Supplementary appendix

Proofs of Theorems

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A Proof of Theorem 1 (Identification)

For deriving the identification of the unconditional effect, we first repeat the definition of the types. In the derivations we also first permit for an undefined type, which will be assumed to have zero probability mass later. Let $\mathcal{N}_\varepsilon$ be a symmetric ε neighbourhood about z_0 and partition $\mathcal{N}_\varepsilon$ into $\mathcal{N}_\varepsilon^+ = \{z : z \geq z_0, z \in \mathcal{N}_\varepsilon\}$ and $\mathcal{N}_\varepsilon^- = \{z : z < z_0, z \in \mathcal{N}_\varepsilon\}$. According to their reaction to the instrument z over $\mathcal{N}_\varepsilon$ we can partition the population into five subpopulations:

$$
\begin{aligned}
\tau_{i,\varepsilon} &= a \quad \text{if} \quad D_i(z) = 1 \quad \forall z \in \mathcal{N}_\varepsilon^- \quad \text{and} \quad D_i(z) = 1 \quad \forall z \in \mathcal{N}_\varepsilon^+ \\
\tau_{i,\varepsilon} &= n \quad \text{if} \quad D_i(z) = 0 \quad \forall z \in \mathcal{N}_\varepsilon^- \quad \text{and} \quad D_i(z) = 0 \quad \forall z \in \mathcal{N}_\varepsilon^+ \\
\tau_{i,\varepsilon} &= c \quad \text{if} \quad D_i(z) = 0 \quad \forall z \in \mathcal{N}_\varepsilon^- \quad \text{and} \quad D_i(z) = 1 \quad \forall z \in \mathcal{N}_\varepsilon^+ \\
\tau_{i,\varepsilon} &= d \quad \text{if} \quad D_i(z) = 1 \quad \forall z \in \mathcal{N}_\varepsilon^- \quad \text{and} \quad D_i(z) = 0 \quad \forall z \in \mathcal{N}_\varepsilon^+ \\
\tau_{i,\varepsilon} &= ind \quad \text{if} \quad D_i(z) \text{ is nonconstant over } \mathcal{N}_\varepsilon^- \text{ or over } \mathcal{N}_\varepsilon^+.
\end{aligned}
$$

A preliminary observation is that $F_{X,Z}$ is permitted to be discontinuous in Z at z_0, whereas F_Z is assumed to be continuous at z_0. A few preliminaries are helpful for later derivations. By Assumption 1, F_Z is differentiable with $f_Z(z_0) > 0$. We can derive[1]

$$
\lim_{\varepsilon \to 0} \Pr\left(Z \geq z_0 | Z \in \mathcal{N}_\varepsilon\right) = \lim_{\varepsilon \to 0} \frac{F_Z(z_0+\varepsilon) - F_Z(z_0)}{F_Z(z_0+\varepsilon) - F_Z(z_0-\varepsilon)} = \lim_{\varepsilon \to 0} \frac{\frac{F_Z(z_0+\varepsilon)-F_Z(z_0)}{\varepsilon}}{\frac{F_Z(z_0+\varepsilon)-F_Z(z_0)}{\varepsilon} - \frac{F_Z(z_0-\varepsilon)-F_Z(z_0)}{\varepsilon}}
$$
$$
= \frac{f_Z(z_0)}{f_Z(z_0)+f_Z(z_0)} = \frac{1}{2}.
$$

Similarly,

$$
\lim_{\varepsilon \to 0} \Pr\left(X \leq x, Z \leq z_0 | Z \in \mathcal{N}_\varepsilon\right) = \lim_{\varepsilon \to 0} \Pr\left(X \leq x | Z \leq z_0, Z \in \mathcal{N}_\varepsilon\right) \Pr\left(Z \leq z_0 | Z \in \mathcal{N}_\varepsilon\right) = \frac{1}{2} F^-(x|z_0)
$$

and

$$
\lim_{\varepsilon \to 0} \Pr\left(X \leq x | Z \in \mathcal{N}_\varepsilon\right) = \lim_{\varepsilon \to 0} \Pr\left(X \leq x | Z \in \mathcal{N}_\varepsilon^-\right) \Pr\left(Z \leq z_0 | Z \in \mathcal{N}_\varepsilon\right) + \Pr\left(X \leq x | Z \in \mathcal{N}_\varepsilon^+\right) \Pr\left(Z > z_0 | Z \in \mathcal{N}_\varepsilon\right)
$$
$$
= \frac{F^-(x|z_0) + F^+(x|z_0)}{2}.
$$

Since by Assumption (1vi) the limit functions $F^-(x|z)$, $F^+(x|z)$ are assumed to be differentiable in x at z_0 we obtain

$$
\lim_{\varepsilon \to 0} \frac{dF_{X|Z \in \mathcal{N}_\varepsilon}(x)}{dx} = \frac{f^-(x|z_0) + f^+(x|z_0)}{2}.
$$

The following derivations make also use of Bayes' theorem in that

$$
\Pr(\tau_{i,\varepsilon} = c | X = x, Z \in \mathcal{N}_\varepsilon) \cdot f_{X|Z \in \mathcal{N}_\varepsilon}(x) = f_{X|\tau_{i,\varepsilon}=c, Z \in \mathcal{N}_\varepsilon}(x) \cdot \Pr(\tau_{i,\varepsilon} = c | Z \in \mathcal{N}_\varepsilon) \tag{1}
$$

We first characterize the denominator of γ. Consider first the partition:

$$
\begin{aligned}
& E\left[D | X, Z \in \mathcal{N}_\varepsilon^+\right] - E\left[D | X, Z \in \mathcal{N}_\varepsilon^-\right] \\
= \; & E\left[D | X, Z \in \mathcal{N}_\varepsilon^+, \tau_\varepsilon = a\right] \Pr\left(\tau_\varepsilon = a | X, Z \in \mathcal{N}_\varepsilon^+\right) - E\left[D | X, Z \in \mathcal{N}_\varepsilon^-, \tau_\varepsilon = a\right] \Pr\left(\tau_\varepsilon = a | X, Z \in \mathcal{N}_\varepsilon^-\right) \\
& + E\left[D | X, Z \in \mathcal{N}_\varepsilon^+, \tau_\varepsilon = n\right] \Pr\left(\tau_\varepsilon = n | X, Z \in \mathcal{N}_\varepsilon^+\right) - E\left[D | X, Z \in \mathcal{N}_\varepsilon^-, \tau_\varepsilon = n\right] \Pr\left(\tau_\varepsilon = n | X, Z \in \mathcal{N}_\varepsilon^-\right) \\
& + E\left[D | X, Z \in \mathcal{N}_\varepsilon^+, \tau_\varepsilon = c\right] \Pr\left(\tau_\varepsilon = c | X, Z \in \mathcal{N}_\varepsilon^+\right) - E\left[D | X, Z \in \mathcal{N}_\varepsilon^-, \tau_\varepsilon = c\right] \Pr\left(\tau_\varepsilon = c | X, Z \in \mathcal{N}_\varepsilon^-\right) \\
& + E\left[D | X, Z \in \mathcal{N}_\varepsilon^+, \tau_\varepsilon = d\right] \Pr\left(\tau_\varepsilon = d | X, Z \in \mathcal{N}_\varepsilon^+\right) - E\left[D | X, Z \in \mathcal{N}_\varepsilon^-, \tau_\varepsilon = d\right] \Pr\left(\tau_\varepsilon = d | X, Z \in \mathcal{N}_\varepsilon^-\right) \\
& + E\left[D | X, Z \in \mathcal{N}_\varepsilon^+, \tau_\varepsilon = i\right] \Pr\left(\tau_\varepsilon = i | X, Z \in \mathcal{N}_\varepsilon^+\right) - E\left[D | X, Z \in \mathcal{N}_\varepsilon^-, \tau_\varepsilon = i\right] \Pr\left(\tau_\varepsilon = i | X, Z \in \mathcal{N}_\varepsilon^-\right),
\end{aligned}
$$

[1] A comment on notation: All these equality signs are of course only valid if all the termwise limits exist. (In other words, the equations should be read from right to left.)

where we define the conditional expectation as zero if the conditioning set has probability mass zero. (E.g. if $\Pr\left(\tau_\varepsilon = i | X, Z \in \mathcal{N}_\varepsilon^+\right) = 0$ then we define $E\left[D | X, Z \in \mathcal{N}_\varepsilon^+, \tau_\varepsilon = i\right] = 0$.) Making use of the definition of the subpopulations this equals

$$
\begin{aligned}
= \;& \Pr\left(\tau_\varepsilon = a | X, Z \in \mathcal{N}_\varepsilon^+\right) - \Pr\left(\tau_\varepsilon = a | X, Z \in \mathcal{N}_\varepsilon^-\right) \\
& + \Pr\left(\tau_\varepsilon = c | X, Z \in \mathcal{N}_\varepsilon^+\right) - \Pr\left(\tau_\varepsilon = d | X, Z \in \mathcal{N}_\varepsilon^-\right) \\
& + E\left[D | X, Z \in \mathcal{N}_\varepsilon^+, \tau_\varepsilon = i\right] \Pr\left(\tau_\varepsilon = i | X, Z \in \mathcal{N}_\varepsilon^+\right) - E\left[D | X, Z \in \mathcal{N}_\varepsilon^-, \tau_\varepsilon = i\right] \Pr\left(\tau_\varepsilon = i | X, Z \in \mathcal{N}_\varepsilon^-\right).
\end{aligned}
$$

Now consider

$$
\int \left(E\left[D | X = x, Z \in \mathcal{N}_\varepsilon^+\right] - E\left[D | X = x, Z \in \mathcal{N}_\varepsilon^-\right]\right) \cdot f_{X | Z \in \mathcal{N}_\varepsilon}(x)\, dx
$$

which we can rewrite by entering the previous derivation

$$
\begin{aligned}
= \;& \int \left\{\Pr\left(\tau_\varepsilon = a | X = x, Z \in \mathcal{N}_\varepsilon^+\right) - \Pr\left(\tau_\varepsilon = a | X = x, Z \in \mathcal{N}_\varepsilon^-\right)\right\} \cdot f_{X | Z \in \mathcal{N}_\varepsilon}(x)\, dx \\
& + \int \Pr\left(\tau_\varepsilon = c | X = x, Z \in \mathcal{N}_\varepsilon^+\right) \cdot f_{X | Z \in \mathcal{N}_\varepsilon}(x)\, dx - \int \Pr\left(\tau_\varepsilon = d | X = x, Z \in \mathcal{N}_\varepsilon^-\right) \cdot f_{X | Z \in \mathcal{N}_\varepsilon}(x)\, dx \\
& + \int E\left[D | X = x, Z \in \mathcal{N}_\varepsilon^+, \tau_\varepsilon = i\right] \Pr\left(\tau_\varepsilon = i | X = x, Z \in \mathcal{N}_\varepsilon^+\right) \cdot f_{X | Z \in \mathcal{N}_\varepsilon}(x)\, dx \\
& - \int E\left[D | X = x, Z \in \mathcal{N}_\varepsilon^-, \tau_\varepsilon = i\right] \Pr\left(\tau_\varepsilon = i | X = x, Z \in \mathcal{N}_\varepsilon^-\right) \cdot f_{X | Z \in \mathcal{N}_\varepsilon}(x)\, dx.
\end{aligned}
$$

By adding and subtracting $\int \Pr\left(\tau_\varepsilon = c | X = x, Z \in \mathcal{N}_\varepsilon\right) \cdot f_{X | Z \in \mathcal{N}_\varepsilon}(x)\, dx$ and making use of (1) we obtain

$$
\begin{aligned}
= \;& \int \left\{\Pr\left(\tau_\varepsilon = a | X = x, Z \in \mathcal{N}_\varepsilon^+\right) - \Pr\left(\tau_\varepsilon = a | X = x, Z \in \mathcal{N}_\varepsilon^-\right)\right\} \cdot f_{X | Z \in \mathcal{N}_\varepsilon}(x)\, dx \\
& - \int \Pr\left(\tau_\varepsilon = d | X = x, Z \in \mathcal{N}_\varepsilon^-\right) \cdot f_{X | Z \in \mathcal{N}_\varepsilon}(x)\, dx \\
& + \int E\left[D | X = x, Z \in \mathcal{N}_\varepsilon^+, \tau_\varepsilon = i\right] \Pr\left(\tau_\varepsilon = i | X = x, Z \in \mathcal{N}_\varepsilon^+\right) \cdot f_{X | Z \in \mathcal{N}_\varepsilon}(x)\, dx \\
& - \int E\left[D | X = x, Z \in \mathcal{N}_\varepsilon^-, \tau_\varepsilon = i\right] \Pr\left(\tau_\varepsilon = i | X = x, Z \in \mathcal{N}_\varepsilon^-\right) \cdot f_{X | Z \in \mathcal{N}_\varepsilon}(x)\, dx \\
& + \int \left\{\Pr\left(\tau_\varepsilon = c | X = x, Z \in \mathcal{N}_\varepsilon^+\right) - \Pr\left(\tau_\varepsilon = c | X = x, Z \in \mathcal{N}_\varepsilon\right)\right\} \cdot f_{X | Z \in \mathcal{N}_\varepsilon}(x)\, dx \\
& + \Pr(\tau_\varepsilon = c | Z \in \mathcal{N}_\varepsilon) \underbrace{\int f_{X | \tau_\varepsilon = c, Z \in \mathcal{N}_\varepsilon}(x)\, dx}_{=1}.
\end{aligned}
$$

Now we consider the limits of each of the terms in the previous expression. Since probabilities as well as the variable D are bounded by zero and one, we can take limits in the integrals. By Assumption (1iii), the first and the fifth term are zero, since Z is independent of the type close to z_0. Also, by Assumption (1ii) $\lim_{\varepsilon \to 0} \Pr\left(\tau_\varepsilon = d | X, Z \in \mathcal{N}_\varepsilon^-\right)$ and $\lim_{\varepsilon \to 0} \Pr\left(\tau_\varepsilon = i | X, Z \in \mathcal{N}_\varepsilon\right)$ are zero. Hence, only the limit of the last term is nonzero and is equal to $\Pr(\tau_\varepsilon = c | Z = z_0)$. Since all terms have well defined limits it follows that

$$
\lim_{\varepsilon \to 0} \int \left(E\left[D | X = x, Z \in \mathcal{N}_\varepsilon^+\right] - E\left[D | X = x, Z \in \mathcal{N}_\varepsilon^-\right]\right) \cdot f_{X | Z \in \mathcal{N}_\varepsilon}(x)\, dx = \Pr(\tau_\varepsilon = c | Z = z_0).
$$

Now consider the limit expression on the left hand side. We can rewrite the left hand side (before taking limits) as

$$
\begin{aligned}
& \int \left(E\left[D | X = x, Z \in \mathcal{N}_\varepsilon^+\right] - E\left[D | X = x, Z \in \mathcal{N}_\varepsilon^-\right]\right) \cdot f_{X | Z \in \mathcal{N}_\varepsilon}(x)\, dx \\
= \;& \int \left(E\left[D | X = x, Z \in \mathcal{N}_\varepsilon^+\right] - E\left[D | X = x, Z \in \mathcal{N}_\varepsilon^-\right] - \left(d^+(x, z_0) - d^-(x, z_0)\right)\right) \cdot f_{X | Z \in \mathcal{N}_\varepsilon}(x)\, dx \\
& + \int \left(d^+(X, z_0) - d^-(X, z_0)\right) \cdot f_{X | Z \in \mathcal{N}_\varepsilon}(x)\, dx - \int \left(d^+(x, z_0) - d^-(x, z_0)\right) \cdot \frac{f^-(x|z_0) + f^+(x|z_0)}{2} dx \\
& + \int \left(d^+(x, z_0) - d^-(x, z_0)\right) \frac{f^-(x|z_0) + f^+(x|z_0)}{2} dx.
\end{aligned}
$$

Since D is bounded, the limits of the first and second term are zero. Hence, we obtain that

$$\lim_{\varepsilon\to0}\int\left(E\left[D|X=x,Z\in\mathcal{N}_\varepsilon^+\right]-E\left[D|X=x,Z\in\mathcal{N}_\varepsilon^-\right]\right)\cdot f_{X|Z\in\mathcal{N}_\varepsilon}(x)\,dx=\int\left(d^+(x,z_0)-d^-(x,z_0)\right)\frac{f^-(x|z_0)+f^+(x|z_0)}{2}dx$$

or altogether that

$$\int\left(d^+(x,z_0)-d^-(x,z_0)\right)\frac{f^-(x|z_0)+f^+(x|z_0)}{2}dx=\lim_{\varepsilon\to0}\Pr(\tau_\varepsilon=c|Z\in\mathcal{N}_\varepsilon)=\Pr(\tau_\varepsilon=c|Z=z_0). \tag{2}$$

Now, the numerator of γ is examined analogously. First consider

$$\begin{aligned}
& E\left[Y|X=x,Z\in\mathcal{N}_\varepsilon^+\right]-E\left[Y|X=x,Z\in\mathcal{N}_\varepsilon^-\right]\\
= \; & E\left[Y^1|X=x,Z\in\mathcal{N}_\varepsilon^+,\tau_\varepsilon=a\right]\Pr\left(\tau_\varepsilon=a|X=x,Z\in\mathcal{N}_\varepsilon^+\right)-E\left[Y^1|X=x,Z\in\mathcal{N}_\varepsilon^-,\tau_\varepsilon=a\right]\Pr\left(\tau_\varepsilon=a|X=x,Z\in\mathcal{N}_\varepsilon^-\right)\\
& +E\left[Y^0|X=x,Z\in\mathcal{N}_\varepsilon^+,\tau_\varepsilon=n\right]\Pr\left(\tau_\varepsilon=n|X=x,Z\in\mathcal{N}_\varepsilon^+\right)-E\left[Y^0|X=x,Z\in\mathcal{N}_\varepsilon^-,\tau_\varepsilon=n\right]\Pr\left(\tau_\varepsilon=n|X=x,Z\in\mathcal{N}_\varepsilon^-\right)\\
& +E\left[Y^1|X=x,Z\in\mathcal{N}_\varepsilon^+,\tau_\varepsilon=c\right]\Pr\left(\tau_\varepsilon=c|X=x,Z\in\mathcal{N}_\varepsilon^+\right)-E\left[Y^0|X=x,Z\in\mathcal{N}_\varepsilon^-,\tau_\varepsilon=c\right]\Pr\left(\tau_\varepsilon=c|X=x,Z\in\mathcal{N}_\varepsilon^-\right)\\
& +E\left[Y^0|X=x,Z\in\mathcal{N}_\varepsilon^+,\tau_\varepsilon=d\right]\Pr\left(\tau_\varepsilon=d|X=x,Z\in\mathcal{N}_\varepsilon^+\right)-E\left[Y^1|X=x,Z\in\mathcal{N}_\varepsilon^-,\tau_\varepsilon=d\right]\Pr\left(\tau_\varepsilon=d|X=x,Z\in\mathcal{N}_\varepsilon^-\right)\\
& +E\left[Y|X=x,Z\in\mathcal{N}_\varepsilon^+,\tau_\varepsilon=i\right]\Pr\left(\tau_\varepsilon=i|X=x,Z\in\mathcal{N}_\varepsilon^+\right)-E\left[Y|X=x,Z\in\mathcal{N}_\varepsilon^-,\tau_\varepsilon=i\right]\Pr\left(\tau_\varepsilon=i|X=x,Z\in\mathcal{N}_\varepsilon^-\right).
\end{aligned}$$

