Online Supporting Materials

Comparison with Other Related Models

First, let us discuss the univariate linear ALT model with time-variant autoregressive parameters, as this model can be considered as one of the most general forms of longitudinal models with SEM framework (Bollen & Curran, 2004). The ALT model is commonly described in SEM framework with intercept (α) and slope (β) factors, but it is possible to express it with mixed-effects model framework, as follows:

$$Y_{jt} = \beta_{0j} + \beta_{1j}X_t + \rho_{t,t-1}Y_{j(t-1)} + e_{jt}, \tag{1}$$

$$\beta_{0j} = \gamma_0 + r_{0j}, \tag{2}$$

$$\beta_{1j} = \gamma_1 + r_{1j}, \tag{3}$$

We used the notations analogous to the standard linear GCM in Equations 1-3 in the manuscript to enhance the comparability of the models. $\rho_{t,t-1}$ is the time-variant autoregressive parameter, representing the impact of the prior value of $Y_{j(t-1)}$ on the current value (Y_{jt}). Intercept and slope factors are represented by β_{0j} and β_{1j}, respectively. Basically, in ALT, parameters and variables can be interpreted in the same way as the standard linear GCM, except that the ALT model considers the conditional distribution of Y_{jt} given the prior value of $Y_{j(t-1)}$. Also, Y_{j1} is usually treated as predetermined in the ALT model and Y_{j1} can be expressed simply by an unconditional mean and an individual deviation from the mean as:

$$Y_{j1} = \nu_1 + e_{j1}. \tag{4}$$

From these equations, it is clear that the ALT model reduces to the standard linear GCM of Equations 1-3 in the manuscript when $\rho_{t,t-1} = 0$. Due to the presence of autoregressive parameters $\rho_{t,t-1}$, the ALT model is generally recognized as a more flexible model that can be fit to many different types of longitudinal data.

However, the above equations of ALT model also make it clear that the model does not include the random effects which vary across time points but are constant across participants within the given time points (i.e. r_t). Indeed, given the absence of common random effects across participants, the ALT model cannot account for the correlations between participants within the same time points (see covariance structure in Equation 23), which is the critical feature of the proposed GCM with time-specific errors. In addition, the ALT model accounts for time-specific effects by the (time-variant) autoregressive parameters $\rho_{t,t-1}$ as well as growth parameters (β_0 and β_1). This means that time-specific effects are accounted for as part of the (true) mean structure of the model, rather than as random errors in covariance structure, making the interpretations of the estimated true growth conflated and difficult.

In general, longitudinal models with the SEM framework that we mentioned above have inherent difficulty in incorporating and accounting for time-specific errors. This is because we can consider only one random unit (i.e. participants) in SEM parametrization. In other words, in standard SEM parametrization, we cannot describe the residual covariance between persons, the critical feature of the current model [1]. Thus, the proposed model is critically different from the ALT model or other related SEM models, with substantially different mean- and covariance structures.

The same point applies to other longitudinal models, including time-series models (e.g., Hamilton, 1994) and continuous-time autoregression models (e.g., Voelkle, Oud, Davidov, & Schmidt, 2012). For example, LC-LSTM (Alessandri, Caprara, & Tisak, 2012) accounts for random events ζ_t as prediction error in predicting constructs by multiple basis curves, but unfortunately it fails to consider covariance between persons within the same time points. The unconditional asset pricing model of Shanken (1990) and the time-varying effect model (e.g., Shiyko, Lanza, Tan, Li, & Shiffman, 2012) do not explicitly incorporate time-specific errors in time-varying intercepts/coefficients. Time varying VAR models (Primiceri, 2005) consider random walks in time-varying autoregressive coefficients, but do not include time-varying (but subject-invariant) intercepts. From a different perspective, in the literature of biometrics, Verbeke and Molenberghs (2000) and Davidian and Giltinan (2003) discussed an interesting model which attempted to decompose residuals into particular realization of observed errors and measurement errors. The observed errors appear similar to time-specific errors, but these random errors are not constant across participants; thus, these errors are not equivalent to time-specific errors that we defined in this manuscript. In econometric literature, time-specific effects are sometimes assumed in the dynamic model (i.e. outcome Y_t is regressed on $Y_{(t-1)}$ and other covariates $X_{(t-1)}$ at previous time point) for panel data to model temporal influence (e.g., economic conditions) on each firm (or individual). However, in the dynamic model, unlike GCM each time point (t) is *not* controlled as a predictor ($x_t = t - 1$). This indicates that this time-specific effects can implicitly include changes of group means. On the other hand, time-specific errors assumed in the proposed GCM is caused by extraneous factors that are irrelevant to such growth/change aspect. Thus, time-specific effects in the dynamic model is not equivalent to time-specific errors that we defined.

