Supplementary Material: Boundary Detection using a Bayesian Hierarchical Model for Multiscale Spatial Data

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1 Introduction

In Appendix A, we provide the proofs of all the technical results stated in the main text of "Boundary Detection using a Bayesian Hierarchical Model for Multiscale Spatial Data" by K. Qu, J. R. Bradley, and X. Niu. In Appendix B, we provide step-by-step instructions on how to implement the BAGE method. In Appendix C, we provide simulations that compare the BAGE boundary detection method to the Canny method for boundary detection. Then, in Appendix D, we give additional simulation results to show that our model is robust to different specifications of the signal-to-noise ratio.

2 Appendix A: Technical Results

Proof of Proposition 1(a):

Let $\boldsymbol{\nu}^{(1)}(\boldsymbol{s}) \equiv (\nu_1^{(1)}(\boldsymbol{s}), ..., \nu_d^{(1)}(\boldsymbol{s}))'$, where $\boldsymbol{s} \in \mathcal{D} \subset \mathbb{R}^d$ and $\nu_m^{(1)}(\boldsymbol{s}) \equiv \frac{\partial \nu(\boldsymbol{s})}{\partial s_m}$. Also, let $\boldsymbol{\nu}^{(1)}(A) \equiv (\nu_{A,1}^{(1)}(\boldsymbol{s}), ..., \nu_{A,d}^{(1)}(\boldsymbol{s}))'$, where $\nu_{A,m}^{(1)}(A) \equiv \frac{1}{|A|} \int_A \nu_m^{(1)}(\boldsymbol{s}) d\boldsymbol{s}$. Similarly, let

the d-dimensional vector $\boldsymbol{\phi_i}^{(1)}(\boldsymbol{s}) \equiv (\phi_{i,1}^{(1)}(\boldsymbol{s}), ..., \phi_{i,d}^{(1)}(\boldsymbol{s}))'$, where $\phi_{i,m}^{(1)} \equiv \frac{\partial \phi_i(\boldsymbol{s})}{\partial s_m}$. Also, let $\boldsymbol{\phi}_{A,i}^{(1)}(A) \equiv (\phi_{A,i,1}^{(1)}(\boldsymbol{s}), ..., \phi_{A,i,d}^{(1)}(\boldsymbol{s}))'$ where $\phi_{A,i,m}^{(1)}(A) \equiv \frac{1}{|A|}\int_A \phi_{i,m}^{(1)}(\boldsymbol{s})d\boldsymbol{s}$. The goal is to show,

$$\zeta_m(A) \equiv E\left(\left|\nu_{A,m}^{(1)}(A) - \sum_{i=1}^{n} \phi_{A,i,m}^{(1)}(A)\alpha_i\right|^2\right); \quad m = 1, ..., d \tag{A.1}$$

Expand (A.1) as follows,

$$\zeta_m(A) = E\left(\nu_{A,m}^{(1)}(A)\right)^2 + E\left(\sum_{i=1}^{n}\sum_{j=1}^{n} \phi_{A,i,m}^{(1)}(A)\phi_{A,j,m}^{(1)}(A)\alpha_i\alpha_j\right) - 2E\left(\nu_{A,m}^{(1)}(A)\sum_{i=1}^{n} \phi_{A,i,m}^{(1)}(A)\alpha_i\right). \tag{A.2}$$

We will simplify each term in (A.2) as follows:

$$\begin{aligned} E\left(\nu_{A,m}^{(1)}(A)\right)^2 &= E\left(\frac{1}{|A|^2}\int_A\int_A \nu_m^{(1)}(\boldsymbol{s})\nu_m^{(1)}(\boldsymbol{u})d\boldsymbol{s}d\boldsymbol{u}\right) \\ &= \frac{1}{|A|^2}\int_A\int_A E\left(\nu_m^{(1)}(\boldsymbol{s})\nu_m^{(1)}(\boldsymbol{u})\right)d\boldsymbol{s}d\boldsymbol{u} \\ &= \frac{1}{|A|^2}\int_A\int_A cov\left(\nu_m^{(1)}(\boldsymbol{s}), \nu_m^{(1)}(\boldsymbol{u})\right)d\boldsymbol{s}d\boldsymbol{u} \\ &= \frac{1}{|A|^2}\int_A\int_A \frac{\partial^2 cov(\nu(\boldsymbol{s}), \nu(\boldsymbol{u}))}{\partial s_m \partial u_m}d\boldsymbol{s}d\boldsymbol{u} \end{aligned} \tag{A.3}$$

$$
\begin{aligned}
E\left(\sum_{i:j=1}^{n}\phi^{(1)}_{A,i,m}(A)\phi^{(1)}_{A,j,m}(A)\alpha_i\alpha_j\right) &= \sum_{i=1}^{n}\sum_{j=1}^{n}\phi^{(1)}_{A,i,m}(A)\phi^{(1)}_{A,j,m}(A)E(\alpha_i\alpha_j) \\
&= \frac{1}{|A|^2}\int_A\int_A\sum_{i=1}^{n}\sum_{j=1}^{n}\phi^{(1)}_{i,m}(\boldsymbol{s})\phi^{(1)}_{j,m}(\boldsymbol{u})\lambda_i\delta_{ij}d\boldsymbol{s}d\boldsymbol{u} \\
&= \frac{1}{|A|^2}\int_A\int_A\sum_{i=1}^{n}\phi^{(1)}_{i,m}(\boldsymbol{s})\phi^{(1)}_{i,m}(\boldsymbol{u})\lambda_i d\boldsymbol{s}d\boldsymbol{u},
\end{aligned}
\tag{A.4}
$$

