Supplementary Document for the Submission of "Estimation in a Semiparametric Panel Data Model with Nonstationarity"

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Notice that in what follows $O(1)$ denotes some constant which may be different at each appearance. Lemmas A.1 to Lemma A.4 introduce some preliminary results. The proofs of the main results are provided in the Appendix A. Appendix B of this file adds some extra simulation studies and the omitted proofs of Lemmas 2.1, A.2-A.4 and Corollary 2.1.

Appendix A: Proofs of the Main Results

Lemma A.1. *Suppose that $g(w)$ is differentiable on $\mathbb{R}$ and $x^{m-j}g^{(j)}(w) \in L^2(\mathbb{R})$ for $j = 0, 1, \ldots, m$ and $m \geq 1$. For the expansion*

$$g(w) = \sum_{j=0}^{\infty} c_j \mathscr{H}_j(w) = Z_k(w)'C + \gamma_k(w), \quad c_j = \int g(w)\mathscr{H}_j(w)dw,$$

$$Z_k(w) = (\mathscr{H}_0(w), \ldots, \mathscr{H}_{k-1}(w))', \quad C = (c_0, \ldots, c_{k-1})$$

the following results hold:

(1) $\int w^2 \mathscr{H}_n^2(w)dw = n + 1/2$; (2) $\max_w |\gamma_k(w)| = O(1)k^{-(m-1)/2-1/12}$; (3) $\int \gamma_k^2(w)dw = O(1)k^{-m}$; (4) $\int \|Z_k(w)\|dw = O(1)k^{11/12}$; (5) $\int \|Z_k(w)\|^2 dw = k$; (6) $\|Z_k(w)\|^2 = O(1)k$ uniformly on $\mathbb{R}$;

(7) $\int |\gamma_k(w)|dw = O(1)k^{-m/2+11/12}$; (8) $\int |\mathscr{H}_n(w)|dw = O(1)n^{5/12}$; (9) $\int |x|^2 \|Z_k(x)\|^2 dx = O(1)k^2$.

Lemma A.1 is a part of Lemma C.1 of the supplementary file of Dong et al. (2016), wherein all the detailed proofs can be found.

Lemma A.2. *Under Assumptions 1 and 2, as $(N, T) \to (\infty, \infty)$ jointly,*
(1) $\left\|\frac{1}{N\sqrt{T}} \sum_{i=1}^{N} \sum_{t=1}^{T} Z_k(u_{it})\gamma_k(u_{it})\right\| = O_P(k^{-(m-1)/2})$;
(2) $\left\|\frac{1}{N\sqrt{T}} \sum_{i=1}^{N} \sum_{t=1}^{T} \phi(u_{it})Z_k(u_{it})'\right\| = O_P(1)$;
(3) $\left\|\frac{1}{N\sqrt{T}} \sum_{i=1}^{N} \sum_{t=1}^{T} v_{it}Z_k(u_{it})'\right\| = O_P\left(\sqrt{\frac{k}{N}}\right)$; (4) $\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} v_{it}v_{it}' = \Sigma_v + O_P\left(\frac{1}{\sqrt{NT}}\right)$;

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(5) $\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} v_{it}\phi(u_{it})' = O_P\left(\frac{1}{\sqrt{NT}}\right)$; (6) $\frac{1}{N\sqrt{T}}\sum_{i=1}^{N}\sum_{t=1}^{T} Z_k(u_{it})e_{it} = O_P\left(\frac{\sqrt{k}}{\sqrt{N}\sqrt[4]{T}}\right)$;

(7) $\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} \phi(u_{it})e_{it} = O_P\left(\frac{1}{\sqrt{N}\sqrt[4]{T^3}}\right)$; (8) $\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} \phi(u_{it})\gamma_k(u_{it}) = O_P\left(\frac{1}{\sqrt{k^m T}}\right)$;

(9) $\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} v_{it}\gamma_k(u_{it}) = O_P\left(\frac{k^{-m/2+5/12}}{\sqrt{NT}}\right)$.

Lemma A.3. *Consider two non-singular symmetric matrices A, B with same dimensions $k \times k$, where k tends to ∞. Suppose that their minimum eigenvalues satisfy that $\lambda_{min}(A) > 0$ and $\lambda_{min}(B) > 0$ uniformly in k. Then $\left\|A^{-1} - B^{-1}\right\| \leq \lambda_{min}^{-1}(A) \cdot \lambda_{min}^{-1}(B)\|A - B\|$.*

Lemma A.4. *Let Assumptions 1 and 2 hold. As $(N,T) \to (\infty,\infty)$ jointly, (1) $\left\|\frac{1}{N\sqrt{T}}\mathcal{Z}\mathcal{E}\right\| = O_P\left(\frac{\sqrt{k}}{\sqrt{N}\sqrt[4]{T}}\right)$; (2) $\left\|\frac{1}{N\sqrt{T}}X'\mathcal{Z}\right\| = O_P(1)$; (3) $\frac{1}{NT}X'\mathcal{E} = O_P\left(\frac{1}{\sqrt{NT}}\right)$; (4) $\frac{1}{NT}X'\gamma = O_P\left(\frac{1}{\sqrt{k^m T}}\right)$; (5) $\left\|\frac{1}{N\sqrt{T}}\mathcal{Z}'\gamma\right\| = O_P(k^{-(m-1)/2})$; (6) $\frac{1}{NT}X'X \to_P \Sigma_v$.*

We are now ready to provide the proofs of the mains results of this paper. All the omitted proofs are stated in the Appendix B.

Proof of Theorem 2.1:

By Lemma 2.1, we have uniformly in k

$$\lambda_{min}\left(\frac{1}{N\sqrt{T}}\mathcal{Z}'\mathcal{Z}\right) = \min_{\|\mu\|=1}\left\{\mu' a_0 I_k \mu + \mu'\left(\frac{1}{N\sqrt{T}}\mathcal{Z}'\mathcal{Z} - a_0 I_k\right)\mu\right\}$$
$$\geq a_0 - \left\|\frac{1}{N\sqrt{T}}\mathcal{Z}'\mathcal{Z} - a_0 I_k\right\| \geq \frac{1}{2}a_0(1 + o_P(1)). \tag{A.1}$$

Therefore, by Lemma A.3

$$\left\|\left(\frac{1}{N\sqrt{T}}\mathcal{Z}'\mathcal{Z}\right)^{-1} - \frac{1}{a_0}I_k\right\| \leq \frac{2(1+o_P(1))}{a_0^2}\left\|\frac{1}{N\sqrt{T}}\mathcal{Z}'\mathcal{Z} - a_0 I_k\right\| = o_P(1). \tag{A.2}$$

Expand $\widehat{\beta}$

$$\widehat{\beta} - \beta_0 = \left(X'M_{\mathcal{Z}}X\right)^{-1}X'M_{\mathcal{Z}}\mathcal{E} + \left(X'M_{\mathcal{Z}}X\right)^{-1}X'M_{\mathcal{Z}}\gamma$$

and observe that

$$\frac{1}{NT}X'M_{\mathcal{Z}}X = \frac{1}{NT}X'X - \frac{1}{N\sqrt{T}}X'\mathcal{Z}\left(\frac{1}{N\sqrt{T}}\mathcal{Z}'\mathcal{Z}\right)^{-1}\frac{1}{NT}\mathcal{Z}'X, \tag{A.3}$$

$$\frac{1}{NT}X'M_{\mathcal{Z}}\mathcal{E} = \frac{1}{NT}X'\mathcal{E} - \frac{1}{N\sqrt{T}}X'\mathcal{Z}\left(\frac{1}{N\sqrt{T}}\mathcal{Z}'\mathcal{Z}\right)^{-1}\frac{1}{NT}\mathcal{Z}'\mathcal{E}, \tag{A.4}$$

$$\frac{1}{NT}X'M_{\mathcal{Z}}\gamma = \frac{1}{NT}X'\gamma - \frac{1}{N\sqrt{T}}X'\mathcal{Z}\left(\frac{1}{N\sqrt{T}}\mathcal{Z}'\mathcal{Z}\right)^{-1}\frac{1}{NT}\mathcal{Z}'\gamma. \tag{A.5}$$

For asymptotic consistency, we consider (A.3)-(A.5) respectively below. Start from (A.3).

$$\frac{1}{NT}X'M_{\mathcal{Z}}X = \frac{1}{NT}X'X - \frac{1}{N\sqrt{T}}X'\mathcal{Z}a_0^{-1}I_k\frac{1}{NT}\mathcal{Z}'X$$

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$$+\frac{1}{N\sqrt{T}}X'\mathcal{Z}\left[a_0^{-1}I_k-\left(\frac{1}{N\sqrt{T}}\mathcal{Z}'\mathcal{Z}\right)^{-1}\right]\frac{1}{NT}\mathcal{Z}'X. \tag{A.6}$$

Notice that

$$\left\|\frac{1}{N\sqrt{T}}X'\mathcal{Z}\left[a_0^{-1}I_k-\left(\frac{1}{N\sqrt{T}}\mathcal{Z}'\mathcal{Z}\right)^{-1}\right]\frac{1}{NT}\mathcal{Z}'X\right\|$$

$$\leq\frac{1}{\sqrt{T}}\left\|\frac{1}{N\sqrt{T}}X'\mathcal{Z}\right\|^2\left\|a_0^{-1}I_k-\left(\frac{1}{N\sqrt{T}}\mathcal{Z}'\mathcal{Z}\right)^{-1}\right\|=o_P\left(\frac{1}{\sqrt{T}}\right),$$

where the last line follows from (A.2) and (2) of Lemma A.4. Similarly, by (2) of Lemma A.4,

$$\left\|\frac{1}{N\sqrt{T}}X'\mathcal{Z}a_0^{-1}I_k\frac{1}{NT}\mathcal{Z}'X\right\|\leq O(1)\left\|\frac{1}{N\sqrt{T}}X'\mathcal{Z}\right\|\left\|\frac{1}{NT}\mathcal{Z}'X\right\|=O_P\left(\frac{1}{\sqrt{T}}\right).$$

In connection with (6) of Lemma A.4, we can further write

$$\frac{1}{NT}X'M_{\mathcal{Z}}X=\frac{1}{NT}X'X+O_P\left(\frac{1}{\sqrt{T}}\right)\to_P\Sigma_v. \tag{A.7}$$

For (A.4), write

$$\frac{1}{NT}X'M_{\mathcal{Z}}\mathcal{E}=\frac{1}{NT}X'\mathcal{E}-\frac{1}{N\sqrt{T}}X'\mathcal{Z}a_0^{-1}I_k\frac{1}{NT}\mathcal{Z}'\mathcal{E}$$

$$+\frac{1}{N\sqrt{T}}X'\mathcal{Z}\left[a_0^{-1}I_k-\left(\frac{1}{N\sqrt{T}}\mathcal{Z}'\mathcal{Z}\right)^{-1}\right]\frac{1}{NT}\mathcal{Z}'\mathcal{E}. \tag{A.8}$$

Notice that

$$\left\|\frac{1}{N\sqrt{T}}X'\mathcal{Z}\left[a_0^{-1}I_k-\left(\frac{1}{N\sqrt{T}}\mathcal{Z}'\mathcal{Z}\right)^{-1}\right]\frac{1}{NT}\mathcal{Z}'\mathcal{E}\right\|$$

$$\leq\left\|\frac{1}{N\sqrt{T}}X'\mathcal{Z}\right\|\left\|a_0^{-1}I_k-\left(\frac{1}{N\sqrt{T}}\mathcal{Z}'\mathcal{Z}\right)^{-1}\right\|\left\|\frac{1}{NT}\mathcal{Z}'\mathcal{E}\right\|=o_P\left(\frac{\sqrt{k}}{\sqrt{N}\sqrt[4]{T^3}}\right),$$

where the last line follows from (A.2) and (1)-(2) of Lemma A.4. Similarly, by (1)-(2) of Lemma A.4,

$$\left\|\frac{1}{N\sqrt{T}}X'\mathcal{Z}a_0^{-1}I_k\frac{1}{NT}\mathcal{Z}'\mathcal{E}\right\|\leq O(1)\left\|\frac{1}{N\sqrt{T}}X'\mathcal{Z}\right\|\left\|\frac{1}{NT}\mathcal{Z}'\mathcal{E}\right\|=O_P\left(\frac{\sqrt{k}}{\sqrt{N}\sqrt[4]{T^3}}\right).$$

Then we can further write (A.8) as

$$\frac{1}{NT}X'M_{\mathcal{Z}}\mathcal{E}=\frac{1}{NT}X'\mathcal{E}+O_P\left(\frac{\sqrt{k}}{\sqrt{N}\sqrt[4]{T^3}}\right)=O_P\left(\frac{1}{\sqrt{NT}}\right), \tag{A.9}$$

where the second equality follows from (3) of Lemma A.4 and Assumption 2.2.

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For (A.5), write

$$\frac{1}{NT}X'M_{\mathcal{Z}}\gamma = \frac{1}{NT}X'\gamma - \frac{1}{N\sqrt{T}}X'\mathcal{Z}a_0^{-1}I_k\frac{1}{NT}\mathcal{Z}'\gamma$$
$$+\frac{1}{N\sqrt{T}}X'\mathcal{Z}\left[a_0^{-1}I_k - \left(\frac{1}{N\sqrt{T}}\mathcal{Z}'\mathcal{Z}\right)^{-1}\right]\frac{1}{NT}\mathcal{Z}'\gamma. \tag{A.10}$$

Notice that

$$\left\|\frac{1}{N\sqrt{T}}X'\mathcal{Z}\left[a_0^{-1}I_k - \left(\frac{1}{N\sqrt{T}}\mathcal{Z}'\mathcal{Z}\right)^{-1}\right]\frac{1}{NT}\mathcal{Z}'\gamma\right\|$$
$$\leq \left\|\frac{1}{N\sqrt{T}}X'\mathcal{Z}\right\|\left\|a_0^{-1}I_k - \left(\frac{1}{N\sqrt{T}}\mathcal{Z}'\mathcal{Z}\right)^{-1}\right\|\left\|\frac{1}{NT}\mathcal{Z}'\gamma\right\| = o_P\left(\frac{1}{\sqrt{k^{m-1}T}}\right),$$

where the last line follows from (A.2), and (2) and (5) of Lemma A.4. Similarly, by (2) and (5) of Lemma A.4,

$$\left\|\frac{1}{N\sqrt{T}}X'\mathcal{Z}a_0^{-1}I_k\frac{1}{NT}\mathcal{Z}'\gamma\right\| \leq O(1)\left\|\frac{1}{N\sqrt{T}}X'\mathcal{Z}\right\|\left\|\frac{1}{NT}\mathcal{Z}'\gamma\right\| = O_P\left(\frac{1}{\sqrt{k^{m-1}T}}\right).$$

Then we can further write (A.10) as

$$\frac{1}{NT}X'M_{\mathcal{Z}}\gamma = \frac{1}{NT}X'\gamma + O_P\left(\frac{1}{\sqrt{k^{m-1}T}}\right) = O_P\left(\frac{1}{\sqrt{k^{m-1}T}}\right), \tag{A.11}$$

where the second equality follows from (4) of Lemma A.4.

The asymptotic consistency follows from (A.7), (A.9) and (A.11) immediately.

Below, we focus on the asymptotic normality:

$$\sqrt{NT}(\widehat{\beta} - \beta_0) = \left(\frac{1}{NT}X'M_{\mathcal{Z}}X\right)^{-1}\frac{1}{\sqrt{NT}}X'M_{\mathcal{Z}}(\gamma + \mathcal{E}) \tag{A.12}$$

By (A.7) and (A.11), it is straightforward to obtain that

$$\left(\frac{1}{NT}X'M_{\mathcal{Z}}X\right)^{-1}\frac{1}{\sqrt{NT}}X'M_{\mathcal{Z}}\gamma = O_P\left(N^{\frac{1}{2}}k^{-\frac{m-1}{2}}\right) = o_P(1),$$

where the second equality follows from the condition of $N/k^{m-1} \to 0$.

Therefore, we need only to consider the next term:

$$\sqrt{NT}(\widehat{\beta} - \beta_0) = \left(\frac{1}{NT}X'M_{\mathcal{Z}}X\right)^{-1}\frac{1}{\sqrt{NT}}X'M_{\mathcal{Z}}\mathcal{E} + o_P(1).$$

By (A.7), $\frac{1}{NT}X'M_{\mathcal{Z}}X \to_P \Sigma_v$. We then focus on $\frac{1}{\sqrt{NT}}X'M_{\mathcal{Z}}\mathcal{E}$ below. Further expand (A.9)

$$\frac{1}{\sqrt{NT}}X'M_{\mathcal{Z}}\mathcal{E} = \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}\tilde{x}_{it}\tilde{e}_{it} + O_P\left(\frac{\sqrt{k}}{\sqrt[4]{T}}\right)$$

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$$= \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=1}^{T} (\phi(u_{it}) + v_{it})e_{it} - \frac{\sqrt{NT}}{NT^2} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} (\phi(u_{it}) + v_{it})e_{is} + O_P\left(\frac{\sqrt{k}}{\sqrt[4]{T}}\right).$$

In the proof of (3) of Lemma A.4, we have shown that $\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} \phi(u_{it})e_{is} = o_P\left(\frac{1}{\sqrt{NT}}\right)$ and $\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} v_{it}e_{is} = o_P\left(\frac{1}{\sqrt{NT}}\right)$. Thus, it is straightforward to obtain that

$$\frac{\sqrt{NT}}{NT^2} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} (\phi(u_{it}) + v_{it})e_{is} = o_P(1).$$

Hence, we can further write

$$\frac{1}{\sqrt{NT}} X' M_{\mathcal{Z}} \mathcal{E} = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=1}^{T} (\phi(u_{it}) + v_{it})e_{it} + o_P(1).$$

By (7) of Lemma A.2, it is easy to know that $\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=1}^{T} \phi(u_{it})e_{it} = o_P(1)$. Therefore, we can write $\frac{1}{\sqrt{NT}} X' M_{\mathcal{Z}} \mathcal{E}$ as

$$\frac{1}{\sqrt{NT}} X' M_{\mathcal{Z}} \mathcal{E} = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=1}^{T} v_{it}e_{it} + o_P(1).$$

Chen et al. (2012) have already shown that $\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=1}^{T} v_{it}e_{it} \to_D N(0, \Sigma_{v,e})$ in their formula (A.44). In connection with $\frac{1}{NT} X' M_{\mathcal{Z}} X \to_P \Sigma_v$, the asymptotic normality follows. ∎

Proof of Lemma 2.2:

Note that

$$\widehat{C} - C = \left(\mathcal{Z}' M_X \mathcal{Z}\right)^{-1} \mathcal{Z}' M_X \gamma + \left(\mathcal{Z}' M_X \mathcal{Z}\right)^{-1} \mathcal{Z}' M_X \mathcal{E},$$

and we normalize each term as

$$\frac{1}{N\sqrt{T}} \mathcal{Z}' M_X \mathcal{Z} = \frac{1}{N\sqrt{T}} \mathcal{Z}' \mathcal{Z} - \frac{1}{N\sqrt{T}} \mathcal{Z}' X \left(\frac{1}{NT} X' X\right)^{-1} \frac{1}{NT} X' \mathcal{Z} \tag{A.13}$$

$$\frac{1}{N\sqrt{T}} \mathcal{Z}' M_X \gamma = \frac{1}{N\sqrt{T}} \mathcal{Z}' \gamma - \frac{1}{N\sqrt{T}} \mathcal{Z}' X \left(\frac{1}{NT} X' X\right)^{-1} \frac{1}{NT} X' \gamma \tag{A.14}$$

$$\frac{1}{N\sqrt{T}} \mathcal{Z}' M_X \mathcal{E} = \frac{1}{N\sqrt{T}} \mathcal{Z}' \mathcal{E} - \frac{1}{N\sqrt{T}} \mathcal{Z}' X \left(\frac{1}{NT} X' X\right)^{-1} \frac{1}{NT} X' \mathcal{E}. \tag{A.15}$$