In summary, although many models in longitudinal data analysis have been proposed in the literature to account for time specific effects, our model represents a new class of model in longitudinal data analysis, in that our model explicitly incorporates random effects which vary across time points but are constant across participants within given time points (i.e. time-specific errors). One important implication is that, as these existing longitudinal models do not take into account time-specific errors, they should suffer from the inflation of Type-1 error rates as discussed in the current manuscript, if we apply these models to the data that include time-specific errors. It is not easy to find an effective solution, especially within the SEM framework, as the residual covariance between individuals within given time points (Equation 23) is not part of standard SEM parametrization. One potential strategy is to re-parametrize the model with a mixed-effects model framework as we did for the ALT model, and explicitly incorporate and estimate time-specific errors using the software for mixed-effects or cross-classified modelling. This strategy, however, may not be possible as the model becomes more complicated, and future research is required to address the issue more effectively.

[1]With a special modelling strategy, it is possible to consider covariance between individuals (Mehta & Neale, 2005). However, we believe the proposed GCM should provide a more direct and flexible way to account for possible time-specific errors in longitudinal design.

Standard Errors of Fixed Effects Estimates

Without loss of generality we set $\gamma_0 = 0$ and $\sigma^2 = 1$, resulting in $\Delta = \gamma_1$, $\rho_0 = \sigma_0^2$ and $\rho_t = \sigma_t^2$ from the relations in Equations 12-13 in the manuscript. The Equation 8 in the manuscript can be expressed in vector notation as:

$$\boldsymbol{Y} = \gamma_1 \boldsymbol{X} + \boldsymbol{\epsilon}. \tag{5}$$

Here, $\boldsymbol{Y}$ is an $(T \times N) \times 1$ outcome vector, and its elements are arranged as $\boldsymbol{Y} = (\boldsymbol{Y_1'}, \ldots, \boldsymbol{Y_j'}, \ldots, \boldsymbol{Y_N'})'$, where $\boldsymbol{Y_j} = (Y_{j1}, \ldots, Y_{jt}, \ldots, Y_{jT})'$. $\boldsymbol{X}$ is a corresponding $(T \times N) \times 1$ vector expressed as $\boldsymbol{X} = \mathbf{1_N} \otimes (0, 1, \ldots, T-1)'$, where $\otimes$ indicates a Kronecker product. $\boldsymbol{\epsilon}$ is also a corresponding $(T \times N) \times 1$ residual vector consisting of $\boldsymbol{\epsilon} = e_{jt} + r_{0j} + r_t$. From Equations 9-11 in the manuscript, it can be shown that $\boldsymbol{\epsilon}$ is distributed as $\boldsymbol{\epsilon} \sim N(\mathbf{0}, \tilde{\boldsymbol{\Sigma}})$, where

$$\tilde{\boldsymbol{\Sigma}} = \sigma_e^2 \boldsymbol{I_{NT}} + \rho_0(\boldsymbol{I_N} \otimes \mathbf{1_T}\mathbf{1_T'}) + \rho_t(\mathbf{1_N}\mathbf{1_N'} \otimes \boldsymbol{I_T}) \tag{6}$$

Here we assume that $\sigma_e^2 \geq 0$, $\rho_0 \geq 0$, and $\rho_t \geq 0$, and that the inverse matrix of $\tilde{\boldsymbol{\Sigma}}$ (denoted as $\tilde{\boldsymbol{\Sigma}}^{-1}$)

exists. When ρ_t, ρ_0 and σ_e^2 are known, $E(\hat{\gamma}_1) = \gamma_1$, because

$$E(\hat{\gamma}_1) = E((\boldsymbol{X}'\tilde{\boldsymbol{\Sigma}}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}'\tilde{\boldsymbol{\Sigma}}^{-1}\boldsymbol{Y}) = (\boldsymbol{X}'\tilde{\boldsymbol{\Sigma}}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}'\tilde{\boldsymbol{\Sigma}}^{-1}E(\boldsymbol{Y}) = (\boldsymbol{X}'\tilde{\boldsymbol{\Sigma}}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}'\tilde{\boldsymbol{\Sigma}}^{-1}E(\boldsymbol{X}\gamma_1) = \gamma_1. \quad (7)$$

This result unchanges regardless of the value of ρ_t, indicating that expected values of linear slope estimates are the same between the standard GCM (i.e., Equations 1-3 in the manuscript) and the GCM assuming time-specific errors (i.e., Equations 5-7 in the manuscript).