and

$$
\begin{aligned}
&E\left(\nu^{(1)}_{A,m}(A)\sum_{i=1}^{n}\phi^{(1)}_{A,i,m}(A)\alpha_i\right) \\
&= \frac{1}{|A|^2}E\left(\int_A\nu^{(1)}_m(\boldsymbol{s})d\boldsymbol{s}\sum_{i=1}^{n}\int_A\phi^{(1)}_{i,m}(\boldsymbol{u})d\boldsymbol{u}\int_D\nu(\boldsymbol{\omega})\phi_{i,m}(\boldsymbol{\omega})d\boldsymbol{\omega}\right) \\
&= \frac{1}{|A|^2}\int_A\int_A\sum_{i=1}^{n}\phi^{(1)}_{i,m}(\boldsymbol{u})\left(\int_D E(\nu^{(1)}_m(\boldsymbol{s})\nu(\boldsymbol{\omega}))\phi_{i,m}(\boldsymbol{\omega})d\boldsymbol{\omega}\right)d\boldsymbol{s}d\boldsymbol{u} \\
&= \frac{1}{|A|^2}\int_A\int_A\sum_{i=1}^{n}\phi^{(1)}_{i,m}(\boldsymbol{u})\left(\int_D\frac{\partial cov(\nu(\boldsymbol{s}),\nu(\boldsymbol{\omega}))}{\partial s_m}\phi_{i,m}(\boldsymbol{\omega})d\boldsymbol{\omega}\right)d\boldsymbol{s}d\boldsymbol{u} \\
&= \frac{1}{|A|^2}\int_A\int_A\sum_{i=1}^{n}\phi^{(1)}_{i,m}(\boldsymbol{s})\phi^{(1)}_{i,m}(\boldsymbol{u})\lambda_i d\boldsymbol{s}d\boldsymbol{u}
\end{aligned}
\tag{A.5}
$$

Substituting (A.3), (A.4), and (A.5) into (A.2) gives

$$
\zeta_m(A) = \frac{1}{|A|^2}\int_A\int_A\frac{\partial^2 cov(\nu(\boldsymbol{s}),\nu(\boldsymbol{u}))}{\partial s_m\partial u_m}d\boldsymbol{s}d\boldsymbol{u} - \frac{1}{|A|^2}\int_A\int_A\sum_{i=1}^{n}\phi^{(1)}_{i,m}(\boldsymbol{s})\phi^{(1)}_{i,m}(\boldsymbol{u})\lambda_i d\boldsymbol{s}d\boldsymbol{u}.
$$

Upon taking the limit we have

$$\lim_{n\to\infty}\zeta_m(A) = \lim_{n\to\infty}\left(\frac{1}{|A|^2}\int_A\int_A \frac{\partial^2 cov(\nu(\boldsymbol{s}),\nu(\boldsymbol{u}))}{\partial s_m \partial u_m} d\boldsymbol{s}d\boldsymbol{u} - \frac{1}{|A|^2}\int_A\int_A \sum_{i=1}^n \phi_{i,m}^{(1)}(\boldsymbol{s})\phi_{i,m}^{(1)}(\boldsymbol{u})\lambda_i d\boldsymbol{s}d\boldsymbol{u}\right)$$
$$= \frac{1}{|A|^2}\int_A\int_A \frac{\partial^2 cov(\nu(\boldsymbol{s}),\nu(\boldsymbol{u}))}{\partial s_m \partial u_m} d\boldsymbol{s}d\boldsymbol{u} - \frac{1}{|A|^2}\int_A\int_A \lim_{n\to\infty}\sum_{i=1}^n \phi_{i,m}^{(1)}(\boldsymbol{s})\phi_{i,m}^{(1)}(\boldsymbol{u})\lambda_i d\boldsymbol{s}d\boldsymbol{u} \tag{A.6}$$

where we have assumed a dominating measure in (A.6). It follows from Kadota [1967] that $\lim_{n\to\infty}\zeta_m(A) = 0$ (A.7) for each $A \subset D_s$.

Proof of Proposition 1(b)

$$cov(\nu_{A,m}^{(1)}(A), \nu_{A,m}^{(1)}(B)) - \sum_{i=1}^n \phi_{A,i,m}^{(1)}(A)\phi_{A,i,m}^{(1)}(B)\lambda_i$$
$$= E\left(\nu_{A,m}^{(1)}(A)\nu_{A,m}^{(1)}(B)\right) - \frac{1}{|A||B|}\int_A\int_B \sum_{i=1}^n \phi_{i,m}^{(1)}(\boldsymbol{s})\phi_{i,m}^{(1)}(\boldsymbol{u})\lambda_i d\boldsymbol{s}d\boldsymbol{u}$$
$$= \frac{1}{|A||B|}\int_A\int_B E\left(\nu_m^{(1)}(\boldsymbol{s})\nu_m^{(1)}(\boldsymbol{u})\right) d\boldsymbol{s}d\boldsymbol{u} - \frac{1}{|A||B|}\int_A\int_B \sum_{i=1}^n \phi_{i,m}^{(1)}(\boldsymbol{s})\phi_{i,m}^{(1)}(\boldsymbol{u})\lambda_i d\boldsymbol{s}d\boldsymbol{u}$$
$$= \frac{1}{|A||B|}\int_A\int_B cov\left(\nu_m^{(1)}(\boldsymbol{s})\nu_m^{(1)}(\boldsymbol{u})\right) d\boldsymbol{s}d\boldsymbol{u} - \frac{1}{|A||B|}\int_A\int_B \sum_{i=1}^n \phi_{i,m}^{(1)}(\boldsymbol{s})\phi_{i,m}^{(1)}(\boldsymbol{u})\lambda_i d\boldsymbol{s}d\boldsymbol{u}$$
$$= \frac{1}{|A||B|}\int_A\int_B \frac{\partial^2 cov(\nu(\boldsymbol{s}),\nu(\boldsymbol{u}))}{\partial s_m \partial u_m} d\boldsymbol{s}d\boldsymbol{u} - \frac{1}{|A||B|}\int_A\int_B \sum_{i=1}^n \phi_{i,m}^{(1)}(\boldsymbol{s})\phi_{i,m}^{(1)}(\boldsymbol{u})\lambda_i d\boldsymbol{s}d\boldsymbol{u}$$
$$= \frac{1}{|A||B|}\int_A\int_B \left(\frac{\partial^2 cov(\nu(\boldsymbol{s}),\nu(\boldsymbol{u}))}{\partial s_m \partial u_m} - \sum_{i=1}^n \phi_{i,m}^{(1)}(\boldsymbol{s})\phi_{i,m}^{(1)}(\boldsymbol{u})\lambda_i\right) d\boldsymbol{s}d\boldsymbol{u} \tag{A.8}$$

It follows from the similar argument below (A.6) section that (A.8) converges to zero.