We now consider (A.13)-(A.15), respectively. Firstly, notice that

$$\left\| \frac{1}{N\sqrt{T}} \mathcal{Z}' M_X \mathcal{Z} - a_0 I_k \right\|$$

$$\leq \left\| \frac{1}{N\sqrt{T}} \mathcal{Z}' \mathcal{Z} - a_0 I_k \right\| + \frac{1}{\sqrt{T}} \left\| \frac{1}{N\sqrt{T}} \mathcal{Z}' X \right\|^2 \left\| \left(\frac{1}{NT} X' X\right)^{-1} \right\| = o_P(1),$$

where the last line follows from Lemma 2.1 as well as (2) and (6) of Lemma A.4 in this paper. Consequently, we obtain that

$$\lambda_{min}\left(\frac{1}{N\sqrt{T}}\mathcal{Z}'M_X\mathcal{Z}\right) = \min_{\|\mu\|=1}\left\{\mu'a_0 I_k\mu + \mu'\left(\frac{1}{N\sqrt{T}}\mathcal{Z}'M_X\mathcal{Z} - a_0 I_k\right)\mu\right\}$$

$$\geq a_0 - \left\|\frac{1}{N\sqrt{T}}\mathcal{Z}'M_X\mathcal{Z} - a_0 I_k\right\| \geq \frac{1}{2}a_0 + o_P(1).$$

For (A.14),

$$\left\|\frac{1}{N\sqrt{T}}\mathcal{Z}'M_X\gamma\right\| \leq \left\|\frac{1}{N\sqrt{T}}\mathcal{Z}'\gamma\right\| + \left\|\frac{1}{N\sqrt{T}}\mathcal{Z}'X\right\|\left\|\left(\frac{1}{NT}X'X\right)^{-1}\right\|\left\|\frac{1}{NT}X'\gamma\right\|$$

$$= O_P\left(k^{-(m-1)/2}\right),$$

where the equality follows from (2), (4), (5) and (6) of Lemma A.4. According to the above, it is easy to obtain that

$$\left\|\left(\mathcal{Z}'M_X\mathcal{Z}\right)^{-1}\mathcal{Z}'M_X\gamma\right\|^2 \leq \lambda_{min}^{-2}\left(\frac{1}{N\sqrt{T}}\mathcal{Z}'M_X\mathcal{Z}\right)\cdot\left\|\frac{1}{N\sqrt{T}}\mathcal{Z}'M_X\gamma\right\|^2 = O_P(k^{-m+1}) \quad \text{(A.16)}$$

For (A.15),

$$\left\|\frac{1}{N\sqrt{T}}\mathcal{Z}'M_X\mathcal{E}\right\| \leq \left\|\frac{1}{N\sqrt{T}}\mathcal{Z}'\mathcal{E}\right\| + \left\|\frac{1}{N\sqrt{T}}\mathcal{Z}'X\right\|\left\|\left(\frac{1}{NT}X'X\right)^{-1}\right\|\left\|\frac{1}{NT}X'\mathcal{E}\right\|$$

$$= O_P\left(\frac{\sqrt{k}}{\sqrt{N}\sqrt[4]{T}}\right),$$

where the equality follows from (1), (2), (3) and (6) of Lemma A.4 in this paper. Similar to (A.16), it is straightforward to obtain that

$$\left\|\left(\mathcal{Z}'M_X\mathcal{Z}\right)^{-1}\mathcal{Z}'M_X\mathcal{E}\right\|^2 \leq \lambda_{min}^{-2}\left(\frac{1}{N\sqrt{T}}\mathcal{Z}'M_X\mathcal{Z}\right)\cdot\left\|\frac{1}{N\sqrt{T}}\mathcal{Z}'M_X\mathcal{E}\right\|^2$$

$$= O_P\left(\frac{k}{N\sqrt{T}}\right). \quad \text{(A.17)}$$

Therefore, the result follows from (A.16) and (A.17) immediately. ∎

Proof of Theorem 2.2:

1) It follows from the orthogonality of the Hermite sequence that

$$\int\left(\widehat{g}(w) - g(w)\right)^2 dw = (\widehat{C} - C)'\int Z_k(w)Z_k(w)'dw(\widehat{C} - C) + \int \gamma_k^2(w)dw$$

$$= \|\widehat{C} - C\|^2 + \int \gamma_k^2(w)dw = O_P\left(\frac{k}{N\sqrt{T}}\right) + O_P\left(k^{-m+1}\right),$$

where Lemmas 2.2 and A.1 are used.

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2) We now focus on the normality. We can write

$$\sqrt{N\sigma_k^{-1}(w)}\sqrt[4]{T}(\widehat{g}(w) - g(w))$$

$$= \sqrt{N\sigma_k^{-1}(w)}\sqrt[4]{T}Z_k(w)'\left(\widehat{C} - C\right) - \sqrt{N\sigma_k^{-1}(w)}\sqrt[4]{T}Z_k(w)'\gamma_k(w)$$

$$= \sqrt{N\sigma_k^{-1}(w)}\sqrt[4]{T}Z_k(w)'(\mathcal{Z}'M_X\mathcal{Z})^{-1}\mathcal{Z}'M_X\mathcal{E}$$

$$\quad + \sqrt{N\sigma_k^{-1}(w)}\sqrt[4]{T}Z_k(w)'(\mathcal{Z}'M_X\mathcal{Z})^{-1}\mathcal{Z}'M_X\gamma - \sqrt{N\sigma_k^{-1}(w)}\sqrt[4]{T}Z_k(w)'\gamma_k(w)$$

$$= \sqrt{N\sigma_k^{-1}(w)}\sqrt[4]{T}Z_k(w)'(\mathcal{Z}'M_X\mathcal{Z})^{-1}\mathcal{Z}'M_X\mathcal{E} + O_P(N^{\frac{1}{2}}T^{\frac{1}{4}}k^{-\frac{m-1}{2}}) + O_P(N^{\frac{1}{2}}T^{\frac{1}{4}}k^{-\frac{m}{2}+\frac{5}{12}})$$

$$= \sqrt{N\sigma_k^{-1}(w)}\sqrt[4]{T}Z_k(w)'\left(\frac{1}{N\sqrt{T}}\mathcal{Z}'M_X\mathcal{Z}\right)^{-1}\frac{1}{N\sqrt{T}}\left(\mathcal{Z}'\mathcal{E} - \mathcal{Z}'X(X'X)^{-1}X'\mathcal{E}\right) + o_P(1)$$

$$= \sqrt{N\sigma_k^{-1}(w)}\sqrt[4]{T}Z_k(w)\left(\left(\frac{1}{N\sqrt{T}}\mathcal{Z}'M_X\mathcal{Z}\right)^{-1} - a_0^{-1}I_k\right)\cdot\frac{1}{N\sqrt{T}}\left(\mathcal{Z}'\mathcal{E} - \mathcal{Z}'X(X'X)^{-1}X'\mathcal{E}\right)$$

$$\quad + \sqrt{N\sigma_k^{-1}(w)}\sqrt[4]{T}Z_k(w)'a_0^{-1}I_k\frac{1}{N\sqrt{T}}\left(\mathcal{Z}'\mathcal{E} - \mathcal{Z}'X(X'X)^{-1}X'\mathcal{E}\right) + o_P(1)$$

$$= \frac{1}{\sqrt{N\sigma_k(w)a_0^2}\sqrt[4]{T}}Z_k(w)'\mathcal{Z}'\mathcal{E} + o_P(1)$$

$$= \frac{1}{\sqrt{N\sigma_k(w)a_0^2}\sqrt[4]{T}}\sum_{i=1}^{N}\sum_{t=1}^{T}Z_k(w)'Z_k(u_{it})e_{it} + o_P(1), \tag{A.18}$$

where the third equality follows from $Z_k(w) = O(\sqrt{k})$, (2) of Lemma A.1 and (A.16); the fourth equality follows from the assumption in the body of this theorem; the sixth equality follows from (2) of Lemma 2.1, (2), (3) and (6) of Lemma A.4 of this paper and Lemma A.3; the last equality follows from the proof for (1) of Lemma A.4.

For notational simplicity, denote $V_{Nk}(t;w) = \frac{1}{\sqrt{N\|Z_k(w)\|^2}}\sum_{i=1}^{N}Z_k(w)'Z_k(u_{it})e_{it}$ and $\tilde{\sigma} = \sqrt{a_0\sigma_e^2}$. We further write

$$\sqrt{N\sigma_k^{-1}(w)}\sqrt[4]{T}(\widehat{g}(w) - g(w)) = \frac{1}{\tilde{\sigma}}\sum_{t=1}^{T}\frac{1}{\sqrt[4]{T}}V_{Nk}(t;w) + o_P(1). \tag{A.19}$$

Notice that $V_{Nk}(t;w)$ is a martingale difference array by Assumption 1. We then use the central limit theorem for martingale difference arrays to show the normality. See Lemma B.1 of Chen et al. (2012) and Corollary 3.1 of Hall and Heyde (1980, p. 58) for reference. Firstly, we verify the conditional Lindeberg condition, i.e., as $(N,T) \to (\infty, \infty)$, for $\forall\epsilon > 0$,

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}E\left[V_{Nk}^2(t;w)I\left(|V_{Nk}(t;w)| \geq \epsilon\sqrt[4]{T}\right)|\mathcal{F}_{Nt-1}\right] = o_P(1). \tag{A.20}$$

To this end, write

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}E\left[V_{Nk}^2(t;w)I\left(|V_{Nk}(t;w)| \geq \epsilon\sqrt[4]{T}\right)|\mathcal{F}_{Nt-1}\right] \leq \frac{1}{\epsilon^2 T}\sum_{t=1}^{T}E\left[V_{Nk}^4(t;w)|\mathcal{F}_{Nt-1}\right]$$

$$\leq \frac{1}{\epsilon^2 T} \sum_{t=1}^{T} \frac{1}{N^2 \|Z_k(w)\|^4} E[|Z_k(w)'Z_k(u_{1t})|^4] \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \sum_{i_3=1}^{N} \sum_{i_4=1}^{N} |E\left[e_{i_1 t}e_{i_2 t}e_{i_3 t}e_{i_4 t}|\mathcal{F}_{Nt-1}\right]|$$

$$\leq \frac{1}{\epsilon^2 T} \sum_{t=1}^{T} E[\|Z_k(u_{1t})\|^4] \cdot \frac{1}{N^2} \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \sum_{i_3=1}^{N} \sum_{i_4=1}^{N} |E\left[e_{i_1 t}e_{i_2 t}e_{i_3 t}e_{i_4 t}|\mathcal{F}_{Nt-1}\right]|$$

$$\leq O_P(1) \frac{1}{\epsilon^2 T} \sum_{t=1}^{T} \frac{1}{d_t} \int \|Z_k(x)\|^4 dx = O_P(1) \frac{k^2}{\epsilon^2 \sqrt{T}} = o_P(1), \tag{A.21}$$

due to the independence of u_{it} and u_{jt} for $i \neq j$, where the first inequality follows from Hölder inequality and Markov inequality; the last line follows from the assumption in the body of this theorem, and $\|Z_k(\cdot)\|^2 = O(k)$, $\int \|Z_k(x)\|^2 dx = k$ as well as the density $f_t(x)$ of $d_t^{-1}u_{1t}$ being bounded uniformly (note that $d_t = |\rho|\sqrt{t}(1 + o(1))$ and see the proof of Lemma 2.1 for more details).

Next, we verify the convergence of the conditional variance of $V_{Nk}(t; w)$. Again, by the independence of u_{it} and u_{jt} for $i \neq j$,

$$\sum_{t=1}^{T} \frac{E[V_{Nk}^2(t; w)|\mathcal{F}_{Nt-1}]}{\sqrt{T}}$$

$$= \frac{1}{\|Z_k(w)\|^2} \frac{1}{N\sqrt{T}} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t=1}^{T} Z_k(w)' E\left[Z_k(u_{it})Z_k(u_{jt})'\right] Z_k(w) \sigma_e(i, j)$$

$$= \frac{1}{\|Z_k(w)\|^2} \frac{1}{N\sqrt{T}} \sum_{i=1}^{N} \sum_{t=1}^{T} Z_k(w)' E\left[Z_k(u_{it})Z_k(u_{it})'\right] Z_k(w) \sigma_e^2$$

$$+ \frac{1}{\|Z_k(w)\|^2} \frac{1}{N\sqrt{T}} \sum_{i \neq j} \sum_{t=1}^{T} Z_k(w)' E\left[Z_k(u_{it})\right] E\left[Z_k(u_{jt})'\right] Z_k(w) \sigma_e(i, j)$$

$$\equiv A_{NT1} + A_{NT2}.$$

By (1) of Lemma 2.1, we have $A_{NT1} \to_P a_0 \sigma_e^2$, and we may show that $A_{NT2} = o_P(1)$. In fact,

$$|A_{NT2}| \leq \frac{1}{N\sqrt{T}} \sum_{i \neq j} \sum_{t=1}^{T} E\left[\|Z_k(u_{it})\|\right] E\left[\|Z_k(u_{jt})\|\right] \cdot |\sigma_e(i, j)|$$

$$= \frac{1}{N} \sum_{i \neq j} |\sigma_e(i, j)| \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left(\int \|Z_k(d_t x)\| f_t(x) dx\right)^2$$

$$\leq O(1) \frac{1}{N} \sum_{i \neq j} |\sigma_e(i, j)| \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \frac{1}{d_t^2} \left(\int \|Z_k(x)\| dx\right)^2$$

$$= O(1) \frac{k^{11/6}}{\sqrt{T}} \sum_{t=1}^{T} \frac{1}{t} \leq O(1) \frac{k^{11/6} \ln(T)}{\sqrt{T}} = o(1),$$

where the first inequality follows from the sub-multiplicativity of Euclidean norm; the second

inequality follows from the uniformly boundedness of $f_t(x)$; the last line follows from (4) of Lemma A.1 and Assumption 1.3.(b).

Therefore, in connection with (A.19), $\sqrt{N\sigma_k^{-1}(w)}\sqrt[4]{T}(\widehat{g}(w) - g(w)) \to_D N(0,1)$. ∎

Appendix B

Appendix B.1: Extra Simulation Results

Firstly, we report the omitted simulations results of the main text. The following QQ-plots (i.e., Figures B.1-B.3) are for the cases with $\rho_e = \rho_v = 0.9$.
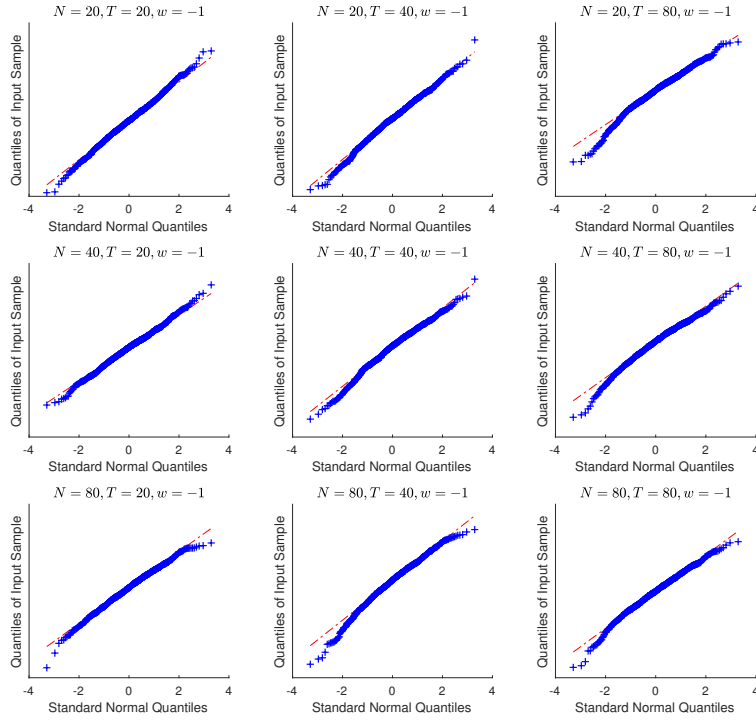

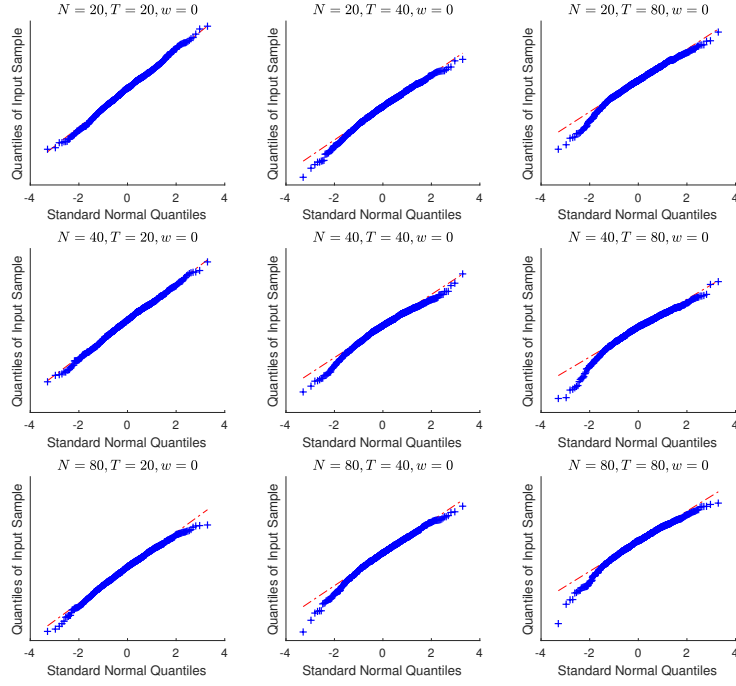
Figure B.1: QQ-plots of $Q_g(w)$ at $w = -1$

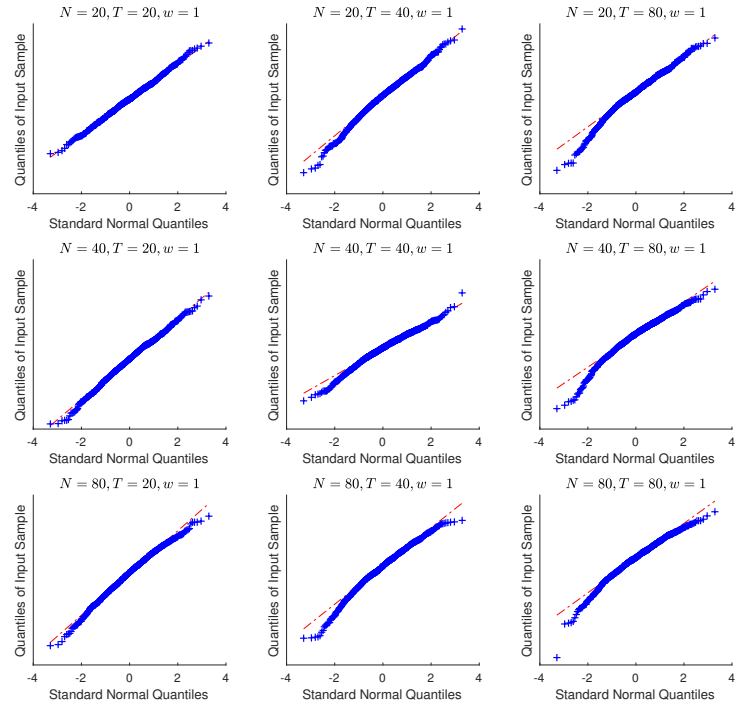Figure B.2: QQ-plots of $Q_g(w)$ at $w = 0$

Figure B.3: QQ-plots of $Q_g(w)$ at $w = 1$

To verify Corollary 2.1, we now implement some extra simulations. The data generating process (DGP) is as follows.

$$y_{it} = x'_{it}\beta_0 + g(u_{it}) + \omega_i + e_{it},$$

$$x_{it} = f(u_{it}) + \lambda_i + v_{it},$$

where $e_{it} \sim i.i.d.$ $N(0,1)$ over i and t. All the other variables are generated in exactly the same way as the main text, and consider the case where $\rho_v = 0.5$ only. To verify Corollary 2.1 and Theorem 2.2, we record the values of

$$Q_\beta = \sqrt{NT} \left(\frac{1}{\widehat{\sigma}_e^2} \widehat{\Sigma}_v \right)^{1/2} (\widehat{\beta} - \beta_0), \tag{B.1}$$

in each replication, and then report the QQ-plots of this quantities against $N(0,1)$ in Figures B.4-B.5 below based on 1000 replications. It is clear that all the QQ-plots strongly suggest that the quantities documented in (B.1) follow $N(0,1)$, which then supports our Corollary 2.1.