Now consider

$$\lim_{\varepsilon\to0}\int\left(E\left[Y|X=x,Z\in\mathcal{N}_\varepsilon^+\right]-E\left[Y|X=x,Z\in\mathcal{N}_\varepsilon^-\right]\right)\cdot f_{X|Z\in\mathcal{N}_\varepsilon}(x)\,dx$$

and insert the previous expression. All terms have well defined limits, mostly zero. The limits of the terms for the $\tau_\varepsilon=a$ and $\tau_\varepsilon=n$ populations are zero by Assumptions (1iii) and (1iv). Since the conditional expectation functions for Y^0 and Y^1 exist and are bounded by Assumption (1vii) from above and below and since conditional probabilities are bounded and since $\lim_{\varepsilon\to0}\Pr\left(\tau_\varepsilon=d|X,Z\in\mathcal{N}_\varepsilon^-\right)$ and $\lim_{\varepsilon\to0}\Pr\left(\tau_\varepsilon=i|X,Z\in\mathcal{N}_\varepsilon\right)$ are zero by Assumption (1ii), also the limits of the terms for the $\tau_\varepsilon=d$ and $\tau_\varepsilon=i$ subpopulations are zero. Hence, it remains

$$=\lim_{\varepsilon\to0}\int\left\{E\left[Y^1|X=x,Z\in\mathcal{N}_\varepsilon^+,\tau_\varepsilon=c\right]\Pr\left(\tau_\varepsilon=c|X=x,Z\in\mathcal{N}_\varepsilon^+\right)-E\left[Y^0|X=x,Z\in\mathcal{N}_\varepsilon^-,\tau_\varepsilon=c\right]\Pr\left(\tau_\varepsilon=c|X=x,Z\in\mathcal{N}_\varepsilon^-\right)\right\}\cdot f$$

by adding and subtracting terms with limit zero we obtain

$$\begin{aligned}
= \; & \lim_{\varepsilon\to0}\int E\left[Y^1|X=x,Z\in\mathcal{N}_\varepsilon^+,\tau_\varepsilon=c\right]\cdot\left\{\Pr\left(\tau_\varepsilon=c|X=x,Z\in\mathcal{N}_\varepsilon^+\right)-\Pr\left(\tau_\varepsilon=c|X=x,Z\in\mathcal{N}_\varepsilon\right)\right\}\cdot f_{X|Z\in\mathcal{N}_\varepsilon}(x)\,dx\\
& +\lim_{\varepsilon\to0}\int\left\{E\left[Y^1|X=x,Z\in\mathcal{N}_\varepsilon^+,\tau_\varepsilon=c\right]-E\left[Y^1|X=x,Z\in\mathcal{N}_\varepsilon,\tau_\varepsilon=c\right]\right\}\cdot\Pr\left(\tau_\varepsilon=c|X=x,Z\in\mathcal{N}_\varepsilon\right)\cdot f_{X|Z\in\mathcal{N}_\varepsilon}(x)\,dx\\
& -\lim_{\varepsilon\to0}\int E\left[Y^0|X=x,Z\in\mathcal{N}_\varepsilon^-,\tau_\varepsilon=c\right]\cdot\left\{\Pr\left(\tau_\varepsilon=c|X=x,Z\in\mathcal{N}_\varepsilon^-\right)-\Pr\left(\tau_\varepsilon=c|X=x,Z\in\mathcal{N}_\varepsilon\right)\right\}\cdot f_{X|Z\in\mathcal{N}_\varepsilon}(x)\,dx\\
& -\lim_{\varepsilon\to0}\int\left\{E\left[Y^0|X=x,Z\in\mathcal{N}_\varepsilon^-,\tau_\varepsilon=c\right]-E\left[Y^0|X=x,Z\in\mathcal{N}_\varepsilon,\tau_\varepsilon=c\right]\right\}\cdot\Pr\left(\tau_\varepsilon=c|X=x,Z\in\mathcal{N}_\varepsilon\right)\cdot f_{X|Z\in\mathcal{N}_\varepsilon}(x)\,dx\\
& +\lim_{\varepsilon\to0}\int E\left[Y^1-Y^0|X=x,Z\in\mathcal{N}_\varepsilon,\tau_\varepsilon=c\right]\Pr\left(\tau_\varepsilon=c|X=x,Z\in\mathcal{N}_\varepsilon\right)\cdot f_{X|Z\in\mathcal{N}_\varepsilon}(x)\,dx.
\end{aligned}$$

By assumption (1iii) the first and third term have limit zero. By assumption (1iv) the second and fourth term have limit zero. By making use of (1), we obtain

$$\begin{aligned}
& \lim_{\varepsilon\to0}\int\left(E\left[Y|X=x,Z\in\mathcal{N}_\varepsilon^+\right]-E\left[Y|X=x,Z\in\mathcal{N}_\varepsilon^-\right]\right)\cdot f_{X|Z\in\mathcal{N}_\varepsilon}(x)\,dx\\
& \qquad =\lim_{\varepsilon\to0}\Pr(\tau_\varepsilon=c|Z\in\mathcal{N}_\varepsilon)\cdot\int E\left[Y^1-Y^0|X=x,Z\in\mathcal{N}_\varepsilon,\tau_\varepsilon=c\right]f_{X|\tau_\varepsilon=c,Z\in\mathcal{N}_\varepsilon}(x)\,dx\\
& \qquad =\lim_{\varepsilon\to0}\Pr(\tau_\varepsilon=c|Z\in\mathcal{N}_\varepsilon)\cdot E\left[Y^1-Y^0|Z\in\mathcal{N}_\varepsilon,\tau_\varepsilon=c\right].
\end{aligned}$$

Now consider the limit expression on the left hand side. We can rewrite the left hand side (before taking limits) as

$$
\begin{aligned}
& \int \left(E\left[Y|X=x, Z\in\mathcal{N}_\varepsilon^+\right] - E\left[Y|X=x, Z\in\mathcal{N}_\varepsilon^-\right]\right)\cdot f_{X|Z\in\mathcal{N}_\varepsilon}(x)\,dx \\
= & \int \left(E\left[Y|X=x, Z\in\mathcal{N}_\varepsilon^+\right] - E\left[Y|X=x, Z\in\mathcal{N}_\varepsilon^-\right] - \left(m^+(X,z_0) - m^-(X,z_0)\right)\right)\cdot f_{X|Z\in\mathcal{N}_\varepsilon}(x)\,dx \\
& + \int \left(m^+(x,z_0) - m^-(x,z_0)\right) f_{X|Z\in\mathcal{N}_\varepsilon}(x)\,dx - \int \left(m^+(x,z_0) - m^-(x,z_0)\right)\frac{f^-(x|z_0)+f^+(x|z_0)}{2}dx \\
& + \int \left(m^+(x,z_0) - m^-(x,z_0)\right)\frac{f^-(x|z_0)+f^+(x|z_0)}{2}dx.
\end{aligned}
$$

Since the conditional expectations of Y exist and are bounded from above and below, the first two terms have limit zero. Hence, we obtain that

$$
\lim_{\varepsilon\to 0}\int \left(E\left[Y|X=x, Z\in\mathcal{N}_\varepsilon^+\right] - E\left[Y|X=x, Z\in\mathcal{N}_\varepsilon^-\right]\right)\cdot f_{X|Z\in\mathcal{N}_\varepsilon}(x)\,dx = \int \left(m^+(x,z_0) - m^-(x,z_0)\right)\frac{f^-(x|z_0)+f^+(x|z_0)}{2}dx.
$$

Putting all these pieces together we obtain, with Assumption (1i) and using (2)

$$
\begin{aligned}
\frac{\int \left(m^+(x,z_0) - m^-(x,z_0)\right)\cdot \frac{f^-(x|z_0)+f^+(x|z_0)}{2}dx}{\int \left(d^+(x,z_0) - d^-(x,z_0)\right)\cdot \frac{f^-(x|z_0)+f^+(x|z_0)}{2}dx} &= \frac{\lim\limits_{\varepsilon\to 0}\Pr(\tau_\varepsilon = c|Z\in\mathcal{N}_\varepsilon)\cdot E\left[Y^1 - Y^0|Z\in\mathcal{N}_\varepsilon, \tau_\varepsilon = c\right]}{\lim\limits_{\varepsilon\to 0}\Pr(\tau_\varepsilon = c|Z\in\mathcal{N}_\varepsilon)} \\
&= \lim_{\varepsilon\to 0} E\left[Y^1 - Y^0|Z\in\mathcal{N}_\varepsilon, \tau_\varepsilon = c\right].
\end{aligned}
$$

What happens if the monotonicity assumption is not valid in the sense that there are defiers (but no individuals of the indefinite type)? For this case, repeating all the previous derivations gives

$$
\begin{aligned}
& \frac{\int \left(m^+(x,z_0) - m^-(x,z_0)\right)\cdot \frac{f^-(x|z_0)+f^+(x|z_0)}{2}dx}{\int \left(d^+(x,z_0) - d^-(x,z_0)\right)\cdot \frac{f^-(x|z_0)+f^+(x|z_0)}{2}dx} \\
& = \frac{\lim\limits_{\varepsilon\to 0}\Pr(\tau_\varepsilon = c|Z\in\mathcal{N}_\varepsilon)\cdot E\left[Y^1 - Y^0|Z\in\mathcal{N}_\varepsilon, \tau_\varepsilon = c\right] - \Pr(\tau_\varepsilon = d|Z\in\mathcal{N}_\varepsilon)\cdot E\left[Y^1 - Y^0|Z\in\mathcal{N}_\varepsilon, \tau_\varepsilon = d\right]}{\lim\limits_{\varepsilon\to 0}\Pr(\tau_\varepsilon = c|Z\in\mathcal{N}_\varepsilon) - \Pr(\tau_\varepsilon = d|Z\in\mathcal{N}_\varepsilon)},
\end{aligned}
$$

provided that Assumptions (1iii) and (1iv) also hold for the defiers. Now, if it is the case that the average treatment effect is the same for the compliers and defiers, then the treatment effect is still identified by the same formula because

$$
= \frac{\lim\limits_{\varepsilon\to 0}\left\{\Pr(\tau_\varepsilon = c|Z\in\mathcal{N}_\varepsilon) - \Pr(\tau_\varepsilon = d|Z\in\mathcal{N}_\varepsilon)\right\}\cdot E\left[Y^1 - Y^0|Z\in\mathcal{N}_\varepsilon, \tau_\varepsilon = c\right]}{\lim\limits_{\varepsilon\to 0}\Pr(\tau_\varepsilon = c|Z\in\mathcal{N}_\varepsilon) - \Pr(\tau_\varepsilon = d|Z\in\mathcal{N}_\varepsilon)} = \lim_{\varepsilon\to 0} E\left[Y^1 - Y^0|Z\in\mathcal{N}_\varepsilon, \tau_\varepsilon = c\right].
$$

Hence, the same estimators can be used for this case.

B Lemma 1: Linearized representation of the local linear estimator

Consider first the estimation of $m^+(x_0, z_0)$ at a location x_0 by local linear regression using only observations to the right of it:

$$
\min_{a,b,c}\sum_{j=1}^{n}\left(Y_j - a - b\left(Z_j - z_0\right) - c'\left(X_j - x_0\right)\right)^2\cdot K_j I_j^+
$$

where $I_j^+ = 1(Z_j > z_0)$ and a product kernel is used

$$
K_j = K_j(x_0, z_0) = \kappa\left(\frac{Z_j - z_0}{h_z}\right)\cdot\prod_{l=1}^{L}\bar{\kappa}\left(\frac{X_{jl} - x_{0l}}{h_x}\right),
$$

where κ is a univariate second order kernel function, which is assumed to be *symmetric* and *integrating to one*. The kernel $\bar{\kappa}$ is a univariate kernel of order λ.

The following kernel constants will be used later: $\mu_l = \int u^l \kappa(u) du$ and $\bar{\mu}_l = \int_0^\infty u^l \kappa(u) du$ and $\tilde{\mu} = \frac{\bar{\mu}_2}{2} - \bar{\mu}_1^2$. (With symmetric kernel $\bar{\mu}_0 = \frac{1}{2}$.) Furthermore define $\ddot{\mu}_l = \int_0^\infty u^l \kappa^2(u) du$. The kernel constants for the higher-order kernel are defined as $\eta_l = \int u^l \bar{\kappa}(u) du$ and $\dot{\eta}_l = \int_{-\infty}^{\infty} u^l \bar{\kappa}^2(u) du$.

To simplify the derivations define

$$Y_j^* = Y_j - m^+(x_0, z_0).$$

The local linear regression estimator can then be written as

$$\min_{a,b,c} \sum_{j=1}^{n} \left(Y_j^* - \left(a - m^+(x_0, z_0)\right) - b\left(Z_j - z_0\right) - c'\left(X_j - x_0\right)\right)^2 \cdot K_j I_j^+$$

or as

$$\min_{\beta} \sum_{j=1}^{n} \left(Y_j^* - \mathbb{X}_j' \beta\right)^2 \cdot K_j I_j^+$$

where $\mathbb{X}$ is the $L+2$ column vector:

$$\mathbb{X}_j = \left(1, \frac{Z_j - z_0}{h_z}, \left(\frac{X_j - x_0}{h_x}\right)'\right)'$$

and $\beta = \{a - m^+(x_0, z_0), h_z b, h_x c'\}'$. The first order condition of the local linear estimator is then

$$\begin{aligned}
\hat{m}^+(x_0, z_0) - m^+(x_0, z_0) &= e_1' \cdot \left[\sum_{j=1}^{n} \mathbb{X}_j \mathbb{X}_j' K_j I_j^+\right]^{-1} \sum_{j=1}^{n} \mathbb{X}_j Y_j^* K_j I_j^+ \\
&= e_1' \cdot \left[\frac{1}{n h_z h_x^L} \sum_{j=1}^{n} \mathbb{X}_j \mathbb{X}_j' K_j I_j^+\right]^{-1} \frac{1}{n h_z h_x^L} \sum_{j=1}^{n} \mathbb{X}_j Y_j^* K_j I_j^+ \\
&= e_1' \cdot \left\{A_+(x_0) + o_p\left(h_x^{\lambda-1} + h_z^2\right) + O_p\left(\frac{1}{\sqrt{n h_z h_x^L}}\right)\right\}^{-1} \frac{1}{n h_z h_x^L} \sum_{j=1}^{n} \mathbb{X}_j Y_j^* K_j I_j^+ \\
&= \frac{1}{n h_z h_x^L} \sum_{j=1}^{n} e_1' A_+^{-1}(x_0) \mathbb{X}_j \left(Y_j - m^+(x_0)\right) K_j I_j^+ \left(1 + o_p(1)\right),
\end{aligned} \quad (3)$$

where e_1 is a column vector of zeros with first element being one, and the symmetric matrix A_+ is given in Lemma 2 below. (Lemma 2 contains the proof of the second equality.)

For $\lambda \geq 3$ we obtain after tedious calculations the following expression for $e_1' A_+^{-1}(x_0)$ retaining only terms up to order h_z and h_x:

$$\frac{1}{f^+(x_0, z_0) \cdot \mathcal{C}} \begin{pmatrix}
\frac{1}{\tilde{\mu}}\left(\bar{\mu}_2 \mathcal{C} + h_z \cdot \left(\bar{\mu}_3 \frac{\partial f^+(x_0,z_0)}{f^+(x_0,z_0) \cdot \partial z} \mathcal{C} + 2\bar{\mu}_2 \bar{\mu}_1 \mathcal{A}\right) + h_x \bar{\mu}_2 \frac{\eta_{\lambda+1}}{\eta_\lambda} \frac{(\lambda-2)!}{(\lambda-1)!} \mathcal{B}\right) \\
-\frac{1}{\tilde{\mu}}\left(\bar{\mu}_1 \mathcal{C} + h_z \cdot \left(\bar{\mu}_2 \frac{\partial f^+(x_0,z_0)}{f^+(x_0,z_0) \cdot \partial z} \mathcal{C} + 2\bar{\mu}_1^2 \mathcal{A}\right) + h_x \bar{\mu}_1 \frac{\eta_{\lambda+1}}{\eta_\lambda} \frac{(\lambda-2)!}{(\lambda-1)!} \mathcal{B}\right) \\
-2h_x \left(\frac{\partial^{\lambda-1} f^+(x_0,z_0)}{\partial x_1^{\lambda-1}} \cdot \mathcal{C}_{\neq 1}\right) \frac{(\lambda-2)!}{(\lambda-1)!} \\
\vdots \\
-2h_x \left(\frac{\partial^{\lambda-1} f^+(x_0,z_0)}{\partial x_L^{\lambda-1}} \cdot \mathcal{C}_{\neq L}\right) \frac{(\lambda-2)!}{(\lambda-1)!}
\end{pmatrix}' \quad (4)$$

where

$$\mathcal{C} = \prod_{l=1}^{L} \frac{\partial^{\lambda-2} f^+(x_0, z_0)}{\partial x_l^{\lambda-2}} \qquad \mathcal{C}_{\neq q} = \prod_{l=1, l \neq q}^{L} \frac{\partial^{\lambda-2} f^+(x_0, z_0)}{\partial x_l^{\lambda-2}}$$

and

$$\mathcal{A} = \sum_{l=1}^{L} \left(\frac{\partial^{\lambda-1} f^+(x_0, z_0)}{\partial x_l^{\lambda-2} \partial z} \cdot \mathcal{C}_{\neq l}\right) \qquad \mathcal{B} = \sum_{l=1}^{L} \left(\frac{\partial^{\lambda-1} f^+(x_0, z_0)}{\partial x_l^{\lambda-1}} \cdot \mathcal{C}_{\neq l}\right).$$

The expression for $\lambda = 2$ with $\eta_3 = 0$, i.e. for a symmetric second-order kernel, is simpler. Then the following expression for $e_1' A_+^{-1}$ is obtained where a few higher order terms have also been retained:

$$\frac{1}{D}\begin{pmatrix} \bar{\mu}_2 + h_z\left(2\bar{\mu}_2\bar{\mu}_1 + \bar{\mu}_3\right)\frac{\partial \ln f^+(x_0,z_0)}{\partial z} + 2\bar{\mu}_1\bar{\mu}_3 h_z^2\left(\frac{\partial \ln f^+(x_0,z_0)}{\partial z}\right)^2 \\ -\bar{\mu}_1 - \left(2\bar{\mu}_1^2 + \bar{\mu}_2\right)h_z\frac{\partial \ln f^+(x_0,z_0)}{\partial z} - 2\bar{\mu}_1\bar{\mu}_2 h_z^2\left(\frac{\partial \ln f^+(x_0,z_0)}{\partial z}\right)^2 \\ \left(-2\tilde{\mu} + \left(2\bar{\mu}_1\bar{\mu}_2 - \bar{\mu}_3\right)h_z\left(\frac{\partial \ln f^+(x_0,z_0)}{\partial z}\right)\right)h_x\frac{\partial \ln f^+(x_0,z_0)}{\partial x_1} \\ \vdots \\ \left(-2\tilde{\mu} + \left(2\bar{\mu}_1\bar{\mu}_2 - \bar{\mu}_3\right)h_z\left(\frac{\partial \ln f^+(x_0,z_0)}{\partial z}\right)\right)h_x\frac{\partial \ln f^+(x_0,z_0)}{\partial x_L} \end{pmatrix}'$$

where

$$D = f^+(x_0,z_0)\left(\tilde{\mu} + \left(\frac{\bar{\mu}_3}{2} - 2\bar{\mu}_1^3\right)h_z\frac{\partial \ln f^+(x_0,z_0)}{\partial z} - \mu_2\tilde{\mu}h_x^2\sum_{l=1}^{L}\left(\frac{\partial \ln f^+(x_0,z_0)}{\partial x_l}\right)^2 + O(h_z^2 + h_x^3)\right).$$

C Lemma 2: Denominator of the local linear estimator

Under the assumption that $nh_z h_x^L \to \infty$, it is to show that for $\lambda \geq 3$

$$\frac{1}{nh_z h_x^L}\sum_{j=1}^{n}\mathbb{X}_j\mathbb{X}_j' K_j I_j^+ = A_+(x_0) + o_p(h_x^{\lambda-1} + h_z^2) + O_p\left(\frac{1}{\sqrt{nh_z h_x^L}}\right) \tag{5}$$

where the *symmetric* matrix $A_+(x_0)$ is

$$\begin{bmatrix} \begin{matrix}\frac{1}{2}f^+(x_0,z_0) + \bar{\mu}_1 h_z\frac{\partial f^+(x_0,z_0)}{\partial z} \\ +\frac{1}{2}\bar{\mu}_2 h_z^2\frac{\partial^2 f^+(x_0,z_0)}{\partial z^2}\end{matrix} & & & & & \\ \begin{matrix}\bar{\mu}_1 f^+(x_0,z_0) + \bar{\mu}_2 h_z\frac{\partial f^+(x_0,z_0)}{\partial z} \\ +\frac{1}{2}\bar{\mu}_3 h_z^2\frac{\partial^2 f^+(x_0,z_0)}{\partial z^2}\end{matrix} & \begin{matrix}\bar{\mu}_2 f^+(x_0,z_0) + \bar{\mu}_3 h_z\frac{\partial f^+(x_0,z_0)}{\partial z} \\ +\frac{1}{2}\bar{\mu}_4 h_z^2\frac{\partial^2 f^+(x_0,z_0)}{\partial z^2}\end{matrix} & & & & \\ \frac{\eta_\lambda h_x^{\lambda-1}}{2(\lambda-1)!}\frac{\partial^{\lambda-1}f^+(x_0,z_0)}{\partial x_1^{\lambda-1}} & \frac{\bar{\mu}_1\eta_\lambda h_x^{\lambda-1}}{(\lambda-1)!}\frac{\partial^{\lambda-1}f^+(x_0,z_0)}{\partial x_1^{\lambda-1}} & \begin{matrix}\frac{\eta_\lambda}{(\lambda-2)!}h_x^{\lambda-2}\begin{pmatrix}\frac{1}{2}\frac{\partial^{\lambda-2}f^+(x_0,z_0)}{\partial x_1^{\lambda-2}} \\ +\bar{\mu}_1 h_z\frac{\partial^{\lambda-1}f^+(x_0,z_0)}{\partial x_1^{\lambda-2}\partial z}\end{pmatrix} \\ +\frac{\eta_{\lambda+1}}{(\lambda-1)!}h_x^{\lambda-1}\left(\frac{1}{2}\frac{\partial^{\lambda-1}f^+(x_0,z_0)}{\partial x_1^{\lambda-1}}\right)\end{matrix} & & & \\ \vdots & \vdots & 0 & \ddots & & \\ \vdots & \vdots & \vdots & 0 & \ddots & \\ \frac{\eta_\lambda h_x^{\lambda-1}}{2(\lambda-1)!}\frac{\partial^{\lambda-1}f^+(x_0,z_0)}{\partial x_L^{\lambda-1}} & \frac{\bar{\mu}_1\eta_\lambda h_x^{\lambda-1}}{(\lambda-1)!}\frac{\partial^{\lambda-1}f^+(x_0,z_0)}{\partial x_L^{\lambda-1}} & 0 & \cdots & 0 & \ddots \end{bmatrix}$$

In the calculations terms of order $h_x^{\lambda-2}, h_z, h_z^2, h_x^{\lambda-1}, h_x^{\lambda-2}\cdot h_z$ and $\left(h_x^{\lambda-2}\right)^2$ were retained, the terms of lower order are ignored.