Proof of Proposition 2(a): We start to by showing that $\phi^{(1)}_{k,m}(\boldsymbol{x}_j) = \phi^{(1)}_{A,k,m}(A_j)$ implies $f(\nu^{(1)}_{s,m}) = f(\nu^{(1)}_{A,m})$ almost surely. We proceed through proof by contradiction. Assume $\nu^{(1)}_m \equiv \nu^{(1)}_{s,m}$ is not almost surely equal to $\nu^{(1)}_{A,m}$, then, for at least one $\boldsymbol{x_i}$ and A_i, there exist $\gamma > 0$ such that

$$\Pr(|\nu^{(1)}_m(\boldsymbol{x_i}) - \nu^{(1)}_{A,m}(A_i)| \geq \gamma) > 0. \tag{A.9}$$

However, from Chebychev's inequality, and the assumption that $\phi^{(1)}_{k,m}(\boldsymbol{x}_j) = \phi^{(1)}_{A,k}(A_j)$ for $j = 1, ..., n_A$,

$$\begin{aligned}
&\Pr(|\nu^{(1)}_m(\boldsymbol{x_i}) - \nu^{(1)}_{A,m}(A_i)| \geq \gamma) \leq \frac{E\left((\nu^{(1)}_m(\boldsymbol{x_i}) - \nu^{(1)}_{A,m}(A_i))^2\right)}{\gamma^2} \\
&\leq \frac{1}{\gamma^2} E\left(\nu^{(1)}_m(\boldsymbol{x_i}) - \sum_{k=1}^{n} \phi^{(1)}_{k,m}(\boldsymbol{x}_i)\alpha_k + \sum_{k=1}^{n} \phi^{(1)}_{A,k,m}(A_i)\alpha_k - \nu^{(1)}_{A,m}(A_i)\right)^2 \\
&= \frac{1}{\gamma^2} E\left(\nu^{(1)}_m(\boldsymbol{x_i}) - \sum_{k=1}^{n} \phi^{(1)}_{k,m}(\boldsymbol{x}_i)\alpha_k\right)^2 + \frac{1}{\gamma^2} E\left(\sum_{k=1}^{n} \phi^{(1)}_{A,k,m}(A_i)\alpha_k - \nu^{(1)}_{A,m}(A_i)\right)^2 \\
&+ \frac{2}{\gamma^2} E\left((\nu^{(1)}_m(\boldsymbol{x_i}) - \sum_{k=1}^{n} \phi^{(1)}_{k,m}(\boldsymbol{x}_i)\alpha_k)(\sum_{k=1}^{n} \phi^{(1)}_{A,k,m}(A_i)\alpha_k - \nu^{(1)}_{A,m}(A_i))\right)
\end{aligned} \tag{A.10}$$

From Kadota [1967] and Proposition 1(a),

$$\begin{aligned}
\lim_{n\to\infty} E\left(\nu^{(1)}_m(\boldsymbol{x_i}) - \sum_{k=1}^{n} \phi^{(1)}_{k,m}(\boldsymbol{x}_i)\alpha_k\right)^2 &= 0 \\
\lim_{n\to\infty} E\left(\sum_{k=1}^{n} \phi^{(1)}_{A,k,m}(A_i)\alpha_k - \nu^{(1)}_{A,m}(A_i)\right)^2 &= 0.
\end{aligned} \tag{A.11}$$

Thus our next step in this proof is to show that

$$\lim_{n\to\infty} \frac{2}{\gamma^2} E\left((\nu_m^{(1)}(\boldsymbol{x_i}) - \sum_{k=1}^{n} \phi_{k,m}^{(1)}(\boldsymbol{x}_i)\alpha_k)(\sum_{k=1}^{n} \phi_{A,k,m}^{(1)}(A_i)\alpha_k - \nu_{A,m}^{(1)}(A_i)) \right) = 0. \tag{A.12}$$

Combining (A.12) and (A.11) in (A.10) contradicts the expression in (A.9), and completes the result. Expand each term on the left hand side of (A.12) as follows,

$$\begin{aligned}
H &\equiv E\left((\nu_m^{(1)}(\boldsymbol{x_i}) - \sum_{k=1}^{n} \phi_{k,m}^{(1)}(\boldsymbol{x}_i)\alpha_k)(\sum_{k=1}^{n} \phi_{A,k,m}^{(1)}(A_i)\alpha_k - \nu_{A,m}^{(1)}(A_i)) \right) \\
&= E\int_{A_i} \sum_{k=1}^{n} \nu_m^{(1)}(\boldsymbol{x_i})\alpha_k\phi_{k,m}^{(1)}(\boldsymbol{s})d\boldsymbol{s} - E\int_{A_i} \left(\nu_m^{(1)}(\boldsymbol{x_i})\nu_m^{(1)}(\boldsymbol{s})d\boldsymbol{s}\right) \\
&- E\sum_{k=1}^{n}\sum_{j=1}^{n} \int_{A_i} \left(\alpha_k\phi_{k,m}^{(1)}(\boldsymbol{x}_i)\alpha_j\phi_{j,m}^{(1)}(\boldsymbol{s})d\boldsymbol{s}\right) + E\sum_{k=1}^{n} \int_{A_i} \left(\alpha_k\phi_{k,m}^{(1)}(\boldsymbol{x}_i)\nu_m^{(1)}(\boldsymbol{s})d\boldsymbol{s}\right) \\
&\equiv H_1 - H_2 - H_3 + H_4,
\end{aligned}$$