After many attempts, we notice that the simulation results are not really sensitive to the choices of $g(\cdot)$ and $\phi(\cdot)$ as long as these functions satisfy the assumptions of the paper. We have also considered, for example, $g(w) = \sqrt{P(w)}$, where $P(w)$ stands for the PDF of the standard Cauchy distribution or the standard normal distribution. The results are quite similar to those presented in the main text and supplementary file of this paper, so we do not repeat them again.
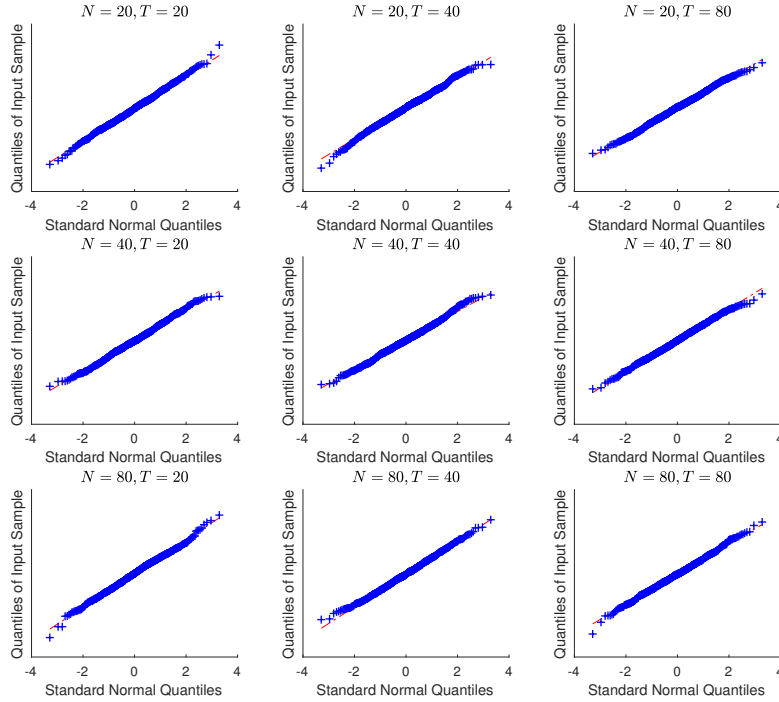

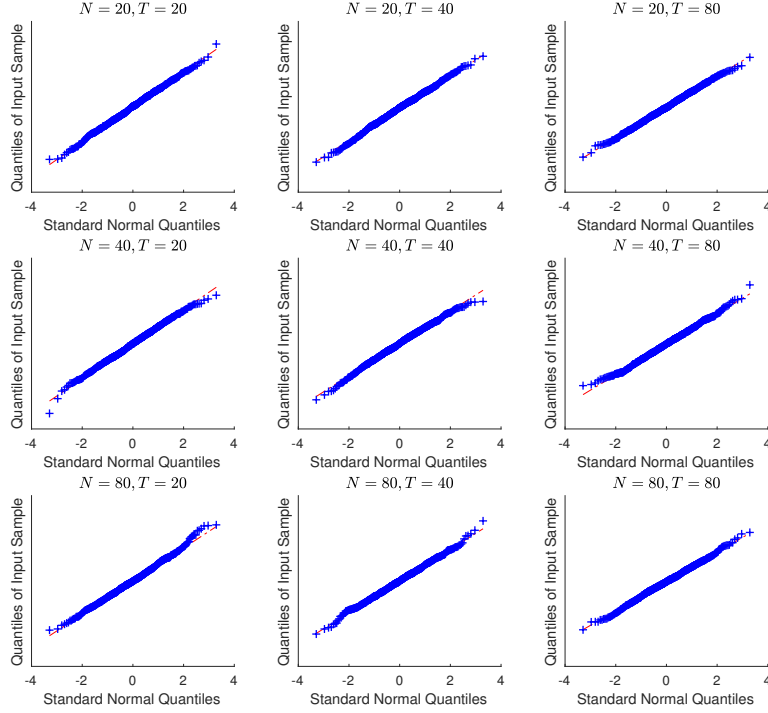Figure B.4: QQ-plots for the first element of Q_β

Figure B.5: QQ-plots for the second element of Q_β

Appendix B.2: Additional Proofs

We start from the proof of Lemma 2.1, which provides some fundamental results and notations used in the rest of this file.

Proof of Lemma 2.1:

It suffices to show that as $(N, T) \to (\infty, \infty)$ jointly,

$$\left\| \frac{1}{N\sqrt{T}} \mathcal{Z}' \mathcal{Z} - \frac{1}{N\sqrt{T}} E[\mathcal{Z}' \mathcal{Z}] \right\| \to_P 0 \quad \text{and} \quad \frac{1}{N\sqrt{T}} E[\mathcal{Z}' \mathcal{Z}] = a_0 I_k.$$

Notice that

$$\frac{1}{N\sqrt{T}} \mathcal{Z}' \mathcal{Z} = \frac{1}{N\sqrt{T}} \sum_{i=1}^{N} \sum_{t=1}^{T} Z_k(u_{it}) Z_k(u_{it})' - \frac{\sqrt{T}}{N} \sum_{i=1}^{N} \bar{Z}_{k,i} \bar{Z}'_{k,i}$$

$$= \frac{1}{N\sqrt{T}} \sum_{i=1}^{N} \sum_{t=1}^{T} Z_k(u_{it}) Z_k(u_{it})' - \frac{1}{NT^{3/2}} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} Z_k(u_{it}) Z_k(u_{is})' \equiv A_{NT} - B_{NT}. \text{(B.2)}$$

Stage One. Calculate the expectation. Note that $\{u_{it}\}$ is i.i.d sequence across i. Therefore, the distribution of u_{it} does not depend on i. Let $d_t = (E[u_{it}^2])^{1/2} = |\rho|\sqrt{t}(1 + o(1))$, where $\rho \neq 0$ is given in Assumption 1. Hence, $d_t^{-1} u_{it}$ has a density $f_t(x)$, which is uniformly bounded over

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x and large t. Meanwhile, as $t \to \infty$, $\max_x |f_t(x) - \varphi(x)| \le Cd_t^{-1}$ for some $C > 0$, where $\varphi(x)$ is the density of a standard normal variable (see Dong and Gao (2017) for more details on the properties of $f_t(x)$). Let $\nu = \nu(T)$ be a function of T such that $\nu \to \infty$ and $k\nu/\sqrt{T} \to 0$ as $T \to \infty$.

$$
\begin{aligned}
E[A_{NT}] =& \frac{1}{N\sqrt{T}} \sum_{i=1}^{N} \sum_{t=1}^{T} E[Z_k(u_{it})Z_k(u_{it})'] \\
=& \frac{1}{N\sqrt{T}} \sum_{i=1}^{N} \sum_{t=1}^{\nu} E[Z_k(u_{it})Z_k(u_{it})'] + \frac{1}{N\sqrt{T}} \sum_{i=1}^{N} \sum_{t=\nu+1}^{T} E[Z_k(u_{it})Z_k(u_{it})'] \\
=& \frac{1}{N\sqrt{T}} \sum_{i=1}^{N} \sum_{t=1}^{\nu} E[Z_k(u_{it})Z_k(u_{it})'] + \frac{1}{N\sqrt{T}} \sum_{i=1}^{N} \sum_{t=\nu+1}^{T} d_t^{-1} \int Z_k(x)Z_k(x)' f_t(d_t^{-1}x)dx \\
=& \frac{1}{\sqrt{T}} \sum_{t=1}^{\nu} E[Z_k(u_{1t})Z_k(u_{1t})'] + \frac{1}{\sqrt{T}} \sum_{t=\nu+1}^{T} d_t^{-1} \int Z_k(x)Z_k(x)' f_t(d_t^{-1}x)dx \\
=& A_{NT,1} + A_{NT,2}.
\end{aligned}
$$

By the construction, it is easy to obtain that for $A_{NT,1}$

$$
\frac{1}{\sqrt{T}} \left\| \sum_{t=1}^{\nu} E[Z_k(u_{1t})Z_k(u_{1t})'] \right\| \le \frac{1}{\sqrt{T}} \sum_{t=1}^{\nu} E[\|Z_k(u_{1t})\|^2] = O(1)\frac{\nu k}{\sqrt{T}} \to 0,
$$

where the equality follows from (6) of Lemma A.1. We then consider $A_{NT,2}$

$$
\begin{aligned}
A_{NT,2} =& \frac{1}{\sqrt{T}} \sum_{t=\nu+1}^{T} d_t^{-1} \int Z_k(x)Z_k(x)' f_t(d_t^{-1}x)dx \\
=& \frac{1}{\sqrt{T}} \sum_{t=\nu+1}^{T} d_t^{-1} \int Z_k(x)Z_k(x)' \left(f_t(d_t^{-1}x) - \varphi(d_t^{-1}x) \right) dx \\
& + \frac{1}{\sqrt{T}} \sum_{t=\nu+1}^{T} d_t^{-1} \int Z_k(x)Z_k(x)' \varphi(d_t^{-1}x)dx \\
=& o(1) + \varphi(0)\frac{1}{\sqrt{T}} \sum_{t=\nu+1}^{T} d_t^{-1} \int Z_k(x)Z_k(x)' dx \\
& + \frac{1}{\sqrt{T}} \sum_{t=\nu+1}^{T} d_t^{-1} \int Z_k(x)Z_k(x)' \left(\varphi(d_t^{-1}x) - \varphi(0) \right) dx \\
=& o(1) + 2\varphi(0)/|\rho|(1 + o(1)) \cdot I_k + \frac{1}{\sqrt{T}} \sum_{t=\nu+1}^{T} d_t^{-1} \int_{|x|<\varepsilon d_t} Z_k(x)Z_k(x)' \left(\varphi(d_t^{-1}x) - \varphi(0) \right) dx \\
& + \frac{1}{\sqrt{T}} \sum_{t=\nu+1}^{T} d_t^{-1} \int_{|x|\ge\varepsilon d_t} Z_k(x)Z_k(x)' \left(\varphi(d_t^{-1}x) - \varphi(0) \right) dx,
\end{aligned}
$$

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where $\varepsilon > 0$ can be as small as we wish; and the second equality follows from

$$\frac{1}{\sqrt{T}} \left\| \sum_{t=\nu+1}^{T} d_t^{-1} \int Z_k(x) Z_k(x)' \left(f_t(d_t^{-1}x) - \varphi(d_t^{-1}x) \right) dx \right\|$$

$$\leq O(1) \frac{1}{\sqrt{T}} \sum_{t=\nu+1}^{T} d_t^{-2} \int \|Z_k(x)\|^2 dx = O(1) \frac{k \ln T}{\sqrt{T}} = o(1).$$

Notice also that

$$\frac{1}{\sqrt{T}} \left\| \sum_{t=\nu+1}^{T} d_t^{-1} \int_{|x|<\varepsilon d_t} Z_k(x) Z_k(x)' \left(\varphi(d_t^{-1}x) - \varphi(0) \right) dx \right\|$$

$$\leq \frac{1}{\sqrt{T}} \sum_{t=\nu+1}^{T} d_t^{-1} \int_{|x|<\varepsilon d_t} \|Z_k(x) Z_k(x)'\| \cdot |\varphi(d_t^{-1}x) - \varphi(0)| dx$$

$$\leq O(1) \frac{1}{\sqrt{T}} \sum_{t=\nu+1}^{T} d_t^{-2} \int_{|x|<\varepsilon d_t} \|Z_k(x) Z_k(x)'\| \cdot |x| dx$$

$$\leq O(1) \frac{\ln(T)}{\sqrt{T}} \left(\int |x|^2 \|Z_k(x)\|^2 dx \int \|Z_k(x)\|^2 dx \right)^{1/2}$$

$$= O(1) \frac{\ln(T)}{\sqrt{T}} \left(k^2 \cdot k \right)^{1/2} = O(1) \frac{k^{3/2} \ln(T)}{\sqrt{T}}, \tag{B.3}$$

where the last line follows from (5) and (9) of Lemma A.1. Moreover,

$$\frac{1}{\sqrt{T}} \left\| \sum_{t=\nu+1}^{T} d_t^{-1} \int_{|x|\geq\varepsilon d_t} Z_k(x) Z_k(x)' \left(\varphi(d_t^{-1}x) - \varphi(0) \right) dx \right\|$$

$$\leq O(1) \frac{1}{\sqrt{T}} \sum_{t=\nu+1}^{T} d_t^{-1} \int_{|x|\geq\varepsilon d_t} \|Z_k(x) Z_k(x)'\| dx$$

$$\leq O(1) \frac{1}{\sqrt{T}} \sum_{t=\nu+1}^{T} \varepsilon^{-1} d_t^{-2} \int_{|x|\geq\varepsilon d_t} \|Z_k(x) Z_k(x)'\| \cdot |x| dx$$

$$\leq O(1) \varepsilon^{-1} \frac{\ln(T)}{\sqrt{T}} \left(\int |x|^2 \|Z_k(x)\|^2 dx \int \|Z_k(x)\|^2 dx \right)^{1/2} = O(1) \frac{k^{3/2} \ln(T)}{\sqrt{T}}. \tag{B.4}$$

In view of Assumption 2, (B.3) and (B.4), we obtain that $E[A_{NT}] = 2\varphi(0)/|\rho| \cdot I_k(1 + o(1))$.

Next, we will show that $E[B_{NT}] = o(1)$. For $t > s$ and $t - s$ is large, note that, without loss of generality letting $u_{i0} = 0$ a.s.

$$u_{it} = \sum_{\ell=1}^{t} \eta_{i\ell} = \sum_{\ell=1}^{t} \sum_{j=-\infty}^{\ell} \rho_{t-j} \epsilon_{ij} = \sum_{j=-\infty}^{t} b_{t,j} \epsilon_{ij}$$

$$= \sum_{j=s+1}^{t} b_{t,j} \epsilon_{ij} + \sum_{j=-\infty}^{s} b_{t,j} \epsilon_{ij} := u_{i,ts} + u_{i,ts}^*,$$

where $b_{t,j} = \sum_{\ell=\max(1,j)}^{t} \rho_{\ell-j}$.

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Similar to the proof of Lemma A.4 of Dong et al. (2016), $\frac{1}{d_{ts}}u_{i,ts}$ has uniformly bounded densities $f_{ts}(w)$ over all t and s, where $d_{ts} = O(1)\sqrt{t-s}$. Without loss of generality, in what follows we abuse the density by neglecting the argument on $\nu = \nu(T)$ as we did before. Let $\mathcal{R}_{is} = \sigma(\ldots, \varepsilon_{i,s-1}, \varepsilon_{is})$ be the sigma field generated by $\varepsilon_{ij}, j \le s$. Then,

$$
\begin{aligned}
E[B_{NT}] =& \frac{1}{NT^{3/2}} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} E\left[Z_k(u_{it})Z_k(u_{is})'\right] \\
=& \frac{1}{NT^{3/2}} \sum_{i=1}^{N} \sum_{t=1}^{T} E\left[Z_k(u_{it})Z_k(u_{it})'\right] + \frac{2}{NT^{3/2}} \sum_{i=1}^{N} \sum_{t=2}^{T} \sum_{s=1}^{t-1} E\left[E[Z_k(u_{it})Z_k(u_{is})'|\mathcal{R}_{is}]\right] \\
=& \frac{1}{NT^{3/2}} \sum_{i=1}^{N} \sum_{t=1}^{T} \int Z_k(d_t w)Z_k(d_t w)' f_t(w) dw \\
& + \frac{2}{NT^{3/2}} \sum_{i=1}^{N} \sum_{t=2}^{T} \sum_{s=1}^{t-1} E \int Z_k(d_{ts} w_1 + u_{i,ts}^*)Z_k(u_{is})' f_{ts}(w_1) dw_1 \\
=& \frac{1}{T^{3/2}} \sum_{t=1}^{T} \frac{1}{d_t} \int Z_k(w)Z_k(w)' f_t(w/d_t) dw \\
& + \frac{2}{T^{3/2}} \sum_{t=2}^{T} \sum_{s=1}^{t-1} \frac{1}{d_{ts}} E \int Z_k(w_1)Z_k(u_{is})' f_{ts}\left(\frac{w_1 - u_{i,ts}^*}{d_{ts}}\right) dw_1.
\end{aligned}
$$

The first term is confined by

$$
\frac{1}{T^{3/2}} \sum_{t=1}^{T} \frac{1}{d_t} \int \|Z_k(w)Z_k(w)'\| f_t(w/d_t) dw \le O(1)\frac{1}{T^{3/2}} \sum_{t=1}^{T} \frac{1}{d_t} \int \|Z_k(w)\|^2 dw = O(1)\frac{k}{T},
$$

while the second term is bounded by

$$
\begin{aligned}
& \frac{2}{T^{3/2}} \sum_{t=2}^{T} \sum_{s=1}^{t-1} \frac{1}{d_{ts}} E \int \|Z_k(w_1)Z_k(u_{is})'\| f_{ts}\left(\frac{w_1 - u_{i,ts}^*}{d_{ts}}\right) dw_1 \\
\le& O(1)\frac{1}{T^{3/2}} \sum_{t=2}^{T} \sum_{s=1}^{t-1} \frac{1}{d_{ts}} \int \|Z_k(w_1)\| dw_1 E\|Z_k(u_{is})\| \\
\le& O(1)\frac{1}{T^{3/2}} \sum_{t=2}^{T} \sum_{s=1}^{t-1} \frac{1}{d_{ts}}\frac{1}{d_s} \left(\int \|Z_k(w_1)\| dw_1\right)^2 = O(1)\frac{k^{11/6}}{T^{1/2}} = o(1),
\end{aligned}
$$

where the last equality follows from (4) of Lemma A.1. The calculation yields $\frac{1}{N\sqrt{T}}E[\mathcal{Z}'\mathcal{Z}] = a_0 I_k(1 + o(1))$.