The relationship (5) is shown via mean square convergence for each element of $A_+(x_0)$. Only the derivations for the $(2,3)$ element are shown here, with the derivations for the other elements being analogous.

Consider the $(2,3)$ element of $\frac{1}{nh_z h_x^L}\sum_{j=1}^{n}\mathbb{X}_j\mathbb{X}_j' K_j I_j^+$ and denote it by ξ

$$\xi = \frac{1}{nh_z h_x^L}\sum_{j=1}^{n}\left(\frac{Z_j - z_0}{h_z}\right)\left(\frac{X_{j1} - x_{01}}{h_x}\right)K_j I_j^+$$

which has the expected value:

$$E[\xi] = \frac{1}{h_z h_x^L}E\left[\left(\frac{Z_j - z_0}{h_z}\right)\left(\frac{X_{j1} - x_{01}}{h_x}\right)K_j I_j^+\right]$$

$$= \frac{1}{h_z h_x^L} \int \cdots \int \left(\frac{z-z_0}{h_z}\right)\left(\frac{x_1-x_{01}}{h_x}\right)\kappa\left(\frac{z-z_0}{h_z}\right)\prod_{l=1}^{L}\bar{\kappa}\left(\frac{x_l-x_{0l}}{h_x}\right)1(z>z_0)f(x,z)dzdx_1\cdots dx_L.$$

With a change in variables: $u=\frac{z-z_0}{h_z}$, $v_l=\frac{x_l-x_{0l}}{h_x}$ and $v=(v_1,...,v_L)'$ and an expansion about the point (x_0,z_0), considering only points to the right of z_0, we obtain

$$\begin{aligned}
&= \int\cdots\int uv_1\kappa(u)\prod_{l=1}^{L}\bar{\kappa}(v_l)\,1(u>0)f(x_0+h_xv,z_0+h_zu)\,dudv \\
&= \frac{1}{(\lambda-1)!}\frac{\partial^{\lambda-1}f^+(x_0,z_0)}{\partial x_1^{\lambda-1}}h_x^{\lambda-1}\int\cdots\int uv_1\kappa(u)\prod_{l=1}^{L}\bar{\kappa}(v_l)\,1(u>0)v_1^{\lambda-1}dudv+O(h_x^{\lambda-1}h_z+h_x^{\lambda}) \\
&= \frac{1}{(\lambda-1)!}\frac{\partial^{\lambda-1}f^+(x_0,z_0)}{\partial x_1^{\lambda-1}}h_x^{\lambda-1}\int_0^{\infty}u\cdot\kappa(u)\,du\int\cdots\int v_1^{\lambda}\prod_{l=1}^{L}\bar{\kappa}(v_l)\,dv+O(h_x^{\lambda-1}h_z+h_x^{\lambda}) \\
&= \frac{\bar{\mu}_1\eta_\lambda}{(\lambda-1)!}\frac{\partial^{\lambda-1}f^+(x_0,z_0)}{\partial x_1^{\lambda-1}}h_x^{\lambda-1}+O(h_x^{\lambda-1}h_z+h_x^{\lambda})
\end{aligned}$$

by bounded convergence.

To show convergence in mean square, it also needs to be shown that $Var(\xi)$ converges to zero:

$$\begin{aligned}
Var(\xi) &= E\left[\xi^2\right]-(E[\xi])^2=E\left[\left(\frac{1}{nh_zh_x^L}\sum_{j=1}^{n}\left(\frac{Z_j-z_0}{h_z}\right)\left(\frac{X_{j1}-x_{01}}{h_x}\right)K_jI_j^+\right)^2\right]-O(h_x^{2\lambda-2}) \\
&= \frac{1}{nh_z^2h_x^{2L}}\int\cdots\int\left(\frac{z-z_0}{h_z}\right)^2\left(\frac{x_1-x_{01}}{h_x}\right)^2\kappa^2\left(\frac{z-z_0}{h_z}\right)\prod_{l=1}^{L}\bar{\kappa}^2\left(\frac{x_l-x_{0l}}{h_x}\right)1(z>z_0)f(x,z)dzdx_1\cdots dx_L-O(h_x^{2\lambda-2}) \\
&= \frac{1}{nh_zh_x^L}\int\cdots\int u^2v_1^2\kappa^2(u)\prod_{l=1}^{L}\bar{\kappa}^2(v_1)\,1(u>0)f(x_0+h_xv,z_0+h_zu)\,dudv-O(h_x^{2\lambda-2}) \\
&= \frac{1}{nh_zh_x^L}f^+(x_0,z_0)\cdot\ddot{\mu}_2\dot{\eta}_2\dot{\eta}_0^{L-1}\left(1+O\left(h_x+h_z\right)\right)-O(h_x^{2\lambda-2})
\end{aligned}$$

where a change in variables: $u=\frac{z-z_0}{h_z}$, $v_l=\frac{x_l-x_{0l}}{h_x}$ and $v=(v_1,...,v_L)'$ and an expansion about the point (x_0,z_0) has been used. As it has been assumed that $nh_zh_x^L\to\infty$, the variance of ξ converges to zero. Hence, mean square convergence has been shown, which implies convergence in probability by Chebyshev's inequality.

D Proof of Proposition 2

The derivation of the asymptotic bias and variance of $\hat{\gamma}_{naive}$ is similar to the results of Theorem 3. Here only a sketch of the derivations is given, with more details in Theorem 3. To derive the asymptotic properties of $\hat{\gamma}$, define $\hat{\gamma} = \frac{\hat{\Delta}}{\hat{\Gamma}}$ and $\gamma = \frac{\Delta}{\Gamma}$. We need to establish the distribution of

$$n^{\frac{1}{3}}\left(\hat{\gamma}-\gamma\right) = n^{\frac{1}{3}}\left(\frac{\hat{\Delta}}{\hat{\Gamma}}-\frac{\Delta}{\Gamma}\right) = n^{\frac{1}{3}}\left(\frac{\frac{1}{n}\sum\limits_{i=1}^{n}\left(\hat{m}^{+}(X_i,z_0)-\hat{m}^{-}(X_i,z_0)\ \right)K_h\left(\frac{Z_i-z_0}{h}\right)}{\frac{1}{n}\sum\limits_{i=1}^{n}\left(\hat{d}^{+}(X_i,z_0)-\hat{d}^{-}(X_i,z_0)\ \right)K_h\left(\frac{Z_i-z_0}{h}\right)} - \frac{\int\left(m^{+}(x)-m^{-}(x)\right)\cdot\frac{f^{-}(x|z_0)+f^{+}(x|z_0)}{2}dx}{\int\left(d^{+}(x)-d^{-}(x)\right)\cdot\frac{f^{-}(x|z_0)+f^{+}(x|z_0)}{2}dx}\right).$$

To ensure that both $\hat{\Delta}$ and $\hat{\Gamma}$ converge to Δ and Γ, respectively, the kernel is scaled by $f(z_0)$ for the remainder of this proof (which does not change the estimator since it appears in the numerator and the denominator of the above expression):

$$K_h\left(\frac{Z_i-z_0}{h}\right) = \frac{\frac{1}{h}\kappa\left(\frac{Z_i-z_0}{h}\right)}{f(z_0)}.$$

Note that $\hat{\gamma}-\gamma$ can be written as

$$(\hat{\gamma}-\gamma) = \frac{\hat{\Delta}}{\hat{\Gamma}}-\frac{\Delta}{\Gamma} = \left(\frac{\hat{\Delta}-\Delta}{\Gamma}-\gamma\frac{\hat{\Gamma}-\Gamma}{\Gamma}\right)\cdot\left(1-\frac{\hat{\Gamma}-\Gamma}{\hat{\Gamma}}\right). \tag{6}$$

The derivation proceeds in two steps. First the term $\hat{\Delta}-\Delta$ is analyzed, with analogous results for $\hat{\Gamma}-\Gamma$. It is shown that bias and variance converge to zero with growing sample size, implying convergence in mean square and thus in probability. This also implies that the last term $\left(1-\frac{\hat{\Gamma}-\Gamma}{\hat{\Gamma}}\right)$ is $1+o_p(1)$. Hence the first-order behaviour of $\hat{\gamma}-\gamma$ is determined by the term $\frac{\hat{\Delta}-\Delta}{\Gamma}-\gamma\frac{\hat{\Gamma}-\Gamma}{\Gamma}$ in

$$n^{\frac{1}{3}}\left(\hat{\gamma}-\gamma\right) = n^{\frac{1}{3}}\left(\frac{\hat{\Delta}-\Delta}{\Gamma}-\gamma\frac{\hat{\Gamma}-\Gamma}{\Gamma}\right)\cdot\left(1+o_p(1)\right). \tag{7}$$

In a preliminary step the term $\hat{\Delta}-\Delta$ is analyzed. (The derivations for $\hat{\Gamma}-\Gamma$ are similar.) Write $\hat{\Delta}-\Delta$ as

$$\hat{\Delta}-\Delta = \frac{1}{n}\sum_{i=1}^{n}\left(\hat{m}^{+}(X_i,z_0)-\hat{m}^{-}(X_i,z_0)\ \right)K_h\left(\frac{Z_i-z_0}{h}\right) - \int\left(m^{+}(x,z_0)-m^{-}(x,z_0)\right)\cdot\frac{f^{-}(x|z_0)+f^{+}(x|z_0)}{2}dx$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left\{\hat{m}^{+}(X_i,z_0)-m^{+}(X_i,z_0)\right\}K_h\left(\frac{Z_i-z_0}{h}\right) - \frac{1}{n}\sum_{i=1}^{n}\left\{\hat{m}^{-}(X_i,z_0)-m^{-}(X_i,z_0)\right\}K_h\left(\frac{Z_i-z_0}{h}\right) \tag{8a}$$

$$+\frac{1}{n}\sum_{i=1}^{n}\left\{m^{+}(X_i,z_0)-m^{-}(X_i,z_0)\right\}K_h\left(\frac{Z_i-z_0}{h}\right) - E\left[\left\{m^{+}(X,z_0)-m^{-}(X,z_0)\right\}K_h\left(\frac{Z-z_0}{h}\right)\right] \tag{8b}$$

$$+E\left[\left\{m^{+}(X,z_0)-m^{-}(X,z_0)\right\}K_h\left(\frac{Z-z_0}{h}\right)\right] - \int\left(m^{+}(x,z_0)-m^{-}(x,z_0)\right)\cdot\frac{f^{-}(x|z_0)+f^{+}(x|z_0)}{2}dx. \tag{8c}$$

D.1 Calculation of the bias term

Consider first the term (8c). This represents the bias if the functions m^{+} and m^{-} were known. To ease notation we ignore the terms containing m^{-} for the moment and retain only those with m^{+}. (The derivations for m^{-} are analogous.)

$$\begin{aligned} &E\left[m^{+}(X,z_0)K_h\left(\frac{Z-z_0}{h}\right)\right] - \int m^{+}(x,z_0)\cdot\frac{f^{-}(x|z_0)+f^{+}(x|z_0)}{2}dx \\ &\quad= \left\{E\left[m^{+}(X,z_0)K_h\left(\frac{Z-z_0}{h}\right)1(Z>z_0)\right] - \int m^{+}(x,z_0)\cdot\frac{f^{+}(x|z_0)}{2}dx\right\} \\ &\qquad+\left\{E\left[m^{+}(X,z_0)K_h\left(\frac{Z-z_0}{h}\right)1(Z\le z_0)\right] - \int m^{+}(x,z_0)\cdot\frac{f^{-}(x|z_0)}{2}dx\right\}. \end{aligned}$$

Concentrate on the first term. (The second is analogous.)

$$
\begin{aligned}
& E\left[m^+(X,z_0)K_h\left(\frac{Z-z_0}{h}\right)1(Z>z_0)\right]-\int m^+(x,z_0)\cdot\frac{f^+(x|z_0)}{2}dx \\
= & \int\cdots\int m^+(x,z_0)\frac{1}{hf(z_0)}\kappa\left(\frac{z-z_0}{h}\right)1(z>z_0)f(x,z)dxdz-\int m^+(x,z_0)\cdot\frac{f^+(x|z_0)}{2}dx \\
= & \int\cdots\int m^+(x,z_0)\frac{\kappa(u)}{f(z_0)}1(u>0)f(x,z_0+uh)dxdu-\int m^+(x,z_0)\cdot\frac{f^+(x|z_0)}{2}dx
\end{aligned}
$$

where $u=\frac{Z-z_0}{h}$. Expanding $f(x,z_0+uh)$ as $f^+(x,z_0)+uh\frac{\partial f^+}{\partial z}(x,z_0)+\frac{1}{2}u^2h^2\frac{\partial^2 f^+}{\partial z^2}(x,z_0)+o(h^2)$ gives

$$
\begin{aligned}
= & \frac{1}{2f(z_0)}\int m^+(x,z_0)f^+(x,z_0)dx+\frac{h\bar{\mu}_1}{f(z_0)}\int m^+(x,z_0)\frac{\partial f^+}{\partial z}(x,z_0)dx \\
& +\frac{h^2\bar{\mu}_2}{f(z_0)}\int\frac{1}{2}m^+(x,z_0)\frac{\partial^2 f^+}{\partial z^2}(x,z_0)dx-\int m^+(x,z_0)\cdot\frac{f^+(x|z_0)}{2}dx+o(h^2).
\end{aligned}
$$

Note that the first term is zero since $f^+(x|z_0)f(z_0)=f^+(x,z_0)$. Repeating the analogous derivations for the terms neglected so far, the bias term (8c) is

$$
\frac{h\bar{\mu}_1}{f(z_0)}\int\left(m^+(x,z_0)-m^-(x,z_0)\right)\left(\frac{\partial f^+(x,z_0)}{\partial z}+\frac{\partial f^-(x,z_0)}{\partial z}\right)dx+O(h^2).
$$

Now consider the first term (8a):

Consider the first part of (8a). (The results for the second part of (8a) are analogous.)

$$
\frac{1}{n}\sum_{i=1}^{n}\left\{\hat{m}^+(X_i,z_0)-m^+(X_i,z_0)\right\}K_h\left(\frac{Z_i-z_0}{h}\right).
$$

The subsequent derivations are exactly identical to those of Theorem 3 until (20). Using (20) we obtain

$$
E\left[\left\{\hat{m}^+(X_i,z_0)-m^+(X_i,z_0)\right\}K_h\left(\frac{Z_i-z_0}{h}\right)\right]
$$

$$
\begin{aligned}
= & E\left[\left\{h_z^2\frac{\bar{\mu}_2^2-\bar{\mu}_1\bar{\mu}_3}{2\tilde{\mu}}\frac{\partial^2 m^+(X_i,z_0)}{\partial z^2}+h_x^\lambda\eta_\lambda\sum_{l=1}^{L}\left\{\frac{\partial^\lambda m^+(X_i,z_0)}{\lambda!\cdot\partial x_l^\lambda}+\sum_{s=1}^{\lambda-1}\frac{\partial^s m^+(X_i,z_0)}{\partial x_l^s}\omega_s^+\right\}\right\}\cdot K_h\left(\frac{Z_i-z_0}{h}\right)\right](1+o_p(1)) \\
= & \int\int\left\{h_z^2\frac{\bar{\mu}_2^2-\bar{\mu}_1\bar{\mu}_3}{2\tilde{\mu}}\frac{\partial^2 m^+(X_i,z_0)}{\partial z^2}+h_x^\lambda\eta_\lambda\sum_{l=1}^{L}\left\{\frac{\partial^\lambda m^+(X_i,z_0)}{\lambda!\cdot\partial x_l^\lambda}+\sum_{s=1}^{\lambda-1}\frac{\partial^s m^+(X_i,z_0)}{\partial x_l^s}\omega_s^+\right\}\right\}\frac{\kappa(u)}{f(z_0)}f(X_i,z_0+uh)dudX_i(1+o_p(1))
\end{aligned}
$$

where $u=\frac{Z_i-z_0}{h}$. We can split the integral into two parts: for $u>0$ and for $u\leq 0$ and expand $f(X_i,z_0+uh)$ about $f(X_i,z_0)$ in the two separate integrals to obtain after some calculations:

$$
=\int\left\{h_z^2\frac{\bar{\mu}_2^2-\bar{\mu}_1\bar{\mu}_3}{2\tilde{\mu}}\frac{\partial^2 m^+(X_i,z_0)}{\partial z^2}+h_x^\lambda\eta_\lambda\sum_{l=1}^{L}\left\{\frac{\partial^\lambda m^+(X_i,z_0)}{\lambda!\cdot\partial x_l^\lambda}+\sum_{s=1}^{\lambda-1}\frac{\partial^s m^+(X_i,z_0)}{\partial x_l^s}\omega_s^+\right\}\right\}\frac{f^-(X_i,z_0)+f^+(X_i,z_0)}{2f(z_0)}dX_i\cdot(1+o_p(1))
$$

Combining this result with the analogous derivations for the second part in (8a) and with the bias term (8c) and repeating the derivations for the term $\hat{\Gamma}-\Gamma$ and combining everything in (7) gives as total bias of the $\hat{\gamma}$ estimator:

$$
\begin{aligned}
& \frac{h\bar{\mu}_1}{\Gamma f(z_0)}\int\left(m^+(x,z_0)-m^-(x,z_0)-\gamma d^+(x,z_0)+\gamma d^-(x,z_0)\right)\left(\frac{\partial f^+(x,z_0)}{\partial z}+\frac{\partial f^-(x,z_0)}{\partial z}\right)dx \\
& +\frac{1}{\Gamma}h_z^2\frac{\bar{\mu}_2^2-\bar{\mu}_1\bar{\mu}_3}{2\tilde{\mu}}\int\left(\frac{\partial^2 m^+(x,z_0)}{\partial z^2}-\frac{\partial^2 m^-(x,z_0)}{\partial z^2}-\gamma\frac{\partial^2 d^+(x,z_0)}{\partial z^2}+\gamma\frac{\partial^2 d^-(x,z_0)}{\partial z^2}\right)\frac{f^-(x,z_0)+f^+(x,z_0)}{2f(z_0)}dx \\
& +\frac{1}{\Gamma}h_x^\lambda\eta_\lambda\int\left\{\sum_{l=1}^{L}\frac{\partial^\lambda m^+(x,z_0)}{\lambda!\cdot\partial x_l^\lambda}+\sum_{s=1}^{\lambda-1}\frac{\partial^s m^+(x,z_0)}{\partial x_l^s}\omega_s^+-\frac{\partial^\lambda m^-(x,z_0)}{\lambda!\cdot\partial x_l^\lambda}-\sum_{s=1}^{\lambda-1}\frac{\partial^s m^-(x,z_0)}{\partial x_l^s}\omega_s^-\right\}\frac{f^-(x,z_0)+f^+(x,z_0)}{2f(z_0)}dx \\
& -\frac{\gamma}{\Gamma}h_x^\lambda\eta_\lambda\int\left\{\sum_{l=1}^{L}\frac{\partial^\lambda d^+(x,z_0)}{\lambda!\cdot\partial x_l^\lambda}+\sum_{s=1}^{\lambda-1}\frac{\partial^s d^+(x,z_0)}{\partial x_l^s}\omega_s^+-\frac{\partial^\lambda d^-(x,z_0)}{\lambda!\cdot\partial x_l^\lambda}-\sum_{s=1}^{\lambda-1}\frac{\partial^s d^-(x,z_0)}{\partial x_l^s}\omega_s^-\right\}\frac{f^-(x,z_0)+f^+(x,z_0)}{2f(z_0)}dx.
\end{aligned}
\tag{9}
$$

D.2 Calculation of the variance term

Remember that we defined the kernel function here by scaling with $f(z_0)$:

$$K_h\left(\frac{Z_i - z_0}{h}\right) = \frac{\kappa\left(\frac{Z_i - z_0}{h}\right)}{hf(z_0)}.$$

For calculating the variance we first examine the first part of (8a). (The results for the second part in (8a) are analogous.)

