Supplementary Material for "Testing for Shifts in Mean with Monotonic Power against Multiple Structural Changes"

Daisuke Yamazaki
Faculty of Economics
Kyushu University

December 2018

Related Lemmas and the Proofs

Lemma 1 *Under Assumption 1,* $\sup_{1\le t\le T}|u_t| = o_p(T^{1/\kappa^*})$.

Proof of Lemma 1

Let c be a positive constant. Then,

$$\begin{aligned}
P\left(T^{-1/\kappa^*}\sup_{1\le t\le T}|u_t| > c\right) &= P\left(\bigcup_{t=1}^{T}\left\{|u_t| > cT^{1/\kappa^*}\right\}\right) \\
&= P\left(\bigcup_{t=1}^{T}\left\{|u_t|^{\kappa^*} > c^{\kappa^*}T\right\}\right) \\
&\le \sum_{t=1}^{T} P\left(|u_t|^{\kappa^*} > c^{\kappa^*}T\right) \\
&\le \sum_{t=1}^{T}\left[\frac{1}{c^{\kappa^*}T}E\left(|u_t|^{\kappa^*}\cdot 1\left\{|u_t|^{\kappa^*} > c^{\kappa^*}T\right\}\right)\right] \\
&\le \frac{1}{c^{\kappa^*}T}\cdot T\sup_{1\le t\le T}E\left(|u_t|^{\kappa^*}\cdot 1\left\{|u_t|^{\kappa^*} > c^{\kappa^*}T\right\}\right) \\
&= o(1),
\end{aligned}$$

because $\sup_{1\le t\le T}E(|u_t|^{\kappa^*}\cdot 1\{|u_t|^{\kappa^*} > c^{\kappa^*}T\}) \to 0$ as $T\to\infty$. ■

Lemma 2 *Let* $\breve{u}_t = \dfrac{\sum_{s=1}^T K(\frac{t-s}{Th})u_s}{\sum_{s=1}^T K(\frac{t-s}{Th})}$ *and* $\breve{\theta}_t = \dfrac{\sum_{s=1}^T K(\frac{t-s}{Th})\left\{\theta\left(\frac{t}{T}\right) - \theta\left(\frac{s}{T}\right)\right\}}{\sum_{s=1}^T K(\frac{t-s}{Th})}$. *Then, under Assumptions 1–6, we have*

$$(a) \quad \frac{1}{T}\sum_{t=j+1}^{T} \breve{u}_t\breve{u}_{t-j} = \tilde{O}_p\left(\frac{1}{Th}\right), \qquad (b) \quad \frac{1}{T}\sum_{t=j+1}^{T} \breve{u}_t\breve{\theta}_{t-j} = \tilde{O}_p\left(\frac{\eta}{\sqrt{Th}}\right),$$

$$(c) \quad \frac{1}{T}\sum_{t=j+1}^{T} u_t\breve{\theta}_{t-j} = \tilde{O}_p\left(\frac{\eta}{\sqrt{T}}\right), \qquad (d) \quad \frac{1}{T}\sum_{t=j+1}^{T} u_t\breve{u}_{t-j} = \tilde{O}_p\left(\frac{1}{Th}\right),$$

$$(e) \quad \frac{1}{T}\sum_{t=j+1}^{T} \breve{\theta}_t\breve{\theta}_{t-j} = O_p(h\eta^2).$$

Proof of Lemma 2

Let us define

$$\breve{\breve{u}}_t = \frac{1}{Th}\sum_{s=1}^{T} K\left(\frac{t-s}{Th}\right)u_s, \tag{1}$$

$$\breve{\breve{\theta}}_t = \frac{1}{Th}\sum_{s=1}^{T} K\left(\frac{t-s}{Th}\right)\left\{\theta\left(\frac{t}{T}\right) - \theta\left(\frac{s}{T}\right)\right\}. \tag{2}$$

Then, we have $\breve{u}_t \sim \breve{\breve{u}}_t$ and $\breve{\theta}_t \sim \breve{\breve{\theta}}_t$ uniformly in t, where "$\sim$" denotes equivalence of the (stochastic) order; that is, $A \sim B$ implies $A/B = O_p(1)$ or $A/B = O(1)$. Therefore, we have

$$(a') \quad \frac{1}{T}\sum_{t=j+1}^{T} \breve{u}_t\breve{u}_{t-j} \sim \frac{1}{T}\sum_{t=j+1}^{T} \breve{\breve{u}}_t\breve{\breve{u}}_{t-j},$$

$$(b') \quad \frac{1}{T}\sum_{t=j+1}^{T} \breve{u}_t\breve{\theta}_{t-j} \sim \frac{1}{T}\sum_{t=j+1}^{T} \breve{\breve{u}}_t\breve{\breve{\theta}}_{t-j},$$

$$(c') \quad \frac{1}{T}\sum_{t=j+1}^{T} u_t\breve{\theta}_{t-j} \sim \frac{1}{T}\sum_{t=j+1}^{T} u_t\breve{\breve{\theta}}_{t-j},$$

$$(d') \quad \frac{1}{T}\sum_{t=j+1}^{T} u_t\breve{u}_{t-j} \sim \frac{1}{T}\sum_{t=j+1}^{T} u_t\breve{\breve{u}}_{t-j},$$

$$(e') \quad \frac{1}{T}\sum_{t=j+1}^{T} \breve{\theta}_t\breve{\theta}_{t-j} \sim \frac{1}{T}\sum_{t=j+1}^{T} \breve{\breve{\theta}}_t\breve{\breve{\theta}}_{t-j}.$$

Proofs of (a) and (d).