where $\boldsymbol{x}_i = (x_1, ..., x_d)' \in D$

$$H_1 = E\int_{A_i}\sum_{k=1}^{n}\nu_m^{(1)}(\boldsymbol{x_i})\alpha_k\phi_{k,m}^{(1)}(\boldsymbol{s})d\boldsymbol{s} = E\int_{A_i}\sum_{k=1}^{n}\nu_m^{(1)}(\boldsymbol{x_i})\left(\int_{D_s}\nu(\boldsymbol{u})\phi_k(\boldsymbol{u})d\boldsymbol{u}\right)\phi_{k,m}^{(1)}(\boldsymbol{s})d\boldsymbol{s}$$
$$= E\int_{A_i}\sum_{k=1}^{n}\frac{\partial}{\partial x_{i,m}}\int_{D_s}\nu(\boldsymbol{x_i})\nu(\boldsymbol{u})\phi_k(\boldsymbol{u})d\boldsymbol{u}\phi_{k,m}^{(1)}(\boldsymbol{s})d\boldsymbol{s}$$
$$= \int_{A_i}\sum_{k=1}^{n}\frac{\partial}{\partial x_{i,m}}\int_{D_s}cov(\nu(\boldsymbol{x_i}),\nu(\boldsymbol{u}))\phi_k(\boldsymbol{u})d\boldsymbol{u}\phi_{k,m}^{(1)}(\boldsymbol{s})d\boldsymbol{s}$$
$$= \int_{A_i}\sum_{k=1}^{n}\frac{\partial}{\partial x_{i,m}}\phi_k(\boldsymbol{x_i})\lambda_k\phi_{k,m}^{(1)}(\boldsymbol{s})d\boldsymbol{s} = \int_{A_i}\sum_{k=1}^{n}\phi_{k,m}^{(1)}(\boldsymbol{s})\phi_{k,m}^{(1)}(\boldsymbol{x_i})\lambda_k d\boldsymbol{s},$$

$$H_2 = E\left(\int_{A_i}\nu_m^{(1)}(\boldsymbol{x_i})\nu_m^{(1)}(\boldsymbol{s})d\boldsymbol{s}\right) = \int_{A_i}\frac{\partial^2 cov(\nu(\boldsymbol{x_i}),\nu(\boldsymbol{s}))}{\partial x_{i,m}\partial s_m}d\boldsymbol{s},$$

$$H_3 = E\left(\sum_{k=1}^{n}\sum_{j=1}^{n}\int_{A_i}\alpha_k\phi_{k,m}^{(1)}(\boldsymbol{x}_i)\alpha_j\phi_{j,m}^{(1)}(\boldsymbol{s})d\boldsymbol{s}\right) = \int_{A_i}\sum_{k=1}^{n}\phi_{k,m}^{(1)}(\boldsymbol{s})\phi_{k,m}^{(1)}(\boldsymbol{x_i})\lambda_k d\boldsymbol{s},$$

and

$$H_4 = E\left(\sum_{k=1}^{n}\int_{A_i}\left(\alpha_k\phi_{k,m}^{(1)}(\boldsymbol{x}_i)\nu_m^{(1)}(\boldsymbol{s})d\boldsymbol{s}\right)\right) = E\left(\sum_{k=1}^{n}\int_{A_i}\left(\int_{D}\nu(\boldsymbol{u})\phi_k(\boldsymbol{u})d\boldsymbol{u}\phi_{k,m}^{(1)}(\boldsymbol{x}_i)\frac{\partial\nu(\boldsymbol{s})}{\partial s_m}d\boldsymbol{s}\right)\right)$$
$$= \sum_{k=1}^{n}\int_{A_i}\frac{\partial}{\partial s_m}\left(\int_{D}cov(\nu(\boldsymbol{u}),\nu(\boldsymbol{s}))\phi_k(\boldsymbol{u})d\boldsymbol{u}\phi_{k,m}^{(1)}(\boldsymbol{x}_i)d\boldsymbol{s}\right) = \sum_{k=1}^{n}\int_{A_i}\frac{\partial}{\partial s_m}\phi_k(\boldsymbol{s})\lambda_k\phi_{k,m}^{(1)}(\boldsymbol{x}_i)d\boldsymbol{s}$$
$$= \int_{A_i}\sum_{k=1}^{n}\phi_{k,m}^{(1)}(\boldsymbol{s})\phi_{k,m}^{(1)}(\boldsymbol{x_i})\lambda_k d\boldsymbol{s}.$$

Thus, it follows from that Proposition 1(b) $\lim_{n\to\infty} H = 0$, which proves the result.

To prove the reverse statement of 2(a), suppose $f(\nu_{s,m}^{(1)}) = f(\nu_{A,m}^{(1)})$ almost surely for any measurable real-valued function f, then the result follows from setting f equal to the identity function.

Proposition 2(b) The proof is similar to Proposition 2(a).

Proposition 2(c): Note that, $\phi^{(1)}_{k,m}(B_j) = \frac{1}{|B_j|}\int_{B_j}\phi^{(1)}_{k,m}(A_j)d\boldsymbol{s} = \phi^{(1)}_{k,m}(A_j) \quad (A.13)$, then it follows that $f(\boldsymbol{\nu}^{(1)}_B) = f(\boldsymbol{\nu}^{(1)}_A)$ by Proposition 2(b).

Proof: BAGE(A)=0 when Eigenfunctions are homogeneous within scale.

If $\phi^{(1)}_{j,m}(\boldsymbol{w}) = \phi^{(1)}_{A,j,m}(A)$ for every j, we have from Proposition 2,

$$\boldsymbol{\nu}^{(1)}(\boldsymbol{w}) = \boldsymbol{\nu}^{(1)}(A) \quad \boldsymbol{w} \in A. \tag{A.14}$$