For the subsequent derivations we define the $L+2$ column vector $\mathbb{X}$ as in Lemma 1 as

$$\mathbb{X}_j = \left(1, \frac{Z_j - z_0}{h_z}, \left(\frac{X_j - x_0}{h_x}\right)'\right)'$$

and define

$$\mathbb{X}_{j,X_i} = \left(1, \frac{Z_j - z_0}{h_z}, \left(\frac{X_j - X_i}{h_x}\right)'\right)'.$$

As a preliminary we will also need the term

$$E\left[\left\{e_1' A_+^{-1}(X_j) \cdot \mathbb{X}_{i,X_j}\left(Y_i - m^+(X_j, z_0)\right) K_{i,X_j}\right\} K_h\left(\frac{Z_j - z_0}{h}\right) | X_i, Y_i, Z_i\right] \tag{10}$$

which we can split into two parts by multiplying with $1(Z_j \leq z_0) + 1(Z_j > z_0) = 1$. Examine first the part

$$E\left[1(Z_j \leq z_0)\left\{e_1' A_+^{-1}(X_j)\mathbb{X}_{i,X_j}\left(Y_i - m^+(X_j, z_0)\right) K_{i,X_j}\right\} \frac{1}{hf(z_0)}\kappa\left(\frac{Z_j - z_0}{h}\right) | X_i, Y_i, Z_i\right],$$

where the derivations for $1(Z_j > z_0)$ are analogous. We obtain

$$= h_x^L \int\int 1(u \leq 0)\left\{e_1' A_+^{-1}(X_i + vh_x)\mathbb{X}_{i,X_i+vh_x}\left(Y_i - m^+(X_i + vh_x, z_0)\right) K_{i,X_i+vh_x}\right\} \frac{\kappa(u)}{f(z_0)} \cdot f(X_i + vh_x, z_0 + uh) dv du,$$

where $v = \frac{X_j - X_i}{h_x}$ and $u = \frac{Z_j - z_0}{h}$, and after inserting the expression for $e_1' A_+^{-1}$

$$= h_x^L \int\int \frac{1(u \leq 0)}{f^+(X_i + vh_x, z_0) \cdot \mathcal{C}} \begin{pmatrix} \frac{1}{\tilde{\mu}}\left(\bar{\mu}_2\mathcal{C} + h_z \cdot \left(\bar{\mu}_3 \frac{\partial f^+(X_i+vh_x,z_0)}{f^+(X_i+vh_x,z_0)\cdot\partial z}\mathcal{C} + 2\bar{\mu}_2\bar{\mu}_1\mathcal{A}\right) + h_x\bar{\mu}_2\frac{\eta_{\lambda+1}}{\eta_\lambda}\frac{(\lambda-2)!}{(\lambda-1)!}\mathcal{B}\right) \\ -\frac{1}{\tilde{\mu}}\left(\bar{\mu}_1\mathcal{C} + h_z \cdot \left(\bar{\mu}_2 \frac{\partial f^+(X_i+vh_x,z_0)}{f^+(X_i+vh_x,z_0)\cdot\partial z}\mathcal{C} + 2\bar{\mu}_1^2\mathcal{A}\right) + h_x\bar{\mu}_1\frac{\eta_{\lambda+1}}{\eta_\lambda}\frac{(\lambda-2)!}{(\lambda-1)!}\mathcal{B}\right) \\ -2h_x\left(\frac{\partial^{\lambda-1}f^+(X_i+vh_x,z_0)}{\partial x_1^{\lambda-1}} \cdot \mathcal{C}_{\neq 1}\right)\frac{(\lambda-2)!}{(\lambda-1)!} \\ \vdots \\ -2h_x\left(\frac{\partial^{\lambda-1}f^+(X_i+vh_x,z_0)}{\partial x_L^{\lambda-1}} \cdot \mathcal{C}_{\neq L}\right)\frac{(\lambda-2)!}{(\lambda-1)!} \end{pmatrix}' \begin{pmatrix} 1 \\ \frac{Z_i - z_0}{h_z} \\ -v_1 \\ \vdots \\ -v_L \end{pmatrix}$$

$$\cdot\left(Y_i - m^+(X_i + vh_x, z_0)\right)\kappa\left(\frac{Z_i - z_0}{h_z}\right) \cdot \prod_{l=1}^{L}\bar{\kappa}(-v_l) \cdot \frac{\kappa(u)}{f(z_0)} \cdot f(X_i + vh_x, z_0 + uh) dv du,$$

and a Taylor series expansion about (X_i, z_0) we obtain:

$$= h_x^L \frac{Y_i - m^+(X_i, z_0)}{2f(z_0)\tilde{\mu}}\kappa\left(\frac{Z_i - z_0}{h_z}\right)\left(\bar{\mu}_2 - \bar{\mu}_1\frac{Z_i - z_0}{h_z}\right)\frac{f^-(X_i, z_0)}{f^+(X_i, z_0)} \cdot (1 + O(h_z + h_x)).$$

Complementing these calculations with those for $1(Z_j > z_0)$ gives an expression for (10) as:

$$= h_x^L \frac{Y_i - m^+(X_i, z_0)}{2\tilde{\mu}f(z_0)}\kappa\left(\frac{Z_i - z_0}{h_z}\right)\left(\bar{\mu}_2 - \bar{\mu}_1\frac{Z_i - z_0}{h_z}\right)\frac{f^-(X_i, z_0) + f^+(X_i, z_0)}{f^+(X_i, z_0)} \cdot (1 + O(h_z + h_x)). \tag{11}$$

Now, if we develop the expressions for the projection theorem, everything is identical to Theorem 2 with the only exception that K_h^* is replaced by K_h. Using the previous result, we obtain the following for the expression $E\left[\varsigma_{ij}+\varsigma_{ji}|X_i,Y_i,Z_i\right]$

$$\begin{aligned} & E\left[\varsigma_{ij}+\varsigma_{ji}|X_i,Y_i,Z_i\right] \\ = \; & \frac{1}{h_z h_x^L} E\left[e_1' A_+^{-1}(X_i)\mathbb{X}_{j,X_i}\left(Y_j - m^+(X_i,z_0)\right) K_{j,X_i} I_j^+ | X_i,Y_i,Z_i\right]\cdot K_h\left(\frac{Z_i-z_0}{h}\right)(1+o_p(1)) \\ & + \frac{1}{h_z h_x^L} E\left[\left\{e_1' A_+^{-1}(X_j)\mathbb{X}_{i,X_j}\left(Y_i - m^+(X_j,z_0)\right) K_{i,X_j}(1+o_p(1))\right\} K_h\left(\frac{Z_j-z_0}{h}\right)|X_i,Y_i,Z_i\right]\cdot I_i^+ \end{aligned}$$

$$\begin{aligned} = \; & \left\{h_z^2 \frac{\bar\mu_2^2 - \bar\mu_1\bar\mu_3}{2\tilde\mu}\frac{\partial^2 m^+(X_i,z_0)}{\partial z^2} + h_x^\lambda \eta_\lambda \sum_{l=1}^{L}\left\{\frac{\partial^\lambda m^+(X_i,z_0)}{\lambda!\cdot\partial x_l^\lambda} + \sum_{s=1}^{\lambda-1}\frac{\partial^s m^+(X_i,z_0)}{\partial x_l^s}\omega_s\right\} + O(h_z^3 + h_z^2 h_x + h_x^{\lambda+1})\right\}\cdot K_h\left(\frac{Z_i-z_0}{h}\right) \\ & + \frac{1}{h_z}\frac{Y_i - m^+(X_i,z_0)}{2\tilde\mu f(z_0)}\kappa\left(\frac{Z_i-z_0}{h_z}\right)\left(\bar\mu_2 - \bar\mu_1\frac{Z_i-z_0}{h_z}\right)\frac{f^-(X_i,z_0)+f^+(X_i,z_0)}{f^+(X_i,z_0)}\cdot\left(1+O\left(h_z+h_x\right)\right)\cdot I_i^+. \end{aligned}$$

For the following variance calculation, the second term, which is mean zero and of order $\frac{1}{h_z}\cdot\left(1+O\left(h_z+h_x\right)\right)$ clearly dominates the first term, such that we can ignore the first term in the following. The second term, however, is identical to the corresponding expression of Theorem 2. It thus follows immediately that

$$\begin{aligned} & Var\left(\frac{2}{n}\sum_{i=1}^{n}\left(E\left[\frac{\varsigma_{ij}+\varsigma_{ji}}{2}|X_i,Y_i,Z_i\right] - E\left[\frac{\varsigma_{ij}+\varsigma_{ji}}{2}\right]\right)\right) \\ = \; & \frac{\bar\mu_2^2\ddot\mu_0 - 2\bar\mu_2\bar\mu_1\ddot\mu_1 + \bar\mu_1^2\ddot\mu_2}{nh_z}\int\frac{\sigma_Y^{2+}(x,z_0)}{4\tilde\mu^2 f^2(z_0)}\frac{\left(f^-(x,z_0)+f^+(x,z_0)\right)^2}{f^+(x,z_0)}dx\cdot\left(1+O\left(h_z+h_x+h\right)\right) \end{aligned}$$

where $\sigma_Y^{2+}(X,z_0) = \lim_{\varepsilon\to 0} E\left[\left(Y - m^+(X,Z)\right)^2 | X, Z = z_0+\varepsilon\right]$.

Collecting all terms (8a), (8b) and (8c), the variance of $\hat\Delta - \Delta$ can be approximated. The term (8c) can be ignored as it is a nonstochastic bias term. We thus obtain that $Var\left(\hat\Delta - \Delta\right)$

$$\begin{aligned} = \; & \frac{1}{n} Var\left\{\left\{\hat m^+(X_i,z_0) - m^+(X_i,z_0)\right\} K_h\left(\frac{Z_i-z_0}{h}\right)\right\} \\ & + \frac{1}{n} Var\left\{\left\{\hat m^-(X_i,z_0) - m^-(X_i,z_0)\right\} K_h\left(\frac{Z_i-z_0}{h}\right)\right\} \\ & + \frac{1}{n} Var\left\{\left\{m^+(X_i,z_0) - m^-(X_i,z_0)\right\} K_h\left(\frac{Z_i-z_0}{h}\right) - E\left[\left\{m^+(X,z_0) - m^-(X,z_0)\right\} K_h\left(\frac{Z-z_0}{h}\right)\right]\right\} \\ & + \frac{1}{n} Cov\left\{\left\{\hat m^+(X_i,z_0) - m^+(X_i,z_0)\right\} K_h\left(\frac{Z_i-z_0}{h}\right), \left\{\hat m^-(X_i,z_0) - m^-(X_i,z_0)\right\} K_h\left(\frac{Z_i-z_0}{h}\right)\right\} \\ & + \frac{1}{n} Cov\left\{\left\{\hat m^+(X_i,z_0) - m^+(X_i,z_0)\right\} K_h\left(\frac{Z_i-z_0}{h}\right), \left\{m^+(X_i,z_0) - m^-(X_i,z_0)\right\} K_h\left(\frac{Z_i-z_0}{h}\right)\right\} \\ & + \frac{1}{n} Cov\left\{\left\{\hat m^-(X_i,z_0) - m^-(X_i,z_0)\right\} K_h\left(\frac{Z_i-z_0}{h}\right), \left\{m^+(X_i,z_0) - m^-(X_i,z_0)\right\} K_h\left(\frac{Z_i-z_0}{h}\right)\right\} \end{aligned}$$

$$\begin{aligned} = \; & \frac{1}{nh_z}\frac{\bar\mu_2^2\ddot\mu_0 - 2\bar\mu_2\bar\mu_1\ddot\mu_1 + \bar\mu_1^2\ddot\mu_2}{4\tilde\mu^2 f^2(z_0)}\int \sigma_Y^{2+}(x,z_0)\frac{\left(f^-(x,z_0)+f^+(x,z_0)\right)^2}{f^+(x,z_0)}dx\cdot\left(1+O\left(h_z+h_x+h\right)\right) \\ & + \frac{1}{nh_z}\frac{\bar\mu_2^2\ddot\mu_0 - 2\bar\mu_2\bar\mu_1\ddot\mu_1 + \bar\mu_1^2\ddot\mu_2}{4\tilde\mu^2 f^2(z_0)}\int \sigma_Y^{2-}(x,z_0)\frac{\left(f^-(x,z_0)+f^+(x,z_0)\right)^2}{f^-(x,z_0)}dx\cdot\left(1+O\left(h_z+h_x+h\right)\right) \\ & + \frac{1}{nh}\frac{\ddot\mu_0}{f^2(z_0)}\int\left(m^+(x,z_0) - m^-(x,z_0)\right)^2\left(f^+(x,z_0)+f^-(x,z_0)\right)dx\cdot(1+O(h)) - O\left(\frac{1}{n}\right) + o\left(\frac{1}{nh_z}\right) \end{aligned} \tag{13}$$

where the following terms have been plugged in

$$\begin{aligned} & \frac{1}{n} Var\left\{\left\{m^+(X_i,z_0) - m^-(X_i,z_0)\right\} K_h\left(\frac{Z_i-z_0}{h}\right) - E\left[\left\{m^+(X,z_0) - m^-(X,z_0)\right\} K_h\left(\frac{Z-z_0}{h}\right)\right]\right\} \\ = \; & \frac{1}{n} E\left[\left\{m^+(X,z_0) - m^-(X,z_0)\right\}^2 K_h^2\left(\frac{Z-z_0}{h}\right)\right] - \frac{1}{n}\left(E\left[\left\{m^+(X,z_0) - m^-(X,z_0)\right\} K_h\left(\frac{Z-z_0}{h}\right)\right]\right)^2 \end{aligned}$$

$$
\begin{aligned}
= & \frac{1}{n}\int\int\left\{m^{+}(x,z_0)-m^{-}(x,z_0)\right\}^2\frac{\kappa^2(u)}{h^2f^2(z_0)}f(x,z_0+uh)hdxdu \\
& -\frac{1}{n}\left(\int\int\left\{m^{+}(x,z_0)-m^{-}(x,z_0)\right\}\frac{\kappa(u)}{hf(z_0)}f(x,z_0+uh)hdxdu\right)^2
\end{aligned}
$$

where $u=\frac{Z_i-z_0}{h}$. Splitting the integrals into the parts $Z>z_0$ and $Z<z_0$ and with a series expansion about (X,z_0) we obtain

$$
=\frac{\ddot{\mu}_0}{nhf^2(z_0)}\int\left\{m^{+}(x,z_0)-m^{-}(x,z_0)\right\}^2\left(f^{+}(x,z_0)+f^{-}(x,z_0)\right)dx+O\left(\frac{1}{n}\right).
$$

Also, the covariance terms are of lower order. First,

$$
Cov\left\{\frac{1}{n}\sum_{i=1}^{n}\left\{\hat{m}^{+}(X_i,z_0)-m^{+}(X_i,z_0)\right\}K_h\left(\frac{Z_i-z_0}{h}\right),\frac{1}{n}\sum_{i=1}^{n}\left\{m^{+}(X_i,z_0)-m^{-}(X_i,z_0)\right\}K_h\left(\frac{Z_i-z_0}{h}\right)\right\}
$$

$$
=Cov(\frac{1}{n}\sum_{i=1}^{n}\left(E\left[\varsigma_{ij}+\varsigma_{ji}|X_i,Y_i,Z_i\right]-E\left[\varsigma_{ij}+\varsigma_{ji}\right]\right)+E\left[\frac{\varsigma_{ij}+\varsigma_{ji}}{2}\right]+o_p(1),\frac{1}{n}\sum_{i=1}^{n}\left\{m^{+}(X_i,z_0)-m^{-}(X_i,z_0)\right\}K_h\left(\frac{Z_i-z_0}{h}\right))
$$

$$
\begin{aligned}
=&\frac{1}{n}E[\left\{\left(E\left[\varsigma_{ij}+\varsigma_{ji}|X_i,Y_i,Z_i\right]-E\left[\varsigma_{ij}+\varsigma_{ji}\right]\right)+E\left[\frac{\varsigma_{ij}+\varsigma_{ji}}{2}\right]\right\} \\
&\cdot\left\{\left\{m^{+}(X_i,z_0)-m^{-}(X_i,z_0)\right\}K_h\left(\frac{Z_i-z_0}{h}\right)-E\left[\left\{m^{+}(X,z_0)-m^{-}(X,z_0)\right\}K_h^{*}\left(\frac{Z-z_0}{h}\right)\right]\right\}](1+o_p(1))
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{1}{nh_z}\int\int\frac{m(X_i,Z_i)-m^{+}(X_i,z_0)}{2\tilde{\mu}f(z_0)}\kappa\left(\frac{Z_i-z_0}{h_z}\right)\left(\bar{\mu}_2-\bar{\mu}_1\frac{Z_i-z_0}{h_z}\right)\frac{f^{-}(X_i,z_0)+f^{+}(X_i,z_0)}{f^{+}(X_i,z_0)}\cdot\left(1+O\left(h_z+h_x\right)\right)\cdot I_i^{+} \\
&\cdot\left\{\left\{m^{+}(X_i,z_0)-m^{-}(X_i,z_0)\right\}K_h\left(\frac{Z_i-z_0}{h}\right)-E\left[\left\{m^{+}(X,z_0)-m^{-}(X,z_0)\right\}K_h\left(\frac{Z-z_0}{h}\right)\right]\right\}f(X_i,Z_i)dX_idZ_i
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{1}{nh_z}\int\int\frac{m(X_i,z_0+uh)-m^{+}(X_i,z_0)}{2\tilde{\mu}f(z_0)}\kappa\left(u\frac{h}{h_z}\right)\left(\bar{\mu}_2-\bar{\mu}_1u\frac{h}{h_z}\right)\frac{f^{-}(X_i,z_0)+f^{+}(X_i,z_0)}{f^{+}(X_i,z_0)}\cdot\left(1+O\left(h_z+h_x\right)\right)\cdot 1\left(u>0\right) \\
&\cdot\left\{\left\{m^{+}(X_i,z_0)-m^{-}(X_i,z_0)\right\}\frac{\kappa(u)}{f(z_0)}-E\left[\left\{m^{+}(X,z_0)-m^{-}(X,z_0)\right\}K_h\left(\frac{Z-z_0}{h}\right)\right]\right\}f(X_i,z_0+uh)dX_ihdu
\end{aligned}
$$

where $u=\frac{Z_i-z_0}{h}$ and a Taylor series expansion about (X_i,z_0) gives

$$
=\frac{1}{nh_z}\left(0+O\left(h+h_z+h_x\right)\right)=o\left(\frac{1}{nh_z}\right).
$$

Also, the following term is of lower order

$$
Cov\left\{\frac{1}{n}\sum_{i=1}^{n}\left\{\hat{m}^{+}(X_i,z_0)-m^{+}(X_i,z_0)\right\}K_h\left(\frac{Z_i-z_0}{h}\right),\frac{1}{n}\sum_{i=1}^{n}\left\{\hat{m}^{-}(X_i,z_0)-m^{-}(X_i,z_0)\right\}K_h\left(\frac{Z_i-z_0}{h}\right)\right\}
$$

$$
\begin{aligned}
= & \frac{1}{n}E\left\{\left\{\hat{m}^{+}(X_i,z_0)-m^{+}(X_i,z_0)\right\}K_h\left(\frac{Z_i-z_0}{h}\right)\left\{\hat{m}^{-}(X_i,z_0)-m^{-}(X_i,z_0)\right\}K_h\left(\frac{Z_i-z_0}{h}\right)\right\} \\
& -\frac{1}{n}E\left[\left\{\hat{m}^{+}(X_i,z_0)-m^{+}(X_i,z_0)\right\}K_h\left(\frac{Z_i-z_0}{h}\right)\right]E\left[\left\{\hat{m}^{-}(X_i,z_0)-m^{-}(X_i,z_0)\right\}K_h\left(\frac{Z_i-z_0}{h}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{1}{n}E[\left(\left(\left(\frac{1}{h_z}\frac{Y_i-m^{+}(X_i,z_0)}{2\tilde{\mu}f(z_0)}\kappa\left(\frac{Z_i-z_0}{h_z}\right)\left(\bar{\mu}_2-\bar{\mu}_1\frac{Z_i-z_0}{h_z}\right)\frac{f^{-}(X_i,z_0)+f^{+}(X_i,z_0)}{f^{+}(X_i,z_0)}\cdot\left(1+O\left(h_z+h_x\right)\right)\cdot I_i^{+}\right)-E\left[\varsigma_{ij}^{+}\right]\right)\right] \\
& \cdot\left(\left(\frac{1}{h_z}\frac{Y_i-m^{-}(X_i,z_0)}{2\tilde{\mu}f(z_0)}\kappa\left(\frac{Z_i-z_0}{h_z}\right)\left(\bar{\mu}_2-\bar{\mu}_1\frac{Z_i-z_0}{h_z}\right)\frac{f^{-}(X_i,z_0)+f^{+}(X_i,z_0)}{f^{-}(X_i,z_0)}\cdot\left(1+O\left(h_z+h_x\right)\right)\cdot I_i^{-}\right)-E\left[\varsigma_{ij}^{-}\right]\right)] \\
& -\frac{1}{n}O\left(h_x^{\lambda}+h_z^2\right)O\left(h_x^{\lambda}+h_z^2\right)
\end{aligned}
$$

$$= 0 + \frac{1}{nh_z} O\left(h_x^\lambda + h_z^2\right) O\left(h_z + h_x\right) - \frac{1}{n} O\left(h_x^\lambda + h_z^2\right) O\left(h_x^\lambda + h_z^2\right) = o\left(\frac{1}{nh_z}\right),$$

because $I_i^+ \cdot I_i^- = 1(Z_i > z_0) \cdot 1(Z_i < z_0) = 0$.

