{m=0}^{p-1} \binom{p-1}{m} \sum_{m_1=0}^{m} \binom{m}{m_1}\right)^{n-p\cdot[n/p]}
\end{aligned}
$$

with $0 \leq n - p \cdot [n/p] \leq p-1$, so that under condition (15), p converging geometric series are achieved.