We also have that $C\{\boldsymbol{w}(t)\} = \frac{1}{\|\boldsymbol{s}_1(t)\|}(s_{12}, \ -s_{11}) \equiv \boldsymbol{s}_1^{\perp}$, where for simplicity we set $d = 2$ and $\boldsymbol{s}_1 = (s_{11}, s_{12})'$. Hence $C\{\boldsymbol{w}(t)\}$ does not depend on $\boldsymbol{s}_0$ or on $\mathcal{T}$. This implies that,

$$\begin{aligned} W(\boldsymbol{s}_0, \alpha, A) &= \frac{1}{N(\mathcal{T})}\int_{\mathcal{T}}\langle\boldsymbol{\nu}^{(1)}(\boldsymbol{w}(t, \boldsymbol{s}_0, \alpha)), C\{\boldsymbol{w}(t, \boldsymbol{s}_0, \alpha)\}\rangle\|\boldsymbol{w}^{(1)}(t, \boldsymbol{s}_0, \alpha)\|dt \\ &= \boldsymbol{\nu}^{(1)}(A)'\left\{\frac{1}{N(\mathcal{T})}\int_{\mathcal{T}} C\{\boldsymbol{w}(t, \boldsymbol{s}_0, \alpha)\}\|\boldsymbol{w}^{(1)}(t, \boldsymbol{s}_0, \alpha)\|dt\right\} = \boldsymbol{\nu}^{(1)}(A)'\boldsymbol{s}_1^{\perp} \end{aligned} \tag{A.15}$$

$$\begin{aligned} W(\alpha, A) &= \frac{1}{|A|}\int_A W(\boldsymbol{s}_0, \alpha, A)d\boldsymbol{s}_0 \\ &= \frac{1}{|A|}\int_A \boldsymbol{\nu}^{(1)}(A)'\left\{\frac{1}{N(\mathcal{T})}\int_{\mathcal{T}} C\{\boldsymbol{w}(t, \boldsymbol{s}_0, \alpha)\}\|\boldsymbol{w}^{(1)}(t, \boldsymbol{s}_0, \alpha)\|dt\right\}d\boldsymbol{s}_0 \\ &= \boldsymbol{\nu}^{(1)}(A)'\frac{1}{|A|}\int_A d\boldsymbol{s}_0\left\{\frac{1}{N(\mathcal{T})}\int_{\mathcal{T}} \boldsymbol{s}_1^{\perp}\|\boldsymbol{w}^{(1)}(t, \boldsymbol{s}_0, \alpha)\|dt\right\} = \boldsymbol{\nu}^{(1)}(A)'\boldsymbol{s}_1^{\perp} \\ &= W(\boldsymbol{s}_0, \alpha, A), \quad \forall\alpha. \end{aligned} \tag{A.16}$$

Thus, $E(BAGE(\alpha, A)) = E\left[\frac{1}{|A|}\int_A(W(\boldsymbol{s}_0, \alpha, A) - W(\alpha, A))^2 d\boldsymbol{s}_0|\boldsymbol{Z}\right] = 0$. The proof of the converse is provided below.

We have shown that the absence of CBF error implies that BAGE($\cdot$) is zero. We are left to show the reverse that when BAGE($\cdot$) is zero CBF error is not present in L_2. Our proof involves showing the contrapositive of this statement, which is logically equivalent. That is, the presence of CBF error in L_2 implies that is BAGE(A) greater than zero for $A \subset D_s$.

We proceed through proof by contradiction. Assume BAGE(A) = 0 and CBF error exists at A. By definition, the trace of $E \int_A \left(\boldsymbol{\phi^{(1)}}(\boldsymbol{s})\boldsymbol{\alpha} - \boldsymbol{\phi^{(1)}}(A)\boldsymbol{\alpha}\right) \cdot \left(\boldsymbol{\phi^{(1)}}(\boldsymbol{s})\boldsymbol{\alpha} - \boldsymbol{\phi^{(1)}}(A)\boldsymbol{\alpha}\right)' d\boldsymbol{s} \quad > 0$ when CBF exists in L_2 at A (See Proposition 2).

Let $\boldsymbol{\mathcal{L}(s)} = \left(\boldsymbol{\phi^{(1)}}(\boldsymbol{s}) - \boldsymbol{\phi^{(1)}}(A)\right)\left(\boldsymbol{\phi^{(1)}}(\boldsymbol{s}) - \boldsymbol{\phi^{(1)}}(A)\right)'$ and the $l_{(i,j)} = \sum_{k=1}^{r}(\phi_{k,i}^{(1)}(\boldsymbol{s}) - \phi_{k,i}^{(1)}(A))(\phi_{k,j}^{(1)}(\boldsymbol{s}) - \phi_{k,j}^{(1)}(A))$ for $(i,j) = 1, ..., d$, then

$$\begin{aligned}
&trace\left(E\int_A \left(\boldsymbol{\phi^{(1)}}(\boldsymbol{s})\boldsymbol{\alpha} - \boldsymbol{\phi^{(1)}}(A)\boldsymbol{\alpha}\right)\left(\boldsymbol{\alpha}'\boldsymbol{\phi^{(1)}}(\boldsymbol{s})' - \boldsymbol{\alpha}'\boldsymbol{\phi^{(1)}}(A)'\right) d\boldsymbol{s}\right) \\
&= E\left(\int_A trace\left(\boldsymbol{\alpha}'\boldsymbol{\mathcal{L}}(\boldsymbol{s})\boldsymbol{\alpha}\right) d\boldsymbol{s}\right) \\
&= E\left(\boldsymbol{\alpha}'\int_A \boldsymbol{\mathcal{L}}(\boldsymbol{s})d\boldsymbol{s}\boldsymbol{\alpha}\right) \\
&= \sum_i^r E(\alpha_i^2)l_{(i,i)} + \sum_{i=1}^r\sum_{j=1}^r E(\alpha_i\alpha_j)l_{(i,j)} \\
&= \sum_i^r var(\alpha_i)l_{(i,i)}
\end{aligned} \tag{A.17}$$

Assuming CBF error exists in L_2, there is at least one $var(\alpha_i)$ greater than zero and the diagonal elements of $L(\boldsymbol{s})$ are greater than zero, since from Proposition 2, $\phi_{k,i}^{(1)}(s) \neq \phi_{k,i}^{(1)}(A)$.