Using (a′) and (d′), we can prove (a) and (d) in the same way as in Lemmas A.1 and A.4 in Juhl and Xiao (2009), so we omit the proofs.

Proof of (b).

Let us define $T_i = [\lambda_i T]$ for $i = 0, \cdots, m+1$. Since $\theta(t/T) = \mu + [c_1 + \sum_{i=1}^{m}(c_{i+1} - c_i) \cdot 1\{t > T_i\}] \cdot \eta$, we have

$$\begin{aligned}
\theta\left(\frac{t}{T}\right) - \theta\left(\frac{s}{T}\right) &= \sum_{i=1}^{m}(c_{i+1} - c_i)\left[1\{t > T_i\} - 1\{s > T_i\}\right]\eta \\
&= \sum_{i=1}^{m}(c_{i+1} - c_i)\left[1\{s \le T_i < t\} - 1\{t \le T_i < s\}\right]\eta.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\breve{\breve{\theta}}_t &= \frac{\eta}{Th}\sum_{s=1}^{T} K\left(\frac{t-s}{Th}\right)\left[\sum_{i=1}^{m}(c_{i-1} - c_i) \cdot 1\left\{\frac{s}{T} \le \lambda_i\right\} \cdot 1\left\{\frac{t}{T} > \lambda_i\right\}\right] \\
&\quad -\frac{\eta}{Th}\sum_{s=1}^{T} K\left(\frac{t-s}{Th}\right)\left[\sum_{i=1}^{m}(c_{i-1} - c_i) \cdot 1\left\{\frac{s}{T} > \lambda_i\right\} \cdot 1\left\{\frac{t}{T} \le \lambda_i\right\}\right] \\
&= \text{(A1-1)} - \text{(A1-2)}, \quad \text{say.}
\end{aligned}$$

First, we have

$$\begin{aligned}
\text{(A1-1)} &= \frac{\eta}{h}\sum_{i=1}^{m}(c_{i+1} - c_i)\left[\frac{1}{T}\sum_{s=1}^{T} K\left(\frac{t-s}{Th}\right) \cdot 1\left\{\frac{s}{T} \le \lambda_i\right\} \cdot 1\left\{\frac{t}{T} > \lambda_i\right\}\right] \\
&= \frac{\eta}{h}\sum_{i=1}^{m}(c_{i+1} - c_i)\left[\left(\int_0^1 K\left(\frac{v-u}{h}\right) 1\{u \le \lambda_i\} du\right) \cdot 1\{v > \lambda_i\} \cdot (1 + o(1))\right] \\
&= \frac{\eta}{h}\sum_{i=1}^{m}(c_{i+1} - c_i)\left[\left(\int_{(v-1)/h}^{v/h} K(w) \cdot 1\{v - hw \le \lambda_i\} \cdot h\, dw\right) \cdot 1\{v > \lambda_i\} \cdot (1 + o(1))\right] \\
&= \eta\left[\sum_{i=1}^{m}(c_{i+1} - c_i)\left(\int_{(v-\lambda_i)/h}^{v/h} K(w) dw\right) \cdot 1\{v > \lambda_i\}\right] \cdot (1 + o(1)),
\end{aligned}$$

where $u = s/T$ and $v = t/T$. Similarly, we have

$$\text{(A1-2)} = \eta\left[\sum_{i=1}^{m}(c_{i+1} - c_i)\left(\int_{(v-1)/h}^{(v-\lambda_i)/h} K(w) dw\right) \cdot 1\{v \le \lambda_i\}\right] \cdot (1 + o(1)).$$

Therefore,

$$\begin{aligned}
\breve{\breve{\theta}}_t &= \eta\left[\sum_{i=1}^{m}(c_{i+1} - c_i)\left\{\left(\int_{(v-\lambda_i)/h}^{v/h} K(w) dw\right) \cdot 1\{v > \lambda_i\}\right.\right. \\
&\qquad \left.\left. - \left(\int_{(v-1)/h}^{(v-\lambda_i)/h} K(w) dw\right) \cdot 1\{v \le \lambda_i\}\right\}\right](1 + o(1)), \qquad (3)
\end{aligned}$$

so that

$$\breve{\breve{\theta}}_t = \tilde{O}(\eta). \tag{4}$$

Let $F = T^{-1}\sum_{t=j+1}^{T} \breve{\breve{u}}_t \breve{\breve{\theta}}_{t-j}$. Then, we have

$$\begin{aligned} E(F^2) &= \frac{1}{T^2}\sum_{t=j+1}^{T}\sum_{t'=j+1}^{T} E(\breve{\breve{u}}_t\breve{\breve{u}}_{t'})\breve{\breve{\theta}}_{t-j}\breve{\breve{\theta}}_{t'-j} \\ &= \tilde{O}\left(\frac{1}{T^2}\cdot T^2 \cdot \frac{\eta^2}{Th}\right) = \tilde{O}\left(\frac{\eta^2}{Th}\right), \end{aligned}$$

because $E(\breve{\breve{u}}_t\breve{\breve{u}}_{t'}) = \tilde{O}((Th)^{-1})$,[1] and (4). Therefore, $F = \tilde{O}_p(\eta/\sqrt{Th})$ and $T^{-1}\sum_{t=j+1}^{T} \breve{u}_t\breve{\theta}_{t-j} = \tilde{O}_p(\eta/\sqrt{Th})$. ■