Stage Two. We shall show that as $(N, T) \to (\infty, \infty)$ jointly

$$
E\left\|\frac{1}{N\sqrt{T}}\mathcal{Z}'\mathcal{Z} - \frac{1}{N\sqrt{T}}E[\mathcal{Z}'\mathcal{Z}]\right\|^2 \to 0.
$$

To do so, $N \to \infty$ and u_{it} being independent with respect to (w.r.t.) i are important. By (B.2)

again,

$$E \left\| \frac{1}{N\sqrt{T}} \mathcal{Z}' \mathcal{Z} - \frac{1}{N\sqrt{T}} E[\mathcal{Z}' \mathcal{Z}] \right\|^2 \tag{B.5}$$

$$\leq \frac{2}{N^2 T} E \left\| \sum_{i=1}^{N} \sum_{t=1}^{T} \{Z_k(u_{it}) Z_k(u_{it})' - E[Z_k(u_{it}) Z_k(u_{it})']\} \right\|^2$$

$$+ \frac{2}{N^2 T^3} E \left\| \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} \{Z_k(u_{it}) Z_k(u_{is})' - E[Z_k(u_{it}) Z_k(u_{is})']\} \right\|^2 \equiv \bar{A}_{NT} + \bar{B}_{NT}.$$

We now consider $\bar{A}_{NT}$ and $\bar{B}_{NT}$ respectively.

$$\bar{A}_{NT} = \frac{2}{N^2 T} E \left\| \sum_{i=1}^{N} \sum_{t=1}^{T} \{Z_k(u_{it}) Z_k(u_{it})' - E[Z_k(u_{it}) Z_k(u_{it})']\} \right\|^2$$

$$= \frac{2}{N^2 T} \sum_{i=1}^{N} E \left\| \sum_{t=1}^{T} \{Z_k(u_{it}) Z_k(u_{it})' - E[Z_k(u_{it}) Z_k(u_{it})']\} \right\|^2$$

$$\leq \frac{2}{N^2 T} \sum_{i=1}^{N} E \left\| \sum_{t=1}^{T} Z_k(u_{it}) Z_k(u_{it})' \right\|^2$$

$$= \frac{2}{N^2 T} \sum_{n=0}^{k-1} \sum_{m=0}^{k-1} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} E\left[\mathscr{H}_n(u_{it}) \mathscr{H}_m(u_{it}) \mathscr{H}_n(u_{is}) \mathscr{H}_m(u_{is})\right]$$

$$= \frac{2}{N^2 T} \sum_{n=0}^{k-1} \sum_{m=0}^{k-1} \sum_{i=1}^{N} \sum_{t=1}^{T} E\left[\mathscr{H}_n^2(u_{it}) \mathscr{H}_m^2(u_{it})\right]$$

$$+ \frac{4}{N^2 T} \sum_{n=0}^{k-1} \sum_{m=0}^{k-1} \sum_{i=1}^{N} \sum_{t=2}^{T} \sum_{s=1}^{t-1} E\left[\mathscr{H}_n(u_{it}) \mathscr{H}_m(u_{it}) \mathscr{H}_n(u_{is}) \mathscr{H}_m(u_{is})\right]$$

$$\equiv \bar{A}_{NT,1} + \bar{A}_{NT,2}.$$

For $\bar{A}_{NT,1}$, write

$$\bar{A}_{NT,1} = \frac{2}{NT} \sum_{n=0}^{k-1} \sum_{m=0}^{k-1} \sum_{t=1}^{T} \int \mathscr{H}_n^2(d_t w) \mathscr{H}_m^2(d_t w) f_t(w) dw$$

$$\leq O(1) \frac{1}{NT} \sum_{n=0}^{k-1} \sum_{m=0}^{k-1} \sum_{t=1}^{T} \frac{1}{d_t} \int \mathscr{H}_n^2(w) f_t(w/d_t) dw$$

$$\leq O(1) \frac{1}{NT} \sum_{n=0}^{k-1} \sum_{m=0}^{k-1} \sum_{t=1}^{T} \frac{1}{d_t} \int \mathscr{H}_n^2(w) dw = O(1) \frac{k^2}{N\sqrt{T}},$$

where the first inequality follows from $\mathscr{H}_n(w)$ being bounded uniformly.

For $\bar{A}_{NT,2}$, write

$$\bar{A}_{NT,2} = \frac{4}{N^2 T} \sum_{n=0}^{k-1} \sum_{i=1}^{N} \sum_{t=2}^{T} \sum_{s=1}^{t-1} E\left[\mathscr{H}_n^2(u_{it}) \mathscr{H}_n^2(u_{is})\right]$$

$$+\frac{8}{N^2T}\sum_{n=1}^{k-1}\sum_{m=0}^{n-1}\sum_{i=1}^{N}\sum_{t=2}^{T}\sum_{s=1}^{t-1}E\left[\mathscr{H}_n(u_{it})\mathscr{H}_m(u_{it})\mathscr{H}_n(u_{is})\mathscr{H}_m(u_{is})\right]\equiv\bar{A}_{NT,21}+\bar{A}_{NT,22}.$$

For $\bar{A}_{NT,21}$, using the conditioning argument again we have

$$\bar{A}_{NT,21}\le O(1)\frac{1}{NT}\sum_{n=0}^{k-1}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\frac{1}{d_{ts}}\frac{1}{d_s}\iint\mathscr{H}_n^2(w_1)\mathscr{H}_n^2(w_2)dw_1dw_2=O(1)\frac{k}{N}.$$

For $\bar{A}_{NT,22}$, we use the decomposition $u_{it}=u_{i,ts}+u_{i,ts}^*$ again. Note that for $1\le i\le N$ and $s<t$, $u_{i,ts}$ includes all the information between time periods $s+1$ and t and $u_{i,ts}^*$ includes all the information up to time period s. As Dong and Gao (2017) show, $\frac{1}{d_{ts}}u_{i,ts}$ has a density $f_{ts}(w)$, which is uniformly bounded on $\mathbb{R}$ and satisfies uniform Lipschitz condition on $\mathbb{R}$, i.e., $\sup_w|f_{ts}(w+v)-f_{ts}(w)|\le C|v|$ for some absolutely constant C. Then we can write

$$\bar{A}_{NT,22}=O(1)\frac{1}{NT}\sum_{n=1}^{k-1}\sum_{m=0}^{n-1}\sum_{t=2}^{T}\sum_{s=1}^{t-1}E\left[\mathscr{H}_n(u_{it})\mathscr{H}_m(u_{it})\mathscr{H}_n(u_{is})\mathscr{H}_m(u_{is})\right]$$

$$=O(1)\frac{1}{NT}\sum_{n=1}^{k-1}\sum_{m=0}^{n-1}\sum_{t=2}^{T}\sum_{s=1}^{t-1}E\left[E\left[\mathscr{H}_n(u_{i,ts}+u_{i,ts}^*)\mathscr{H}_m(u_{i,ts}+u_{i,ts}^*)|\mathcal{R}_{is}\right]\mathscr{H}_n(u_{is})\mathscr{H}_m(u_{is})\right]$$

$$=O(1)\frac{1}{NT}\sum_{n=1}^{k-1}\sum_{m=0}^{n-1}\sum_{t=2}^{T}\sum_{s=1}^{t-1}E\left[\int\mathscr{H}_n(d_{ts}w+u_{is}^*)\mathscr{H}_m(d_{ts}w+u_{is}^*)f_{ts}(w)dw\cdot\mathscr{H}_n(u_{is})\mathscr{H}_m(u_{is})\right]$$

$$=O(1)\frac{1}{NT}\sum_{n=1}^{k-1}\sum_{m=0}^{n-1}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\frac{1}{d_{ts}}E\left[\int\mathscr{H}_n(w)\mathscr{H}_m(w)f_{ts}\left(\frac{w-u_{is}^*}{d_{ts}}\right)dw\cdot\mathscr{H}_n(u_{is})\mathscr{H}_m(u_{is})\right]$$

$$=O(1)\frac{1}{NT}\sum_{n=1}^{k-1}\sum_{m=0}^{n-1}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\frac{1}{d_{ts}}$$
$$\cdot E\left[\int\mathscr{H}_n(w)\mathscr{H}_m(w)\left[f_{ts}\left(\frac{w-u_{is}^*}{d_{ts}}\right)-f_{ts}\left(\frac{-u_{is}^*}{d_{ts}}\right)\right]dw\cdot\mathscr{H}_n(u_{is})\mathscr{H}_m(u_{is})\right],$$

where the last line follows from the fact that $\int\mathscr{H}_n(w)\mathscr{H}_m(w)dw=0$ for $m\neq n$. By the uniform Lipschitz condition of f_{ts}, we then obtain that

$$|\bar{A}_{NT,22}|\le O(1)\frac{1}{NT}\sum_{n=1}^{k-1}\sum_{m=0}^{n-1}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\frac{1}{d_{ts}^2}\int|w\mathscr{H}_n(w)\mathscr{H}_m(w)|dw\cdot E\left[|\mathscr{H}_n(u_{is})\mathscr{H}_m(u_{is})|\right]$$

$$=O(1)\frac{1}{NT}\sum_{n=1}^{k-1}\sum_{m=0}^{n-1}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\frac{1}{d_{ts}^2}\int|w\mathscr{H}_n(w)\mathscr{H}_m(w)|dw\cdot\int|\mathscr{H}_n(d_sw)\mathscr{H}_m(d_sw)|f_s(w)dw$$

$$\le O(1)\frac{1}{NT}\sum_{n=1}^{k-1}\sum_{m=0}^{n-1}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\frac{1}{d_{ts}^2}\frac{1}{d_s}\int|w\mathscr{H}_n(w)\mathscr{H}_m(w)|dw\cdot\int|\mathscr{H}_n(w)\mathscr{H}_m(w)|dw$$

$$\le O(1)\frac{1}{NT}\sum_{n=1}^{k-1}\sum_{m=0}^{n-1}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\frac{1}{d_{ts}^2}\frac{1}{d_s}\left\{\int\mathscr{H}_n^2(w)dw\int w^2\mathscr{H}_m^2(w)dw\right\}^{1/2}$$

$$\cdot\left\{\int\mathscr{H}_n^2(w)dw\int\mathscr{H}_m^2(w)dw\right\}^{1/2}$$

$$\leq O(1)\frac{1}{NT}\sum_{n=1}^{k-1}\sum_{m=0}^{n-1}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\frac{1}{d_{ts}^2}\frac{1}{d_s}\sqrt{m} = O\left(\frac{k^{5/2}\ln T}{N\sqrt{T}}\right) = o(1).$$

By the calculation of $\bar{A}_{NT,1}$ and $\bar{A}_{NT,2}$, we have shown that $\bar{A}_{NT} = o(1)$.

For $\bar{B}_{NT}$, by the independence across i of $\{u_{i1},\ldots,u_{iT}\}$, write

$$\bar{B}_{NT} = \frac{2}{N^2T^3}\sum_{i=1}^{N} E\left\|\sum_{t=1}^{T}\sum_{s=1}^{T}\{Z_k(u_{it})Z_k(u_{is})' - E[Z_k(u_{it})Z_k(u_{is})']\}\right\|^2$$

$$\leq \frac{O(1)}{N^2T^3}\sum_{i=1}^{N} E\left\|\sum_{t=1}^{T}\sum_{s=1}^{T}Z_k(u_{it})Z_k(u_{is})'\right\|^2 \leq \frac{O(1)}{N^2T^3}\sum_{i=1}^{N} E\left[\sum_{t=1}^{T}\sum_{s=1}^{T}\|Z_k(u_{it})\|\|Z_k(u_{is})\|\right]^2$$

$$= O(1)\frac{1}{N^2T^3}\sum_{i=1}^{N} E\left[\sum_{t_1=1}^{T}\sum_{t_2=1}^{T}\sum_{t_3=1}^{T}\sum_{t_4=1}^{T}\|Z_k(u_{it_1})\|\|Z_k(u_{it_2})\|\|Z_k(u_{it_3})\|\|Z_k(u_{it_4})\|\right]$$

$$= O(1)\frac{1}{N^2T^3}\sum_{i=1}^{N} E\left[\sum_{\text{all }t_1,t_2,t_3,t_4\text{different}}\|Z_k(u_{it_1})\|\|Z_k(u_{it_2})\|\|Z_k(u_{it_3})\|\|Z_k(u_{it_4})\|\right]$$

$$+O(1)\frac{1}{N^2T^3}\sum_{i=1}^{N} E\left[\sum_{\text{two of }t_1,t_2,t_3,t_4\text{same}}\|Z_k(u_{it_1})\|\|Z_k(u_{it_2})\|\|Z_k(u_{it_3})\|\|Z_k(u_{it_4})\|\right]$$

$$+O(1)\frac{1}{N^2T^3}\sum_{i=1}^{N} E\left[\sum_{\text{three of }t_1,t_2,t_3,t_4\text{same}}\|Z_k(u_{it_1})\|\|Z_k(u_{it_2})\|\|Z_k(u_{it_3})\|\|Z_k(u_{it_4})\|\right]$$

$$+O(1)\frac{1}{N^2T^3}\sum_{i=1}^{N} E\left[\sum_{t=1}^{T}\|Z_k(u_{it})\|^4\right] \equiv \bar{B}_{NT,1} + \bar{B}_{NT,2} + \bar{B}_{NT,3} + \bar{B}_{NT,4}.$$

For $\bar{B}_{NT,1}$, without loss of generality, assume that $t_1 > t_2 > t_3 > t_4$. Then, by the conditioning argument,

$$\bar{B}_{NT,1} = \frac{O(1)}{N^2T^3}\sum_{i=1}^{N}\sum_{t_1=4}^{T}\sum_{t_2=3}^{t_1-1}\sum_{t_3=2}^{t_2-1}\sum_{t_4=1}^{t_3-1} E[\|Z_k(u_{it_1})\|\|Z_k(u_{it_2})\|\|Z_k(u_{it_3})\|\|Z_k(u_{it_4})\|]$$

$$\leq \frac{O(1)}{N^2T^3}\sum_{i=1}^{N}\sum_{t_1=4}^{T}\sum_{t_2=3}^{t_1-1}\sum_{t_3=2}^{t_2-1}\sum_{t_4=1}^{t_3-1}\frac{1}{d_{t_1t_2}}\frac{1}{d_{t_2t_3}}\frac{1}{d_{t_3t_4}}\frac{1}{d_{t_4}}$$

$$\cdot\iiiint\|Z_k(w_1)\|\|Z_k(w_2)\|\|Z_k(w_3)\|\|Z_k(w_4)\|dw_1dw_2dw_3dw_4$$

$$= \frac{O(1)}{NT}\left(\int\|Z_k(w)\|dw\right)^4 = O\left(\frac{k^{11/3}}{NT}\right) = o(1),$$

where the last line follows from (4) of Lemma A.1 and Assumption 2.2.

For $\bar{B}_{NT,2}$, without loss of generality, assume that $t_1 = t_2 > t_3 > t_4$. Then write

$$\bar{B}_{NT,2} = \frac{O(1)}{N^2T^3}\sum_{i=1}^{N}\sum_{t_1=3}^{T}\sum_{t_3=2}^{t_1-1}\sum_{t_4=1}^{t_3-1} E[\|Z_k(u_{it_1})\|^2\|Z_k(u_{it_3})\|\|Z_k(u_{it_4})\|]$$

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$$\leq \frac{O(1)}{N^2 T^3} \sum_{i=1}^{N} \sum_{t_1=3}^{T} \sum_{t_3=2}^{t_1-1} \sum_{t_4=1}^{t_3-1} \frac{1}{d_{t_1 t_3}} \frac{1}{d_{t_3 d_4}} \frac{1}{d_{t_4}}$$

$$\cdot \iiint \|Z_k(w_1)\|^2 \|Z_k(w_2)\| \|Z_k(w_3)\| dw_1 dw_2 dw_3$$

$$\leq \frac{O(1)}{N T^{3/2}} \int \|Z_k(w)\|^2 dw \left(\int \|Z_k(w)\| dw \right)^2 = O\left(\frac{k^{17/6}}{N T^{3/2}} \right) = o(1),$$

where the last line follows from (4)-(5) of Lemma A.1 and Assumption 2.2.

For $\bar{B}_{NT,3}$, without loss of generality, assume that $t_1 = t_2 = t_3 > t_4$. Then write

$$\bar{B}_{NT,3} = \frac{O(1)}{N^2 T^3} \sum_{i=1}^{N} \sum_{t_1=2}^{T} \sum_{t_4=1}^{t_1-1} E[\|Z_k(u_{it_1})\|^3 \|Z_k(u_{it_4})\|]$$

$$\leq \frac{O(1)}{N^2 T^3} \sum_{i=1}^{N} \sum_{t_1=2}^{T} \sum_{t_4=1}^{t_1-1} \frac{1}{d_{t_1 d_4}} \frac{1}{d_{t_4}} \iint \|Z_k(w_1)\|^3 \|Z_k(w_2)\| dw_1 dw_2$$

$$\leq O\left(\frac{k}{N T^2} \right) \left(\int \|Z_k(w)\| dw \right)^2 = O\left(\frac{k^{17/6}}{N T^2} \right) = o(1),$$

where the last line follows from (4) and (6) of Lemma A.1 and Assumption 2.2.

For $\bar{B}_{NT,4}$, write

$$\bar{B}_{NT,4} = \frac{1}{N^2 T^3} \sum_{i=1}^{N} \sum_{t=1}^{T} E\|Z_k(w)\|^4 = \frac{1}{N^2 T^3} \sum_{i=1}^{N} \sum_{t=1}^{T} \int \|Z_k(d_t w)\|^4 f_t(w) dw$$

$$\leq O\left(\frac{1}{N^2 T^3} \right) \sum_{i=1}^{N} \sum_{t=1}^{T} \frac{1}{d_t} \int \|Z_k(w)\|^4 dw \leq O\left(\frac{k}{N T^{5/2}} \right) \int \|Z_k(w)\|^2 dw = O\left(\frac{k^2}{N T^{5/2}} \right) = o(1).$$

Combining $\bar{B}_{NT,1}$, $\bar{B}_{NT,2}$, $\bar{B}_{NT,3}$ and $\bar{B}_{NT,4}$ together, we know that $\bar{B}_{NT} = o(1)$.

Therefore, we have shown that $\left\| \frac{1}{N\sqrt{T}} \mathcal{Z}' \mathcal{Z} - \sqrt{\frac{2}{\pi |\rho|^2}} I_k \right\| = o_P(1)$. We now complete the proof of the first result of this lemma. ∎

Proof of Lemma A.2:

1) Write

$$E \left\| \frac{1}{N\sqrt{T}} \sum_{i=1}^{N} \sum_{t=1}^{T} Z_k(u_{it}) \gamma_k(u_{it}) \right\|^2$$

$$= \frac{1}{N^2 T} E \left[\sum_{i=1}^{N} \left(\sum_{t=1}^{T} \|Z_k(u_{it})\|^2 \gamma_k^2(u_{it}) + 2 \sum_{t=2}^{T} \sum_{s=1}^{t-1} Z_k(u_{it})' Z_k(u_{is}) \gamma_k(u_{it}) \gamma_k(u_{is}) \right) \right]$$

$$+ \frac{2}{N^2 T} E \left[\sum_{i=2}^{N} \sum_{j=1}^{i-1} \sum_{t=1}^{T} Z_k(u_{it})' Z_k(u_{jt}) \gamma_k(u_{it}) \gamma_k(u_{jt}) \right]$$

$$+ \frac{4}{N^2 T} E \left[\sum_{i=2}^{N} \sum_{j=1}^{i-1} \sum_{t=2}^{T} \sum_{s=1}^{t-1} Z_k(u_{it})' Z_k(u_{js}) \gamma_k(u_{it}) \gamma_k(u_{js}) \right] \equiv A_1 + 2A_2 + 4A_3.$$

Notice that

$$A_1 = \frac{1}{N^2T} \sum_{i=1}^{N} \sum_{t=1}^{T} E\left[\|Z_k(u_{it})\|^2 \gamma_k^2(u_{it})\right]$$
$$+ \frac{2}{N^2T} \sum_{i=1}^{N} \sum_{t=2}^{T} \sum_{s=1}^{t-1} E\left[Z_k(u_{it})'Z_k(u_{is})\gamma_k(u_{it})\gamma_k(u_{is})\right]. \tag{B.6}$$

The first term on RHS of (B.6) can be written as

$$\frac{1}{N^2T} \sum_{i=1}^{N} \sum_{t=1}^{T} E[\|Z_k(u_{it})\|^2 \gamma_k^2(u_{it})] \leq O\left(\frac{k^{-m+5/6}}{N^2T}\right) \sum_{i=1}^{N} \sum_{t=1}^{T} \int \|Z_k(d_t w)\|^2 f_t(w)dw$$

$$= O\left(\frac{k^{-m+5/6}}{N^2T}\right) \sum_{i=1}^{N} \sum_{t=1}^{T} \int \frac{1}{d_t} \|Z_k(w)\|^2 f_t(w/d_t)dw$$

$$\leq O\left(\frac{k^{-m+5/6}}{N^2T}\right) \sum_{i=1}^{N} \sum_{t=1}^{T} \frac{1}{d_t} \int \|Z_k(w)\|^2 dw \leq O\left(\frac{k^{-m+11/6}}{N\sqrt{T}}\right),$$

where the first inequality follows from (2) of Lemma A.1 and the second inequality follows from $f_t(w)$ being bounded uniformly.