The proof will now involve writing E(BAGE(A)) as a quadratic form of $\boldsymbol{\alpha}$, which will only be equal to zero when $var(\alpha_i) = 0$ for all i. This is a contradiction

of the existence of CBF error at A. By definition,

$$E\{BAGE(\boldsymbol{\alpha}, A)\} = E\Big(\frac{1}{|A|}\int_A (W(\boldsymbol{s}, \boldsymbol{\alpha}, A) - W(\boldsymbol{\alpha}, A))^2 d\boldsymbol{s}|\boldsymbol{Z}\Big) \tag{A.18}$$

$$= E\Big(\frac{1}{|A|}\int_A \Big(W(\boldsymbol{s}, \boldsymbol{\alpha}, A) - \frac{1}{|A|}\int_A W(\boldsymbol{s}, \boldsymbol{\alpha}, A)d\boldsymbol{s}\Big)^2 d\boldsymbol{s}|\boldsymbol{Z}\Big)$$

$$= E\Big(\frac{1}{|A|}\int_A \Big(\frac{1}{|A|}\int_{\mathcal{T}} \boldsymbol{\nu}^{(1)}(\boldsymbol{w}(t, \boldsymbol{s}_0, \boldsymbol{s}_1))' C\{\boldsymbol{w}(t, \boldsymbol{s}_0, \boldsymbol{s}_1)\}\|\boldsymbol{w}^{(1)}(t, \boldsymbol{s}_0, \boldsymbol{s}_1)\|dt$$

$$-\frac{1}{|A|}\int_A \frac{1}{|A|}\int_{\mathcal{T}} \boldsymbol{\nu}^{(1)}(\boldsymbol{w}(t, \boldsymbol{s}_0, \boldsymbol{s}_1))' C\{\boldsymbol{w}(t, \boldsymbol{s}_0, \boldsymbol{s}_1)\}\|\boldsymbol{w}^{(1)}(t, \boldsymbol{s}_0, \boldsymbol{s}_1)\|d(t)d\boldsymbol{s}\Big)^2 d\boldsymbol{s}|\boldsymbol{Z}\Big)$$

$$= E\Big(\frac{1}{|A|}\int_A \boldsymbol{\alpha}' \Big(\frac{1}{|A|}\int_{\mathcal{T}} \boldsymbol{\phi}^{(1)}(\boldsymbol{w}(t, \boldsymbol{s}_0, \boldsymbol{s}_1))' C\{\boldsymbol{w}(t, \boldsymbol{s}_0, \boldsymbol{s}_1)\}\|\boldsymbol{w}^{(1)}(t, \boldsymbol{s}_0, \boldsymbol{s}_1)\|dt$$

$$-\frac{1}{|A|}\int_A \frac{1}{|A|}\int_{\mathcal{T}} \boldsymbol{\phi}^{(1)}(\boldsymbol{w}(t, \boldsymbol{s}_0, \boldsymbol{s}_1))' C\{\boldsymbol{w}(t, \boldsymbol{s}_0, \boldsymbol{s}_1)\}\|\boldsymbol{w}^{(1)}(t, \boldsymbol{s}_0, \boldsymbol{s}_1)\|d(t)d\boldsymbol{s}\Big)^2 \boldsymbol{\alpha} d\boldsymbol{s}|\boldsymbol{Z}\Big)$$

Let $\boldsymbol{W} = \int_A \Big(\frac{1}{|A|}\int_{\mathcal{T}} \boldsymbol{\phi}^{(1)}(\boldsymbol{w}(t, \boldsymbol{s}_0, \boldsymbol{s}_1))' C\{\boldsymbol{w}(t, \boldsymbol{s}_0, \boldsymbol{s}_1)\}\|\boldsymbol{w}^{(1)}(t, \boldsymbol{s}_0, \boldsymbol{s}_1)\|dtd\boldsymbol{s}$ $-\frac{1}{|A|}\int_A \frac{1}{|A|}\int_{\mathcal{T}} \boldsymbol{\phi}^{(1)}(\boldsymbol{s})' C\{\boldsymbol{w}(t, \boldsymbol{s}_0, \boldsymbol{s}_1)\}\|\boldsymbol{w}^{(1)}(t, \boldsymbol{s}_0, \boldsymbol{s}_1)\|d(t)d\boldsymbol{s}\Big)^2$ and denote the i-th element of this r dimensional vector with w_i for $i = 1, ..., r$. Then

$$\begin{aligned} E(BAGE(A)) &= E\left(\int_A \boldsymbol{\alpha}'\boldsymbol{W}\boldsymbol{W}'\boldsymbol{\alpha} d\boldsymbol{s}|\boldsymbol{Z}\right) \\ &= E\int_A \left(\sum_{i=1}^r \sum_{j=1}^r \alpha_i w_i w_j \alpha_j\right) d\boldsymbol{s}|\boldsymbol{Z} \\ &= \int_A \left(\sum_{i=1}^r E(\alpha_i^2) w_i^2 + \sum_{i=1}^r \sum_{j=1}^r E(\alpha_i \alpha_j) w_i w_i\right) d\boldsymbol{s}|\boldsymbol{Z} \\ &= \int_A \left(\sum_{i=1}^r var(\alpha_i) w_i^2 + 0\right) d\boldsymbol{s}|\boldsymbol{Z}. \end{aligned} \tag{A.19}$$

Since BAGE(A) is assumed zero and $w_i^2 > 0$ (this condition can be checked for the choice of $\boldsymbol{\phi}(\boldsymbol{s})$) then $var(\alpha_i) = 0$ for $i = 1, ..., r$. However this contradicts the above result that there is at least one of $var(\alpha_i)$ greater than zero when CBF error

exists. This proves the result.