Proof of (c). Let $G = T^{-1}\sum_{t=j+1}^{T} u_t\breve{\breve{\theta}}_{t-j}$. Then,

$$\begin{aligned} E(G^2) &= \frac{1}{T^2}\sum_{t=j+1}^{T} E(u_t^2)\breve{\breve{\theta}}_{t-j}^2 + \frac{1}{T^2}\sum_{t=j+1}^{T}\sum_{t'=j+1,\, t\neq t'}^{T} E(u_t u_{t'})\breve{\breve{\theta}}_{t-j}\breve{\breve{\theta}}_{t'-j} \\ &= G_1 + G_2, \quad \text{say.} \end{aligned}$$

First, we can easily see that $G_1 = \tilde{O}(T^{-2}\cdot T\cdot \eta^2) = \tilde{O}(\eta^2/T)$ using (4). For G_2, following Juhl and Xiao (2009), we have

$$\begin{aligned} G_2 &\le \frac{1}{T^2}\sum_{t=j+1}^{T}\sum_{t'=j+1,\, t\neq t'}^{T} |E(u_t u_{t'})| \cdot \sup_{t,t'}\left|\breve{\breve{\theta}}_{t-j}\breve{\breve{\theta}}_{t'-j}\right| \\ &\le \frac{1}{T^2}\sum_{t=j+1}^{T}\sum_{t'=j+1,\, t\neq t'}^{T} \beta_{t-t'}^{\frac{\delta}{1+\delta}} M^{\frac{1}{1+\delta}} \cdot O(\eta^2) \\ &= O\left(\frac{1}{T^2}\cdot T\cdot \eta^2\right) = O\left(\frac{\eta^2}{T}\right). \end{aligned}$$

Therefore, we have $E(G^2) = \tilde{O}(\eta^2/T)$, so that $G = \tilde{O}_p(\eta/\sqrt{T})$ and $T^{-1}\sum_{t=j+1}^{T} u_t\breve{\theta}_{t-j} = \tilde{O}_p(\eta/\sqrt{T})$. ■

[1] This is proved in Juhl and Xiao (2009).

Proof of (e). Using (3), we have

$$
\begin{aligned}
\breve{\theta}_t^2 \;=\; & \eta^2 \Bigg[\sum_{i=1}^{m} (c_{i+1}-c_i)^2 \left\{ \left(\int_{(v-\lambda_i)/h}^{v/h} K(w)dw \right)^2 \cdot 1\{v>\lambda_i\} + \left(\int_{(v-1)/h}^{(v-\lambda_i)/h} K(w)dw \right)^2 \cdot 1\{v \le \lambda_i\} \right\} \\
& + \sum_{i=1}^{m} \sum_{j=1, i\neq j}^{m} (c_{i+1}-c_i)(c_{j+1}-c_j) \\
& \quad \cdot \left\{ \left(\int_{(v-\lambda_i)/h}^{v/h} K(w)dw \right) \left(\int_{(v-\lambda_j)/h}^{v/h} K(w)dw \right) 1\{v>\lambda_i\} 1\{v>\lambda_j\} \right. \\
& \quad - \left(\int_{(v-\lambda_i)/h}^{v/h} K(w)dw \right) \left(\int_{(v-1)/h}^{(v-\lambda_j)/h} K(w)dw \right) 1\{v>\lambda_i\} 1\{v\le\lambda_j\} \\
& \quad - \left(\int_{(v-1)/h}^{(v-\lambda_i)/h} K(w)dw \right) \left(\int_{(v-\lambda_j)/h}^{v/h} K(w)dw \right) 1\{v\le\lambda_i\} 1\{v>\lambda_j\} \\
& \quad \left. + \left(\int_{(v-1)/h}^{(v-\lambda_i)/h} K(w)dw \right) \left(\int_{(v-1)/h}^{(v-\lambda_j)/h} K(w)dw \right) 1\{v\le\lambda_i\} 1\{v\le\lambda_j\} \right\} \Bigg] (1+o(1)),
\end{aligned}
$$

so that

$$
\begin{aligned}
& \frac{1}{T}\sum_{t=j+1}^{T}\breve{\ddot{\theta}}_t\breve{\ddot{\theta}}_{t-j} \\
\sim\ & \frac{1}{T}\sum_{t=1}^{T}\breve{\ddot{\theta}}_t^2 \\
=\ & \eta^2\sum_{i=1}^{m}(c_{i+1}-c_i)^2\left\{\frac{1}{T}\sum_{t=1}^{T}\left(\int_{(v-\lambda_i)/h}^{v/h}K(w)dw\right)^2 1\{v>\lambda_i\}\right. \\
& \qquad\qquad \left.+\frac{1}{T}\sum_{t=1}^{T}\left(\int_{(v-1)/h}^{(v-\lambda_i)/h}K(w)dw\right)^2 1\{v\le\lambda_i\}\right\} \\
& +\eta^2\sum_{i=1}^{m}\sum_{j=1,i\neq j}^{m}(c_{i+1}-c_i)(c_{j+1}-c_j) \\
& \quad\cdot\left\{\frac{1}{T}\sum_{t=1}^{T}\left(\int_{(v-\lambda_i)/h}^{v/h}K(w)dw\right)\left(\int_{(v-\lambda_j)/h}^{v/h}K(w)dw\right)1\{v>\lambda_i\}1\{v>\lambda_j\}\right. \\
& \quad -\frac{1}{T}\sum_{t=1}^{T}\left(\int_{(v-\lambda_i)/h}^{v/h}K(w)dw\right)\left(\int_{(v-1)/h}^{(v-\lambda_j)/h}K(w)dw\right)1\{v>\lambda_i\}1\{v\le\lambda_j\} \\
& \quad -\frac{1}{T}\sum_{t=1}^{T}\left(\int_{(v-1)/h}^{(v-\lambda_i)/h}K(w)dw\right)\left(\int_{(v-\lambda_j)/h}^{v/h}K(w)dw\right)1\{v\le\lambda_i\}1\{v>\lambda_j\} \\
& \quad \left.+\frac{1}{T}\sum_{t=1}^{T}\left(\int_{(v-1)/h}^{(v-\lambda_i)/h}K(w)dw\right)\left(\int_{(v-1)/h}^{(v-\lambda_j)/h}K(w)dw\right)1\{v\le\lambda_i\}1\{v\le\lambda_j\}\right\} \\
=\ & (\text{A2-1})+\{(\text{A2-2})-(\text{A2-3})-(\text{A2-4})+(\text{A2-5})\},\quad \text{say.}
\end{aligned}
$$