The variance $Var(\hat{\Gamma} - \Gamma)$ can be derived analogously to the variance $Var(\hat{\Delta} - \Delta)$, and is also of order $O(\frac{1}{nh_z} + \frac{1}{nh})$. The covariance terms can be computed similarly to Theorem 2 and are also of order $O(\frac{1}{nh_z} + \frac{1}{nh})$.

E Proof of Theorem 3

To derive the asymptotic distribution of $\hat{\gamma}_{RDD}$, define $\hat{\gamma}_{RDD} = \frac{\hat{\Delta}}{\hat{\Gamma}}$ and $\gamma = \frac{\Delta}{\Gamma}$. We need to establish the distribution of

$$\begin{aligned}
n^{\frac{2}{5}}\left(\hat{\gamma}_{RDD} - \gamma\right) &= n^{\frac{2}{5}}\left(\frac{\hat{\Delta}}{\hat{\Gamma}} - \frac{\Delta}{\Gamma}\right) \\
&= n^{\frac{2}{5}}\left(\frac{\frac{1}{n}\sum_{i=1}^{n}\left(\hat{m}^+(X_i, z_0) - \hat{m}^-(X_i, z_0)\ \right) K_h^*\left(\frac{Z_i - z_0}{h}\right)}{\frac{1}{n}\sum_{i=1}^{n}\left(\hat{d}^+(X_i, z_0) - \hat{d}^-(X_i, z_0)\ \right) K_h^*\left(\frac{Z_i - z_0}{h}\right)} - \frac{\int\left(m^+(x, z_0) - m^-(x, z_0)\right) \cdot \frac{f^-(x|z_0) + f^+(x|z_0)}{2} dx}{\int\left(d^+(x, z_0) - d^-(x, z_0)\right) \cdot \frac{f^-(x|z_0) + f^+(x|z_0)}{2} dx}\right),
\end{aligned}$$

where

$$K_h^*(u) = \frac{\bar{\mu}_2 - \bar{\mu}_1 |u|}{2\tilde{\mu} f(z_0)} K_h(u).$$

To derive the asymptotic distribution of $\hat{\gamma}_{RDD}$, note that $\hat{\gamma}_{RDD} - \gamma$ can be written as

$$\left(\hat{\gamma}_{RDD} - \gamma\right) = \frac{\hat{\Delta}}{\hat{\Gamma}} - \frac{\Delta}{\Gamma} = \left(\frac{\hat{\Delta} - \Delta}{\Gamma} - \gamma\frac{\hat{\Gamma} - \Gamma}{\Gamma}\right) \cdot \left(1 - \frac{\hat{\Gamma} - \Gamma}{\hat{\Gamma}}\right). \tag{15}$$

The derivation proceeds in two steps. First the term $\hat{\Delta} - \Delta$ is analyzed, with analogous results for $\hat{\Gamma} - \Gamma$. It is shown that bias and variance converge to zero with growing sample size, implying convergence in mean square and thus in probability. This also implies that the last term $\left(1 - \frac{\hat{\Gamma} - \Gamma}{\hat{\Gamma}}\right)$ is $1 + o_p(1)$. Hence the first-order behaviour of $\hat{\gamma}_{RDD} - \gamma$ is determined by the term $\frac{\hat{\Delta} - \Delta}{\Gamma} - \gamma\frac{\hat{\Gamma} - \Gamma}{\Gamma}$ in

$$\hat{\gamma}_{RDD} - \gamma = \left(\frac{\hat{\Delta} - \Delta}{\Gamma} - \gamma\frac{\hat{\Gamma} - \Gamma}{\Gamma}\right) \cdot \left(1 + o_p(1)\right). \tag{16}$$

In a preliminary step the term $\hat{\Delta} - \Delta$ is analyzed. (The derivations for $\hat{\Gamma} - \Gamma$ are analogous.) Write $\hat{\Delta} - \Delta$ as

$$\hat{\Delta} - \Delta = \frac{1}{n}\sum_{i=1}^{n}\left(\hat{m}^+(X_i, z_0) - \hat{m}^-(X_i, z_0)\ \right) K_h^*\left(\frac{Z_i - z_0}{h}\right) - \int\left(m^+(x, z_0) - m^-(x, z_0)\right) \cdot \frac{f^-(x|z_0) + f^+(x|z_0)}{2} dx$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left\{\hat{m}^+(X_i, z_0) - m^+(X_i, z_0)\right\} K_h^*\left(\frac{Z_i - z_0}{h}\right) - \frac{1}{n}\sum_{i=1}^{n}\left\{\hat{m}^-(X_i, z_0) - m^-(X_i, z_0)\right\} K_h^*\left(\frac{Z_i - z_0}{h}\right) \tag{17a}$$

$$+ \frac{1}{n}\sum_{i=1}^{n}\left\{m^+(X_i, z_0) - m^-(X_i, z_0)\right\} K_h^*\left(\frac{Z_i - z_0}{h}\right) - E\left[\left\{m^+(X, z_0) - m^-(X, z_0)\right\} K_h^*\left(\frac{Z - z_0}{h}\right)\right] \tag{17b}$$

$$+ E\left[\left\{m^+(X, z_0) - m^-(X, z_0)\right\} K_h^*\left(\frac{Z - z_0}{h}\right)\right] - \int\left(m^+(x, z_0) - m^-(x, z_0)\right) \cdot \frac{f^-(x|z_0) + f^+(x|z_0)}{2} dx. \tag{17c}$$

E.1 Calculation of the bias term

The calculation of the bias term is done in several steps. Consider first the term (17c). This represents the bias if the functions m^+ and m^- were known. To ease notation we ignore the terms containing m^- for the moment and retain only those with m^+. (The derivations for m^- are analogous.)

$$E\left[m^+(X,z_0)K_h^*\left(\frac{Z-z_0}{h}\right)\right]-\int m^+(x,z_0)\cdot\frac{f^-(x|z_0)+f^+(x|z_0)}{2}dx$$
$$=\left\{E\left[m^+(X,z_0)K_h^*\left(\frac{Z-z_0}{h}\right)1(Z>z_0)\right]-\int m^+(x,z_0)\frac{f^+(x|z_0)}{2}dx\right\}$$
$$+\left\{E\left[m^+(X,z_0)K_h^*\left(\frac{Z-z_0}{h}\right)1(Z\le z_0)\right]-\int m^+(x,z_0)\frac{f^-(x|z_0)}{2}dx\right\}.$$

Concentrate on the first term. (The second is analogous.) By inserting the expression for the boundary kernel we obtain

$$\begin{aligned}
&E\left[m^+(X,z_0)K_h^*\left(\frac{Z-z_0}{h}\right)1(Z>z_0)\right]-\int m^+(x,z_0)\frac{f^+(x|z_0)}{2}dx\\
=\;&\int\cdots\int m^+(x,z_0)K_h^*\left(\frac{z-z_0}{h}\right)1(z>z_0)f(x,z)dxdz-\int m^+(x,z_0)\frac{f^+(x|z_0)}{2}dx\\
=\;&\int\cdots\int m^+(x,z_0)\frac{\bar\mu_2-\bar\mu_1\left|\frac{z-z_0}{h}\right|}{2\tilde\mu f(z_0)}\frac{1}{h}\kappa\left(\frac{z-z_0}{h}\right)1(z>z_0)f(x,z)dxdz-\int m^+(x,z_0)\frac{f^+(x|z_0)}{2}dx\\
=\;&\int\cdots\int m^+(x,z_0)\frac{\bar\mu_2-\bar\mu_1|u|}{2\tilde\mu f(z_0)}\kappa(u)\,1(u>0)f(x,z_0+uh)dxdu-\int m^+(x,z_0)\frac{f^+(x|z_0)}{2}dx
\end{aligned}$$

where $u=\frac{z-z_0}{h}$. Expanding $f(x,z_0+uh)$ as $f^+(x,z_0)+uh\frac{\partial f^+}{\partial z}(x,z_0)+\frac{1}{2}u^2h^2\frac{\partial^2 f^+}{\partial z^2}(x,z_0)+o(h^2)$ gives

$$\begin{aligned}
=\;&\int m^+(x,z_0)\frac{f^+(x,z_0)}{2f(z_0)}dx-\int m^+(x,z_0)\frac{f^+(x|z_0)}{2}dx\\
&+\int\frac{m^+(x,z_0)}{2\tilde\mu f(z_0)}h\frac{\partial f^+}{\partial z}(x,z_0)\underbrace{(\bar\mu_2\bar\mu_1-\bar\mu_1\bar\mu_2)}_{=0}dx\\
&+h^2\frac{\bar\mu_2^2-\bar\mu_1\bar\mu_3}{4\tilde\mu f(z_0)}\int m^+(x,z_0)\frac{\partial^2 f^+}{\partial z^2}(x,z_0)dx+o(h^2)
\end{aligned}$$

Note that the first term is zero since

$$f^+(x|z_0)f(z_0)=f^+(x,z_0)$$

and the second term is also zero. Repeating the analogous derivations for the terms neglected so far, the bias term (17c) is

$$h^2\frac{\bar\mu_2^2-\bar\mu_1\bar\mu_3}{4\tilde\mu f(z_0)}\int\left(m^+(x,z_0)-m^-(x,z_0)\right)\left(\frac{\partial^2 f^+}{\partial z^2}(x,z_0)+\frac{\partial^2 f^-}{\partial z^2}(x,z_0)\right)dx+O_p(h^3).$$

In contrast to Proposition 2, this term is of order $O(h^2)$.

Now consider the first term (17a):

markus ab hier alles noch nachrechnen Consider the first part of (17a). (The results for the second part of (17a) are analogous.)

$$\frac{1}{n}\sum_{i=1}^n\left\{\hat m^+(X_i,z_0)-m^+(X_i,z_0)\right\}K_h^*\left(\frac{Z_i-z_0}{h}\right).$$

By inserting the expression from Lemma 1 for the nonparametric regression estimator $\hat m^+(x,z_0)$ we obtain

$$=\frac{1}{n}\sum_{i=1}^n\left\{\frac{1}{nh_zh_x^L}\sum_{j=1}^n e_1'A_+^{-1}(X_i)\mathbb{X}_{j,X_i}\cdot\left(Y_j-m^+(X_i,z_0)\right)K_{j,X_i}I_j^+\left(1+o_p(1)\right)\right\}K_h^*\left(\frac{Z_i-z_0}{h}\right)$$

where

$$K_{j,X_i}I_j^+=\kappa\left(\frac{Z_j-z_0}{h_z}\right)\cdot\prod_{l=1}^L\bar\kappa\left(\frac{X_{jl}-X_{il}}{h_x}\right)\cdot 1(Z_j>z_0).$$

We can rewrite this expression as:

$$= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\varsigma_{ij}+\varsigma_{ji}}{2} = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\varsigma_{ij}+\varsigma_{ji}}{2} - E\left[\frac{\varsigma_{ij}+\varsigma_{ji}}{2}\right]\right) + E\left[\frac{\varsigma_{ij}+\varsigma_{ji}}{2}\right] \tag{18}$$

where $\varsigma_{ij} = \left\{\frac{1}{h_z h_x^L} e_1' A_+^{-1}(X_i)\mathbb{X}_{j,X_i}\left(Y_j - m^+(X_i,z_0)\right) K_{j,X_i} I_j^+ \left(1+o_p(1)\right)\right\} K_h^*\left(\frac{Z_i - z_0}{h}\right)$. The term $\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\varsigma_{ij}+\varsigma_{ji}}{2}$ is a nondegenerate von Mises statistic to which a projection theorem can be applied. This requires that $E\left[\left(\frac{\varsigma_{ij}+\varsigma_{ji}}{2}\right)^2\right] < o(n)$, see Serfling (1980, p.190). Notice that $E\left[\left(\frac{\varsigma_{ij}+\varsigma_{ji}}{2}\right)^2\right] = E\left[\left(\frac{\varsigma_{ij}+\varsigma_{ji}}{2}\right)\right]^2 + Var\left(\frac{\varsigma_{ij}+\varsigma_{ji}}{2}\right)$. It is shown further below that $E\left[\varsigma_{ij}\right] = O(h_z^2 + h_x^\lambda)$ and that $Var\left(\frac{\varsigma_{ij}+\varsigma_{ji}}{2}\right) = O(\frac{1}{nh_z})$. Both terms are $o(1)$ by Assumption 2. Hence, the projection theorem can be applied. The von Mises statistic is asymptotically equivalent to the corresponding U-statistic, and its projection is

$$= \frac{2}{n}\sum_{i=1}^{n}\left(E\left[\frac{\varsigma_{ij}+\varsigma_{ji}}{2}|X_i,Y_i,Z_i\right] - E\left[\frac{\varsigma_{ij}+\varsigma_{ji}}{2}\right]\right) + E\left[\frac{\varsigma_{ij}+\varsigma_{ji}}{2}\right] + o_p(1). \tag{19}$$

The first term determines the variance of (17a) while the latter part determines its bias.

Calculate first the **bias term**. For this the following expression will be needed:

$$E\left[e_1' A_+^{-1}(X_i)\cdot \mathbb{X}_{j,X_i}\left(Y_j - m^+(X_i,z_0)\right) K_{j,X_i} I_j^+ \ |X_i,Y_i,Z_i\right]$$

$$= \int e_1' A_+^{-1}(X_i)\begin{pmatrix} 1 \\ \frac{Z_j - z_0}{h_z} \\ \frac{X_{j1}-X_{i1}}{h_x} \\ \vdots \\ \frac{X_{jL}-X_{iL}}{h_x}\end{pmatrix}\left(m(X_j,Z_j) - m^+(X_i,z_0)\right) K_{j,X_i} I_j^+ \cdot f(X_j,Z_j)dX_j dZ_j$$

$$= h_z h_x^L \int\cdots\int e_1' A_+^{-1}(X_i)\cdot \begin{pmatrix} 1 \\ u \\ v_1 \\ \vdots \\ v_L\end{pmatrix}\left(m(X_i + vh_x, z_0 + uh_z) - m^+(X_i,z_0)\right)\cdot \kappa(u)\prod_{l=1}^{L}\bar\kappa(v_l)\,1(u>0)\cdot f(X_i + vh_x, z_0+uh_z)dvdu$$

where $u = \frac{Z_j - z_0}{h_z}$ and $v = \frac{X_j - X_i}{h_x}$. Because of the term $1(u>0)$ the integral is evaluated only when u is positive, such that we can expand $m(X_i + vh_x, z_0+uh_z)$ about $m^+(X_i,z_0)$ and expand $f(X_i+vh_x,z_0+uh_z)$ about $f^+(X_i,z_0)$. Consider a Taylor series expansion of $m(\cdot)f(\cdot)$ up to order λ. Due to the structure of the vector $A_+^{-1}(X_i)$ and since $\bar\kappa$ is a kernel of order λ all terms where powers of v_l lower than $\lambda-1$ appear will be zero. Only the $\lambda-1$ and the λ terms of the series expansion will remain as well as the first two terms with respect to z as well as higher order terms:

$$= h_z h_x^L \int\cdots\int e_1' A_+^{-1}(X_i)\cdot \begin{pmatrix} 1 \\ u \\ v_1 \\ \vdots \\ v_L\end{pmatrix}\kappa(u)\prod_{l=1}^{L}\bar\kappa(v_l)\,1(u>0)$$

$$\times\left\{\begin{array}{c} \frac{\partial m^+(X_i,z_0)}{\partial z} f^+(X_i,z_0)uh_z + \frac{\partial^2 m^+(X_i,z_0)}{\partial z^2}\frac{f^+(X_i,z_0)}{2}u^2h_z^2 + \frac{\partial m^+(X_i,z_0)}{\partial z}\frac{\partial f^+(X_i,z_0)}{\partial z}u^2 h_z^2 \\ +\sum_{l=1}^{L}\frac{1}{\lambda!}v_l^\lambda h_x^\lambda\left(\sum_{s=1}^{\lambda}\binom{\lambda}{s}\frac{\partial^s m^+(X_i,z_0)}{\partial x_l^s}\frac{\partial^{\lambda-s} f^+(X_i,z_0)}{\partial x_l^{\lambda-s}}\right) \\ +\sum_{l=1}^{L}\frac{1}{(\lambda-1)!}v_l^{\lambda-1}h_x^{\lambda-1}\left(\sum_{s=1}^{\lambda-1}\binom{\lambda-1}{s}\frac{\partial^s m^+(X_i,z_0)}{\partial x_l^s}\frac{\partial^{\lambda-1-s} f^+(X_i,z_0)}{\partial x_l^{\lambda-1-s}}\right) + O(h_z^3 + h_x^{\lambda+1}) \end{array}\right\}\times dvdu,$$

where $\binom{\lambda}{s} = \frac{\lambda!}{s!(\lambda-s!)}$. Notice that the summation of the partial derivatives in $\sum_{s=1}^{\lambda}$ and $\sum_{s=1}^{\lambda-1}$ starts with $s=1$ and not with $s=0$ since the expression $(m(X_i+vh_x, z_0+uh_z) - m^+(X_i, z_0))\, f(X_i+vh_x, z_0+uh_z)$ is zero at $v=u=0$ when only derivatives with respect to f but not with respect to m are taken in the Taylor series expansion. Now entering the expression (4) for $e_1' A_+^{-1}(X_i)$ gives after some tedious calculations:

$$= \frac{h_z h_x^L}{f^+(X_i,z_0)} \frac{1}{\tilde{\mu}} h_z^2 \left(\bar{\mu}_2^2 - \bar{\mu}_1\bar{\mu}_3\right) \left\{ -\frac{\partial m^+(X_i,z_0)}{\partial z}\frac{\partial f^+(x_0,z_0)}{\partial z} + \frac{\partial^2 m^+(X_i,z_0)}{\partial z^2}\frac{f^+(X_i,z_0)}{2} + \frac{\partial m^+(X_i,z_0)}{\partial z}\frac{\partial f^+(X_i,z_0)}{\partial z} + O(h_z+h_x) \right\}$$

$$+ \frac{1}{f^+(X_i,z_0)\cdot\mathcal{C}} h_x^\lambda \eta_\lambda h_z h_x^L \sum_{l=1}^{L} \left\{ \begin{array}{c} \mathcal{C}\frac{1}{\lambda!}\left(\sum_{s=1}^{\lambda}\binom{\lambda}{s}\frac{\partial^s m^+(X_i,z_0)}{\partial x_l^s}\frac{\partial^{\lambda-s} f^+(X_i,z_0)}{\partial x_l^{\lambda-s}}\right) \\ -\left(\frac{\partial^{\lambda-1} f^+(x_0,z_0)}{\partial x_1^{\lambda-1}}\cdot\mathcal{C}_{\neq 1}\right)\frac{(\lambda-2)!}{(\lambda-1)!}\frac{1}{(\lambda-1)!}\left(\sum_{s=1}^{\lambda-1}\binom{\lambda-1}{s}\frac{\partial^s m^+(X_i,z_0)}{\partial x_l^s}\frac{\partial^{\lambda-1-s} f^+(X_i,z_0)}{\partial x_l^{\lambda-1-s}}\right) \end{array} \right\}$$

With some further calculations these terms simplify to

$$= h_z h_x^L \left\{ h_z^2 \frac{\bar{\mu}_2^2 - \bar{\mu}_1\bar{\mu}_3}{2\tilde{\mu}} \frac{\partial^2 m^+(X_i,z_0)}{\partial z^2} + h_x^\lambda \eta_\lambda \sum_{l=1}^{L} \left\{ \frac{\partial^\lambda m^+(X_i,z_0)}{\lambda!\cdot\partial x_l^\lambda} + \sum_{s=1}^{\lambda-1} \frac{\partial^s m^+(X_i,z_0)}{\partial x_l^s}\omega_s^+ \right\} + O(h_z^3 + h_z^2 h_x + h_x^{\lambda+1}) \right\} \tag{20}$$

where $\omega_s^+ = \left\{ \frac{\partial^{\lambda-s} f^+(X_i,z_0)}{s!(\lambda-s)!\cdot\partial x_l^{\lambda-s}} - \frac{\partial^{\lambda-1} f^+(x_0,z_0)}{\partial x_1^{\lambda-1}} \cdot \left(\frac{\partial^{\lambda-2} f^+(x_0,z_0)}{\partial x_l^{\lambda-2}}\right)^{-1} \frac{(\lambda-2)!}{(\lambda-1)!s!(\lambda-1-s)!} \frac{\partial^{\lambda-1-s} f^+(X_i,z_0)}{\partial x_l^{\lambda-1-s}} \right\} / f^+(X_i,z_0)$. Notice that $\omega_1 = 0$.