3 Appendix B: Step-by-Step Implementation of BAGE

Given posterior MCMC replicates from the model in the main text, we can compute BAGE(A), as follows:

1. Randomly sample $\tilde{\boldsymbol{s}}_1, ..., \tilde{\boldsymbol{s}}_m$ with $\tilde{\boldsymbol{s}}_j = (\tilde{\boldsymbol{s}}_{1j}, \tilde{\boldsymbol{s}}_{2j})$. Define an angle α.
2. For each b=1,...,B and j=1,...,m, based on (15), compute of the
   $\frac{\partial \nu(\tilde{\boldsymbol{s}})^{[b]}}{\partial \tilde{s}_{1j}} = \frac{-4}{\phi}(\tilde{s}_{1j} - c_{i1}) exp\left(\frac{-2}{\phi}\{(\tilde{s}_{1j} - c_{i1})^2 + (\tilde{s}_{2j} - c_{i2})^2\}\right) \boldsymbol{\eta}^{[b]}$, and
   $\frac{\partial \nu(\tilde{\boldsymbol{s}})^{[b]}}{\partial \tilde{s}_{2j}} = \frac{-4}{\phi}(\tilde{s}_{2j} - c_{i2}) exp\left(\frac{-2}{\phi}\{(\tilde{s}_{1j} - c_{i1})^2 + (\tilde{s}_{2j} - c_{i2})^2\}\right) \boldsymbol{\eta}^{[b]}$,
   where $\tilde{\boldsymbol{s}} = (\tilde{s}_1, \tilde{s}_2)$ and $\{\boldsymbol{c}_i.\}$ are knot points.
3. For each b=1,...,B and j=1,...,m, compute
   $W(\tilde{\boldsymbol{s}}_j, \alpha, A)^{[b]} = \frac{1}{N(\mathcal{T})} \int_{\mathcal{T}} \langle \frac{\partial \nu(\tilde{\boldsymbol{s}}_j)^{[b]}}{\partial \tilde{\boldsymbol{s}}_{.j}}, C\{\boldsymbol{w}(t, \tilde{\boldsymbol{s}}_j, \alpha)\}\rangle \|\boldsymbol{w}^{(1)}(t, \tilde{\boldsymbol{s}}_j, \alpha)\| dt$ using the `integral` function in `Curbature R` package.
4. For each j=1,...,m, compute $W(\alpha, A)^{[b]} = \frac{1}{m}\sum_{j=1}^{m} W(\tilde{\boldsymbol{s}}_j, \alpha, A)^{[b]}$.
5. For each b, compute $BAGE(\alpha, A)^{[b]} = \frac{1}{|A|}\sum_{j=1}^{m}(W(\tilde{\boldsymbol{s}}_j, \alpha, A)^{[b]} - W(\alpha, A)^{[b]})^2$.
6. Compute $BAGE(\alpha, A) = \frac{1}{B}\sum_{b=1}^{B} BAGE(\alpha, A)^{[b]}$.
7. Repeat steps 1 - 6 for every A and α under consideration.
8. Produce a map of $\{BAGE(\alpha, A)\}$.

4 Appendix C: A Simulation Study Comparing the BAGE and Canny Methods

In general, it is difficult to use the simulations to validate boundary detection methods for continuous spatial fields. This is because there is no "true" boundaries that exist in spatial fields, just "large" gradients that suggest a boundary. Consequently, there is no "gold standard" to compare to. See Banerjee and Gelfand [2006] for additional discussions on this issue. Our setting, however, is different because we take a latent Gaussian process approach. That is, one can perform two sets of a standard boundary detection algorithms, one on the field $\{Y(\boldsymbol{s}) : \boldsymbol{s} \in D_s\}$ and another on the observed data $\{Z(\boldsymbol{s}) : \boldsymbol{s} \in D_s\}$. Since we are interested in inference on $\{Y(\boldsymbol{s}) : \boldsymbol{s} \in D_s\}$ and $\{Y(\boldsymbol{s}) : \boldsymbol{s} \in D_s\}$ is unobserved, chosen boundaries based on $\{Y(\boldsymbol{s}) : \boldsymbol{s} \in D_s\}$ can be treated as the "gold standard" to compare competing boundary detection methods. Thus, the first step in our comparison is to define a "gold standard" to compare to. Specifically the Canny boundary detection algorithm (`R` `edge.detect` function from the *R wvtool* package) is applied to a Gaussian random field $\{Y(\boldsymbol{s}) : \boldsymbol{s} \in D_s\}$ defined on a 32×32 grid. The Canny-selected boundaries on $\{Y(\boldsymbol{s}) : \boldsymbol{s} \in D_s\}$ are then aggregated to a 16×16 grid. These aggregated set of boundaries on the 16×16 are referred to the "gold standard." In particular, we compare the agreement between the Canny method that replaces $\{Y(\boldsymbol{s}) : \boldsymbol{s} \in D_s\}$ with $\{Z(\boldsymbol{s}) : \boldsymbol{s} \in D_s\}$ versus the gold standard. Similarly, we compare the agreement between calibrating BAGE in Section 4.2 and the gold standard. The "agreements" (accuracies) are computed using,

$$\text{Accuracy} = \frac{\sum_{i=1}^{n_A} I\{q_m(A_{16}) = q_G(A_{16})\}}{n_A}; \quad m = 1, 2 \tag{C.1}$$

where $q_G(A)$ is equal to one provided the "gold standard" classified a boundary in region A (and zero otherwise), the Canny method classifier that uses the data $\{Z(\boldsymbol{s}) : \boldsymbol{s} \in D_s\}$ as an input (instead of $\{Y(\boldsymbol{s}) : \boldsymbol{s} \in D_s\}$), is denoted with $q_2(A)$. The BAGE method classifier $q_1(\cdot)$ is defined as,

$$q_1(A) = \mathrm{I}[\max_{\alpha} \{BAGE(\alpha, A)\} > f]; \quad A \subset D \tag{C.2}$$

which is equal to one when we classify a boundary in A and is equal to zero when then no boundary detected in A. We choose the value of f in (C.2) to maximize the agreement between the BAGE method classifier and the Canny method classifier that uses the estimated $\{E(Y(\boldsymbol{s})|\boldsymbol{Z} : \boldsymbol{s} \in D_s\}$ instead of $\{Y(\boldsymbol{s}) : \boldsymbol{s} \in D_s\}$. Here, the n-dimensional data vector is defined as $\boldsymbol{Z} = (Z(\boldsymbol{s}_1), ..., Z(\boldsymbol{s}_n))'$. Note that both $q_1(\cdot)$ and $q_2(\cdot)$ are completely data driven. Thus the method with large values of (C.1) implies a higher agreement with the gold standard. These accuracy calculations are repeated with 30 times with simulated data sets having small signal to noise ratio (SNR) of 0.2. Boxplot of 30 replicates of the accuracies in (C.1) are presented in Figure 1.