Let us consider (A2-1). Since

$$
\begin{aligned}
& \frac{1}{T}\sum_{t=1}^{T}\left(\int_{(v-\lambda_i)/h}^{v/h}K(w)dw\right)^2 1\{v>\lambda_i\}+\frac{1}{T}\sum_{t=1}^{T}\left(\int_{(v-1)/h}^{(v-\lambda_i)/h}K(w)dw\right)^2 1\{v\le\lambda_i\} \\
=\ & \left[\int_{\lambda_i}^{1}\left(\int_{(v-\lambda_i)/h}^{v/h}K(w)dw\right)^2 dv+\int_0^{\lambda_i}\left(\int_{(v-1)/h}^{(v-\lambda_i)/h}K(w)dw\right)^2 dv\right](1+o(1)) \\
=\ & h\left[\int_0^{(1-\lambda_i)/h}\left(\int_x^{x+\lambda_i/h}K(w)dw\right)^2 dx+\int_{-\lambda_i/h}^{0}\left(\int_{x-(1-\lambda_i)/h}^{x}K(w)dw\right)^2 dx\right](1+o(1)) \\
=\ & h\left[\int_0^{\infty}\left(\int_x^{\infty}K(w)dw\right)^2 dx+\int_{-\infty}^{0}\left(\int_{-\infty}^{x}K(w)dw\right)^2 dx\right](1+o(1)),
\end{aligned}
$$

we have

$$
\begin{aligned}
(\text{A2-1}) &= h\eta^2\sum_{i=1}^{m}(c_{i+1}-c_i)^2\left[\int_0^{\infty}\left(\int_x^{\infty}K(w)dw\right)^2 dx+\int_{-\infty}^{0}\left(\int_{-\infty}^{x}K(w)dw\right)^2 dx\right](1+o(1)) \\
&= O(h\eta^2).
\end{aligned}
$$

Next, consider (A2-2). When $i < j$,

$$\begin{aligned}
& \frac{1}{T}\sum_{t=1}^{T}\left(\int_{(v-\lambda_i)/h}^{v/h} K(w)dw\right)\left(\int_{(v-\lambda_j)/h}^{v/h} K(w)dw\right) 1\{v > \lambda_i\}1\{v > \lambda_j\} \\
= & \frac{1}{T}\sum_{t=1}^{T}\left(\int_{(v-\lambda_i)/h}^{v/h} K(w)dw\right)\left(\int_{(v-\lambda_j)/h}^{v/h} K(w)dw\right) 1\{v > \lambda_j\} \\
= & \left\{\int_{\lambda_j}^{1}\left(\int_{(v-\lambda_i)/h}^{v/h} K(w)dw\right)\left(\int_{(v-\lambda_j)/h}^{v/h} K(w)dw\right) dv\right\}(1+o(1)) \\
= & h\left\{\int_{0}^{(1-\lambda_j)/h}\left(\int_{x-(\lambda_j-\lambda_i)/h}^{x+\lambda_j/h} K(w)dw\right)\left(\int_{x}^{x+\lambda_j/h} K(w)dw\right) dx\right\}(1+o(1)) \\
= & h\left\{\int_{0}^{\infty}\left(\int_{-\infty}^{\infty} K(w)dw\right)\left(\int_{x}^{\infty} K(w)dw\right) dx\right\}(1+o(1)) \\
= & h\left\{\int_{0}^{\infty}\left(\int_{x}^{\infty} K(w)dw\right) dx\right\}(1+o(1)),
\end{aligned}$$

and similarly when $i > j$,

$$\begin{aligned}
& \frac{1}{T}\sum_{t=1}^{T}\left(\int_{(v-\lambda_i)/h}^{v/h} K(w)dw\right)\left(\int_{(v-\lambda_j)/h}^{v/h} K(w)dw\right) 1\{v > \lambda_i\}1\{v > \lambda_j\} \\
= & h\left\{\int_{0}^{\infty}\left(\int_{x}^{\infty} K(w)dw\right) dx\right\}(1+o(1)).
\end{aligned}$$

Using the results above, we get

$$\begin{aligned}
\text{(A2-2)} &= h\eta^2\left\{\sum_{i=1}^{m}\sum_{j=1,i\neq j}^{m}(c_{i+1}-c_i)(c_{j+1}-c_j)\int_{0}^{\infty}\left(\int_{x}^{\infty} K(w)dw\right) dx\right\}(1+o(1)) \\
&= O(h\eta^2).
\end{aligned}$$