Let the diagonal elements of $\tilde{\boldsymbol{\Sigma}}^{-1}$ be a, off-diagonal elements that share the same participant be b, off-diagonal elements that share the same time point be c, and other elements be d. Comparing the left and right sides of the identity $\tilde{\boldsymbol{\Sigma}}\tilde{\boldsymbol{\Sigma}}^{-1} = \boldsymbol{I}$, the following relations are obtained:

$$a + (T-1)b\rho_0 + (N-1)\rho_t c = 1 \quad (8)$$

$$b + \rho_0 a + (T-2)\rho_0 b + (N-1)\rho_t d = 0 \quad (9)$$

$$c + (T-1)d\rho_0 + \rho_t a + (N-2)\rho_t c = 0 \quad (10)$$

$$d + \rho_0 c + (T-2)\rho_0 d + \rho_t b + (N-2)\rho_t d = 0 \quad (11)$$

Simple calculation of the above simultaneous equations provide the following result (see also Hsiao

(2014) for a different expression of $\tilde{\boldsymbol{\Sigma}}^{-1}$ that explicitly denotes determinant):

$$a = -\frac{AA}{(\rho_t + \rho_0 - 1)(\rho_t N - \rho_t - \rho_0 + 1)(\rho_0 T - \rho_t - \rho_0 + 1)(\rho_0 T + \rho_t N - \rho_t - \rho_0 + 1)} \tag{12}$$

$$b = \frac{\rho_0(\rho_0\rho_t NT - 2\rho_0\rho_t T - \rho_0^2 T + \rho_0 T + \rho_t^2 N^2 - 3\rho_t^2 N - 2\rho_0\rho_t N + 2\rho_t N + 3\rho_t^2 + 4\rho_0\rho_t - 4\rho_t + \rho_0^2 - 2\rho_0 + 1)}{(\rho_t + \rho_0 - 1)(\rho_t N - \rho_t - \rho_0 + 1)(\rho_0 T - \rho_t - \rho_0 + 1)(\rho_0 T + \rho_t N - \rho_t - \rho_0 + 1)} \tag{13}$$

$$c = \frac{\rho_t(\rho_0^2 T^2 + \rho_0\rho_t NT - 2\rho_0\rho_t T - 3\rho_0^2 T + 2\rho_0 T - \rho_t^2 N - 2\rho_0\rho_t N + \rho_t N + \rho_t^2 + 4\rho_0\rho_t - 2\rho_t + 3\rho_0^2 - 4\rho_0 + 1)}{(\rho_t + \rho_0 - 1)(\rho_t N - \rho_t - \rho_0 + 1)(\rho_0 T - \rho_t - \rho_0 + 1)(\rho_0 T + \rho_t N - \rho_t - \rho_0 + 1)} \tag{14}$$

$$d = -\frac{\rho_0\rho_t(\rho_0 T + \rho_t N - 2\rho_t - 2\rho_0 + 2)}{(\rho_t + \rho_0 - 1)(\rho_t N - \rho_t - \rho_0 + 1)(\rho_0 T - \rho_t - \rho_0 + 1)(\rho_0 T + \rho_t N - \rho_t - \rho_0 + 1))}. \tag{15}$$

Here,

$$\begin{aligned} AA = {} & \rho_0^2\rho_t NT^2 - 2\rho_0^2\rho_t T^2 - \rho_0^3 T^2 + \rho_0^2 T^2 + \rho_0\rho_t^2 N^2 T - 4\rho_0\rho_t^2 NT - 4\rho_0^2\rho_t NT + 3\rho_0\rho_t NT + 4\rho_0\rho_t^2 T \\ & + 8\rho_0^2\rho_t T - 6\rho_0\rho_t T + 3\rho_0^3 T - 5\rho_0^2 T + 2\rho_0 T - \rho_t^3 N^2 - 2\rho_0\rho_t^2 N^2 + \rho_t^2 N^2 + 3\rho_t^3 N + 8\rho_0\rho_t^2 N - 5\rho_t^2 N \\ & + 4\rho_0^2\rho_t N - 6\rho_0\rho_t N + 2\rho_t N - 2\rho_t^3 - 8\rho_0\rho_t^2 + 5\rho_t^2 - 8\rho_0^2\rho_t + 12\rho_0\rho_t - 4\rho_t - 2\rho_0^3 + 5\rho_0^2 - 4\rho_0 + 1. \end{aligned} \tag{16}$$

Using the generalized least squares estimators, a sample distribution of $\hat{\gamma}_1$ can be expressed as

$$\hat{\gamma}_1 \sim N((\boldsymbol{X}'\tilde{\boldsymbol{\Sigma}}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}'\tilde{\boldsymbol{\Sigma}}^{-1}\boldsymbol{Y}, (\boldsymbol{X}'\tilde{\boldsymbol{\Sigma}}^{-1}\boldsymbol{X})^{-1}), \tag{17}$$