For the second term on RHS of (B.6),

$$\left| \frac{1}{N^2T} \sum_{i=1}^{N} \sum_{t=2}^{T} \sum_{s=1}^{t-1} E[Z_k(u_{it})'Z_k(u_{is})\gamma_k(u_{it})\gamma_k(u_{is})] \right|$$

$$\leq \frac{1}{N^2T} \sum_{i=1}^{N} \sum_{t=2}^{T} \sum_{s=1}^{t-1} E[\|Z_k(u_{it})\|\|Z_k(u_{is})\||\gamma_k(u_{it})||\gamma_k(u_{is})|]$$

$$\leq O\left(\frac{1}{N^2T}\right) \sum_{i=1}^{N} \sum_{t=2}^{T} \sum_{s=1}^{t-1} \iint \frac{1}{d_{ts}} \frac{1}{d_s} \|Z_k(w_1)\|\|Z_k(w_2)\||\gamma_k(w_1)||\gamma_k(w_2)|dw_1 dw_2$$

$$\leq O\left(\frac{1}{NT}\right) \sum_{t=2}^{T} \sum_{s=1}^{t-1} \frac{1}{d_{ts}} \frac{1}{d_s} \iint \|Z_k(w_1)\|\|Z_k(w_2)\||\gamma_k(w_1)||\gamma_k(w_2)|dw_1 dw_2$$

$$\leq O\left(\frac{1}{N}\right) \left(\int \|Z_k(w)\||\gamma_k(w)|dw\right)^2 \leq O\left(\frac{1}{N}\right) \int \|Z_k(w)\|^2 dw \int |\gamma_k(w)|^2 dw = O\left(\frac{k^{-m+1}}{N}\right).$$

Therefore, $A_1 = O\left(\frac{k^{-m+1}}{N}\right)$.

For A_2, by virtue of $Z_k(w) = (\mathscr{H}_0(w), \ldots, \mathscr{H}_{k-1}(w))'$

$$|A_2| = \left| \frac{1}{N^2T} \sum_{i=2}^{N} \sum_{j=1}^{i-1} \sum_{t=1}^{T} \sum_{n=0}^{k-1} E\left[\mathscr{H}_n(u_{it})\gamma_k(u_{it})\right] \cdot E\left[\mathscr{H}_n(u_{jt})\gamma_k(u_{jt})\right] \right|$$

$$= \left| \frac{1}{N^2T} \sum_{i=2}^{N} \sum_{j=1}^{i-1} \sum_{t=1}^{T} \sum_{n=0}^{k-1} \int \mathscr{H}_n(d_t w)\gamma_k(d_t w)f_t(w)dw \int \mathscr{H}_n(d_t w)\gamma_k(d_t w)f_t(w)dw \right|$$

$$\leq O\left(\frac{1}{N^2T}\right) \sum_{i=2}^{N} \sum_{j=1}^{i-1} \sum_{t=1}^{T} \frac{1}{d_t^2} \sum_{n=0}^{k-1} \left(\int |\mathscr{H}_n(w)\gamma_k(w)|dw\right)^2$$

$$\leq O\left(\frac{\ln T}{T}\right)\int\|Z_k(w)\|^2 dw\int|\gamma_k(w)|^2 dw = o\left(k^{-m+1}\right).$$

Similar to A_2, for A_3 we write

$$|A_3| = \left|\frac{1}{N^2 T}\sum_{i=2}^{N}\sum_{j=1}^{i-1}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\sum_{n=0}^{k-1} E\left[\mathscr{H}_n(u_{it})\gamma_k(u_{it})\right]\cdot E\left[\mathscr{H}_n(u_{js})\gamma_k(u_{js})\right]\right|$$

$$\leq O\left(\frac{1}{T}\right)\sum_{t=2}^{T}\sum_{s=1}^{t-1}\frac{1}{d_t}\frac{1}{d_s}\sum_{n=0}^{k-1}\left(\int|\mathscr{H}_n(w)\gamma_k(w)|dw\right)^2$$

$$\leq O(1)\int\|Z_k(w)\|^2 dw\int|\gamma_k(w)|^2 dw = O(k^{-m+1}).$$

Thus, the result follows. ∎

2) Write

$$E\left\|\frac{1}{N\sqrt{T}}\sum_{i=1}^{N}\sum_{t=1}^{T}\phi(u_{it})Z_k(u_{it})'\right\|^2$$

$$=\frac{1}{N^2 T}\sum_{m=1}^{d}\sum_{n=0}^{k-1}\sum_{i=1}^{N}\sum_{t=1}^{T}E[\phi_m^2(u_{it})\mathscr{H}_n^2(u_{it})]$$

$$+\frac{2}{N^2 T}\sum_{m=1}^{d}\sum_{n=0}^{k-1}\sum_{i=1}^{N}\sum_{t=2}^{T}\sum_{s=1}^{t-1}E[\phi_m(u_{it})\mathscr{H}_n(u_{it})\phi_m(u_{is})\mathscr{H}_n(u_{is})]$$

$$+\frac{2}{N^2 T}\sum_{m=1}^{d}\sum_{n=0}^{k-1}\sum_{i=2}^{N}\sum_{j=1}^{i-1}\sum_{t=1}^{T}\sum_{s=1}^{T}E[\phi_m(u_{it})\mathscr{H}_n(u_{it})]E[\phi_m(u_{js})\mathscr{H}_n(u_{js})]$$

$$\leq O\left(\frac{1}{N^2 T}\right)\sum_{m=1}^{d}\sum_{n=0}^{k-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\frac{1}{d_t}\int\mathscr{H}_n^2(w)dw$$

$$+O\left(\frac{2}{N^2 T}\right)\sum_{m=1}^{d}\sum_{n=0}^{k-1}\sum_{i=1}^{N}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\frac{1}{d_{ts}}\frac{1}{d_s}\iint\phi_m(w_1)\mathscr{H}_n(w_1)\phi_m(w_2)\mathscr{H}_n(w_2)dw_1 dw_2$$

$$+O\left(\frac{2}{N^2 T}\right)\sum_{m=1}^{d}\sum_{n=0}^{k-1}\sum_{i=2}^{N}\sum_{j=1}^{i-1}\sum_{t=1}^{T}\sum_{s=1}^{T}\frac{1}{d_t}\frac{1}{d_s}\left(\int\phi_m(w)\mathscr{H}_n(w)dw\right)^2$$

$$\leq O\left(\frac{k}{N\sqrt{T}}\right)+\frac{2}{N^2}\sum_{m=1}^{d}\sum_{n=0}^{k-1}\sum_{i=1}^{N}\left(\int\phi_m(w)\mathscr{H}_n(w)dw\right)^2$$

$$+\frac{2}{N^2}\sum_{m=1}^{d}\sum_{n=0}^{k-1}\sum_{i=2}^{N}\sum_{j=1}^{i-1}\left(\int\phi_m(w)\mathscr{H}_n(w)dw\right)^2$$

$$= o\left(1\right)+\frac{2}{N^2}\sum_{m=1}^{d}\sum_{n=0}^{k-1}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\int\phi_m(w)\mathscr{H}_n(w)dw\right)^2$$

$$\leq o\left(1\right)+\frac{2}{N^2}\sum_{m=1}^{d}\sum_{i=1}^{N}\sum_{j=1}^{N}\int\phi_m^2(w)dw = O(1),$$

where the first equality is due to Assumption 1.4; the first inequality follows from that $f_t(w)$ is bounded uniformly and $\phi_m(w)$ is also bounded uniformly on $\mathbb{R}$ for $m = 1,\ldots,d$; the last

inequality follows from that $\phi_m(w) \in L^2(\mathbb{R})$ (such that $\phi_m(w) = \sum_{n=0}^{\infty} c_{m,n}\mathscr{H}_n(w)$ for $m = 1,\ldots,d$, $c_{m,n} = \int \phi_m(w)\mathscr{H}_n(w)dw$ for $n = 0,\ldots,\infty$ and $\sum_{n=0}^{\infty} c_{m,n}^2 = \int \phi_m^2(w)dw$).

The proof is then completed. ∎

3) Let v_{it,n_1} denote the n_1^{th} element of v_{it}. Write

$$E\left\|\sum_{i=1}^{N}\sum_{t=1}^{T} v_{it}Z_k(u_{it})'\right\|^2 = \sum_{n_1=1}^{d}\sum_{n_2=0}^{k-1} E\left[\sum_{i=1}^{N}\sum_{t=1}^{T} v_{it,n_1}\mathscr{H}_{n_2}(u_{it})\right]^2$$

$$= \sum_{n_1=1}^{d}\sum_{n_2=0}^{k-1}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T} E\left[v_{it,n_1}v_{js,n_1}\right]E\left[\mathscr{H}_{n_2}(u_{it})\mathscr{H}_{n_2}(u_{js})\right]$$

$$\leq O(k)\sum_{n_1=1}^{d}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T} c_\delta(\alpha_{ij}(|t-s|))^{\delta/(4+\delta)}\left(E|v_{it,n_1}|^{2+\delta/2}\right)^{2/(4+\delta)}\left(E|v_{js,n_1}|^{2+\delta/2}\right)^{2/(4+\delta)}$$

$$\leq O(k)\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}(\alpha_{ij}(|t-s|))^{\delta/(4+\delta)} = O(kNT),$$

where $c_\delta = 2^{(4+2\delta)/(4+\delta)} \cdot (4+\delta)/\delta$; the second equality follows from Assumption 1.4; the first inequality follows from Davydov inequality (cf., pages 19-20 in Bosq (1996) and the supplementary of Su and Jin (2012)) and the fact that $\mathscr{H}_n(w)$ is bounded uniformly (cf., Nevai (1986)); the second inequality follows from Assumption 1.3.(a). Thus, the result follows. ∎

4) Let Σ_{v,n_1n_2} denote the $(n_1,n_2)^{th}$ element of Σ_v. Write

$$E\left\|\sum_{i=1}^{N}\sum_{t=1}^{T}(v_{it}v_{it}' - \Sigma_v)\right\|^2 = \sum_{n_1=1}^{d}\sum_{n_2=1}^{d} E\left[\sum_{i=1}^{N}\sum_{t=1}^{T}(v_{it,n_1}v_{it,n_2} - \Sigma_{v,n_1n_2})\right]^2$$

$$\leq \sum_{n_1=1}^{d}\sum_{n_2=1}^{d}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T} c_\delta(\alpha_{ij}(|t-s|))^{\delta/(4+\delta)}$$

$$\cdot \left(E|v_{it,n_1}v_{it,n_2}|^{2+\delta/2}\right)^{2/(4+\delta)}\left(E|v_{js,n_1}v_{js,n_2}|^{2+\delta/2}\right)^{2/(4+\delta)}$$

$$\leq \frac{c_\delta}{2}\sum_{n_1=1}^{d}\sum_{n_2=1}^{d}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}(\alpha_{ij}(|t-s|))^{\delta/(4+\delta)} \cdot \left(E|v_{it,n_1}v_{it,n_2}|^{2+\delta/2}\right)^{4/(4+\delta)}$$

$$+\frac{c_\delta}{2}\sum_{n_1=1}^{d}\sum_{n_2=1}^{d}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}(\alpha_{ij}(|t-s|))^{\delta/(4+\delta)} \cdot \left(E|v_{js,n_1}v_{js,n_2}|^{2+\delta/2}\right)^{4/(4+\delta)}$$

$$\leq O(1)\sum_{n_1=1}^{d}\sum_{n_2=1}^{d}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}(\alpha_{ij}(|t-s|))^{\delta/(4+\delta)} \cdot \left(E|v_{it,n_1}|^{4+\delta} \cdot E|v_{it,n_2}|^{4+\delta}\right)^{2/(4+\delta)}$$

$$\leq O(1)\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}(\alpha_{ij}(|t-s|))^{\delta/(4+\delta)} = O(NT),$$

where $c_\delta = 2^{(4+2\delta)/(4+\delta)} \cdot (4+\delta)/\delta$; the first inequality follows from Davydov inequality; the third inequality follows from Cauchy-Schwarz inequality; the last line follows from Assumption 1.3.(a). Therefore, the result follows. ∎

5) Write

$$E\left\|\sum_{i=1}^{N}\sum_{t=1}^{T}v_{it}\phi(u_{it})'\right\|^2 = \sum_{n_1=1}^{d}\sum_{n_2=1}^{d}E\left[\sum_{i=1}^{N}\sum_{t=1}^{T}v_{it,n_1}\phi_{n_2}(u_{it})\right]^2$$

$$= \sum_{n_1=1}^{d}\sum_{n_2=1}^{d}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}E\left[v_{it,n_1}v_{js,n_1}\right]E\left[\phi_{n_2}(u_{it})\phi_{n_2}(u_{js})\right]$$

$$\le O(1)\sum_{n=1}^{d}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}(\alpha_{ij}(|t-s|))^{\delta/(4+\delta)}\left(E|v_{it,n}|^{2+\delta/2}\right)^{2/(4+\delta)}\left(E|v_{js,n}|^{2+\delta/2}\right)^{2/(4+\delta)}$$

$$\le O(1)\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}(\alpha_{ij}(|t-s|))^{\delta/(4+\delta)} = O(NT),$$

where the second equality follows from Assumption 1.4; the first inequality follows from Davydov inequality and the uniform boundedness of $\phi_n(w)$ on $\mathbb{R}$ for $n = 1,\dots,d$. Therefore, the result follows immediately. ∎

6) By Assumptions 1.1 and 1.4,

$$E\left\|\frac{1}{N\sqrt{T}}\sum_{i=1}^{N}\sum_{t=1}^{T}Z_k(u_{it})e_{it}\right\|^2 = \frac{1}{N^2T}E\left[\left(\sum_{i=1}^{N}\sum_{t=1}^{T}Z_k(u_{it})e_{it}\right)'\left(\sum_{i=1}^{N}\sum_{t=1}^{T}Z_k(u_{it})e_{it}\right)\right]$$

$$= \frac{1}{N^2T}\sum_{i=1}^{N}\sum_{t=1}^{T}E\left[\|Z_k(u_{it})\|^2\right]E\left[e_{it}^2\right] + \frac{2}{N^2T}\sum_{i=2}^{N}\sum_{j=1}^{i-1}\sum_{t=1}^{T}E\left[Z_k(u_{it})'\right]E\left[Z_k(u_{jt})\right]E[e_{it}e_{jt}]$$

$$\equiv B_1 + 2B_2.$$

For B_1, write

$$B_1 = \frac{1}{N^2T}\sum_{i=1}^{N}\sum_{t=1}^{T}\int\|Z_k(d_tw)\|^2 f_t(w)dw \cdot \sigma_e^2$$

$$\le O\left(\frac{1}{NT}\right)\sum_{t=1}^{T}\frac{1}{d_t}\int\|Z_k(w)\|^2 dw = O\left(\frac{k}{N\sqrt{T}}\right),$$

where the second line follow from that $f_t(w)$ being bounded uniformly.

For B_2,

$$|B_2| = \left|\frac{1}{N^2T}\sum_{n=0}^{k-1}\sum_{i=2}^{N}\sum_{j=1}^{i-1}\sum_{t=1}^{T}\int\mathscr{H}_n(d_tw)f_t(w)dw\int\mathscr{H}_n(d_tw)f_t(w)dw \cdot \sigma_e(i,j)\right|$$

$$\le \frac{1}{N^2T}\sum_{n=0}^{k-1}\sum_{i=2}^{N}\sum_{j=1}^{i-1}\sum_{t=1}^{T}\int\mathscr{H}_n^2(d_tw)dw\int f_t^2(w)dw\,|\sigma_e(i,j)|$$

$$\le O\left(\frac{1}{N^2T}\right)\sum_{n=0}^{k-1}\sum_{t=1}^{T}\frac{1}{d_t}\int\mathscr{H}_n^2(w)dw\sum_{i=2}^{N}\sum_{j=1}^{i-1}|\sigma_e(i,j)| \le O\left(\frac{k}{N\sqrt{T}}\right),$$

where the first inequality follows from Cauchy-Schwarz inequality; the second inequality follows from $f_t(w)$ being bounded uniformly; the third inequality follows from Assumption 1.3.(b). In connection with $B_1 = O\left(\frac{k}{N\sqrt{T}}\right)$, we obtain that $\frac{1}{N\sqrt{T}} \sum_{i=1}^{N} \sum_{t=1}^{T} Z_k(u_{it})e_{it} = O_P\left(\frac{\sqrt{k}}{\sqrt{N}\sqrt[4]{T}}\right)$. ∎

7) Write

$$E\left\|\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\phi(u_{it})e_{it}\right\|^2 = \frac{1}{N^2T^2}\sum_{n=1}^{d}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}E\left[\phi_n(u_{it})\phi_n(u_{jt})\right]E\left[e_{it}e_{jt}\right]$$

$$= \frac{\sigma_e^2}{N^2T^2}\sum_{n=1}^{d}\sum_{i=1}^{N}\sum_{t=1}^{T}\int \phi_n^2(d_t w)f_t(w)dw$$

$$+ \frac{2}{N^2T^2}\sum_{n=1}^{d}\sum_{i=2}^{N}\sum_{j=1}^{i-1}\sum_{t=1}^{T}\int \phi_n(d_t w)f_t(w)dw \int \phi_n(d_t w)f_t(w)dw \cdot \sigma_e(i,j)$$

$$\leq O\left(\frac{1}{NT^2}\right)\sum_{n=1}^{d}\sum_{t=1}^{T}\frac{1}{d_t}\int \phi_n^2(w)dw$$

$$+ O\left(\frac{2}{N^2T^2}\right)\sum_{n=1}^{d}\sum_{i=2}^{N}\sum_{j=1}^{i-1}\sum_{t=1}^{T}\int \phi_n^2(d_t w)dw \int f_t^2(w)dw|\sigma_e(i,j)|$$

$$\leq O\left(\frac{1}{NT^{3/2}}\right) + O\left(\frac{1}{NT^{3/2}}\right) = O\left(\frac{1}{NT^{3/2}}\right),$$

where the first equality follows from Assumption 1.4; the first inequality follows from $f_t(w)$ being bounded uniformly and $\phi_n(w) \in L^2(\mathbb{R})$; the second inequality follows from $\phi_n(w)$ being integrable and Assumption 1.3. Therefore, the result follows. ∎