Then, let us consider (A2-3). When $i < j$, we have

$$\begin{aligned}
& \frac{1}{T}\sum_{t=1}^{T}\left(\int_{(v-\lambda_i)/h}^{v/h} K(w)dw\right)\left(\int_{(v-1)/h}^{(v-\lambda_j)/h} K(w)dw\right) 1\{v > \lambda_i\}1\{v \leq \lambda_j\} \\
= & \frac{1}{T}\sum_{t=1}^{T}\left(\int_{(v-\lambda_i)/h}^{v/h} K(w)dw\right)\left(\int_{(v-1)/h}^{(v-\lambda_j)/h} K(w)dw\right) 1\{\lambda_i < v \leq \lambda_j\} \\
= & h\left\{\int_{\lambda_i}^{\lambda_j}\left(\int_{(v-\lambda_i)/h}^{v/h} K(w)dw\right)\left(\int_{(v-1)/h}^{(v-\lambda_j)/h} K(w)dw\right) dx\right\}(1+o(1)) \\
= & h\left\{\int_{0}^{(\lambda_j-\lambda_i)/h}\left(\int_{x}^{x+\lambda_i/h} K(w)dw\right)\left(\int_{x-(1-\lambda_i)/h}^{x-(\lambda_j-\lambda_i)/h} K(w)dw\right) dx\right\}(1+o(1)) \\
= & h\left\{\int_{0}^{\infty}\left(\int_{x}^{\infty} K(w)dw\right) r(x)dx\right\}(1+o(1)) \\
= & o(h),
\end{aligned}$$

where $r(x) \to 0$ as $T \to \infty$. In addition, when $i > j$,

$$\frac{1}{T}\sum_{t=1}^{T}\left(\int_{(v-\lambda_i)/h}^{v/h} K(w)dw\right)\left(\int_{(v-1)/h}^{(v-\lambda_j)/h} K(w)dw\right) 1\{v > \lambda_i\}1\{v \leq \lambda_j\} = 0,$$

because $\lambda_i > \lambda_j$. Therefore, (A2-3) $= o(h\eta^2)$.

Similarly, we can prove that (A2-4) $= o(h\eta^2)$ and (A2-5) $= O(h\eta^2)$, so that $T^{-1}\sum_{t=j+1}^{T}\breve{\theta}_t\breve{\theta}_{t-j} = O(h\eta^2)$. ■

Lemma 3 *Under Assumptions 1–6, we have*

$$\begin{aligned}
(a)\quad & \frac{1}{T}\sum_{t=2}^{T} u_t\breve{\theta}_{t-1} - \frac{1}{T}\sum_{t=2}^{T} u_{t-1}\breve{\theta}_{t-1} = O_p\left(\frac{\eta}{T}\right),\\
(b)\quad & \frac{1}{T}\sum_{t=2}^{T} \breve{\theta}_t u_{t-1} - \frac{1}{T}\sum_{t=2}^{T} \breve{\theta}_{t-1} u_{t-1} = O_p\left(\frac{\eta}{T}\right),\\
(c)\quad & \frac{1}{T}\sum_{t=2}^{T} \breve{u}_t\breve{\theta}_{t-1} - \frac{1}{T}\sum_{t=2}^{T} \breve{u}_{t-1}\breve{\theta}_{t-1} = \tilde{O}_p\left(\frac{\eta}{T^{3/2}h^{3/2}}\right),\\
(d)\quad & \frac{1}{T}\sum_{t=2}^{T} \breve{\theta}_t\breve{u}_{t-1} - \frac{1}{T}\sum_{t=2}^{T} \breve{\theta}_{t-1}\breve{u}_{t-1} = \tilde{O}_p\left(\frac{\eta}{T^{3/2}h^{3/2}}\right),\\
(e)\quad & \frac{1}{T}\sum_{t=2}^{T} \breve{\theta}_t\breve{\theta}_{t-1} - \frac{1}{T}\sum_{t=2}^{T} \breve{\theta}_{t-1}^2 = O_p\left(\frac{\eta^2}{T}\right).
\end{aligned}$$

Proof of Lemma 3

Proofs of (a) and (b).

For (a), we have

$$\begin{aligned}
\breve{\breve{\theta}}_t - \breve{\breve{\theta}}_{t-1} &= \frac{1}{Th}\sum_{s=1}^{T}\left[K\left(\frac{t-s}{Th}\right)\left\{\theta\left(\frac{t}{T}\right) - \theta\left(\frac{s}{T}\right)\right\} - K\left(\frac{t-1-s}{Th}\right)\left\{\theta\left(\frac{t-1}{T}\right) - \theta\left(\frac{s}{T}\right)\right\}\right]\\
&= \frac{1}{Th}\sum_{s=1}^{T} K\left(\frac{t-s}{Th}\right)\left\{\theta\left(\frac{t}{T}\right) - \theta\left(\frac{t-1}{T}\right)\right\}\\
&\quad + \frac{1}{Th}\sum_{s=1}^{T}\theta\left(\frac{t-1}{T}\right)\left\{K\left(\frac{t-s}{Th}\right) - K\left(\frac{t-1-s}{Th}\right)\right\}\\
&\quad - \frac{1}{Th}\sum_{s=1}^{T}\theta\left(\frac{s}{T}\right)\left\{K\left(\frac{t-s}{Th}\right) - K\left(\frac{t-1-s}{Th}\right)\right\}. \qquad (5)
\end{aligned}$$