$se(\hat{\gamma}_1)$ corresponds to a squareroot of $se^2(\hat{\gamma}_1) = 1/(\boldsymbol{X'}\tilde{\boldsymbol{\Sigma}}^{-1}\boldsymbol{X})$. Here, $\boldsymbol{X'}\tilde{\boldsymbol{\Sigma}}^{-1}\boldsymbol{X}$ can be calculated as:

$$\begin{aligned}
\boldsymbol{X'}\tilde{\boldsymbol{\Sigma}}^{-1}\boldsymbol{X} &= Na\sum_{t=0}^{T-1} t^2 + Nb\left(\left[\sum_{t=0}^{T-1} t\right]^2 - \sum_{t=0}^{T-1} t^2\right) + N(N-1)c\sum_{t=0}^{T-1} t^2 + N(N-1)d\left(\left[\sum_{t=0}^{T-1} t\right]^2 - \sum_{t=0}^{T-1} t^2\right)\\
&= Na\frac{T(T-1)(2T-1)}{6} + Nb\left(\left[\frac{T(T-1)}{2}\right]^2 - \frac{T(T-1)(2T-1)}{6}\right) + N(N-1)c\frac{T(T-1)(2T-1)}{6}\\
&\quad +N(N-1)d\left(\left[\frac{T(T-1)}{2}\right]^2 - \frac{T(T-1)(2T-1)}{6}\right)\\
&= \frac{NT(T-1)}{12}[2a(2T-1) + b(T-2)(3T-1) + 2c(N-1)(2T-1) + d(N-1)(T-2)(3T-1)].
\end{aligned} \tag{18}$$

From the Equations 12-16, the above equation can be further simplified as

$$\frac{N(T-1)T(\rho_0 T^2 + 4\rho_t NT - 4\rho_t T - 3\rho_0 T + 4T - 2\rho_t N + 2\rho_t + 2\rho_0 - 2)}{12(\rho_t N - \rho_t - \rho_0 + 1)(\rho_0 T + \rho_t N - \rho_t - \rho_0 + 1)} \tag{19}$$

Thus, $se(\hat{\gamma}_1)$ can now be expressed as

$$se(\hat{\gamma}_1) = \sqrt{\frac{12(\rho_t N - \rho_t - \rho_0 + 1)(\rho_0 T + \rho_t N - \rho_t - \rho_0 + 1)}{N(T-1)T(\rho_0 T^2 + 4\rho_t NT - 4\rho_t T - 3\rho_0 T + 4T - 2\rho_t N + 2\rho_t + 2\rho_0 - 2)}}, \tag{20}$$

by the functions of ρ_t, ρ_0, N and T.

Let $se(\hat{\gamma}_{1mis})$ be a similar standard error of estimate of overall slope mean when we wrongly use the standard GCM (i.e., Equations 1-3 in the manuscript with assuming $\sigma_1^2 = 0$) but time-specific errors actually exist (i.e., $\rho_t > 0$). $se(\hat{\gamma}_{1mis})$ can be expressed by setting $\rho_t = 0$ and substituting

$\rho_0' = \rho_0/(1-\rho_t)$ into ρ_0 in Equation 20 to satisfy the relation $\sigma^2 = 1$ as:

$$\begin{aligned} se(\hat{\gamma}_{1mis}) &= \sqrt{\frac{12(1-\rho_0')(\rho_0'T-\rho_0'+1)}{N(T-1)T(\rho_0'T^2-3\rho_0'T+4T+2\rho_0'-2)}} \\ &= \sqrt{\frac{12(1-\rho_t-\rho_0)(\rho_0T-\rho_t-\rho_0+1)}{(1-\rho_t)N(T-1)T(\rho_0T^2-4\rho_tT-3\rho_0T+4T+2\rho_t+2\rho_0-2)}}, \end{aligned} \tag{21}$$

From Equations 20-21, the shrinkage factor $\frac{se(\hat{\gamma}_{1mis})}{se(\hat{\gamma}_1)}$ is calculated as

$$\sqrt{\frac{(1-\rho_t-\rho_0)(\rho_0T-\rho_t-\rho_0+1)(\rho_0T^2+4\rho_tNT-4\rho_tT-3\rho_0T+4T-2\rho_tN+2\rho_t+2\rho_0-2)}{(1-\rho_t)(\rho_tN-\rho_t-\rho_0+1)(\rho_0T+\rho_tN-\rho_t-\rho_0+1)(\rho_0T^2-4\rho_tT-3\rho_0T+4T+2\rho_t+2\rho_0-2)}}. \tag{22}$$

In the quadratic GCM (Equations 16-19 in the manuscript), a matrix expression analogous to Equation 5 becomes

$$\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{\epsilon}. \tag{23}$$

$\boldsymbol{\beta} = (\gamma_1, \gamma_2)'$ and $\boldsymbol{X} = (\boldsymbol{X_1}, \boldsymbol{X_2})$, where $\boldsymbol{X_1} = \mathbf{1_N} \otimes (0, 1, \ldots, T-1)'$ and $\boldsymbol{X_2} = \mathbf{1_N} \otimes (0^2, 1^2, \ldots, (T-1)^2)'$. Then precision matrix of regression coefficients $\boldsymbol{P} = \boldsymbol{X'\tilde{\Sigma}^{-1}X}$ can be evaluated by the similar manner as the linear GCM. Specifically, (1,1), (1,2) and (2,2) elements of $\boldsymbol{P}$ (P_{11}, P_{12} and P_{22}) can be evaluated as