8) Write

$$E\left\|\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\phi(u_{it})\gamma_k(u_{it})\right\|^2 = \frac{1}{N^2T^2}\sum_{n=1}^{d}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}E\left[\phi_n(u_{it})\gamma_k(u_{it})\phi_n(u_{js})\gamma_k(u_{js})\right]$$

$$= \frac{1}{N^2T^2}\sum_{n=1}^{d}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}E\left[\phi_n(u_{it})\gamma_k(u_{it})\phi_n(u_{is})\gamma_k(u_{is})\right]$$

$$+ \frac{2}{N^2T^2}\sum_{n=1}^{d}\sum_{i=2}^{N}\sum_{j=1}^{i-1}\sum_{t=1}^{T}\sum_{s=1}^{T}E\left[\phi_n(u_{it})\gamma_k(u_{it})\right]E\left[\phi_n(u_{js})\gamma_k(u_{js})\right]$$

$$= \frac{1}{N^2T^2}\sum_{n=1}^{d}\sum_{i=1}^{N}\sum_{t=1}^{T}E\left[\phi_n^2(u_{it})\gamma_k^2(u_{it})\right] + \frac{2}{N^2T^2}\sum_{n=1}^{d}\sum_{i=1}^{N}\sum_{t=2}^{T}\sum_{s=1}^{t-1}E\left[\phi_n(u_{it})\gamma_k(u_{it})\phi_n(u_{is})\gamma_k(u_{is})\right]$$

$$+ \frac{2}{N^2T}\sum_{n=1}^{d}\sum_{i=2}^{N}\sum_{j=1}^{i-1}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}E\left[\phi_n(u_{it})\gamma_k(u_{it})\right]\frac{1}{\sqrt{T}}\sum_{s=1}^{T}E\left[\phi_n(u_{js})\gamma_k(u_{js})\right]$$

$$\equiv C_1 + 2C_2 + 2C_3.$$

By (2) of Lemma A.1,

$$C_1 = O\left(\frac{k^{-m+5/6}}{N^2T^2}\right)\sum_{n=1}^{d}\sum_{i=1}^{N}\sum_{t=1}^{T}\int \phi_n^2(d_t w)f_t(w)dw$$

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$$\leq O\left(\frac{k^{-m+5/6}}{N^2T^2}\right)\sum_{n=1}^{d}\sum_{i=1}^{N}\sum_{t=1}^{T}\frac{1}{d_t}\int \phi_n^2(w)dw \leq O\left(\frac{k^{-m+5/6}}{NT^{3/2}}\right).$$

For C_2, write

$$|C_2| = \left|\frac{1}{N^2T^2}\sum_{n=1}^{d}\sum_{i=1}^{N}\sum_{t=2}^{T}\sum_{s=1}^{t-1}E\left[\phi_n(u_{it})\gamma_k(u_{it})\phi_n(u_{is})\gamma_k(u_{is})\right]\right|$$

$$\leq O\left(\frac{1}{N^2T^2}\right)\sum_{n=1}^{d}\sum_{i=1}^{N}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\frac{1}{d_{ts}}\frac{1}{d_s}\iint|\phi_n(w_1)\gamma_k(w_1)\phi_n(w_2)\gamma_k(w_2)|dw_1dw_2$$

$$\leq O\left(\frac{1}{NT}\right)\sum_{n=1}^{d}\left(\int|\phi_n(w)\gamma_k(w)|dw\right)^2 \leq O\left(\frac{1}{NT}\right)\sum_{n=1}^{d}\int \phi_n^2(w)dw\int\gamma_k^2(w)dw = O\left(\frac{k^{-m}}{NT}\right).$$

For C_3, write

$$\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}E\left[\phi_n(u_{it})\gamma_k(u_{it})\right]\right| \leq \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\int|\phi_n(d_tw)\gamma_k(d_tw)|f_t(w)dw$$

$$\leq O(1)\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\frac{1}{d_t}\int|\phi_n(w)\gamma_k(w)|dw \leq O(1)\left\{\int\phi_n^2(w)dw\int\gamma_k^2(w)dw\right\}^{1/2} = O(k^{-m/2}).$$

Thus, $|C_3| \leq \frac{1}{N^2T}\sum_{n=1}^{d}\sum_{i=2}^{N}\sum_{j=1}^{i-1}O(k^{-m}) = O\left(\frac{1}{k^mT}\right)$. In connection with the analysis for C_1 and C_2, $\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\phi(u_{it})\gamma_k(u_{it}) = O_P\left(\frac{1}{\sqrt{k^mT}}\right)$. Then the proof is complete. ∎

9) Write

$$E\left\|\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}v_{it}\gamma_k(u_{it})\right\|^2 = \frac{1}{N^2T^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}E\left[v'_{it}v_{js}\gamma_k(u_{it})\gamma_k(u_{js})\right]$$

$$= \frac{1}{N^2T^2}\sum_{n=1}^{d}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}E\left[v_{it,n}v_{js,n}\right]E\left[\gamma_k(u_{it})\gamma_k(u_{js})\right]$$

$$\leq O\left(\frac{k^{-m+5/6}}{N^2T^2}\right)\sum_{n=1}^{d}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}(\alpha_{ij}(|t-s|))^{\delta/(4+\delta)}$$

$$\cdot\left(E\left[|v_{it,n}|^{2+\delta/2}\right]\cdot E\left[|v_{js,n}|^{2+\delta/2}\right]\right)^{2/(4+\delta)}$$

$$\leq O\left(\frac{k^{-m+5/6}}{N^2T^2}\right)\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}(\alpha_{ij}(|t-s|))^{\delta/(4+\delta)} = O\left(\frac{k^{-m+5/6}}{NT}\right),$$

where the second equality follows from Assumption 1.4; the first inequality follows from (2) of Lemma A.1 and Davydov inequality; the last line follows from Assumption 1.3.(a). Then the result follows. ∎

Proof of Lemma A.3:

For two non-singular symmetric matrices A and B with same dimensions, we observe that

$$\left\|A^{-1} - B^{-1}\right\| = \left\|B^{-1}(B-A)A^{-1}\right\| = \left\|vec\left(B^{-1}(B-A)A^{-1}\right)\right\|$$

$$= \left\| \left(A^{-1} \otimes B^{-1} \right) \text{vec} \left(B - A \right) \right\| \leq \lambda_{min}^{-1} \left(A \otimes B \right) \left\| \text{vec} \left(B - A \right) \right\|$$

$$= \lambda_{min}^{-1} \left(A \right) \cdot \lambda_{min}^{-1} \left(B \right) \left\| A - B \right\|.$$

The above calculation is straightforward and all the necessary theorems can be found in Magnus and Neudecker (2007). ∎

Proof of Lemma A.4:

1)

$$\frac{1}{N\sqrt{T}} \mathcal{Z}' \mathcal{E} = \frac{1}{N\sqrt{T}} \sum_{i=1}^{N} \sum_{t=1}^{T} Z_k(u_{it}) e_{it} - \frac{1}{NT^{3/2}} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} Z_k(u_{it}) e_{is} \equiv A_1 - A_2.$$

We have shown that $A_1 = O_P \left(\frac{\sqrt{k}}{\sqrt{N}\sqrt[4]{T}} \right)$ in (6) of Lemma A.2. We then just focus on A_2. By Assumptions 1.1, 1.3.(b) and 1.4, write

$$E \left\| A_2 \right\|^2 = \frac{1}{N^2 T^3} \sum_{i=1}^{N} \sum_{t_1=1}^{T} \sum_{t_2=1}^{T} E \left[Z_k(u_{it_1})' Z_k(u_{it_2}) \right] \sum_{s=1}^{T} E[e_{is}^2]$$

$$+ \frac{2}{N^2 T^3} \sum_{i=2}^{N} \sum_{j=1}^{i-1} \sum_{t_1=1}^{T} \sum_{t_2=1}^{T} E \left[Z_k(u_{it_1})' \right] E \left[Z_k(u_{jt_2}) \right] \sum_{s=1}^{T} E[e_{is} e_{js}]$$

$$= \frac{\sigma_e^2}{N^2 T^2} \sum_{i=1}^{N} \sum_{t_1=1}^{T} \sum_{t_2=1}^{T} E \left[Z_k(u_{it_1})' Z_k(u_{it_2}) \right]$$

$$+ \frac{2}{N^2 T^2} \sum_{i=2}^{N} \sum_{j=1}^{i-1} \sum_{t_1=1}^{T} \sum_{t_2=1}^{T} E \left[Z_k(u_{it_1})' \right] E \left[Z_k(u_{jt_2}) \right] \sigma_e(i,j) \equiv A_{21} + 2A_{22}.$$

For A_{21}, write

$$|A_{21}| \leq O \left(\frac{1}{N^2 T^2} \right) \sum_{i=1}^{N} \sum_{t_1=1}^{T} \sum_{t_2=1}^{T} E \left[\| Z_k(u_{it_1}) \| \| Z_k(u_{it_2}) \| \right] \leq O \left(\frac{T k^{\frac{11}{6}}}{N T^2} \right) = o \left(\frac{k}{N\sqrt{T}} \right),$$

where the second inequality has been provided in the proof of Lemma 2.1 of this paper and the last equality follows from Assumption 2.2.

For A_{22}, write

$$|A_{22}| \leq \frac{1}{N^2 T^2} \sum_{n=0}^{k-1} \sum_{i=2}^{N} \sum_{j=1}^{i-1} \sum_{t_1=1}^{T} \sum_{t_2=1}^{T} \left| \int \mathscr{H}_n(d_{t_1} w) f_{t_1}(w) dw \right| \left| \int \mathscr{H}_n(d_{t_2} w) f_{t_2}(w) dw \right| |\sigma_e(i,j)|$$

$$\leq O \left(\frac{1}{N^2 T^2} \right) \sum_{n=0}^{k-1} \sum_{t_1=1}^{T} \sum_{t_2=1}^{T} \int \frac{1}{d_{t_1}} |\mathscr{H}_n(w)| \, dw \int \frac{1}{d_{t_2}} |\mathscr{H}_n(w)| \, dw \sum_{i=2}^{N} \sum_{j=1}^{i-1} |\sigma_e(i,j)|$$

$$\leq O \left(\frac{\sum_{n=0}^{k-1} n^{\frac{5}{6}}}{NT} \right) \leq O \left(\frac{k^2}{NT} \right) = o \left(\frac{k}{N\sqrt{T}} \right),$$

where the last line follows from (8) of Lemma A.1 and Assumption 2.2.

By the derivations for the asymptotic orders of A_{21} and A_{22}, we obtain $A_2 = o_P\left(\frac{\sqrt{k}}{\sqrt{N}\sqrt[4]{T}}\right)$. In connection with that $A_1 = O_P\left(\frac{\sqrt{k}}{\sqrt{N}\sqrt[4]{T}}\right)$, the result follows. ∎

2)

$$\frac{1}{N\sqrt{T}}X'\mathcal{Z} = \frac{1}{N\sqrt{T}}\sum_{i=1}^{N}\sum_{t=1}^{T}(\phi(u_{it})+v_{it})Z_k(u_{it})' - \frac{1}{NT^{3/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}(\phi(u_{it})+v_{it})Z_k(u_{is})'$$

$$\equiv B_1 - B_2,$$

where $\|B_1\| = O_P(1)$ follows from (2) and (3) of Lemma A.2 of this paper immediately. Then we just need to focus on B_2 below.

$$B_2 = \frac{1}{NT^{3/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}\phi(u_{it})Z_k(u_{is})' + \frac{1}{NT^{3/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}v_{it}Z_k(u_{is})' \equiv B_{21} + B_{22}$$

For B_{21}, write

$$E\|B_{21}\|^2 = \frac{1}{N^2T^3}\sum_{n_1=1}^{d}\sum_{n_2=0}^{k-1}\sum_{i,j=1}^{N}\sum_{t_1,t_2,t_3,t_4=1}^{T}E[\phi_{n_1}(u_{it_1})\mathscr{H}_{n_2}(u_{it_2})\phi_{n_1}(u_{jt_3})\mathscr{H}_{n_2}(u_{jt_4})]$$

$$= \frac{1}{N^2T^3}\sum_{n_1=1}^{d}\sum_{n_2=0}^{k-1}\sum_{i=1}^{N}\sum_{t_1,t_2,t_3,t_4=1}^{T}E[\phi_{n_1}(u_{it_1})\mathscr{H}_{n_2}(u_{it_2})\phi_{n_1}(u_{it_3})\mathscr{H}_{n_2}(u_{it_4})]$$

$$+ \frac{2}{N^2T^3}\sum_{n_1=1}^{d}\sum_{n_2=0}^{k-1}\sum_{i=2}^{N}\sum_{j=1}^{i-1}\sum_{t_1,t_2,t_3,t_4=1}^{T}E[\phi_{n_1}(u_{it_1})\mathscr{H}_{n_2}(u_{it_2})]E[\phi_{n_1}(u_{jt_3})\mathscr{H}_{n_2}(u_{jt_4})]$$

$$\equiv B_{211} + 2B_{212}.$$

For B_{211}, we write

$$B_{211} = \frac{1}{N^2T^3}\sum_{n_1=1}^{d}\sum_{n_2=0}^{k-1}\sum_{i=1}^{N}\sum_{t_1,t_2,t_3,t_4=1}^{T}E[\phi_{n_1}(u_{it_1})\mathscr{H}_{n_2}(u_{it_2})\phi_{n_1}(u_{it_3})\mathscr{H}_{n_2}(u_{it_4})]$$

$$= \frac{1}{N^2T^3}\sum_{n_1=1}^{d}\sum_{n_2=0}^{k-1}\sum_{i=1}^{N}\sum_{\text{all }t_1,t_2,t_3,t_4\text{ are different}}E[\phi_{n_1}(u_{it_1})\mathscr{H}_{n_2}(u_{it_2})\phi_{n_1}(u_{it_3})\mathscr{H}_{n_2}(u_{it_4})]$$

$$+ \frac{1}{N^2T^3}\sum_{n_1=1}^{d}\sum_{n_2=0}^{k-1}\sum_{i=1}^{N}\sum_{\text{only two of }t_1,t_2,t_3,t_4\text{ are same}}E[\phi_{n_1}(u_{it_1})\mathscr{H}_{n_2}(u_{it_2})\phi_{n_1}(u_{it_3})\mathscr{H}_{n_2}(u_{it_4})]$$

$$+ \frac{1}{N^2T^3}\sum_{n_1=1}^{d}\sum_{n_2=0}^{k-1}\sum_{i=1}^{N}\sum_{\text{only three of }t_1,t_2,t_3,t_4\text{ are same}}E[\phi_{n_1}(u_{it_1})\mathscr{H}_{n_2}(u_{it_2})\phi_{n_1}(u_{it_3})\mathscr{H}_{n_2}(u_{it_4})]$$

$$+ \frac{1}{N^2T^3}\sum_{n_1=1}^{d}\sum_{n_2=0}^{k-1}\sum_{i=1}^{N}\sum_{\text{four of }t_1,t_2,t_3,t_4\text{ are same}}E[\phi_{n_1}(u_{it_1})\mathscr{H}_{n_2}(u_{it_2})\phi_{n_1}(u_{it_3})\mathscr{H}_{n_2}(u_{it_4})]$$

$$\equiv B_{2111} + B_{2112} + B_{2113} + B_{2114}.$$

For B_{2111}, without loss of generality, assume that $t_1 > t_2 > t_3 > t_4$. For other cases, for example $t_2 > t_3 > t_1 > t_4$, the analysis will be same and the order will remain same. Then

$$\frac{1}{N^2T^3}\sum_{n_1=1}^{d}\sum_{n_2=0}^{k-1}\sum_{i=1}^{N}\sum_{t_1=4}^{T}\sum_{t_2=3}^{t_1-1}\sum_{t_3=2}^{t_2-1}\sum_{t_4=1}^{t_3-1}|E[\phi_{n_1}(u_{it_1})\mathscr{H}_{n_2}(u_{it_2})\phi_{n_1}(u_{it_3})\mathscr{H}_{n_2}(u_{it_4})]|$$

$$\leq O\left(\frac{1}{N^2T^3}\right)\sum_{n_1=1}^{d}\sum_{n_2=0}^{k-1}\sum_{i=1}^{N}\sum_{t_1=4}^{T}\sum_{t_2=3}^{t_1-1}\sum_{t_3=2}^{t_2-1}\sum_{t_4=1}^{t_3-1}\frac{1}{d_{t_1t_2}}\frac{1}{d_{t_2t_3}}\frac{1}{d_{t_3t_4}}\frac{1}{d_{t_4}}$$

$$\cdot\iiiint|\phi_{n_1}(w_1)\mathscr{H}_{n_2}(w_2)\phi_{n_1}(w_3)\mathscr{H}_{n_2}(w_4)|dw_1dw_2dw_3dw_4$$

$$\leq O\left(\frac{1}{N^2T^3}\right)\sum_{n_1=1}^{d}\sum_{n_2=0}^{k-1}\sum_{i=1}^{N}\sum_{t_1=4}^{T}\sum_{t_2=3}^{t_1-1}\sum_{t_3=2}^{t_2-1}\sum_{t_4=1}^{t_3-1}\frac{1}{d_{t_1t_2}}\frac{1}{d_{t_2t_3}}\frac{1}{d_{t_3t_4}}\frac{1}{d_{t_4}}$$

$$\cdot\left(\int|\phi_{n_1}(w)|dw\right)^2\left(\int|\mathscr{H}_{n_2}(w)|dw\right)^2$$

$$\leq O\left(\frac{1}{NT}\right)\sum_{n_2=0}^{k-1}n_2^{5/6}\leq O\left(\frac{k^2}{NT}\right)=o(1),$$

where the last line follows from (8) of Lemma A.1, $\phi_n(w)$ being integrable function on $\mathbb{R}$ for $n=1,\ldots,d$ and Assumption 2.2.

For B_{2112}, without loss of generality, we assume that $t_1=t_2>t_3>t_4$. For other cases, for example $t_1=t_3>t_2>t_4$, the analysis will be even simpler and the order will remain same. Then write

$$\frac{1}{N^2T^3}\sum_{n_1=1}^{d}\sum_{n_2=0}^{k-1}\sum_{i=1}^{N}\sum_{t_1=3}^{T}\sum_{t_3=2}^{t_1-1}\sum_{t_4=1}^{t_3-1}|E[\phi_{n_1}(u_{it_1})\mathscr{H}_{n_2}(u_{it_1})\phi_{n_1}(u_{it_3})\mathscr{H}_{n_2}(u_{it_4})]|$$

$$\leq O\left(\frac{1}{N^2T^3}\right)\sum_{n_1=1}^{d}\sum_{n_2=0}^{k-1}\sum_{i=1}^{N}\sum_{t_1=3}^{T}\sum_{t_3=2}^{t_1-1}\sum_{t_4=1}^{t_3-1}\frac{1}{d_{t_1t_3}}\frac{1}{d_{t_3t_4}}\frac{1}{d_{t_4}}$$

$$\cdot\iiint|\phi_{n_1}(w_1)\mathscr{H}_{n_2}(w_1)\phi_{n_1}(w_2)\mathscr{H}_{n_2}(w_3)|dw_1dw_2dw_3$$

$$\leq O\left(\frac{1}{NT^{3/2}}\right)\sum_{n_1=1}^{d}\sum_{n_2=0}^{k-1}\left\{\int\phi_{n_1}^2(w)dw\int\mathscr{H}_{n_2}^2(w)dw\right\}^{1/2}\int|\phi_{n_1}(w)|dw\int|\mathscr{H}_{n_2}(w)|dw$$

$$\leq O\left(\frac{1}{NT^{3/2}}\right)\sum_{n_2=0}^{k-1}n_2^{5/12}\leq O\left(\frac{k^2}{NT^{3/2}}\right)=o(1),$$

where the last line follows from (8) of Lemma A.1, $\phi_n(w)$ being integrable function on $\mathbb{R}$ for $n=1,\ldots,d$ and Assumption 2.2.