Therefore, we have

$$
\begin{aligned}
& \frac{1}{T}\sum_{t=2}^{T} u_t\breve{\theta}_{t-1} - \frac{1}{T}\sum_{t=2}^{T} u_{t-1}\breve{\theta}_{t-1} \\
= \quad & -\frac{1}{T}\sum_{t=2}^{T} u_t(\breve{\theta}_t - \breve{\theta}_{t-1}) + \frac{1}{T}u_T\breve{\theta}_T - \frac{1}{T}u_1\breve{\theta}_1 \\
\sim \quad & -\frac{1}{T}\sum_{t=2}^{T} u_t(\breve{\breve{\theta}}_t - \breve{\breve{\theta}}_{t-1}) + \frac{1}{T}u_T\breve{\breve{\theta}}_T - \frac{1}{T}u_1\breve{\breve{\theta}}_1 \\
= \quad & -\frac{1}{T}\sum_{t=2}^{T} u_t\left[\frac{1}{Th}\sum_{s=1}^{T} K\left(\frac{t-s}{Th}\right)\left\{\theta\left(\frac{t}{T}\right) - \theta\left(\frac{t-1}{T}\right)\right\}\right] \\
& -\frac{1}{T}\sum_{t=2}^{T} u_t\left[\frac{1}{Th}\sum_{s=1}^{T} \theta\left(\frac{t-1}{T}\right)\left\{K\left(\frac{t-s}{Th}\right) - K\left(\frac{t-1-s}{Th}\right)\right\}\right] \\
& +\frac{1}{T}\sum_{t=2}^{T} u_t\left[\frac{1}{Th}\sum_{s=1}^{T} \theta\left(\frac{s}{T}\right)\left\{K\left(\frac{t-s}{Th}\right) - K\left(\frac{t-1-s}{Th}\right)\right\}\right] \\
& +\frac{1}{T}u_T\breve{\breve{\theta}}_T - \frac{1}{T}u_1\breve{\breve{\theta}}_1 \\
= \quad & -(\text{A3-1}) - (\text{A3-2}) + (\text{A3-3}) + (\text{A3-4}) - (\text{A3-5}), \quad \text{say.}
\end{aligned}
$$

First, we have (A3-1) $= O_p(\eta/T)$ because

$$
\frac{1}{Th}\sum_{s=1}^{T} K\left(\frac{t-s}{Th}\right)\left\{\theta\left(\frac{t}{T}\right) - \theta\left(\frac{t-1}{T}\right)\right\} = \begin{cases} O(\eta) & \text{for } t = T_i + 1 \ \ (i = 1, \cdots, m), \\ 0 & \text{otherwise.} \end{cases} \tag{6}
$$

Next, we prove that (A3-2) $= \tilde{O}_p(\eta/(T^{3/2}h))$. We have

$$
\begin{aligned}
(\text{A3-2}) \quad = \quad & \frac{1}{T}\sum_{t=2}^{T} u_t\left[\frac{1}{Th}\sum_{s=1}^{T} \theta\left(\frac{t-1}{T}\right)\left\{K\left(\frac{t-s}{Th}\right) - K\left(\frac{t-1-s}{Th}\right)\right\}\right] \\
= \quad & \sum_{i=1}^{m+1}\left[\frac{1}{T}\sum_{t=T_{i-1}+2}^{T_i+1} u_t \cdot \frac{c_i\eta}{Th}\left\{K\left(\frac{t-1}{Th}\right) - K\left(\frac{t-1-T}{Th}\right)\right\}\right] \\
& -\frac{1}{T}u_{T+1}\cdot\frac{c_{m+1}\eta}{Th}\left[K\left(\frac{1}{h}\right) - K(0)\right] \\
= \quad & (\text{A3-2-1}) - (\text{A3-2-2}), \quad \text{say.}
\end{aligned}
$$

Here we consider (A3-2-1). Let us define

$$
W_{1i} = \frac{1}{T}\sum_{t=T_{i-1}+2}^{T_i+1} u_t \cdot \frac{c_i\eta}{Th}\left\{K\left(\frac{t-1}{Th}\right) - K\left(\frac{t-1-T}{Th}\right)\right\}.
$$

Then,

$$
\begin{aligned}
W_{1i}^2 \quad \leq \quad & \frac{1}{T^2} \sum_{t=T_{i-1}+2}^{T_i+1} u_t^2 \cdot \frac{c_i^2 \eta^2}{T^2 h^2} \left\{ K\left(\frac{t-1}{Th}\right) - K\left(\frac{t-1-T}{Th}\right) \right\}^2 \\
& + \frac{1}{T^2} \sum_{t=T_{i-1}+2}^{T_i+1} \sum_{t'=T_{i-1}+2, t \neq t'}^{T_i+1} |u_t u_{t'}| \\
& \cdot \frac{c_i^2 \eta^2}{T^2 h^2} \left| \left\{ K\left(\frac{t-1}{Th}\right) - K\left(\frac{t-1-T}{Th}\right) \right\} \left\{ K\left(\frac{t'-1}{Th}\right) - K\left(\frac{t'-1-T}{Th}\right) \right\} \right|,
\end{aligned}
$$