$$P_{11} = aN\sum_{t=0}^{T-1} t^2 + bN\left(\sum_{t=0}^{T-1} t \sum_{t=0}^{T-1} t - \sum_{t=0}^{T-1} t^2\right) + cN(N-1)\sum_{t=0}^{T-1} t^2 + dN(N-1)\left(\sum_{t=0}^{T-1} t \sum_{t=0}^{T-1} t - \sum_{t=0}^{T-1} t^2\right) \tag{24}$$

$$P_{12} = aN\sum_{t=0}^{T-1} t^3 + bN\left(\sum_{t=0}^{T-1} t \sum_{t=0}^{T-1} t^2 - \sum_{t=0}^{T-1} t^3\right) + cN(N-1)\sum_{t=0}^{T-1} t^3 + dN(N-1)\left(\sum_{t=0}^{T-1} t \sum_{t=0}^{T-1} t^2 - \sum_{t=0}^{T-1} t^3\right) \tag{25}$$

$$P_{22} = aN\sum_{t=0}^{T-1} t^4 + bN\left(\sum_{t=0}^{T-1} t^2 \sum_{t=0}^{T-1} t^2 - \sum_{t=0}^{T-1} t^4\right) + cN(N-1)\sum_{t=0}^{T-1} t^4 + dN(N-1)\left(\sum_{t=0}^{T-1} t^2 \sum_{t=0}^{T-1} t^2 - \sum_{t=0}^{T-1} t^4\right) \tag{26}$$

Using the Equations 12-16, standard errors $se(\hat{\gamma}_1)$ and $se(\hat{\gamma}_2)$ can then be evaluated as:

$$\begin{aligned} se(\hat{\gamma}_1) &= \sqrt{\frac{P_{22}}{P_{11}P_{22} - P_{12}^2}} \\ &= \sqrt{\frac{12(\rho_t N - \rho_t - \rho_0 + 1)(2T-1)(8\rho_0 T^3 + 18\rho_t N T^2 - 18\rho_t T^2 - 21\rho_0 T^2 + 18T^2 - 18\rho_t NT + 18\rho_t T + 7\rho_0 T - 18T - 6\rho_t N + 6\rho_t + 6\rho_0 - 6)}{N(T-2)(T-1)T(T+1)(\rho_0 T^3 + 9\rho_t N T^2 - 9\rho_t T^2 - 6\rho_0 T^2 + 9T^2 - 9\rho_t NT + 9\rho_t T + 11\rho_0 T - 9T + 6\rho_t N - 6\rho_t - 6\rho_0 + 6)}} \end{aligned} \tag{27}$$

$$\begin{aligned} se(\hat{\gamma}_2) &= \sqrt{\frac{P_{11}}{P_{11}P_{22} - P_{12}^2}} \\ &= \sqrt{\frac{180(\rho_t N - \rho_t - \rho_0 + 1)(\rho_0 T^2 + 4\rho_t NT - 4\rho_t T - 3\rho_0 T + 4T - 2\rho_t N + 2\rho_t + 2\rho_0 - 2)}{N(T-2)(T-1)T(T+1)(\rho_0 T^3 + 9\rho_t N T^2 - 9\rho_t T^2 - 6\rho_0 T^2 + 9T^2 - 9\rho_t NT + 9\rho_t T + 11\rho_0 T - 9T + 6\rho_t N - 6\rho_t - 6\rho_0 + 6)}} \end{aligned} \tag{28}$$

Let $\hat{\gamma}_{1mis}$ and $\hat{\gamma}_{2mis}$ be parameter estimates of slopes for linear and quadratic changes when we wrongly use the standard quadratic GCM (i.e., setting $r_t = 0$ for all t in Equation 17 in the manuscript). Their standard errors $se(\hat{\gamma}_{1mis})$ and $se(\hat{\gamma}_{2mis})$ can be expressed by setting $\rho_t = 0$ and substituting $\rho_0' = \rho_0/(1-\rho_t)$ into ρ_0 in Equations 27-28 to satisfy the relation $\sigma^2 = 1$ as:

$$se(\hat{\gamma}_{1mis}) = \sqrt{\frac{12(1-\rho_0)(2T-1)(8\rho_0 T^3 - 21\rho_0 T^2 + 18T^2 + 7\rho_0 T - 18T + 6\rho_0 - 6)}{N(T-2)(T-1)T(T+1)(\rho_0 T^3 - 6\rho_0 T^2 + 9T^2 + 11\rho_0 T - 9T - 6\rho_0 + 6)}}, \tag{29}$$

$$se(\hat{\gamma}_{2mis}) = \sqrt{\frac{180(1-\rho_0)(\rho_0 T^2 - 3\rho_0 T + 4T + 2\rho_0 - 2)}{N(T-2)(T-1)T(T+1)(\rho_0 T^3 - 6\rho_0 T^2 + 9T^2 + 11\rho_0 T - 9T - 6\rho_0 + 6)}}. \tag{30}$$