For B_{2113}, without loss of generality, assume that $t_1=t_2=t_3>t_4$. For other cases, for example $t_1=t_3=t_4>t_2$, the analysis will be same and the order will remain same. Then write

$$\frac{1}{N^2T^3}\sum_{n_1=1}^{d}\sum_{n_2=0}^{k-1}\sum_{i=1}^{N}\sum_{t_1=2}^{T}\sum_{t_4=1}^{t_1-1}|E[\phi_{n_1}(u_{it_1})\mathscr{H}_{n_2}(u_{it_1})\phi_{n_1}(u_{it_1})\mathscr{H}_{n_2}(u_{it_4})]|$$

$$\leq O\left(\frac{1}{N^2T^3}\right)\sum_{n_1=1}^{d}\sum_{n_2=0}^{k-1}\sum_{i=1}^{N}\sum_{t_1=2}^{T}\sum_{t_4=1}^{t_1-1}\frac{1}{d_{t_1t_4}}\frac{1}{d_{t_4}}$$

$$\cdot \iint |\phi_{n_1}(w_1)\mathscr{H}_{n_2}(w_1)\phi_{n_1}(w_1)\mathscr{H}_{n_2}(w_2)|dw_1dw_2$$

$$\leq O\left(\frac{1}{NT^2}\right)\sum_{n_1=1}^{d}\sum_{n_2=0}^{k-1}\int|\phi_{n_1}(w)|dw\int|\mathscr{H}_{n_2}(w)|dw = o(1),$$

where the last line follows from $\mathscr{H}_j(w)$ and $\phi_j(w)$ being bounded uniformly, (8) of Lemma A.1 and Assumption 2.2.

For B_{2114}, write

$$B_{2114} = \frac{1}{N^2T^3}\sum_{n_1=1}^{d}\sum_{n_2=0}^{k-1}\sum_{i=1}^{N}\sum_{t=1}^{T}E[\phi_{n_1}^2(u_{it})\mathscr{H}_{n_2}^2(u_{it})]$$

$$= \frac{1}{N^2T^3}\sum_{n_1=1}^{d}\sum_{n_2=0}^{k-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\int\phi_{n_1}^2(d_tw)\mathscr{H}_{n_2}^2(d_tw)f_t(w)dw$$

$$\leq O\left(\frac{1}{N^2T^3}\right)\sum_{n_1=1}^{d}\sum_{n_2=0}^{k-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\frac{1}{d_t}\int\phi_{n_1}^2(w)dw \leq O\left(\frac{k}{NT^{5/2}}\right) = o(1).$$

Combining B_{2111}, B_{2112}, B_{2113} and B_{2114} together, we obtain that $B_{211} = o(1)$.

For B_{212}, write

$$\left|E\left[\sum_{t_1=1}^{T}\sum_{t_2=1}^{T}\phi_{n_1}(u_{it_1})\mathscr{H}_{n_2}(u_{it_2})\right]\right|$$

$$\leq \sum_{t=1}^{T}E[|\phi_{n_1}(u_{it})\mathscr{H}_{n_2}(u_{it})|] + \sum_{t=2}^{T}\sum_{s=1}^{t-1}E[|\phi_{n_1}(u_{it})\mathscr{H}_{n_2}(u_{is})|] + \sum_{t=2}^{T}\sum_{s=1}^{t-1}E[|\mathscr{H}_{n_2}(u_{it})\phi_{n_1}(u_{is})|]$$

$$\leq O(1)\sum_{t=1}^{T}\frac{1}{d_t}\int|\mathscr{H}_{n_2}(w)|dw + O(1)\sum_{t=2}^{T}\sum_{s=1}^{t-1}\frac{1}{d_{ts}}\frac{1}{d_s}\int|\mathscr{H}_{n_2}(w)|dw$$

$$\leq O(\sqrt{T}n_2^{5/12}) + O(Tn_2^{5/12}) = O(Tn_2^{5/12}),$$

where the last line follows from (8) of Lemma A.1. Therefore,

$$|B_{212}| \leq O\left(\frac{1}{N^2T^3}\right)\sum_{n_1=1}^{d}\sum_{n_2=0}^{k-1}\sum_{i=2}^{N}\sum_{j=1}^{i-1}T^2n_2^{5/6} \leq O\left(\frac{k^2}{T}\right) = o(1).$$

Since $|B_{211}| = o(1)$ and $|B_{212}| = o(1)$, then $B_{21} = o_P(1)$.

Below, we focus on B_{22}.

$$E\|B_{22}\|^2 = E\left\|\frac{1}{NT^{3/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}v_{it}Z_k(u_{is})'\right\|^2$$

$$= \frac{1}{N^2T^3}\sum_{n_1=1}^{d}\sum_{n_2=0}^{k-1}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t_1=1}^{T}\sum_{t_2=1}^{T}\sum_{t_3=1}^{T}\sum_{t_4=1}^{T}E[v_{it_1,n_1}\mathscr{H}_{n_2}(u_{it_2})v_{jt_3,n_1}\mathscr{H}_{n_2}(u_{jt_4})]$$

$$= \frac{1}{N^2T^3}\sum_{n_1=1}^{d}\sum_{n_2=0}^{k-1}\sum_{i=1}^{N}\sum_{t_1=1}^{T}\sum_{t_2=1}^{T}\sum_{t_3=1}^{T}\sum_{t_4=1}^{T}E[v_{it_1,n_1}v_{it_3,n_1}]E[\mathscr{H}_{n_2}(u_{it_2})\mathscr{H}_{n_2}(u_{it_4})]$$

$$+\frac{2}{N^2T^3}\sum_{n_1=1}^{d}\sum_{n_2=0}^{k-1}\sum_{i=2}^{N}\sum_{j=1}^{i-1}\sum_{t_1=1}^{T}\sum_{t_2=1}^{T}\sum_{t_3=1}^{T}\sum_{t_4=1}^{T}E[v_{it_1,n_1}v_{jt_3,n_1}]E[\mathscr{H}_{n_2}(u_{it_2})]E[\mathscr{H}_{n_2}(u_{jt_4})]$$

$$\equiv B_{221}+2B_{222}.$$

For B_{221}, write

$$|B_{221}|\leq\frac{1}{N^2T^3}\sum_{n_1=1}^{d}\sum_{n_2=0}^{k-1}\sum_{i=1}^{N}\sum_{t_1=1}^{T}\sum_{t_2=1}^{T}\sum_{t_3=1}^{T}\sum_{t_4=1}^{T}|E[v_{it_1,n_1}v_{it_3,n_1}]E[\mathscr{H}_{n_2}(u_{it_2})\mathscr{H}_{n_2}(u_{it_4})]|$$

$$\leq O\left(\frac{1}{N^2T^3}\right)\sum_{n_1=1}^{d}\sum_{n_2=0}^{k-1}\sum_{i=1}^{N}\sum_{t_1=1}^{T}\sum_{t_3=1}^{T}|E[v_{it_1,n_1}v_{it_3,n_1}]|\sum_{t_2=1}^{T}\frac{1}{d_{t_2}}\int\mathscr{H}_{n_2}^2(w)dw$$

$$+O\left(\frac{1}{N^2T^3}\right)\sum_{n_1=1}^{d}\sum_{n_2=0}^{k-1}\sum_{i=1}^{N}\sum_{t_1=1}^{T}\sum_{t_3=1}^{T}|E[v_{it_1,n_1}v_{it_3,n_1}]|\sum_{t_2=2}^{T}\sum_{t_4=1}^{t_2-1}\frac{1}{d_{t_2t_4}}\frac{1}{d_{t_4}}\left(\int|\mathscr{H}_{n_2}(w)|dw\right)^2$$

$$\leq O\left(\frac{k}{N^2T^3}\right)\sum_{i=1}^{N}T^{3/2}+O\left(\frac{1}{N^2T^3}\right)\sum_{n_2=0}^{k-1}\sum_{i=1}^{N}T^2n_2^{5/6}\leq O\left(\frac{k}{NT^{3/2}}\right)+O\left(\frac{k^2}{NT}\right)=o(1),$$

where the last line follows from Davydov inequality, Assumption 1.3 and (8) of Lemma A.1.

For B_{222}, write

$$|B_{222}|\leq\frac{1}{N^2T^3}\sum_{n_1=1}^{d}\sum_{n_2=0}^{k-1}\sum_{i=2}^{N}\sum_{j=1}^{i-1}\sum_{t_1=1}^{T}\sum_{t_3=1}^{T}|E[v_{it_1,n_1}v_{jt_3,n_1}]|$$

$$\cdot\sum_{t_2=1}^{T}\int|\mathscr{H}_{n_2}(d_{t_2}w)|f_{t_2}(w)dw\sum_{t_4=1}^{T}\int|\mathscr{H}_{n_2}(d_{t_4}w)|f_{t_4}(w)dw$$

$$\leq O\left(\frac{1}{N^2T^3}\right)\sum_{n_1=1}^{d}\sum_{n_2=0}^{k-1}\sum_{i=2}^{N}\sum_{j=1}^{i-1}\sum_{t_1=1}^{T}\sum_{t_3=1}^{T}c_{\delta}(\alpha_{ij}(|t_1-t_3|))^{\delta/(4+\delta)}\cdot\left(E[|v_{it_1,n_1}|^{2+\delta/2}]\right)^{2/(4+\delta)}$$

$$\cdot\left(E[|v_{jt_3,n_1}|^{2+\delta/2}]\right)^{2/(4+\delta)}\sum_{t_2=1}^{T}\frac{1}{d_{t_2}}\int|\mathscr{H}_{n_2}(w)|dw\sum_{t_4=1}^{T}\frac{1}{d_{t_4}}\int|\mathscr{H}_{n_2}(w)|dw$$

$$\leq O\left(\frac{1}{N^2T^2}\right)\sum_{n_2=0}^{k-1}n_2^{5/6}\sum_{i=2}^{N}\sum_{j=1}^{i-1}\sum_{t_1=1}^{T}\sum_{t_3=1}^{T}(\alpha_{ij}(|t_1-t_3|))^{\delta/(4+\delta)}\leq O\left(\frac{k^2}{NT}\right)=o(1),$$

where the second inequality follows form Davydov inequality and $c_{\delta}=2^{(4+2\delta)/(4+\delta)}\cdot(4+\delta)/\delta$; the third inequality follows from Assumption 1.3 and (8) of Lemma A.1.

Since $|B_{221}|=o(1)$ and $|B_{222}|=o(1)$, we know that $B_{22}=o_P(1)$. We therefore complete this part. $\blacksquare$

3)

$$\frac{1}{NT}X'\mathcal{E}=\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}(\phi(u_{it})+v_{it})e_{it}-\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}(\phi(u_{it})+v_{it})e_{is}\equiv C_1-C_2$$

Expand C_1 as

$$C_1 = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \phi(u_{it})e_{it} + \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} v_{it}e_{it} \equiv C_{11} + C_{12}.$$

We have shown that $C_{11} = O_P\left(\frac{1}{\sqrt{N}\sqrt[4]{T^3}}\right)$ in (7) of Lemma A.2. Moreover, by Assumption 1.3.(b)

$$E\|C_{12}\|^2 = \frac{1}{N^2T^2} \sum_{n=1}^{d} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t=1}^{T} E\left[v_{it,n}v_{jt,n}E[e_{it}e_{jt}|\mathcal{F}_{Nt-1}]\right] = O\left(\frac{1}{NT}\right).$$

Thus, $C_{12} = O_P\left(\frac{1}{\sqrt{NT}}\right)$. In connection with $C_{11} = O_P\left(\frac{1}{\sqrt{N}\sqrt[4]{T^3}}\right)$, we obtain $C_1 = O_P\left(\frac{1}{\sqrt{NT}}\right)$.

Then we focus on C_2 below and write

$$C_2 = \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} \phi(u_{it})e_{is} + \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} v_{it}e_{is} = C_{21} + C_{22}.$$

For C_{21},

$$E\|C_{21}\|^2 = \frac{1}{N^2T^4} \sum_{n=1}^{d} \sum_{i=1}^{N} \sum_{t=1}^{T} E[\phi_n^2(u_{it})] \sum_{s=1}^{T} E[e_{is}^2]$$

$$+ \frac{2}{N^2T^4} \sum_{n=1}^{d} \sum_{i=1}^{N} \sum_{t_1=2}^{T} \sum_{t_2=1}^{t_1-1} E[\phi_n(u_{it_1})\phi_n(u_{it_2})] \sum_{s=1}^{T} E[e_{is}^2]$$

$$+ \frac{2}{N^2T^4} \sum_{n=1}^{d} \sum_{i=2}^{N} \sum_{j=1}^{i-1} \sum_{t_1=1}^{T} \sum_{t_2=1}^{T} E[\phi_n(u_{it_1})\phi_n(u_{jt_2})] \sum_{s=1}^{T} E[e_{is}^2]$$

$$\equiv C_{211} + 2C_{212} + 2C_{213}.$$

For C_{211},

$$C_{211} = O\left(\frac{1}{N^2T^3}\right) \sum_{n=1}^{d} \sum_{i=1}^{N} \sum_{t=1}^{T} \int \phi_n^2(d_t w) f_t(w) dw \leq O\left(\frac{1}{NT^3}\right) \sum_{n=1}^{d} \sum_{t=1}^{T} \frac{1}{d_t} \int \phi_n^2(w) dw$$

$$= O\left(\frac{1}{NT^{5/2}}\right) = o\left(\frac{1}{NT}\right).$$

Similarly, for C_{212}

$$|C_{212}| \leq O\left(\frac{1}{N^2T^3}\right) \sum_{n=1}^{d} \sum_{i=1}^{N} \sum_{t_1=2}^{T} \sum_{t_2=1}^{t_1-1} E[|\phi_n(u_{it_1})\phi_n(u_{it_2})|]$$

$$\leq O\left(\frac{1}{N^2T^3}\right) \sum_{n=1}^{d} \sum_{i=1}^{N} \sum_{t_1=2}^{T} \sum_{t_2=1}^{t_1-1} \frac{1}{d_{t_1 t_2}} \frac{1}{d_{t_2}} \int |\phi_n(w)| dw \int |\phi_n(w)| dw$$

$$\leq O\left(\frac{1}{NT^2}\right) = o\left(\frac{1}{NT}\right),$$

where the last inequality follows from $f_{t_1 t_2}(w)$ and $f_{t_2}(w)$ being bounded uniformly and $\phi_n(w)$ being integrable for $n = 1, \ldots, d$.

For C_{213}, note that

$$|C_{213}| \leq \frac{1}{N^2 T^3} \sum_{n=1}^{d} \sum_{t_1=1}^{T} \sum_{t_2=1}^{T} \sum_{i=2}^{N} \sum_{j=1}^{i-1} |E[\phi_n(u_{it_1})]E[\phi_n(u_{jt_2})]| \cdot |\sigma_e(i,j)|$$

$$\leq \frac{1}{N^2 T^3} \sum_{n=1}^{d} \sum_{t_1=1}^{T} \sum_{t_2=1}^{T} \sum_{i=2}^{N} \sum_{j=1}^{i-1} \frac{1}{d_{t_1}} \int |\phi_n(w)|dw \cdot \frac{1}{d_{t_2}} \int |\phi_n(w)|dw \cdot |\sigma_e(i,j)|$$

$$\leq O\left(\frac{1}{N^2 T^3}\right) \sum_{n=1}^{d} \sum_{t_1=1}^{T} \frac{1}{d_{t_1}} \sum_{t_2=1}^{T} \frac{1}{d_{t_2}} \sum_{i=2}^{N} \sum_{j=1}^{i-1} |\sigma_e(i,j)| = O\left(\frac{1}{NT^2}\right) = o\left(\frac{1}{NT}\right),$$

where the last line follows form f_{t_1} and f_{t_2} being bounded uniformly, $\phi_n(w)$ being integrable for $n = 1, \ldots, d$ and Assumption 1.3.(b).

Since $|C_{211}| = o\left(\frac{1}{NT}\right)$, $|C_{212}| = o\left(\frac{1}{NT}\right)$ and $|C_{213}| = o\left(\frac{1}{NT}\right)$, we obtain that $C_{21} = o_P\left(\frac{1}{\sqrt{NT}}\right)$.

By Assumption 1.3.(c), it is straightforward to obtain that

$$E\|C_{22}\|^2 = \frac{1}{N^2 T^4} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t_1=1}^{T} \sum_{t_2=1}^{T} \sum_{t_3=1}^{T} \sum_{t_4=1}^{T} E[v'_{it_1} e_{it_2} v_{jt_3} e_{jt_4}] = O\left(\frac{1}{NT^2}\right).$$

Thus, $C_{22} = o_P\left(\frac{1}{\sqrt{NT}}\right)$. Since $C_{21} = o_P\left(\frac{1}{\sqrt{NT}}\right)$ and $C_{22} = o_P\left(\frac{1}{\sqrt{NT}}\right)$, then we have $C_2 = o_P\left(\frac{1}{\sqrt{NT}}\right)$. In connection with that $C_1 = O_P\left(\frac{1}{\sqrt{NT}}\right)$, the result follows. ∎

4)

$$\frac{1}{NT} X'\gamma = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} (\phi(u_{it}) + v_{it})\gamma_k(u_{it}) - \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} (\phi(u_{it}) + v_{it})\gamma_k(u_{is}) \equiv D_1 - D_2$$

By (8) and (9) of Lemma A.2, $D_1 = O_P\left(\frac{1}{\sqrt{k^m T}}\right)$ follows immediately. D_2 can be expanded as

$$D_2 = \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} \phi(u_{it})\gamma_k(u_{is}) + \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} v_{it}\gamma_k(u_{is}) \equiv D_{21} + D_{22}.$$

For D_{21},

$$\|D_{21}\| \leq \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} \|\phi(u_{it})\gamma_k(u_{is})\| \leq O(1)\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} |\gamma_k(u_{it})|,$$

where the second inequality follows from $\phi_n(w)$ being bounded uniformly for $n = 1, \ldots, d$. For the summation on RHS above,

$$E\left|\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} |\gamma_k(u_{it})|\right|^2 = \frac{1}{N^2 T^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} E[|\gamma_k(u_{it})||\gamma_k(u_{js})|]$$

$$= \frac{1}{N^2 T^2} \sum_{i=1}^{N} \sum_{t=1}^{T} E[\gamma_k^2(u_{it})] + \frac{2}{N^2 T^2} \sum_{i=1}^{N} \sum_{t=2}^{T} \sum_{s=1}^{t-1} E[|\gamma_k(u_{it})||\gamma_k(u_{is})|]$$

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$$+\frac{2}{N^2T^2}\sum_{i=2}^{N}\sum_{j=1}^{i-1}\sum_{t=1}^{T}\sum_{s=1}^{T}E[|\gamma_k(u_{it})|]E[|\gamma_k(u_{js})|]$$

$$\leq O\left(\frac{1}{N^2T^2}\right)\sum_{i=1}^{N}\sum_{t=1}^{T}\frac{1}{d_t}\int\gamma_k^2(w)dw$$

$$+O\left(\frac{1}{N^2T^2}\right)\sum_{i=1}^{N}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\frac{1}{d_{ts}}\frac{1}{d_s}\int|\gamma_k(w)|^2dw\left\{\int f_{ts}^2(w)dw\right\}^{1/2}\left\{\int f_s^2(w)dw\right\}^{1/2}$$

$$+O\left(\frac{1}{N^2T^2}\right)\sum_{i=2}^{N}\sum_{j=1}^{i-1}\sum_{t=1}^{T}\sum_{s=1}^{T}\frac{1}{d_t}\frac{1}{d_s}\int|\gamma_k(w)|^2dw\left\{\int f_t^2(w)dw\right\}^{1/2}\left\{\int f_s^2(w)dw\right\}^{1/2}$$

$$\leq O\left(\frac{1}{NT^{3/2}}\right)+O\left(\frac{k^{-m}}{NT}\right)+O\left(\frac{k^{-m}}{T}\right),\tag{B.7}$$

where the first inequality follows from $f_t(w)$ being bounded uniformly and Cauchy-Schwarz inequality; the second inequality follows from (3) of Lemma A.1 and the fact that $f_t(w)$ and $f_{ts}(w)$ are both bounded uniformly. Then we have shown that $D_{21} = O_P\left(\frac{1}{\sqrt{k^mT}}\right)$.