so that

$$
\begin{aligned}
E(W_{1i}^2) \quad & \leq \quad \frac{1}{T^2} \sum_{t=T_{i-1}+2}^{T_i+1} E(u_t^2) \cdot \frac{4C^2 \cdot c_i^2 \eta^2}{T^2 h^2} + \frac{1}{T^2} \sum_{t=T_{i-1}+2}^{T_i+1} \sum_{t'=T_{i-1}+2, t \neq t'}^{T_i+1} E|u_t u_{t'}| \cdot \frac{4C^2 \cdot c_i^2 \eta^2}{T^2 h^2} \\
& \leq \quad \frac{1}{T^2} \sum_{t=T_{i-1}+2}^{T_i+1} E(u_t^2) \cdot \frac{4C^2 \cdot c_i^2 \eta^2}{T^2 h^2} + \frac{1}{T^2} \sum_{t=T_{i-1}+2}^{T_i+1} \sum_{t'=T_{i-1}+2, t \neq t'}^{T_i+1} \beta_{t-t'}^{\frac{\delta}{1+\delta}} M^{\frac{1}{1+\delta}} \cdot \frac{4C^2 \cdot c_i^2 \eta^2}{T^2 h^2} \\
& = \quad O\left(\frac{\eta^2}{T^3 h^2}\right),
\end{aligned}
$$

because $\sup_x |K(x)| \leq C$ by Assumption 3 (b). Therefore, we have (A3-2-1) $= \tilde{O}_p(\eta/(T^{3/2}h))$.

In addition,

$$
\begin{aligned}
|\text{(A3-2-2)}| \quad & \leq \quad \frac{1}{T} \cdot \left| \frac{c_{m+1}\eta}{Th} \right| \cdot |u_{T+1}| \cdot 2C \\
& = \quad O_p\left(\frac{\eta}{T^2 h}\right),
\end{aligned}
$$

so that (A3-2) $= \tilde{O}_p(\eta/(T^{3/2}h))$.

Similarly, we can prove that (A3-3) $= \tilde{O}_p(\eta/(T^{3/2}h))$. We can easily see that (A3-4) $= \tilde{O}_p(\eta/T)$ and (A3-5) $= \tilde{O}_p(\eta/T)$.

Therefore, we have $T^{-1}\sum_{t=2}^{T} u_t \breve{\theta}_{t-1} - T^{-1}\sum_{t=2}^{T} u_{t-1}\breve{\theta}_{t-1} = O_p(\eta/T) + \tilde{O}_p(\eta/(T^{3/2}h)) = O_p(\eta/T)$ because $T^{1/4}h \to \infty$ as $T \to \infty$ by Assumption 5. In the same way, we can prove (b). ■

Proofs of (c) and (d). Using (5), we have

$$
\begin{aligned}
&\frac{1}{T}\sum_{t=2}^{T}\breve{u}_t\breve{\theta}_{t-1}-\frac{1}{T}\sum_{t=2}^{T}\breve{u}_{t-1}\breve{\theta}_{t-1}\\
\sim\ & -\frac{1}{T}\sum_{t=2}^{T}\breve{\breve{u}}_t\left[\frac{1}{Th}\sum_{s=1}^{T}K\left(\frac{t-s}{Th}\right)\left\{\theta\left(\frac{t}{T}\right)-\theta\left(\frac{t-1}{T}\right)\right\}\right]\\
&-\frac{1}{T}\sum_{t=2}^{T}\breve{\breve{u}}_t\left[\frac{1}{Th}\sum_{s=1}^{T}\theta\left(\frac{t-1}{T}\right)\left\{K\left(\frac{t-s}{Th}\right)-K\left(\frac{t-1-s}{Th}\right)\right\}\right]\\
&+\frac{1}{T}\sum_{t=2}^{T}\breve{\breve{u}}_t\left[\frac{1}{Th}\sum_{s=1}^{T}\theta\left(\frac{s}{T}\right)\left\{K\left(\frac{t-s}{Th}\right)-K\left(\frac{t-1-s}{Th}\right)\right\}\right]\\
&+\frac{1}{T}\breve{\breve{u}}_T\breve{\breve{\theta}}_T-\frac{1}{T}\breve{\breve{u}}_1\breve{\breve{\theta}}_1\\
=\ & -(\text{A4-1})-(\text{A4-2})+(\text{A4-3})+(\text{A4-4})-(\text{A4-5}),\quad \text{say.}
\end{aligned}
$$

First, (A4-1) $=\tilde{O}_p(\eta/(T^{3/2}h^{1/2}))$ because $\breve{\breve{u}}_t=\tilde{O}_p(1/\sqrt{Th})$ and (6). We also obtain (A4-2) $=\tilde{O}_p(\eta/(T^{3/2}h^{3/2}))$, (A4-3) $=\tilde{O}_p(\eta/(T^{3/2}h^{3/2}))$, (A4-4) $=\tilde{O}_p(\eta/(T^{3/2}h^{1/2}))$ and (A4-5) $=\tilde{O}_p(\eta/(T^{3/2}h^{1/2}))$ in the same way as in the proof of (a). Therefore, we have $T^{-1}\sum_{t=2}^{T}\breve{u}_t\breve{\theta}_{t-1}-T^{-1}\sum_{t=2}^{T}\breve{u}_{t-1}\breve{\theta}_{t-1}=\tilde{O}_p(\eta/(T^{3/2}h^{1/2}))+\tilde{O}_p(\eta/(T^{3/2}h^{3/2}))=\tilde{O}_p(\eta/(T^{3/2}h^{3/2}))$. (d) can be proved similarly. ■

Proof of (e).