When a second order kernel $\lambda=2$ is used throughout, the expression simplifies to

$$h_z h_x^L \left\{ h_z^2 \frac{\bar{\mu}_2^2 - \bar{\mu}_1\bar{\mu}_3}{2\tilde{\mu}} \frac{\partial^2 m^+(X_i,z_0)}{\partial z^2} + h_x^2 \mu_2 \sum_{l=1}^{L} \frac{1}{2}\frac{\partial^2 m^+(X_i,z_0)}{\partial x_l^2} + O(h_z^3 + h_z^2 h_x + h_x^3) \right\}. \tag{21}$$

With this intermediate result, the total bias can be computed as

$$E\left[\varsigma_{ij}\right] = EE\left[\varsigma_{ij}|X_i, Y_i, Z_i\right]$$

$$= \frac{1}{h_z h_x^L} E\left[h_z h_x^L \left\{ h_z^2 \frac{\bar{\mu}_2^2 - \bar{\mu}_1\bar{\mu}_3}{2\tilde{\mu}} \frac{\partial^2 m^+(X_i,z_0)}{\partial z^2} + h_x^\lambda \eta_\lambda \sum_{l=1}^{L} \left\{ \frac{\partial^\lambda m^+(X_i,z_0)}{\lambda!\cdot\partial x_l^\lambda} + \sum_{s=1}^{\lambda-1} \frac{\partial^s m^+(X_i,z_0)}{\partial x_l^s}\omega_s^+ \right\} \right\} \cdot K_h^*\left(\frac{Z_i - z_0}{h}\right) \right] (1 + o_p(1))$$

$$= \int\int \left\{ h_z^2 \frac{\bar{\mu}_2^2 - \bar{\mu}_1\bar{\mu}_3}{2\tilde{\mu}} \frac{\partial^2 m^+(X_i,z_0)}{\partial z^2} + h_x^\lambda \eta_\lambda \sum_{l=1}^{L} \left\{ \frac{\partial^\lambda m^+(X_i,z_0)}{\lambda!\cdot\partial x_l^\lambda} + \sum_{s=1}^{\lambda-1} \frac{\partial^s m^+(X_i,z_0)}{\partial x_l^s}\omega_s^+ \right\} \right\} \cdot K_h^*\left(\frac{Z_i - z_0}{h}\right) f(X_i, Z_i) dX_i dZ_i$$

$$= \int\int \left\{ h_z^2 \frac{\bar{\mu}_2^2 - \bar{\mu}_1\bar{\mu}_3}{2\tilde{\mu}} \frac{\partial^2 m^+(X_i,z_0)}{\partial z^2} + h_x^\lambda \eta_\lambda \sum_{l=1}^{L} \left\{ \frac{\partial^\lambda m^+(X_i,z_0)}{\lambda!\cdot\partial x_l^\lambda} + \sum_{s=1}^{\lambda-1} \frac{\partial^s m^+(X_i,z_0)}{\partial x_l^s}\omega_s^+ \right\} \right\} \cdot \frac{\bar{\mu}_2 - \bar{\mu}_1 |u|}{2\tilde{\mu} f(z_0)} \kappa(u) f(X_i, z_0 + uh) du dX_i,$$

where $u = \frac{Z_i - z_0}{h}$. We can split the integral into two parts: for $u > 0$ and for $u \le 0$ and expand $f(X_i, z_0 + uh)$ about $f(X_i, z_0)$ in the two separate integrals to obtain after some calculations:

$$= \int \left\{ h_z^2 \frac{\bar{\mu}_2^2 - \bar{\mu}_1\bar{\mu}_3}{2\tilde{\mu}} \frac{\partial^2 m^+(X_i,z_0)}{\partial z^2} + h_x^\lambda \eta_\lambda \sum_{l=1}^{L} \left\{ \frac{\partial^\lambda m^+(X_i,z_0)}{\lambda!\cdot\partial x_l^\lambda} + \sum_{s=1}^{\lambda-1} \frac{\partial^s m^+(X_i,z_0)}{\partial x_l^s}\omega_s^+ \right\} \right\} \cdot \frac{f^-(X_i,z_0) + f^+(X_i,z_0)}{2f(z_0)} dX_i.$$

Combining this result with the analogous derivations for the second part in (17a) and with the bias term (17c) and repeating the derivations for the term $\hat{\Gamma} - \Gamma$ and combining everything in (15) gives as total bias of the $\hat{\gamma}_{RDD}$ estimator:

$$\frac{1}{\Gamma} h^2 \frac{\bar{\mu}_2^2 - \bar{\mu}_1\bar{\mu}_3}{4\tilde{\mu} f(z_0)} \int \left(m^+(x,z_0) - m^-(x,z_0) - \gamma\left(d^+(x,z_0) - d^-(x,z_0)\right)\right) \left(\frac{\partial^2 f^+}{\partial z^2}(x,z_0) + \frac{\partial^2 f^-}{\partial z^2}(x,z_0)\right) dx \tag{22}$$

$$+ \frac{1}{\Gamma} h_z^2 \frac{\bar{\mu}_2^2 - \bar{\mu}_1\bar{\mu}_3}{2\tilde{\mu}} \int \left(\frac{\partial^2 m^+(x,z_0)}{\partial z^2} - \frac{\partial^2 m^-(x,z_0)}{\partial z^2} - \gamma\frac{\partial^2 d^+(x,z_0)}{\partial z^2} + \gamma\frac{\partial^2 d^-(x,z_0)}{\partial z^2} \right) \cdot \frac{f^-(x,z_0) + f^+(x,z_0)}{2f(z_0)} dx$$

$$+ \frac{1}{\Gamma} h_x^\lambda \eta_\lambda \int \sum_{l=1}^{L} \left\{ \frac{\partial^\lambda m^+(x,z_0)}{\lambda!\cdot\partial x_l^\lambda} + \sum_{s=1}^{\lambda-1} \frac{\partial^s m^+(x,z_0)}{\partial x_l^s}\omega_s^+ - \frac{\partial^\lambda m^-(x,z_0)}{\lambda!\cdot\partial x_l^\lambda} - \sum_{s=1}^{\lambda-1} \frac{\partial^s m^-(x,z_0)}{\partial x_l^s}\omega_s^- \right\} \cdot \frac{f^-(x,z_0) + f^+(x,z_0)}{2f(z_0)} dx$$

$$- \frac{\gamma}{\Gamma} h_x^\lambda \eta_\lambda \int \sum_{l=1}^{L} \left\{ \frac{\partial^\lambda d^+(x,z_0)}{\lambda!\cdot\partial x_l^\lambda} + \sum_{s=1}^{\lambda-1} \frac{\partial^s d^+(x,z_0)}{\partial x_l^s}\omega_s^+ - \frac{\partial^\lambda d^-(x,z_0)}{\lambda!\cdot\partial x_l^\lambda} - \sum_{s=1}^{\lambda-1} \frac{\partial^s d^-(x,z_0)}{\partial x_l^s}\omega_s^- \right\} \cdot \frac{f^-(x,z_0) + f^+(x,z_0)}{2f(z_0)} dx.$$

E.2 Calculation of the variance term

For the subsequent derivations we define the $L+2$ column vector $\mathbb{X}$ as in Lemma 1 as

$$\mathbb{X}_j = \left(1, \frac{Z_j - z_0}{h_z}, \left(\frac{X_j - x_0}{h_x}\right)'\right)'$$

and define

$$\mathbb{X}_{j,X_i} = \left(1, \frac{Z_j - z_0}{h_z}, \left(\frac{X_j - X_i}{h_x}\right)'\right)'.$$

For calculating the variance we first examine the first part of (17a). (The results for the second part in (17a) are analogous.) As a preliminary we will also need the term

$$E\left[\left\{e_1' A_+^{-1}(X_j) \cdot \mathbb{X}_{i,X_j} \left(Y_i - m^+(X_j, z_0)\right) K_{i,X_j}\right\} K_h^*\left(\frac{Z_j - z_0}{h}\right) | X_i, Y_i, Z_i\right] \tag{23}$$

which we can split into two parts by multiplying with $1(Z_j \leq z_0) + 1(Z_j > z_0) = 1$. Examine first the part

$$E\left[1(Z_j \leq z_0)\left\{e_1' A_+^{-1}(X_j)\mathbb{X}_{i,X_j}\left(Y_i - m^+(X_j, z_0)\right) K_{i,X_j}\right\} K_h^*\left(\frac{Z_j - z_0}{h}\right) | X_i, Y_i, Z_i\right],$$

where the derivations for $1(Z_j > z_0)$ are analogous. We obtain

$$= h_x^L \int\int 1(u \leq 0)\left\{e_1' A_+^{-1}(X_i + vh_x)\mathbb{X}_{i,X_i+vh_x}\left(Y_i - m^+(X_i + vh_x, z_0)\right) K_{i,X_i+vh_x}\right\} \frac{\bar{\mu}_2 - \bar{\mu}_1 |u|}{2\tilde{\mu} f(z_0)} \kappa(u) \cdot f(X_i + vh_x, z_0 + uh) dv du,$$

where $v = \frac{X_j - X_i}{h_x}$ and $u = \frac{Z_j - z_0}{h}$, and after inserting the expression for $e_1' A_+^{-1}$

$$= \quad h_x^L \int\int \frac{1(u \leq 0)}{f^+(X_i + vh_x, z_0) \cdot \mathcal{C}} \begin{pmatrix} \frac{1}{\tilde{\mu}}\left(\bar{\mu}_2 \mathcal{C} + h_z \cdot \left(\bar{\mu}_3 \frac{\partial f^+(X_i + vh_x, z_0)}{f^+(X_i + vh_x, z_0) \cdot \partial z} \mathcal{C} + 2\bar{\mu}_2 \bar{\mu}_1 \mathcal{A}\right) + h_x \bar{\mu}_2 \frac{\eta_{\lambda+1}}{\eta_\lambda} \frac{(\lambda-2)!}{(\lambda-1)!} \mathcal{B}\right) \\ -\frac{1}{\tilde{\mu}}\left(\bar{\mu}_1 \mathcal{C} + h_z \cdot \left(\bar{\mu}_2 \frac{\partial f^+(X_i + vh_x, z_0)}{f^+(X_i + vh_x, z_0) \cdot \partial z} \mathcal{C} + 2\bar{\mu}_1^2 \mathcal{A}\right) + h_x \bar{\mu}_1 \frac{\eta_{\lambda+1}}{\eta_\lambda} \frac{(\lambda-2)!}{(\lambda-1)!} \mathcal{B}\right) \\ -2h_x \left(\frac{\partial^{\lambda-1} f^+(X_i + vh_x, z_0)}{\partial x_1^{\lambda-1}} \cdot \mathcal{C}_{\neq 1}\right) \frac{(\lambda-2)!}{(\lambda-1)!} \\ \vdots \\ -2h_x \left(\frac{\partial^{\lambda-1} f^+(X_i + vh_x, z_0)}{\partial x_L^{\lambda-1}} \cdot \mathcal{C}_{\neq L}\right) \frac{(\lambda-2)!}{(\lambda-1)!} \end{pmatrix}' \begin{pmatrix} 1 \\ \frac{Z_i - z_0}{h_z} \\ -v_1 \\ \vdots \\ -v_L \end{pmatrix}$$

$$\cdot \left(Y_i - m^+(X_i + vh_x, z_0)\right) \kappa\left(\frac{Z_i - z_0}{h_z}\right) \cdot \prod_{l=1}^{L} \bar{\kappa}(-v_l) \cdot \frac{\bar{\mu}_2 - \bar{\mu}_1 |u|}{2\tilde{\mu} f(z_0)} \kappa(u) \cdot f(X_i + vh_x, z_0 + uh) dv du,$$

and a Taylor series expansion about (X_i, z_0) we obtain:

$$= h_x^L \frac{Y_i - m^+(X_i, z_0)}{2\tilde{\mu} f(z_0)} \kappa\left(\frac{Z_i - z_0}{h_z}\right)\left(\bar{\mu}_2 - \bar{\mu}_1 \left|\frac{Z_i - z_0}{h_z}\right|\right) \frac{f^-(X_i, z_0)}{f^+(X_i, z_0)} \cdot (1 + O(h_z + h_x)).$$

Complementing these calculations with those for $1(Z_j > z_0)$ gives an expression for (23) as:

$$= h_x^L \frac{Y_i - m^+(X_i, z_0)}{2\tilde{\mu} f(z_0)} \kappa\left(\frac{Z_i - z_0}{h_z}\right)\left(\bar{\mu}_2 - \bar{\mu}_1 \left|\frac{Z_i - z_0}{h_z}\right|\right) \frac{f^-(X_i, z_0) + f^+(X_i, z_0)}{f^+(X_i, z_0)} \cdot (1 + O(h_z + h_x)).$$

With these results we obtain the following for the expression $E[\varsigma_{ij} + \varsigma_{ji} | X_i, Y_i, Z_i]$ in (19)

$$\begin{aligned} & E[\varsigma_{ij} + \varsigma_{ji} | X_i, Y_i, Z_i] \\ = \; & \frac{1}{h_z h_x^L} E\left[e_1' A_+^{-1}(X_i)\mathbb{X}_{j,X_i}\left(Y_j - m^+(X_i, z_0)\right) K_{j,X_i} I_j^+ | X_i, Y_i, Z_i\right] \cdot K_h^*\left(\frac{Z_i - z_0}{h}\right)(1 + o_p(1)) \\ & + \frac{1}{h_z h_x^L} E\left[\left\{e_1' A_+^{-1}(X_j)\mathbb{X}_{i,X_j}\left(Y_i - m^+(X_j, z_0)\right) K_{i,X_j}(1 + o_p(1))\right\} K_h^*\left(\frac{Z_j - z_0}{h}\right) | X_i, Y_i, Z_i\right] \cdot I_i^+ \end{aligned}$$

$$= \left\{ h_z^2 \frac{\bar{\mu}_2^2 - \bar{\mu}_1\bar{\mu}_3}{2\tilde{\mu}} \frac{\partial^2 m^+(X_i, z_0)}{\partial z^2} + h_x^\lambda \eta_\lambda \sum_{l=1}^{L} \left\{ \frac{\partial^\lambda m^+(X_i, z_0)}{\lambda! \cdot \partial x_l^\lambda} + \sum_{s=1}^{\lambda-1} \frac{\partial^s m^+(X_i, z_0)}{\partial x_l^s} \omega_s \right\} + O(h_z^3 + h_z^2 h_x + h_x^{\lambda+1}) \right\} \cdot K_h^* \left(\frac{Z_i - z_0}{h} \right)$$
$$+ \frac{1}{h_z} \frac{Y_i - m^+(X_i, z_0)}{2\tilde{\mu} f(z_0)} \kappa \left(\frac{Z_i - z_0}{h_z} \right) \left(\bar{\mu}_2 - \bar{\mu}_1 \left| \frac{Z_i - z_0}{h_z} \right| \right) \frac{f^-(X_i, z_0) + f^+(X_i, z_0)}{f^+(X_i, z_0)} \cdot (1 + O(h_z + h_x)) \cdot I_i^+.$$

For the following variance calculation, the second term, which is mean zero and of order $\frac{1}{h_z} \cdot (1 + O(h_z + h_x))$ clearly dominates the first term, such that we can ignore the first term in the following.

Now the variance of (19) can be approximated as follows:

$$Var \left(\frac{2}{n} \sum_{i=1}^{n} \left(E\left[\frac{\varsigma_{ij} + \varsigma_{ji}}{2} | X_i, Y_i, Z_i \right] - E\left[\frac{\varsigma_{ij} + \varsigma_{ji}}{2} \right] \right) \right)$$
$$= \frac{1}{n} Var \left(E\left[\varsigma_{ij} + \varsigma_{ji} | X_i, Y_i, Z_i \right] - E\left[\varsigma_{ij} + \varsigma_{ji} \right] \right)$$
$$= \frac{1}{n} E\left[\left(\frac{1}{h_z} \frac{Y_i - m^+(X_i, z_0)}{2\tilde{\mu} f(z_0)} \kappa \left(\frac{Z_i - z_0}{h_z} \right) \left(\bar{\mu}_2 - \bar{\mu}_1 \left| \frac{Z_i - z_0}{h_z} \right| \right) \frac{f^-(X_i, z_0) + f^+(X_i, z_0)}{f^+(X_i, z_0)} \cdot (1 + O(h_z + h_x)) \cdot I_i^+ \right)^2 \right]$$
$$= E\left[\frac{(Y_i - m^+(X_i, Z_i) + m^+(X_i, Z_i) - m^+(X_i, z_0))^2}{n h_z^2 \cdot 4\tilde{\mu}^2 f^2(z_0)} \kappa^2 \left(\frac{Z_i - z_0}{h_z} \right) \frac{(f^-(X_i, z_0) + f^+(X_i, z_0))^2}{f^+(X_i, z_0)^2} \left(\bar{\mu}_2 - \bar{\mu}_1 \frac{Z_i - z_0}{h_z} \right)^2 I_i^+ \right]$$
$$\times (1 + O(h_z + h_x))$$

$$= \frac{1}{n h_z^2} h_z \int\int \frac{E\left[(Y_i - m^+(X_i, Z_i))^2 | X_i, Z_i = z_0 + uh \right] + (m^+(X_i, z_0 + uh) - m^+(X_i, z_0))^2}{4\tilde{\mu}^2 f^2(z_0)} \kappa^2(u) \frac{(f^-(X_i, z_0) + f^+(X_i, z_0))^2}{f^+(X_i, z_0)^2}$$
$$(\bar{\mu}_2 - \bar{\mu}_1 u)^2 \cdot 1(u > 0) \cdot f(X_i, z_0 + uh) du dX_i \cdot (1 + O(h_z + h_x))$$

where $u = \frac{Z_i - z_0}{h_z}$ and with a series expansion about z_0 we obtain

$$= \frac{\bar{\mu}_2^2 \ddot{\mu}_0 - 2\bar{\mu}_2 \bar{\mu}_1 \ddot{\mu}_1 + \bar{\mu}_1^2 \ddot{\mu}_2}{n h_z} \int \frac{\sigma_Y^{2+}(x, z_0)}{4\tilde{\mu}^2 f^2(z_0)} \frac{(f^-(x, z_0) + f^+(x, z_0))^2}{f^+(x, z_0)} dx \cdot (1 + O(h_z + h_x + h))$$

where $\sigma_Y^{2+}(X, z_0) = \lim_{\varepsilon \to 0} E\left[(Y - m^+(X, Z))^2 | X, Z = z_0 + \varepsilon \right]$.