Thus, shrinkage factors $se(\hat{\gamma}_{1mis})/se(\hat{\gamma}_1)$ and $se(\hat{\gamma}_{2mis})/se(\hat{\gamma}_2)$ can be calculated as:

$$\frac{se(\hat{\gamma}_{1mis})}{se(\hat{\gamma}_1)} = \sqrt{\frac{(1-\rho_t)(\rho_t N - \rho_t - \rho_0 + 1)(\rho_0 T^3 - 9\rho_t T^2 - 6\rho_0 T^2 + 9T^2 + 9\rho_t T + 11\rho_0 T - 9T - 6\rho_t - 6\rho_0 + 6)(8\rho_0 T^3 + 18\rho_t N T^2 - 18\rho_t T^2 - 21\rho_0 T^2 + 18T^2 - 18\rho_t NT + 18\rho_t T + 7\rho_0 T - 18T - 6\rho_t N + 6\rho_t + 6\rho_0 - 6)}{(1-\rho_t-\rho_0)(\rho_0 T^3 + 9\rho_t N T^2 - 9\rho_t T^2 - 6\rho_0 T^2 + 9T^2 - 9\rho_t NT + 9\rho_t T + 11\rho_0 T - 9T + 6\rho_t N - 6\rho_t - 6\rho_0 + 6)(8\rho_0 T^3 - 18\rho_t T^2 - 21\rho_0 T^2 + 18T^2 + 18\rho_t T + 7\rho_0 T - 18T + 6\rho_t + 6\rho_0 - 6)}}, \tag{31}$$

$$\frac{se(\hat{\gamma}_{2mis})}{se(\hat{\gamma}_2)} = \sqrt{\frac{(1-\rho_t)(\rho_t N - \rho_t - \rho_0 + 1)(\rho_0 T^2 + 4\rho_t NT - 4\rho_t T - 3\rho_0 T + 4T - 2\rho_t N + 2\rho_t + 2\rho_0 - 2)(\rho_0 T^3 - 9\rho_t T^2 - 6\rho_0 T^2 + 9T^2 + 9\rho_t T + 11\rho_0 T - 9T - 6\rho_t - 6\rho_0 + 6)}{(1-\rho_t-\rho_0)(\rho_0 T^2 - 4\rho_t T - \rho_0 T + 4T + 2\rho_t + 2\rho_0 - 2)(\rho_0 T^3 + 9\rho_t N T^2 - 9\rho_t T^2 - 6\rho_0 T^2 + 9T^2 - 9\rho_t NT + 9\rho_t T + 11\rho_0 T - 9T + 6\rho_t N - 6\rho_t - 6\rho_0 + 6)}}. \tag{32}$$

Relative Influences of ρ_0 and ρ_t on Standard Error of Linear Slope Effect Estimates

From Equation 20, the first derivations of error variance of fixed linear slope estimate (i.e., $se^2(\hat{\gamma}_1)$) with respect to ρ_0 and ρ_t (i.e., $\frac{\partial se^2(\hat{\gamma}_1)}{\partial \rho_0}$ and $\frac{\partial se^2(\hat{\gamma}_1)}{\partial \rho_t}$) can be evaluated. The difference of derivation results $\frac{\partial se^2(\hat{\gamma}_1)}{\partial \rho_t} - \frac{\partial se^2(\hat{\gamma}_1)}{\partial \rho_0}$ becomes

$$\frac{\partial se^2(\hat{\gamma}_1)}{\partial \rho_t} - \frac{\partial se^2(\hat{\gamma}_1)}{\partial \rho_0} = \frac{12[(T-1)(NT^2-3T+2N)\rho_0^2 + 2(\rho_t N - \rho_t + 1)(T-1)(NT+3T-2N)\rho_0 - (\rho_t N - \rho_t + 1)^2(3T^2-4NT-3T+2N)]}{N(T-1)T(\rho_0 T^2 + 4\rho_t NT - 4\rho_t T - 3\rho_0 T + 4T - 2\rho_t N + 2\rho_t + 2\rho_0 - 2)^2} \tag{33}$$

When $T \geq 1$ and $N \geq 1$, $(T-1)(NT^2-3T+2N) \geq 0$. So, a function $K = f(\rho_0) = (T-1)(NT^2 - 3T + 2N)\rho_0^2 + 2(\rho_t N - \rho_t + 1)(T-1)(NT+3T-2N)\rho_0 - (\rho_t N - \rho_t + 1)^2(3T^2 - 4NT - 3T + 2N)$, which appears in the numerator of the above equation, takes minimum values K_{min} as:

$$K_{min} = \frac{3N(\rho_t N - \rho_t + 1)^2 T^2 (T+1)(N-T)}{NT^2 - 3T + 2N}. \tag{34}$$

Therefore, when $N \geq T$, the relation $K_{min} \geq 0$ is satisfied, indicating the result:

$$\frac{\partial se^2(\hat{\gamma}_1)}{\partial \rho_t} \geq \frac{\partial se^2(\hat{\gamma}_1)}{\partial \rho_0}. \tag{35}$$

Thus, increasing ρ_t is always more influential than ρ_0 on $se^2(\hat{\gamma}_1)$ (or $se(\hat{\gamma}_1)$).