$$
\begin{aligned}
&\frac{1}{T}\sum_{t=2}^{T}\breve{\theta}_t\breve{\theta}_{t-1}-\frac{1}{T}\sum_{t=2}^{T}\breve{\theta}_{t-1}^2\\
\sim\ & \frac{1}{T}\sum_{t=1}^{T}\breve{\breve{\theta}}_t\breve{\breve{\theta}}_{t-1}-\frac{1}{T}\sum_{t=1}^{T}\breve{\breve{\theta}}_{t-1}^2\\
=\ & -\frac{1}{T}\sum_{t=1}^{T}\breve{\breve{\theta}}_{t-1}\left[\frac{1}{Th}\sum_{s=1}^{T}K\left(\frac{t-s}{Th}\right)\left\{\theta\left(\frac{t}{T}\right)-\theta\left(\frac{t-1}{T}\right)\right\}\right]\\
&-\frac{1}{T}\sum_{t=1}^{T}\breve{\breve{\theta}}_{t-1}\left[\frac{1}{Th}\sum_{s=1}^{T}\theta\left(\frac{t-1}{T}\right)\left\{K\left(\frac{t-s}{Th}\right)-K\left(\frac{t-1-s}{Th}\right)\right\}\right]\\
&+\frac{1}{T}\sum_{t=1}^{T}\breve{\breve{\theta}}_{t-1}\left[\frac{1}{Th}\sum_{s=1}^{T}\theta\left(\frac{s}{T}\right)\left\{K\left(\frac{t-s}{Th}\right)-K\left(\frac{t-1-s}{Th}\right)\right\}\right]\\
=\ & -(\text{A5-1})-(\text{A5-2})+(\text{A5-3}),\quad \text{say.}
\end{aligned}
$$

First, (A5-1) $=O(\eta^2/T)$ because $\breve{\breve{\theta}}_{T_i+1}=O(\eta)$ for $i=1,\cdots,m$ and (6). For (A5-2), we have

$$
\begin{aligned}
\text{(A5-2)} &= \sum_{i=1}^{m+1}\left[\frac{1}{T}\sum_{t=T_{i-1}+2}^{T_i+1}\breve{\breve{\theta}}_{t-1}\left[\frac{c_i\eta}{Th}\left\{K\left(\frac{t-1}{Th}\right)-K\left(\frac{t-1-T}{Th}\right)\right\}\right]\right] \\
&\quad+\left[\frac{1}{T}\breve{\breve{\theta}}_1\cdot\frac{c_1\eta}{Th}\left\{K(0)-K\left(-\frac{1}{h}\right)\right\}-\frac{1}{T}\breve{\breve{\theta}}_{T+1}\cdot\frac{c_{m+1}}{Th}\left\{K\left(\frac{1}{h}\right)-K(0)\right\}\right] \\
&= \text{(A5-2-1)}+\text{(A5-2-2)},\quad \text{say.}
\end{aligned}
$$

First, consider (A5-2-1). Here we define

$$
W_{2i}=\frac{1}{T}\sum_{t=T_{i-1}+2}^{T_i+1}\breve{\breve{\theta}}_{t-1}\left[\frac{c_i\eta}{Th}\left\{K\left(\frac{t-1}{Th}\right)-K\left(\frac{t-1-T}{Th}\right)\right\}\right].
$$

Then,

$$
\begin{aligned}
|W_{2i}| &\leq \frac{1}{T}\sum_{t=T_{i-1}+2}^{T_i+1}\left|\breve{\breve{\theta}}_{t-1}\right|\cdot\left|\frac{c_i\eta}{Th}\right|\cdot\left|K\left(\frac{t-1}{Th}\right)-K\left(\frac{t-1-T}{Th}\right)\right| \\
&\leq \left(h\eta\int_{-\infty}^{\infty}|f(x)|dx\right)(1+o(1))\cdot\left|\frac{c_i\eta}{Th}\right|\cdot 2C \\
&= O\left(\frac{\eta^2}{T}\right),
\end{aligned}
$$

where $f(x)=\sum_{i=1}^{m}(c_{i+1}-c_i)\left[\left(\int_{-\infty}^{x}K(w)dw\right)\cdot 1\{x<0\}-\left(\int_{x}^{\infty}K(w)dw\right)\cdot 1\{x\geq 0\}\right]$. Therefore, (A5-2-1) $=\sum_{i=1}^{m+1}W_{2i}=\tilde{O}(\eta^2/T)$. Moreover,

$$
\begin{aligned}
|\text{(A5-2-2)}| &\leq \frac{1}{T}\cdot|\breve{\breve{\theta}}_1|\cdot\left|\frac{c_1\eta}{Th}\right|\cdot 2C+\frac{1}{T}\cdot|\breve{\breve{\theta}}_{T+1}|\cdot\left|\frac{c_{m+1}\eta}{Th}\right|\cdot 2C \\
&= \tilde{O}\left(\frac{\eta^2}{T^2h}\right),
\end{aligned}
$$

so that (A5-2) $=\tilde{O}(\eta^2/T)$. Similarly we obtain (A5-3) $=\tilde{O}(\eta^2/T)$, so that $T^{-1}\sum_{t=2}^{T}\breve{\theta}_t\breve{\theta}_{t-1}-T^{-1}\sum_{t=2}^{T}\breve{\theta}_{t-1}^2=O(\eta^2/T)$. ■

References

[1] Juhl, T. and Z. Xiao (2009), "Tests for Changing Mean with Monotonic Power," *Journal of Econometrics* 148, 14–24.