The BAGE criterion can be used to perform multiscale inference on the unobserved process $\{Y(\boldsymbol{s}) : \boldsymbol{s} \in D_s\}$. However, the simulated data set (that defines $\boldsymbol{Z}$) in this section only used a single scale of the data. This is done partially because the standard computational package that implement Canny can not handle multiple overlapping scales of the data.

To make our comparisons fair, when Canny is applied to $\{Y(\boldsymbol{s}) : \boldsymbol{s} \in D_s\}$, $\boldsymbol{Z}$, and $\{E(Y(\boldsymbol{s})|\boldsymbol{Z} : \boldsymbol{s} \in D_s\}$, we use the same default specification of the low threshold, high threshold, and bandwidth parameters from the R package *wvtool*.

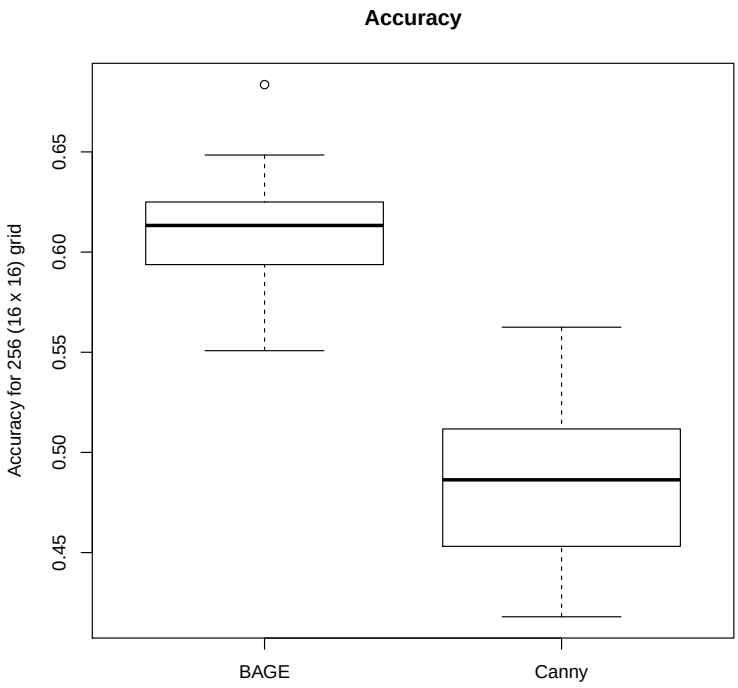

Figure 1: Boxplot of accuracy over 30 replicates for an aggregation (16×16 grid)

These numerical simulation results support that BAGE is practically useful in boundary classification. In particular, for realistic signal to noise ratios, the BAGE method performs better (in term of (C.1)) than the Canny method because our approach can appropriately account for uncertainty in the data. Furthermore, the results from Section 5.1 show that our method can provide a measure of uncertainty (i.e., BAGE), while the deterministic Canny can not.

5 Appendix D: Model Performance by Signal-to-noise Ratio

The signal-to-noise ratio (SNR) is defined to be,

$$\frac{\sum_{i=1}^{n} Var(Y_i)}{\sum_{i=1}^{n} Var(\varepsilon_i)} \tag{C.1}$$

In Tables 1 and 2 we present the estimates by SNR and their true values. Here we see that we can reasonably recover the truth in each setting.

Table 1: Posterior quantiles of σ_ε^2 by SNR and various specifications of the true β and σ_ε^2 in the simulations

SNR	β(True)	σ_ε^2**(True)**	Est. 2.5% (σ_ε^2)	**Est. 50.0% (σ_ε^2)**	Est. 97.5% (σ_ε^2)
0.5	1	**2**	1.6503	**1.7999**	1.9782
1	1	**1**	0.8269	**0.9022**	0.9929
3	1	**0.3333**	0.2773	**0.3028**	0.3341
5	1	**0.2**	0.1686	**0.1828**	0.2013
7	1	**0.1429**	0.1219	**0.1324**	0.1444
0.5	3	**2**	1.6508	**1.7995**	1.9717
1	3	**1**	0.8267	**0.9051**	1.0000
3	3	**0.3333**	0.2765	**0.3021**	0.3329
5	3	**0.2**	0.1665	**0.1826**	0.1999
7	3	**0.1429**	0.1204	**0.1315**	0.1449
0.5	5	**2**	1.6346	**1.7939**	1.9670
1	5	**1**	0.8319	**0.9032**	0.9862
3	5	**0.3333**	0.2771	**0.3026**	0.3334
5	5	**0.2**	0.1684	**0.1828**	0.2011
7	5	**0.1429**	0.1202	**0.1316**	0.1454

Table 2: Posterior quantiles of β by SNR and various specifications of the true β and σ_ε^2 in the simulations

SNR	β**(True)**	σ_ε^2(True)	Est. 2.5% (β)	**Est. 50.0% (β)**	Est. 97.5% (β)
0.5	**1**	2	0.5024	**0.8121**	1.1150
1	**1**	1	0.4810	**0.7308**	1.1046
3	**1**	0.3333	0.6086	**0.8024**	0.9369
5	**1**	0.2	0.3620	**0.7239**	1.0198
7	**1**	0.1429	0.5232	**0.7369**	0.9080
0.5	**3**	2	2.4710	**2.7597**	3.0300
1	**3**	1	2.4680	**2.7569**	3.0245
3	**3**	0.3333	2.5754	**2.7851**	3.0470
5	**3**	0.2	2.3755	**2.6863**	3.0715
7	**3**	0.1429	2.5357	**2.7000**	2.9358
0.5	**5**	2	4.3228	**4.7875**	5.0949
1	**5**	1	4.4809	**4.7737**	5.1241
3	**5**	0.3333	4.4652	**4.6981**	5.0064
5	**5**	0.2	4.4759	**4.7818**	4.9628
7	**5**	0.1429	4.5747	**4.7469**	4.9321

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