We now focus on D_{22}.

$$E\|D_{22}\|^2 = \frac{1}{N^2T^4}\sum_{n=1}^{d}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t_1=1}^{T}\sum_{t_2=1}^{T}\sum_{t_3=1}^{T}\sum_{t_4=1}^{T}E[v_{it_1,n}v_{jt_3,n}]E[\gamma_k(u_{it_2})\gamma_k(u_{jt_4})]$$

$$\leq O(1)\frac{1}{N^2T^2}\sum_{n=1}^{d}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t_2=1}^{T}\sum_{t_4=1}^{T}|E[\gamma_k(u_{it_2})\gamma_k(u_{jt_4})]| \leq O\left(\frac{k^{-m}}{T}\right),$$

where the first equality follows from Assumption 1.4; the first inequality follows from Assumption 1.3.(a); the second inequality follows from (B.7). Then $D_{22} = O_P\left(\frac{1}{\sqrt{k^mT}}\right)$.

Based on the above, the result follows. ∎

5)

$$\frac{1}{N\sqrt{T}}\mathcal{Z}'\gamma = \frac{1}{N\sqrt{T}}\sum_{i=1}^{N}\sum_{t=1}^{T}Z_k(u_{it})\gamma_k(u_{it}) - \frac{\sqrt{T}}{N}\sum_{i=1}^{N}\bar{Z}_{k,i}\bar{\gamma}_{k,i} \equiv E_1 - E_2$$

In (1) of Lemma A.2, we have shown that $\|E_1\| = O_P(k^{-(m-1)/2})$. Then we just need to focus on E_2 below and write

$$\|E_2\| \leq \frac{1}{NT^{3/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}\|Z_k(u_{it})\gamma_k(u_{is})\| \leq O(k^{1/2})\frac{1}{N\sqrt{T}}\sum_{i=1}^{N}\sum_{t=1}^{T}|\gamma_k(u_{it})|,$$

where the second inequality follows from (6) of Lemma A.1. In (B.7) of this lemma, we have shown $\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}|\gamma_k(u_{it})| = O_P\left(\frac{k^{-m/2}}{\sqrt{T}}\right)$, so we easily obtain $\frac{1}{N\sqrt{T}}\sum_{i=1}^{N}\sum_{t=1}^{T}|\gamma_k(u_{it})| = O_P\left(k^{-m/2}\right)$. Based on the above, it further implies that $\|E_2\| = O_P(k^{-(m-1)/2})$. Then the result follows. ∎

6) For the first result, write

$$\frac{1}{NT}X'X = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}(\phi(u_{it}) + v_{it})(\phi(u_{it}) + v_{it})'$$

$$-\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}(\phi(u_{it}) + v_{it})(\phi(u_{is}) + v_{is})' \equiv F_1 - F_2.$$

By going through a procedure similar to (B.7), it is easy to show that

$$\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\phi(u_{it})\phi(u_{it})' \to_P 0.$$

In connection with (4) and (5) of Lemma A.2, we obtain that $F_1 \to_P \Sigma_v$ immediately.

We just need to focus on F_2 below.

$$F_2 = \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}\phi(u_{it})\phi(u_{is})' + \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}\phi(u_{it})v_{is}'$$

$$+\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}v_{it}\phi(u_{is})' + \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}v_{it}v_{is}'$$

$$\equiv F_{21} + F_{22} + F_{23} + F_{24}.$$

Notice that $F_{24} = o_P(1)$ follows from Assumption 1.3.(a) straightaway. We then focus on F_{21} below and write

$$E\|F_{21}\|^2 = \frac{1}{N^2T^4}\sum_{n_1=1}^{d}\sum_{n_2=1}^{d}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t_1=1}^{T}\sum_{t_2=1}^{T}\sum_{t_3=1}^{T}\sum_{t_4=1}^{T}E[\phi_{n_1}(u_{it_1})\phi_{n_2}(u_{it_2})\phi_{n_1}(u_{jt_3})\phi_{n_2}(u_{jt_4})]$$

$$= \frac{1}{N^2T^4}\sum_{n_1=1}^{d}\sum_{n_2=1}^{d}\sum_{i=1}^{N}\sum_{t_1=1}^{T}\sum_{t_2=1}^{T}\sum_{t_3=1}^{T}\sum_{t_4=1}^{T}E[\phi_{n_1}(u_{it_1})\phi_{n_2}(u_{it_2})\phi_{n_1}(u_{it_3})\phi_{n_2}(u_{it_4})]$$

$$+\frac{2}{N^2T^4}\sum_{n_1=1}^{d}\sum_{n_2=1}^{d}\sum_{i=2}^{N}\sum_{j=1}^{i-1}\sum_{t_1=1}^{T}\sum_{t_2=1}^{T}\sum_{t_3=1}^{T}\sum_{t_4=1}^{T}E[\phi_{n_1}(u_{it_1})\phi_{n_2}(u_{it_2})\phi_{n_1}(u_{jt_3})\phi_{n_2}(u_{jt_4})]$$

$$= F_{211} + 2F_{212}.$$

For F_{211}, write

$$F_{211} = \frac{1}{N^2T^4}\sum_{n_1=1}^{d}\sum_{n_2=1}^{d}\sum_{i=1}^{N}\sum_{t_1=1}^{T}\sum_{t_2=1}^{T}\sum_{t_3=1}^{T}\sum_{t_4=1}^{T}E[\phi_{n_1}(u_{it_1})\phi_{n_2}(u_{it_2})\phi_{n_1}(u_{it_3})\phi_{n_2}(u_{it_4})]$$

$$= \frac{1}{N^2T^4}\sum_{n_1=1}^{d}\sum_{n_2=1}^{d}\sum_{i=1}^{N}\sum_{\text{all } t_1,t_2,t_3,t_4 \text{ are different}}E[\phi_{n_1}(u_{it_1})\phi_{n_2}(u_{it_2})\phi_{n_1}(u_{it_3})\phi_{n_2}(u_{it_4})]$$

$$+\frac{1}{N^2T^4}\sum_{n_1=1}^{d}\sum_{n_2=1}^{d}\sum_{i=1}^{N}\sum_{\text{only two of } t_1,t_2,t_3,t_4 \text{ are same}}E[\phi_{n_1}(u_{it_1})\phi_{n_2}(u_{it_2})\phi_{n_1}(u_{it_3})\phi_{n_2}(u_{it_4})]$$

$$+\frac{1}{N^2T^4}\sum_{n_1=1}^{d}\sum_{n_2=1}^{d}\sum_{i=1}^{N}\sum_{\text{only three of } t_1,t_2,t_3,t_4 \text{ are same}}E[\phi_{n_1}(u_{it_1})\phi_{n_2}(u_{it_2})\phi_{n_1}(u_{it_3})\phi_{n_2}(u_{it_4})]$$

$$+\frac{1}{N^2T^4}\sum_{n_1=1}^{d}\sum_{n_2=1}^{d}\sum_{i=1}^{N}\sum_{\text{four of }t_1,t_2,t_3,t_4\text{ are same}}E[\phi_{n_1}(u_{it_1})\phi_{n_2}(u_{it_2})\phi_{n_1}(u_{it_3})\phi_{n_2}(u_{it_4})]$$

$$\equiv F_{2111}+F_{2112}+F_{2113}+F_{2114}.$$

For F_{2111}, without loss of generality, assume that $t_1>t_2>t_3>t_4$. For other cases, for example $t_2>t_3>t_1>t_4$, the analysis will be same and the order will remain same. Then

$$\frac{1}{N^2T^4}\sum_{n_1=1}^{d}\sum_{n_2=1}^{d}\sum_{i=1}^{N}\sum_{t_1=4}^{T}\sum_{t_2=3}^{t_1-1}\sum_{t_3=2}^{t_2-1}\sum_{t_4=1}^{t_3-1}|E[\phi_{n_1}(u_{it_1})\phi_{n_2}(u_{it_2})\phi_{n_1}(u_{it_3})\phi_{n_2}(u_{it_4})]|$$

$$\leq O\left(\frac{1}{N^2T^4}\right)\sum_{n_1=1}^{d}\sum_{n_2=1}^{d}\sum_{i=1}^{N}\sum_{t_1=4}^{T}\sum_{t_2=3}^{t_1-1}\sum_{t_3=2}^{t_2-1}\sum_{t_4=1}^{t_3-1}\frac{1}{d_{t_1t_2}}\frac{1}{d_{t_2t_3}}\frac{1}{d_{t_3t_4}}\frac{1}{d_{t_4}}$$

$$\cdot\iiiint|\phi_{n_1}(w_1)\phi_{n_2}(w_2)\phi_{n_1}(w_3)\phi_{n_2}(w_4)|dw_1dw_2dw_3dw_4$$

$$\leq O\left(\frac{1}{NT^2}\right)\sum_{n_1=1}^{d}\sum_{n_2=1}^{d}\left(\int|\phi_{n_1}(w)|dw\right)^2\left(\int|\phi_{n_2}(w)|dw\right)^2\leq O\left(\frac{1}{NT^2}\right)=o(1),$$

where the last inequality follows from $\phi_n(w)$ being integrable for $n=1,\ldots,d$.

For F_{2112}, we focus on the case of $t_1=t_2>t_3>t_4$. For other cases, for example $t_1=t_3>t_2>t_4$, the analysis will be even simpler and the order will remain same. Then write

$$\frac{1}{N^2T^4}\sum_{n_1=1}^{d}\sum_{n_2=1}^{d}\sum_{i=1}^{N}\sum_{t_1=3}^{T}\sum_{t_3=2}^{t_1-1}\sum_{t_4=1}^{t_3-1}|E[\phi_{n_1}(u_{it_1})\phi_{n_2}(u_{it_1})\phi_{n_1}(u_{it_3})\phi_{n_2}(u_{it_4})]|$$

$$\leq O\left(\frac{1}{N^2T^4}\right)\sum_{n_1=1}^{d}\sum_{n_2=1}^{d}\sum_{i=1}^{N}\sum_{t_1=3}^{T}\sum_{t_3=2}^{t_1-1}\sum_{t_4=1}^{t_3-1}\frac{1}{d_{t_1t_3}}\frac{1}{d_{t_3t_4}}\frac{1}{d_{t_4}}$$

$$\cdot\iiint|\phi_{n_1}(w_1)\phi_{n_2}(w_1)\phi_{n_1}(w_2)\phi_{n_2}(w_3)|dw_1dw_2dw_3$$

$$\leq O\left(\frac{1}{NT^{5/2}}\right)\sum_{n_1=1}^{d}\sum_{n_2=1}^{d}\left\{\int\phi_{n_1}^2(w)dw\int\phi_{n_2}^2(w)dw\right\}^{1/2}\int|\phi_{n_1}(w)|dw\int|\phi_{n_2}(w)|dw$$

$$\leq O\left(\frac{1}{NT^{5/2}}\right),$$

where the last line follows from $\phi_n(w)$ being integrable and $\phi_n(w)\in L^2(\mathbb{R})$ for $n=1,\ldots,d$.

For F_{2113}, we also focus on the case of $t_1=t_2=t_3>t_4$. For other cases, for example $t_1=t_3=t_4>t_2$, the analysis will be same and the order will remain same. Then write

$$\frac{1}{N^2T^4}\sum_{n_1=1}^{d}\sum_{n_2=1}^{d}\sum_{i=1}^{N}\sum_{t_1=2}^{T}\sum_{t_4=1}^{t_1-1}|E[\phi_{n_1}(u_{it_1})\phi_{n_2}(u_{it_1})\phi_{n_1}(u_{it_1})\phi_{n_2}(u_{it_4})]|$$

$$\leq O\left(\frac{1}{N^2T^4}\right)\sum_{n_1=1}^{d}\sum_{n_2=1}^{d}\sum_{i=1}^{N}\sum_{t_1=2}^{T}\sum_{t_4=1}^{t_1-1}\frac{1}{d_{t_1t_4}}\frac{1}{d_{t_4}}$$

$$\cdot\iint|\phi_{n_1}(w_1)\phi_{n_2}(w_1)\phi_{n_1}(w_1)\phi_{n_2}(w_2)|dw_1dw_2$$

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$$\leq O\left(\frac{1}{NT^3}\right) \sum_{n_1=1}^{d} \sum_{n_2=1}^{d} \int \phi_{n_1}^2(w)dw \int |\phi_{n_2}(w)|dw = O\left(\frac{1}{NT^3}\right),$$

where the last line follows from $\phi_n(w)$ being integrable and $\phi_n(w) \in L^2(\mathbb{R})$ for $n = 1, \ldots, d$.

For F_{2114}, write

$$F_{2114} = \frac{1}{N^2T^4} \sum_{n_1=1}^{d} \sum_{n_2=1}^{d} \sum_{i=1}^{N} \sum_{t=1}^{T} \int \phi_{n_1}^2(d_t w)\phi_{n_2}^2(d_t w)f_t(w)dw$$

$$\leq O\left(\frac{1}{N^2T^4}\right) \sum_{i=1}^{N} \sum_{t=1}^{T} \frac{1}{d_t} \leq O\left(\frac{1}{NT^{7/2}}\right),$$

where the first inequality follows from $\phi_n(w)$ being bounded uniformly and $\phi_n(w) \in L^2(\mathbb{R})$ for $n = 1, \ldots, d$.

Combining F_{2111}, F_{2112}, F_{2113} and F_{2114} together, we obtain that $F_{211} = o(1)$.

We now turn to F_{212} and write

$$\left| E\left[\sum_{t_1=1}^{T} \sum_{t_2=1}^{T} \phi_{n_1}(u_{it_1})\phi_{n_2}(u_{it_2}) \right] \right|$$

$$\leq \sum_{t=1}^{T} E[|\phi_{n_1}(u_{it})\phi_{n_2}(u_{it})|] + \sum_{t=2}^{T} \sum_{s=1}^{t-1} E[|\phi_{n_1}(u_{it})\phi_{n_2}(u_{is})|] + \sum_{t=2}^{T} \sum_{s=1}^{t-1} E[|\phi_{n_2}(u_{it})\phi_{n_1}(u_{is})|]$$

$$\leq O(1) \sum_{t=1}^{T} \frac{1}{d_t} \int |\phi_{n_1}(w)|dw + O(1) \sum_{t=2}^{T} \sum_{s=1}^{t-1} \frac{1}{d_{ts}} \frac{1}{d_s} \int |\phi_{n_1}(w)|dw$$

$$\leq O(1)\sqrt{T} + O(1)T = O(T),$$

where the last inequality follows from $\phi_n(w)$ being integrable for $n = 1, \ldots, d$. Therefore,

$$|F_{212}| \leq O\left(\frac{1}{N^2T^4}\right) \sum_{n=1}^{d} \sum_{i=2}^{N} \sum_{j=1}^{i-1} T^2 = o(1).$$

Since $F_{211} = o(1)$ and $F_{212} = o(1)$, we have shown that $\|F_{21}\| = o_P(1)$. Similarly, we can show that $\|F_{22}\| = o_P(1)$ and $\|F_{23}\| = o_P(1)$. Therefore, the result follows. ∎

Proof of Corollary 2.1:

We need only to prove the first result of this corollary. The second result then follows immediately.

1) By (6) of Lemma A.4, $\widehat{\Sigma}_v = \frac{1}{NT}X'X \to_P \Sigma_v$. Thus, we just need to focus on $\widehat{\sigma}_e^2$, where

$$\widehat{\sigma}_e^2 = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} (\tilde{X}_{it}'(\beta_0 - \widehat{\beta}) + \tilde{Z}_k(u_{it})'(C - \widehat{C}) + \tilde{\gamma}_k(u_{it}) + \tilde{e}_{it})^2. \tag{B.8}$$

Now denote that $A_1 = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} (\tilde{X}_{it}'(\beta_0 - \widehat{\beta}))^2$, $A_2 = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} (\tilde{Z}_k(u_{it})'(C - \widehat{C}))^2$, $A_3 = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \tilde{\gamma}_k^2(u_{it})$ and $A_4 = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \tilde{e}_{it}^2$.

For A_1, write

$$|A_1| \leq \left\| \beta_0 - \widehat{\beta} \right\|^2 \cdot \left\| \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \tilde{X}_{it} \tilde{X}_{it}' \right\| = o_P(1),$$

where the last equality follows from Theorem 3.1 and (6) of Lemma A.4.

For A_2, write

$$|A_2| \leq \left\| C - \widehat{C} \right\|^2 \cdot \frac{1}{\sqrt{T}} \left\| \frac{1}{N\sqrt{T}} \sum_{i=1}^{N} \sum_{t=1}^{T} \tilde{Z}_k(u_{it}) \tilde{Z}_k(u_{it})' \right\| = o_P(1),$$

where the last equality follows from Lemma 3.1, Lemma 2.1 and Assumption 2.2.

For A_3, by (2) of Lemma A.1, $\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \tilde{\gamma}_k^2(u_{it}) = O(k^{-m+5/6}) = o(1)$.

For A_4, write

$$A_4 = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} e_{it}^2 - \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} e_{it} e_{is} \equiv A_{41} - A_{42}.$$

For A_{41}, Assumption 1.3.(b),

$$E\left[A_{41}^2 - \sigma_e^2\right]^2 = \frac{1}{N^2 T^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} E[(e_{it}^2 - \sigma_e^2)(e_{js}^2 - \sigma_e^2)] = o\left(1\right).$$

For A_{42}

$$E\left[A_{42}^2\right] = \frac{1}{N^2 T^4} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t_1=1}^{T} \sum_{t_2=1}^{T} \sum_{t_3=1}^{T} \sum_{t_4=1}^{T} E[e_{it_1} e_{it_2} e_{jt_3} e_{jt_4}] = o\left(1\right),$$

where the RHS follows from e_t being martingale difference sequence (cf., Assumption 1.3.(b)).

Therefore, we have shown that $A_1 \to_P 0$, $A_2 \to_P 0$, $A_3 \to_P 0$ and $A_4 \to_P \sigma_e^2$. Based on the above, all the interaction terms generated by $\tilde{X}_{it}'(\beta_0 - \widehat{\beta})$, $\tilde{Z}_k(u_{it})'(C - \widehat{C})$ and $\tilde{\gamma}_k(u_{it})$ from the expansion of (B.8) can be shown converging to 0 in probability easily. For example,

$$\left| \frac{2}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \tilde{X}_{it}'(\beta_0 - \widehat{\beta}) \tilde{Z}_k(u_{it})'(C - \widehat{C}) \right| \leq \frac{2}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \left| \tilde{X}_{it}'(\beta_0 - \widehat{\beta}) \tilde{Z}_k(u_{it})'(C - \widehat{C}) \right|$$

$$\leq \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \left| \tilde{X}_{it}'(\beta_0 - \widehat{\beta}) \right|^2 + \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \left| \tilde{Z}_k(u_{it})'(C - \widehat{C}) \right|^2 = A_1 + A_2 = o_P(1).$$

We now focus on the interaction terms generated by $\tilde{e}_{it}$.

Firstly,

$$\left| \frac{2}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \tilde{X}_{it}'(\beta_0 - \widehat{\beta}) \tilde{e}_{it} \right| \leq \left\| \beta_0 - \widehat{\beta} \right\| \cdot \left\| \frac{2}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \tilde{X}_{it} \tilde{e}_{it} \right\| = o_P(1),$$

where the last equality follows from Theorem 3.1 and (3) of Lemma A.4.

Secondly,

$$\left| \frac{2}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \tilde{Z}_k(u_{it})'(C - \widehat{C}) \tilde{e}_{it} \right| \leq \left\| C - \widehat{C} \right\| \cdot \left\| \frac{2}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \tilde{Z}_k(u_{it}) \tilde{e}_{it} \right\| = o_P(1),$$

where the last equality follows from Lemma 3.1 and (1) of Lemma A.4.

Thirdly, by similar approach to (9) of Lemma A.2, $\left| \frac{2}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \tilde{\gamma}_k(u_{it}) \tilde{e}_{it} \right| = o_P(1)$.

Therefore, based on the above, the result follows. ∎

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