Notes on Estimation

Time-specific errors explain the random fluctuation of outcomes caused by the time-point of sampling. In other words, time-specific errors represent time-specific residuals, to which all participants are subjected, around the true growth trajectory (see Figure 1). Our simulations showed that the proposed model can effectively extract the information about time-specific errors from data (see Table and Figures in Online Supporting Material). To further clarify how the model can separately estimate the fixed components of the true growth trajectory (e.g., quadratic slope γ_2) and random participant intercepts (σ_0^2) from time-specific errors (σ_t^2), we present the mean and covariance structures of the proposed quadratic GCM (Equations 16-19 in the manuscript). For example, when $T = 3$ and $N = 2$ and we only assume random participant intercepts (i.e., setting $r_{1j} = r_{2j} = 0$) , from the relations Equations 8-11 and Equation 20 in the manuscript, mean $\boldsymbol{\mu}$ and covariance structures $\boldsymbol{\Sigma}$ for $\boldsymbol{Y}$=$(Y_{11}, Y_{12}, Y_{13}, Y_{21}, Y_{22}, Y_{23})'$ can be expressed as:

$$\boldsymbol{\mu} = (\gamma_0, \gamma_0 + \gamma_1 + \gamma_2, \gamma_0 + 2\gamma_1 + 4\gamma_2, \gamma_0, \gamma_0 + \gamma_1 + \gamma_2, \gamma_0 + 2\gamma_1 + 4\gamma_2)', \tag{36}$$

and

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_t^2 + \sigma_0^2 + \sigma_e^2 & \sigma_0^2 & \sigma_0^2 & \sigma_t^2 & 0 & 0 \\ \sigma_0^2 & \sigma_t^2 + \sigma_0^2 + \sigma_e^2 & \sigma_0^2 & 0 & \sigma_t^2 & 0 \\ \sigma_0^2 & \sigma_0^2 & \sigma_t^2 + \sigma_0^2 + \sigma_e^2 & 0 & 0 & \sigma_t^2 \\ \sigma_t^2 & 0 & 0 & \sigma_t^2 + \sigma_0^2 + \sigma_e^2 & \sigma_0^2 & \sigma_0^2 \\ 0 & \sigma_t^2 & 0 & \sigma_0^2 & \sigma_t^2 + \sigma_0^2 + \sigma_e^2 & \sigma_0^2 \\ 0 & 0 & \sigma_t^2 & \sigma_0^2 & \sigma_0^2 & \sigma_t^2 + \sigma_0^2 + \sigma_e^2 \end{pmatrix}. \tag{37}$$

Time-specific errors σ_t^2 appear only in convariance structure $\boldsymbol{\Sigma}$, indicating that the estimations of quadratic slope γ_2 and σ_t^2 use different sources of information. In $\boldsymbol{\Sigma}$, σ_t^2 appears not only in diagonal elements (i.e., $(1, 1), (2, 2), \ldots, (6, 6)$ elements), but also in block diagonal elements that share the same time points (i.e., $(1, 4), (2, 5), (3, 6), (4, 1), (5, 2), (6, 3)$ elements). This is how random participant intercepts σ_0^2 and time-specific errors σ_t^2 are separately estimated.

Even when we loosen the model assumption by setting $\sigma_1^2 \neq 0$, this basic observation does not change. In this case mean structure $\boldsymbol{\mu}^*$ remains unchanged ($\boldsymbol{\mu}^* = \boldsymbol{\mu}$), and covariance structure $\boldsymbol{\Sigma}^*$ becomes:

$$\boldsymbol{\Sigma}^* = \begin{pmatrix} \sigma_t^2+\sigma_0^2+\sigma_e^2 & \sigma_0^2+\sigma_{01} & \sigma_0^2+2\sigma_{01} & \sigma_t^2 & 0 & 0 \\ \sigma_0^2+\sigma_{01} & \sigma_t^2+\sigma_0^2+\sigma_1^2+2\sigma_{01}+\sigma_e^2 & \sigma_0^2+2\sigma_1^2+3\sigma_{01} & 0 & \sigma_t^2 & 0 \\ \sigma_0^2+2\sigma_{01} & \sigma_0^2+2\sigma_1^2+3\sigma_{01} & \sigma_t^2+\sigma_0^2+4\sigma_1^2+4\sigma_{01}+\sigma_e^2 & 0 & 0 & \sigma_t^2 \\ \sigma_t^2 & 0 & 0 & \sigma_t^2+\sigma_0^2+\sigma_e^2 & \sigma_0^2+\sigma_{01} & \sigma_0^2+2\sigma_{01} \\ 0 & \sigma_t^2 & 0 & \sigma_0^2+\sigma_{01} & \sigma_t^2+\sigma_0^2+\sigma_1^2+2\sigma_{01}+\sigma_e^2 & \sigma_0^2+2\sigma_1^2+3\sigma_{01} \\ 0 & 0 & \sigma_t^2 & \sigma_0^2+2\sigma_{01} & \sigma_0^2+2\sigma_1^2+3\sigma_{01} & \sigma_t^2+\sigma_0^2+4\sigma_1^2+4\sigma_{01}+\sigma_e^2 \end{pmatrix}. \tag{38}$$

These means and covariance structures clarify the sources of information to estimate time-specific errors and other fixed and random effects. The fact that time-specific errors are represented in the covariance between participants from the same time points (i.e., $(1,4), (2,5), (3,6), (4,1), (5,2), (6,3)$ elements) means that the proposed GCM (with time-specific errors) allows data from different participants within each time point to be correlated. In standard GCM, on the other hand, participants are posited to be independent from each other (after accounting for fixed effects).

Effectiveness of the proposed model

To demonstrate that the proposed GCMs can effectively address the inflation of Type-1 error rates, we conducted sets of Monte Carlo simulations that applied a GCM with time-specific errors to generated data. Specifically, we applied a linear GCM and quadratic GCM with time-specific errors (i.e., Equations 5-7 and Equations 16-19 in the manuscript with $r_{2j} = 0$, respectively), to the data generated from a linear GCM and quadratic GCM with and without time-specific errors, respectively. The results are displayed in Figure S3 and S4 in Online Supporting Materials . As expected, the proposed mixed-effects models with time-specific errors kept closely to the nominal Type-1 error rates, except for a very slight deviation with a small number of time points (i.e., $T = 5$) and a small number of participants (i.e., $N = 200$). It should be noted that the proposed model with time-specific errors kept closely to the nominal Type-1 error rates even when $\sigma_t^2 = 0$ (i.e. the model was overparameterized).

To further demonstrate the effectiveness of the proposed mixed-effects models, we ran another set of Monte Carlo simulations to examine whether the correct GCMs can effectively recover true parameters without bias. In this set of simulations, data were generated (replication = 1,000 for each condition) by a quadratic GCM with time-specific errors (i.e., Equations 16-19 in the manuscript with $r_{2j} = 0$). We assumed the presence of a positive quadratic effect ($\gamma_2 = 0.01$), and systematically changed the number of participants (N= 200, 500, and 1,500), the number of time points (equally spaced, $T = 5$, 10, and 15), the variance of random participant slope ($\sigma_1^2 = 0.01, 0.05$ and 0.20), and the variance of time-specific errors ($\sigma_t^2 = 0, 0.01, 0.03$, and 0.05). Random participant intercept variance was fixed to 0.5 (i.e., $\sigma_0^2 = 0.5$).

For each generated dataset, we applied the mixed-effects model with time-specific errors (i.e., Equations 16-19 in the manuscript with $r_{2j} = 0$), and calculated bias and root mean squared errors (RMSE) for the fixed quadratic effect (γ_2) and time-specific errors (σ_t^2) as such:

$$Bias = \frac{1}{1000}\sum_{s=1}^{S=1000}(\hat{b}_s - b), \quad RMSE = \sqrt{\frac{1}{1000}\sum_{s=1}^{S=1000}(\hat{b}_s - b)^2}. \tag{39}$$

b denotes the parameters expressing fixed and random effects and $\hat{b}_s$ is its estimate at the s-th repetition, respectively.

Table S1 in Online Supporting Materials reports the bias and RMSE for the coefficient of the quadratic term (γ_2) and time-specific errors (ρ_t). Overall, parameters were accurately estimated without substantial bias. When there are only a few time points (i.e., $T = 5$), GCM with time-specific errors showed slight bias in estimating time-specific errors as well as the fixed quadratic effect. This could explain the slight deviation of Type-1 error rates (from 5%) with a small number of time points observed in Figure S4 in Online Supporting Materials. Increasing the number of time points reduced parameter bias and RMSE of both effects, but increasing the number of participants does not seem to reduce the bias. Note again that parameters were accurately estimated without substantial bias and RMSE even when time-specific errors are not present (i.e.,$\sigma_t^2 = 0$), indicating that misspecified time-specific errors do not influence other parameter estimates. These results suggest that GCM with time-specific errors can effectively recover true parameters.