Collecting all terms (17a), (17b) and (17c), the variance of $\hat{\Delta} - \Delta$ can be approximated. The term (17c) can be ignored as it is a nonstochastic bias term. We thus obtain that $Var\left(\hat{\Delta} - \Delta\right)$

$$= \frac{1}{n} Var \left\{ \left\{ \hat{m}^+(X_i, z_0) - m^+(X_i, z_0) \right\} K_h^* \left(\frac{Z_i - z_0}{h} \right) \right\}$$
$$+ \frac{1}{n} Var \left\{ \left\{ \hat{m}^-(X_i, z_0) - m^-(X_i, z_0) \right\} K_h^* \left(\frac{Z_i - z_0}{h} \right) \right\}$$
$$+ \frac{1}{n} Var \left\{ \left\{ m^+(X_i, z_0) - m^-(X_i, z_0) \right\} K_h^* \left(\frac{Z_i - z_0}{h} \right) - E\left[\left\{ m^+(X, z_0) - m^-(X, z_0) \right\} K_h^* \left(\frac{Z - z_0}{h} \right) \right] \right\}$$
$$+ \frac{1}{n} Cov \left\{ \left\{ \hat{m}^+(X_i, z_0) - m^+(X_i, z_0) \right\} K_h^* \left(\frac{Z_i - z_0}{h} \right), \left\{ \hat{m}^-(X_i, z_0) - m^-(X_i, z_0) \right\} K_h^* \left(\frac{Z_i - z_0}{h} \right) \right\}$$
$$+ \frac{1}{n} Cov \left\{ \left\{ \hat{m}^+(X_i, z_0) - m^+(X_i, z_0) \right\} K_h^* \left(\frac{Z_i - z_0}{h} \right), \left\{ m^+(X_i, z_0) - m^-(X_i, z_0) \right\} K_h^* \left(\frac{Z_i - z_0}{h} \right) \right\}$$
$$+ \frac{1}{n} Cov \left\{ \left\{ \hat{m}^-(X_i, z_0) - m^-(X_i, z_0) \right\} K_h^* \left(\frac{Z_i - z_0}{h} \right), \left\{ m^+(X_i, z_0) - m^-(X_i, z_0) \right\} K_h^* \left(\frac{Z_i - z_0}{h} \right) \right\}$$

$$
\begin{aligned}
= & \frac{1}{nh_z}\frac{\bar{\mu}_2^2\ddot{\mu}_0-2\bar{\mu}_2\bar{\mu}_1\ddot{\mu}_1+\bar{\mu}_1^2\ddot{\mu}_2}{4\tilde{\mu}^2 f^2(z_0)}\int\sigma_Y^{2+}(x,z_0)\frac{\left(f^-(x,z_0)+f^+(x,z_0)\right)^2}{f^+(x,z_0)}dx\cdot\left(1+O\left(h_z+h_x+h\right)\right) \qquad (25)\\
&+\frac{1}{nh_z}\frac{\bar{\mu}_2^2\ddot{\mu}_0-2\bar{\mu}_2\bar{\mu}_1\ddot{\mu}_1+\bar{\mu}_1^2\ddot{\mu}_2}{4\tilde{\mu}^2 f^2(z_0)}\int\sigma_Y^{2-}(x,z_0)\frac{\left(f^-(x,z_0)+f^+(x,z_0)\right)^2}{f^-(x,z_0)}dx\cdot\left(1+O\left(h_z+h_x+h\right)\right)\\
&+\frac{1}{nh}\frac{\bar{\mu}_2^2\ddot{\mu}_0-2\bar{\mu}_2\bar{\mu}_1\ddot{\mu}_1+\bar{\mu}_1^2\ddot{\mu}_2}{4\tilde{\mu}^2 f^2(z_0)}\int\left(m^+(x,z_0)-m^-(x,z_0)\right)^2\left(f^+(x,z_0)+f^-(x,z_0)\right)dx\cdot(1+O(h))-O\left(\frac{1}{n}\right)+o\left(\frac{1}{nh_z}\right)
\end{aligned}
$$

where the following terms have been plugged in

$$
\begin{aligned}
&\frac{1}{n}Var\left\{\left\{m^+(X_i,z_0)-m^-(X_i,z_0)\right\}K_h^*\left(\frac{Z_i-z_0}{h}\right)-E\left[\left\{m^+(X,z_0)-m^-(X,z_0)\right\}K_h^*\left(\frac{Z-z_0}{h}\right)\right]\right\}\\
=\ &\frac{1}{n}E\left[\left\{m^+(X_i,z_0)-m^-(X_i,z_0)\right\}^2K_h^*\left(\frac{Z_i-z_0}{h}\right)^2\right]-\frac{1}{n}\left(E\left[\left\{m^+(X,z_0)-m^-(X,z_0)\right\}K_h^*\left(\frac{Z-z_0}{h}\right)\right]\right)^2
\end{aligned}
$$

$$
\begin{aligned}
=\ &\frac{1}{n}E\left[\left(\left\{m^+(X_i,z_0)-m^-(X_i,z_0)\right\}\frac{\bar{\mu}_2-\bar{\mu}_1\left|\frac{Z_i-z_0}{h}\right|}{2\tilde{\mu}f(z_0)}K_h\left(\frac{Z_i-z_0}{h}\right)\right)^2\right]\\
&-\frac{1}{n}\left(E\left[\left\{m^+(X,z_0)-m^-(X,z_0)\right\}\frac{\bar{\mu}_2-\bar{\mu}_1\left|\frac{Z_i-z_0}{h}\right|}{2\tilde{\mu}f(z_0)}K_h\left(\frac{Z_i-z_0}{h}\right)\right]\right)^2
\end{aligned}
$$

$$
\begin{aligned}
=\ &\frac{1}{n}\int\int\left\{m^+(X_i,z_0)-m^-(X_i,z_0)\right\}^2\left(\frac{\bar{\mu}_2-\bar{\mu}_1|u|}{2\tilde{\mu}f(z_0)}\right)^2\frac{1}{h^2}h\kappa^2(u)\,f(X_i,z_0+uh)dudX_i\\
&-\frac{1}{n}\left(\int\int\left(\left\{m^+(X,z_0)-m^-(X,z_0)\right\}\frac{\bar{\mu}_2-\bar{\mu}_1|u|}{2\tilde{\mu}f(z_0)}\frac{1}{h}h\kappa(u)\,f(X,z_0+uh)dudX\right)\right)^2
\end{aligned}
$$

where $u=\frac{Z_i-z_0}{h}$. Splitting the integrals into the parts $Z>z_0$ and $Z<z_0$ and with a series expansion about (X,z_0) we obtain

$$
=\frac{1}{nh}\frac{\left(\bar{\mu}_2^2\ddot{\mu}_0-2\bar{\mu}_2\bar{\mu}_1\ddot{\mu}_1+\bar{\mu}_1^2\ddot{\mu}_2\right)}{4\tilde{\mu}^2 f^2(z_0)}\int\left(m^+(X_i,z_0)-m^-(X_i,z_0)\right)^2\left(f^+(X_i,z_0)+f^-(X_i,z_0)\right)dX_i\cdot(1+O(h))-O\left(\frac{1}{n}\right)
$$

Further, the following two terms are of lower order:

$$
Cov\left\{\frac{1}{n}\sum_{i=1}^n\left\{\hat{m}^+(X_i,z_0)-m^+(X_i,z_0)\right\}K_h^*\left(\frac{Z_i-z_0}{h}\right),\frac{1}{n}\sum_{i=1}^n\left\{m^+(X_i,z_0)-m^-(X_i,z_0)\right\}K_h^*\left(\frac{Z_i-z_0}{h}\right)\right\}
$$

$$
=Cov\Big(\frac{1}{n}\sum_{i=1}^n\left(E\left[\varsigma_{ij}+\varsigma_{ji}|X_i,Y_i,Z_i\right]-E\left[\varsigma_{ij}+\varsigma_{ji}\right]\right)+E\left[\frac{\varsigma_{ij}+\varsigma_{ji}}{2}\right]+o_p(1),\frac{1}{n}\sum_{i=1}^n\left\{m^+(X_i,z_0)-m^-(X_i,z_0)\right\}K_h^*\left(\frac{Z_i-z_0}{h}\right)\Big)
$$

$$
\begin{aligned}
=\ &\frac{1}{n}E\Big[\left\{\left(E\left[\varsigma_{ij}+\varsigma_{ji}|X_i,Y_i,Z_i\right]-E\left[\varsigma_{ij}+\varsigma_{ji}\right]\right)+E\left[\frac{\varsigma_{ij}+\varsigma_{ji}}{2}\right]\right\}\\
&\cdot\left\{\left\{m^+(X_i,z_0)-m^-(X_i,z_0)\right\}K_h^*\left(\frac{Z_i-z_0}{h}\right)-E\left[\left\{m^+(X,z_0)-m^-(X,z_0)\right\}K_h^*\left(\frac{Z-z_0}{h}\right)\right]\right\}\Big]\left(1+o_p(1)\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{n}E\left[\left(E\left[\varsigma_{ij}+\varsigma_{ji}|X_i,Y_i,Z_i\right]-E\left[\varsigma_{ij}+\varsigma_{ji}\right]\right)+E\left[\frac{\varsigma_{ij}+\varsigma_{ji}}{2}\right]\right]\\
&\qquad E\left[\left\{m^+(X_i,z_0)-m^-(X_i,z_0)\right\}K_h^*\left(\frac{Z_i-z_0}{h}\right)-E\left[\left\{m^+(X,z_0)-m^-(X,z_0)\right\}K_h^*\left(\frac{Z-z_0}{h}\right)\right]\right]
\end{aligned}
$$

$$
\begin{aligned}
=\ &\frac{1}{nh_z}\int\int\frac{m(X_i,Z_i)-m^+(X_i,z_0)}{2\tilde{\mu}f(z_0)}\kappa\left(\frac{Z_i-z_0}{h_z}\right)\left(\bar{\mu}_2-\bar{\mu}_1\left|\frac{Z_i-z_0}{h_z}\right|\right)\frac{f^-(X_i,z_0)+f^+(X_i,z_0)}{f^+(X_i,z_0)}\cdot\left(1+O\left(h_z+h_x\right)\right)\cdot I_i^+\\
&\cdot\left\{\left\{m^+(X_i,z_0)-m^-(X_i,z_0)\right\}K_h^*\left(\frac{Z_i-z_0}{h}\right)-E\left[\left\{m^+(X,z_0)-m^-(X,z_0)\right\}K_h^*\left(\frac{Z-z_0}{h}\right)\right]\right\}f(X_i,Z_i)dX_idZ_i
\end{aligned}
$$

$$= \frac{1}{nh_z} \int \int \frac{m(X_i, z_0 + uh) - m^+(X_i, z_0)}{2\tilde{\mu} f(z_0)} \kappa \left(u \frac{h}{h_z} \right) \left(\bar{\mu}_2 - \bar{\mu}_1 |u| \frac{h}{h_z} \right) \frac{f^-(X_i, z_0) + f^+(X_i, z_0)}{f^+(X_i, z_0)} \cdot (1 + O(h_z + h_x)) \cdot 1(u > 0)$$
$$\cdot \left\{ \{ m^+(X_i, z_0) - m^-(X_i, z_0) \} \frac{\bar{\mu}_2 - \bar{\mu}_1 |u|}{2\tilde{\mu} f(z_0)} \frac{1}{h} \kappa(u) - E\left[\{ m^+(X, z_0) - m^-(X, z_0) \} K_h^* \left(\frac{Z - z_0}{h} \right) \right] \right\} f(X_i, z_0 + uh) dX_i h du$$

where $u = \frac{Z_i - z_0}{h}$ and a Taylor series expansion about (X_i, z_0) gives

$$= \frac{1}{nh_z} (0 + O(h + h_z + h_x)) = o\left(\frac{1}{nh_z} \right).$$

Also, the following term is of lower order

$$Cov\left\{ \frac{1}{n} \sum_{i=1}^n \{ \hat{m}^+(X_i, z_0) - m^+(X_i, z_0) \} K_h^* \left(\frac{Z_i - z_0}{h} \right), \frac{1}{n} \sum_{i=1}^n \{ \hat{m}^-(X_i, z_0) - m^-(X_i, z_0) \} K_h^* \left(\frac{Z_i - z_0}{h} \right) \right\}$$

$$= \frac{1}{n} E\left\{ \{ \hat{m}^+(X_i, z_0) - m^+(X_i, z_0) \} K_h^* \left(\frac{Z_i - z_0}{h} \right) \{ \hat{m}^-(X_i, z_0) - m^-(X_i, z_0) \} K_h^* \left(\frac{Z_i - z_0}{h} \right) \right\}$$
$$- \frac{1}{n} E\left[\{ \hat{m}^+(X_i, z_0) - m^+(X_i, z_0) \} K_h^* \left(\frac{Z_i - z_0}{h} \right) \right] E\left[\{ \hat{m}^-(X_i, z_0) - m^-(X_i, z_0) \} K_h^* \left(\frac{Z_i - z_0}{h} \right) \right]$$

$$= \frac{1}{n} E[\left[\left(\left(\frac{1}{h_z} \frac{Y_i - m^+(X_i, z_0)}{2\tilde{\mu} f(z_0)} \kappa \left(\frac{Z_i - z_0}{h_z} \right) \left(\bar{\mu}_2 - \bar{\mu}_1 \left| \frac{Z_i - z_0}{h_z} \right| \right) \frac{f^-(X_i, z_0) + f^+(X_i, z_0)}{f^+(X_i, z_0)} \cdot (1 + O(h_z + h_x)) \cdot I_i^+ \right) - E\left[\varsigma_{ij}^+ \right] \right) \right]$$
$$\cdot \left(\left(\frac{1}{h_z} \frac{Y_i - m^-(X_i, z_0)}{2\tilde{\mu} f(z_0)} \kappa \left(\frac{Z_i - z_0}{h_z} \right) \left(\bar{\mu}_2 - \bar{\mu}_1 \left| \frac{Z_i - z_0}{h_z} \right| \right) \frac{f^-(X_i, z_0) + f^+(X_i, z_0)}{f^-(X_i, z_0)} \cdot (1 + O(h_z + h_x)) \cdot I_i^- \right) - E\left[\varsigma_{ij}^- \right] \right)]$$
$$- \frac{1}{n} O\left(h_x^\lambda + h_z^2 \right) O\left(h_x^\lambda + h_z^2 \right)$$

$$= 0 + \frac{1}{nh_z} O\left(h_x^\lambda + h_z^2 \right) O(h_z + h_x) - \frac{1}{n} O\left(h_x^\lambda + h_z^2 \right) O\left(h_x^\lambda + h_z^2 \right) = o\left(\frac{1}{nh_z} \right),$$

because $I_i^+ \cdot I_i^- = 1(Z_i > z_0) \cdot 1(Z_i < z_0) = 0$.

The variance $Var(\hat{\Gamma} - \Gamma)$ can be derived analogously to the variance $Var(\hat{\Delta} - \Delta)$. The missing piece to the total variance term is the covariance between $(\hat{\Delta} - \Delta)$ and $(\hat{\Gamma} - \Gamma)$, which is derived in the following. Using the terms (17a) and (17b) and the corresponding terms for $\hat{\Gamma} - \Gamma$, the covariance is

$$Cov\left(\hat{\Delta} - \Delta, -\gamma \left(\hat{\Gamma} - \Gamma \right) \right)$$

$$
\begin{aligned}
= \quad & -\gamma Cov\left(\frac{1}{n}\sum_{i=1}^{n}\left\{\hat{m}^{+}(X_i,z_0)-m^{+}(X_i,z_0)\right\}K_h^*\left(\frac{Z_i-z_0}{h}\right),\frac{1}{n}\sum_{i=1}^{n}\left\{\hat{d}^{+}(X_i,z_0)-d^{+}(X_i,z_0)\right\}K_h^*\left(\frac{Z_i-z_0}{h}\right)\right)\\
& +\gamma Cov\left(\frac{1}{n}\sum_{i=1}^{n}\left\{\hat{m}^{+}(X_i,z_0)-m^{+}(X_i,z_0)\right\}K_h^*\left(\frac{Z_i-z_0}{h}\right),\frac{1}{n}\sum_{i=1}^{n}\left\{\hat{d}^{-}(X_i,z_0)-d^{-}(X_i,z_0)\right\}K_h^*\left(\frac{Z_i-z_0}{h}\right)\right)\\
& -\gamma Cov\left(\frac{1}{n}\sum_{i=1}^{n}\left\{\hat{m}^{+}(X_i,z_0)-m^{+}(X_i,z_0)\right\}K_h^*\left(\frac{Z_i-z_0}{h}\right),\frac{1}{n}\sum_{i=1}^{n}\left\{d^{+}(X_i,z_0)-d^{-}(X_i,z_0)\right\}K_h^*\left(\frac{Z_i-z_0}{h}\right)\right)\\
& +\gamma Cov\left(\frac{1}{n}\sum_{i=1}^{n}\left\{\hat{m}^{-}(X_i,z_0)-m^{-}(X_i,z_0)\right\}K_h^*\left(\frac{Z_i-z_0}{h}\right),\frac{1}{n}\sum_{i=1}^{n}\left\{\hat{d}^{+}(X_i,z_0)-d^{+}(X_i,z_0)\right\}K_h^*\left(\frac{Z_i-z_0}{h}\right)\right)\\
& -\gamma Cov\left(\frac{1}{n}\sum_{i=1}^{n}\left\{\hat{m}^{-}(X_i,z_0)-m^{-}(X_i,z_0)\right\}K_h^*\left(\frac{Z_i-z_0}{h}\right),\frac{1}{n}\sum_{i=1}^{n}\left\{\hat{d}^{-}(X_i,z_0)-d^{-}(X_i,z_0)\right\}K_h^*\left(\frac{Z_i-z_0}{h}\right)\right)\\
& +\gamma Cov\left(\frac{1}{n}\sum_{i=1}^{n}\left\{\hat{m}^{-}(X_i,z_0)-m^{-}(X_i,z_0)\right\}K_h^*\left(\frac{Z_i-z_0}{h}\right),\frac{1}{n}\sum_{i=1}^{n}\left\{d^{+}(X_i,z_0)-d^{-}(X_i,z_0)\right\}K_h^*\left(\frac{Z_i-z_0}{h}\right)\right)\\
& -\gamma Cov\left(\frac{1}{n}\sum_{i=1}^{n}\left\{m^{+}(X_i,z_0)-m^{-}(X_i,z_0)\right\}K_h^*\left(\frac{Z_i-z_0}{h}\right),\frac{1}{n}\sum_{i=1}^{n}\left\{\hat{d}^{+}(X_i,z_0)-d^{+}(X_i,z_0)\right\}K_h^*\left(\frac{Z_i-z_0}{h}\right)\right)\\
& +\gamma Cov\left(\frac{1}{n}\sum_{i=1}^{n}\left\{m^{+}(X_i,z_0)-m^{-}(X_i,z_0)\right\}K_h^*\left(\frac{Z_i-z_0}{h}\right),\frac{1}{n}\sum_{i=1}^{n}\left\{\hat{d}^{-}(X_i,z_0)-d^{-}(X_i,z_0)\right\}K_h^*\left(\frac{Z_i-z_0}{h}\right)\right)\\
& -\gamma Cov\left(\frac{1}{n}\sum_{i=1}^{n}\left\{m^{+}(X_i,z_0)-m^{-}(X_i,z_0)\right\}K_h^*\left(\frac{Z_i-z_0}{h}\right),\frac{1}{n}\sum_{i=1}^{n}\left\{d^{+}(X_i,z_0)-d^{-}(X_i,z_0)\right\}K_h^*\left(\frac{Z_i-z_0}{h}\right)\right).
\end{aligned}
$$

These terms are calculated in the following:

Term 1:

$$
\begin{aligned}
& -\gamma Cov\left(\frac{1}{n}\sum_{i=1}^{n}\left\{\hat{m}^{+}(X_i,z_0)-m^{+}(X_i,z_0)\right\}K_h^*\left(\frac{Z_i-z_0}{h}\right),\frac{1}{n}\sum_{i=1}^{n}\left\{\hat{d}^{+}(X_i,z_0)-d^{+}(X_i,z_0)\right\}K_h^*\left(\frac{Z_i-z_0}{h}\right)\right)\\
&= -\gamma\frac{1}{n}E\left[\left\{\hat{m}^{+}(X_i,z_0)-m^{+}(X_i,z_0)\right\}K_h^*\left(\frac{Z_i-z_0}{h}\right)\left\{\hat{d}^{+}(X_i,z_0)-d^{+}(X_i,z_0)\right\}K_h^*\left(\frac{Z_i-z_0}{h}\right)\right]+\frac{1}{n}O(h_z^2+h_x^{\lambda})O(h_z^2+h_x^{\lambda})
\end{aligned}
$$

$$
\begin{aligned}
&= -\gamma\frac{1}{n}E[\frac{1}{h_z}\frac{Y_i-m^{+}(X_i,z_0)}{2\tilde{\mu}f(z_0)}\kappa\left(\frac{Z_i-z_0}{h_z}\right)\left(\bar{\mu}_2-\bar{\mu}_1\left|\frac{Z_i-z_0}{h_z}\right|\right)\frac{f^{-}(X_i,z_0)+f^{+}(X_i,z_0)}{f^{+}(X_i,z_0)}\cdot\left(1+O\left(h_z+h_x\right)\right)\cdot I_i^{+}\\
&\quad\cdot\frac{1}{h_z}\frac{D_i-d^{+}(X_i,z_0)}{2\tilde{\mu}f(z_0)}\kappa\left(\frac{Z_i-z_0}{h_z}\right)\left(\bar{\mu}_2-\bar{\mu}_1\left|\frac{Z_i-z_0}{h_z}\right|\right)\frac{f^{-}(X_i,z_0)+f^{+}(X_i,z_0)}{f^{+}(X_i,z_0)}\cdot\left(1+O\left(h_z+h_x\right)\right)\cdot I_i^{+}]+\frac{1}{n}O(h_z^2+h_x^{\lambda})^2\\
&= -\gamma\frac{1}{n}\frac{1}{h_z^2}E\left[\left(Y_i-m^{+}(X_i,z_0)\right)\frac{D_i-d^{+}(X_i,z_0)}{4\tilde{\mu}^2f^2(z_0)}\kappa^2\left(\frac{Z_i-z_0}{h_z}\right)\left(\bar{\mu}_2-\bar{\mu}_1\left|\frac{Z_i-z_0}{h_z}\right|\right)^2\left(\frac{f^{-}(X_i,z_0)+f^{+}(X_i,z_0)}{f^{+}(X_i,z_0)}\right)^2 I_i^{+}\right]
\end{aligned}
$$

plus terms of lower order. Further derivations give

$$
\begin{aligned}
&= -\gamma\frac{1}{n}\frac{1}{h_z^2}E[\left(Y_i-m^{+}(X_i,Z_i)+m^{+}(X_i,Z_i)-m^{+}(X_i,z_0)\right)\\
&\qquad\frac{D_i-d^{+}(X_i,Z_i)+d^{+}(X_i,Z_i)-d^{+}(X_i,z_0)}{4\tilde{\mu}^2f^2(z_0)}\kappa^2\left(\frac{Z_i-z_0}{h_z}\right)\left(\bar{\mu}_2-\bar{\mu}_1\left|\frac{Z_i-z_0}{h_z}\right|\right)^2\left(\frac{f^{-}(X_i,z_0)+f^{+}(X_i,z_0)}{f^{+}(X_i,z_0)}\right)^2 I_i^{+}]
\end{aligned}
$$

$$
\begin{aligned}
&= -\gamma\frac{1}{n}\frac{1}{h_z^2}E[\left(\frac{E\left[\left(Y-m^{+}(X,Z)\right)\left(D-d^{+}(X,Z)\right)|X,Z\right]}{4\tilde{\mu}^2f^2(z_0)}+\left(m^{+}(X_i,Z_i)-m^{+}(X_i,z_0)\right)\frac{d^{+}(X_i,Z_i)-d^{+}(X_i,z_0)}{4\tilde{\mu}^2f^2(z_0)}\right)\\
&\qquad\kappa^2\left(\frac{Z_i-z_0}{h_z}\right)\left(\bar{\mu}_2-\bar{\mu}_1\left|\frac{Z_i-z_0}{h_z}\right|\right)^2\left(\frac{f^{-}(X_i,z_0)+f^{+}(X_i,z_0)}{f^{+}(X_i,z_0)}\right)^2 I_i^{+}]
\end{aligned}
$$

$$
\begin{aligned}
&= \int\int\left(\frac{E\left[\left(Y-m^{+}(X,Z)\right)\left(D-d^{+}(X,Z)\right)|X,Z=z_0+uh\right]}{4\tilde{\mu}^2f^2(z_0)}+\left(m^{+}(X_i,z_0+uh)-m^{+}(X_i,z_0)\right)\frac{d^{+}(X_i,z_0+uh)-d^{+}(X_i,z_0)}{4\tilde{\mu}^2f^2(z_0)}\right)\\
&\qquad\cdot\frac{-\gamma}{nh_z}\cdot\kappa^2\left(u\right)\left(\bar{\mu}_2-\bar{\mu}_1\left|u\right|\right)^2\left(\frac{f^{-}(X_i,z_0)+f^{+}(X_i,z_0)}{f^{+}(X_i,z_0)}\right)^2\cdot 1\left(u>0\right)f(X_i,z_0+uh)dudX_i
\end{aligned}
$$

where $u = \frac{Z_i - z_0}{h_z}$ and with a series expansion about z_0 we obtain

$$= -\gamma \frac{\bar{\mu}_2^2 \ddot{\mu}_0 - 2\bar{\mu}_2 \bar{\mu}_1 \ddot{\mu}_1 + \bar{\mu}_1^2 \ddot{\mu}_2}{nh_z} \int \frac{\sigma_{YD}^{2+}(X, z_0)}{4\tilde{\mu}^2 f^2(z_0)} \frac{\left(f^-(X_i, z_0) + f^+(X_i, z_0)\right)^2}{f^+(X_i, z_0)} dX_i,$$

where $\sigma_{YD}^{2+}(X, z_0) = \lim_{\varepsilon \to 0} E\left[(Y - m^+(X, Z))\left(D - d^+(X, Z)\right)|X, Z = z_0 + \varepsilon\right]$.

Term 2:

$$\gamma Cov\left(\frac{1}{n}\sum_{i=1}^{n}\left\{\hat{m}^+(X_i, z_0) - m^+(X_i, z_0)\right\} K_h^*\left(\frac{Z_i - z_0}{h}\right), \frac{1}{n}\sum_{i=1}^{n}\left\{\hat{d}^-(X_i, z_0) - d^-(X_i, z_0)\right\} K_h^*\left(\frac{Z_i - z_0}{h}\right)\right)$$

$$= \gamma \frac{1}{n} E\left[\left\{\hat{m}^+(X_i, z_0) - m^+(X_i, z_0)\right\} K_h^*\left(\frac{Z_i - z_0}{h}\right) \cdot \left\{\hat{d}^-(X_i, z_0) - d^-(X_i, z_0)\right\} K_h^*\left(\frac{Z_i - z_0}{h}\right)\right] - \frac{1}{n} O\left(h_x^\lambda + h_z^2\right)^2$$

$$= 0 + \frac{1}{n} O(h_z^2 + h_x^\lambda) \frac{1}{h_z} O\left(h_z + h_x\right) - \frac{1}{n} O\left(h_x^\lambda + h_z^2\right)^2 = o\left(\frac{1}{nh_z}\right),$$

because $I_i^+ \cdot I_i^- = 1(Z_i > z_0) \cdot 1(Z_i < z_0) = 0$.

Term 3:

$$-\gamma Cov\left(\frac{1}{n}\sum_{i=1}^{n}\left\{\hat{m}^+(X_i, z_0) - m^+(X_i, z_0)\right\} K_h^*\left(\frac{Z_i - z_0}{h}\right), \frac{1}{n}\sum_{i=1}^{n}\left\{d^+(X_i, z_0) - d^-(X_i, z_0)\right\} K_h^*\left(\frac{Z_i - z_0}{h}\right)\right)$$

$$= \frac{-\gamma}{n} E[\left\{\hat{m}^+(X_i, z_0) - m^+(X_i, z_0)\right\} K_h^*\left(\frac{Z_i - z_0}{h}\right)$$
$$\cdot \left\{\left\{d^+(X_i, z_0) - d^-(X_i, z_0)\right\} K_h^*\left(\frac{Z_i - z_0}{h}\right) - E\left[\left\{d^+(X_i, z_0) - d^-(X_i, z_0)\right\} K_h^*\left(\frac{Z_i - z_0}{h}\right)\right]\right\}]$$

$$= \frac{-\gamma}{n} E[(E\left[\varsigma_{ij} + \varsigma_{ji}|X_i, Y_i, Z_i\right] - E\left[\varsigma_{ij}\right])$$
$$\cdot \left\{\left\{d^+(X_i, z_0) - d^-(X_i, z_0)\right\} K_h^*\left(\frac{Z_i - z_0}{h}\right) - E\left[\left\{d^+(X_i, z_0) - d^-(X_i, z_0)\right\} K_h^*\left(\frac{Z_i - z_0}{h}\right)\right]\right\}]$$

$$= \frac{-\gamma}{nh_z} \int\int \frac{m(X_i, Z_i) - m^+(X_i, z_0)}{2\tilde{\mu} f(z_0)} \kappa\left(\frac{Z_i - z_0}{h_z}\right)\left(\bar{\mu}_2 - \bar{\mu}_1\left|\frac{Z_i - z_0}{h_z}\right|\right) \frac{f^-(X_i, z_0) + f^+(X_i, z_0)}{f^+(X_i, z_0)} \cdot \left(1 + O\left(h_z + h_x\right)\right) \cdot I_i^+$$
$$\cdot \left\{\left\{d^+(X_i, z_0) - d^-(X_i, z_0)\right\} K_h^*\left(\frac{Z_i - z_0}{h}\right) - E\left[\left\{d^+(X_i, z_0) - d^-(X_i, z_0)\right\} K_h^*\left(\frac{Z_i - z_0}{h}\right)\right]\right\} \cdot f(X_i, Z_i) dX_i dZ_i$$

$$= \frac{-\gamma}{nh_z} \int\int \frac{m(X_i, z_0 + uh) - m^+(X_i, z_0)}{2\tilde{\mu} f(z_0)} \kappa\left(u\frac{h}{h_z}\right)\left(\bar{\mu}_2 - \bar{\mu}_1 |u| \frac{h}{h_z}\right) \frac{f^-(X_i, z_0) + f^+(X_i, z_0)}{f^+(X_i, z_0)} \cdot \left(1 + O\left(h_z + h_x\right)\right) \cdot 1\left(u > 0\right)$$
$$\cdot \left\{\left\{d^+(X_i, z_0) - d^-(X_i, z_0)\right\} \frac{\bar{\mu}_2 - \bar{\mu}_1 u}{2\tilde{\mu} f(z_0)} \frac{1}{h} \kappa\left(u\right) - E\left[\left\{d^+(X_i, z_0) - d^-(X_i, z_0)\right\} K_h^*\left(\frac{Z_i - z_0}{h}\right)\right]\right\} \cdot f(X_i, z_0 + uh) dX_i h du$$

where $u = \frac{Z_i - z_0}{h}$ and a Taylor series expansion about (X_i, z_0) gives

$$= \frac{-\gamma}{nh_z}\left(0 + O\left(h + h_z + h_x\right)\right) = o\left(\frac{1}{nh_z}\right).$$

Term 4: The derivations are analogous to Term 2.

Term 5: The derivations are analogous to Term 1 and yield the expression

$$= -\gamma \frac{\bar{\mu}_2^2 \ddot{\mu}_0 - 2\bar{\mu}_2 \bar{\mu}_1 \ddot{\mu}_1 + \bar{\mu}_1^2 \ddot{\mu}_2}{nh_z} \int \frac{\sigma_{YD}^{2-}(X, z_0)}{4\tilde{\mu}^2 f^2(z_0)} \frac{\left(f^-(X_i, z_0) + f^+(X_i, z_0)\right)^2}{f^-(X_i, z_0)} dX_i,$$

Term 6,7 and 8: The derivations are analogous to Term 3.

Term 9:

$$-\gamma Cov\left(\frac{1}{n}\sum_{i=1}^{n}\left\{m^{+}(X_i,z_0)-m^{-}(X_i,z_0)\right\}K_h^{*}\left(\frac{Z_i-z_0}{h}\right),\frac{1}{n}\sum_{i=1}^{n}\left\{d^{+}(X_i,z_0)-d^{-}(X_i,z_0)\right\}K_h^{*}\left(\frac{Z_i-z_0}{h}\right)\right)$$

$$=-\gamma\frac{1}{n}Cov\left(\left\{m^{+}(X_i,z_0)-m^{-}(X_i,z_0)\right\}K_h^{*}\left(\frac{Z_i-z_0}{h}\right),\left\{d^{+}(X_i,z_0)-d^{-}(X_i,z_0)\right\}K_h^{*}\left(\frac{Z_i-z_0}{h}\right)\right)$$

$$=-\gamma\frac{1}{n}E\left[\left\{m^{+}(X_i,z_0)-m^{-}(X_i,z_0)\right\}K_h^{*}\left(\frac{Z_i-z_0}{h}\right)\cdot\left\{d^{+}(X_i,z_0)-d^{-}(X_i,z_0)\right\}K_h^{*}\left(\frac{Z_i-z_0}{h}\right)\right]$$
$$+\gamma\frac{1}{n}E\left[\left\{m^{+}(X_i,z_0)-m^{-}(X_i,z_0)\right\}K_h^{*}\left(\frac{Z_i-z_0}{h}\right)\right]E\left[\left\{d^{+}(X_i,z_0)-d^{-}(X_i,z_0)\right\}K_h^{*}\left(\frac{Z_i-z_0}{h}\right)\right]$$

$$=-\gamma\frac{1}{n}\int\int\left\{m^{+}(X_i,z_0)-m^{-}(X_i,z_0)\right\}\left\{d^{+}(X_i,z_0)-d^{-}(X_i,z_0)\right\}\frac{(\bar{\mu}_2-\bar{\mu}_1\left|u\right|)^2}{4\tilde{\mu}^2f^2(z_0)}\frac{1}{h^2}\kappa^2(u)f(X_i,z_0+uh)dX_ihdu$$
$$+\gamma\frac{1}{n}\left(\int\int\left\{m^{+}(X_i,z_0)-m^{-}(X_i,z_0)\right\}\frac{\bar{\mu}_2-\bar{\mu}_1\left|u\right|}{2\tilde{\mu}f(z_0)}\frac{1}{h}\kappa(u)f(X_i,z_0+uh)dX_ihdu\right)$$
$$\left(\int\int\left\{d^{+}(X_i,z_0)-d^{-}(X_i,z_0)\right\}\frac{\bar{\mu}_2-\bar{\mu}_1\left|u\right|}{2\tilde{\mu}f(z_0)}\frac{1}{h}\kappa(u)f(X_i,z_0+uh)dX_ihdu\right)$$

where $u=\frac{Z_i-z_0}{h}$. Splitting the integrals into the part $1(Z>z_0)$ and $1(Z<z_0)$ and using a Taylor series expansion about (X_i,z_0) gives

$$=\frac{-\gamma}{nh}\int\int\left\{m^{+}(X_i,z_0)-m^{-}(X_i,z_0)\right\}\left\{d^{+}(X_i,z_0)-d^{-}(X_i,z_0)\right\}\frac{\bar{\mu}_2^2\ddot{\mu}_0-2\bar{\mu}_1\bar{\mu}_2\ddot{\mu}_1+\bar{\mu}_1^2\ddot{\mu}_2}{4\tilde{\mu}^2f^2(z_0)}\left(f^{-}(X_i,z_0)+f^{+}(X_i,z_0)\right)dX_i+O\left(\frac{1}{n}\right).$$

Having computed all variance and covariance terms, we obtain the first order term of the variance of $\hat{\gamma}_{RDD}-\gamma$ using (16) as

$$Var(\hat{\gamma}_{RDD})=\frac{1}{\Gamma^2}\left(Var\left(\hat{\Delta}-\Delta\right)-2\gamma Cov\left(\hat{\Delta}-\Delta,\hat{\Gamma}-\Gamma\right)+\gamma^2Var\left(\hat{\Gamma}-\Gamma\right)\right)\cdot(1+o_p(1))$$

$$=\frac{\bar{\mu}_2^2\ddot{\mu}_0-2\bar{\mu}_2\bar{\mu}_1\ddot{\mu}_1+\bar{\mu}_1^2\ddot{\mu}_2}{\Gamma^2 4\tilde{\mu}^2f^2(z_0)}$$
$$\times\Bigg(\frac{1}{nh_z}\int\left(\frac{\sigma_Y^{2+}(x,z_0)-2\gamma\sigma_{YD}^{2+}(x,z_0)+\gamma^2\sigma_D^{2+}(x,z_0)}{f^{+}(x,z_0)}+\frac{\sigma_Y^{2-}(x,z_0)-2\gamma\sigma_{YD}^{2-}(x,z_0)+\gamma^2\sigma_D^{2-}(x,z_0)}{f^{-}(x,z_0)}\right)\left(f^{+}(x,z_0)+f^{-}(x,z_0)\right)^2dx$$
$$+\frac{1}{nh}\int\left\{m^{+}(x,z_0)-\gamma d^{+}(x,z_0)-m^{-}(x,z_0)+\gamma d^{-}(x,z_0)\right\}^2\cdot\left(f^{+}(x,z_0)+f^{-}(x,z_0)\right)dx\Bigg),$$

where $\sigma_{YD}^{2+}(X,z_0)=\lim_{\varepsilon\to 0}E\left[\left(Y-m^{+}(X,Z)\right)\left(D-d^{+}(X,Z)\right)|X,Z=z_0+\varepsilon\right]$
and $\sigma_D^{2+}(X,z_0)=\lim_{\varepsilon\to 0}E\left[\left(D-d^{+}(X,Z)\right)^2|X,Z=z_0+\varepsilon\right]$.

Having derived asymptotic bias and variance, the asymptotic normality follows from Theorem A of Serfling (1980, page 192) for U and V-statistics. This follows as we have shown above that $E\left[\left(\varsigma_{ij}+\varsigma_{ji}\right)^2\right]<\infty$ and that the U-statistic is not degenerate, such that by the projection theorem the first order term can be written as a sum of independent and identically distributed random variables.

F Proof of Theorem 4

To simplify the derivations, it is helpful to define a random variable $W = Y - \gamma D$ and define $w^+(X,z) = \lim_{\varepsilon \to 0} E\left[W|X, Z = z + \varepsilon\right]$ and

$$\sigma_W^{2+}(X,z) = \lim_{\varepsilon \to 0} E\left[\left(W - w^+(X,Z)\right)^2 |X, Z = z + \varepsilon\right],$$

and analogously for w^- and σ_W^{2-}. The asymptotic variance can then be written as

$$\mathcal{V}_{RDD} = \frac{\bar{\mu}_2^2\ddot{\mu}_0 - 2\bar{\mu}_2\bar{\mu}_1\ddot{\mu}_1 + \bar{\mu}_1^2\ddot{\mu}_2}{\Gamma^2 4\tilde{\mu}^2 f^2(z_0)} \times \Big(\frac{1}{r_z}\int \left(\frac{\sigma_W^{2+}(x,z_0)}{f^+(x,z_0)} + \frac{\sigma_W^{2-}(x,z_0)}{f^-(x,z_0)}\right)\left(f^+(x,z_0) + f^-(x,z_0)\right)^2 dx$$
$$+ \int \left\{w^+(x,z_0) - w^-(x,z_0)\right\}^2 \cdot \left(f^+(x,z_0) + f^-(x,z_0)\right) dx \quad \Big).$$

Now consider the asymptotic distribution when not controlling for X. Obviously, if Assumption 1 is only valid with conditioning on X, the estimator would clearly be inconsistent. Now, suppose the instrumental variables assumptions hold with and without conditioning on X, and consider the asymptotic distribution without controlling for any X regressors. In the following, all functions *without* an x argument are defined as the respective conditional expectation without conditioning on X, e.g. $w^+(z) = \lim_{\varepsilon \to 0} E\left[W|Z = z + \varepsilon\right]$. First, if Assumption 1 is valid when X is the empty set, this implies that $w^+(z_0) - w^-(z_0) = 0$ as can be shown by a few simple calculations. Furthermore, continuity of $f(z)$ near z_0 implies that $f^+(z_0) = f^-(z_0)$. Repeating all the previous derivations of Theorem 3 for this case without regressors, one would obtain the following asymptotic variance matrix

$$\mathcal{V}_{no\,X} = \frac{\bar{\mu}_2^2\ddot{\mu}_0 - 2\bar{\mu}_2\bar{\mu}_1\ddot{\mu}_1 + \bar{\mu}_1^2\ddot{\mu}_2}{\Gamma^2\tilde{\mu}^2 f(z_0) r_z} \cdot \left(\sigma_W^{2+}(z_0) + \sigma_W^{2-}(z_0)\right).$$

The asymptotic bias term would be:

$$\mathcal{B}_{no\,X} = \frac{r r_z^2}{\Gamma}\frac{\bar{\mu}_2^2 - \bar{\mu}_1\bar{\mu}_3}{2\tilde{\mu}}\left(\frac{\partial^2 w^+(z_0)}{\partial z^2} - \frac{\partial^2 w^-(z_0)}{\partial z^2}\right).$$

Now we can compare $\mathcal{V}_{no\,X}$ to $\mathcal{V}_{RDD}$. Note that we can write

$$\begin{aligned} E\left[\left(W - w^+(Z)\right)^2 |Z\right] &= E\left[E\left[\left(W - w^+(X,Z) + w^+(X,Z) - w^+(Z)\right)^2 |X,Z\right] |Z\right] \\ &= E\left[E\left[\left(W - w^+(X,Z)\right)^2 |X,Z\right] |Z\right] + E\left[E\left[\left(w^+(X,Z) - w^+(Z)\right)^2 |X,Z\right] |Z\right] \end{aligned}$$

which, after taking limits and using intermediate results from the proof of Theorem 1, gives

$$\sigma_W^{2+}(z_0) = \int \sigma_W^{2+}(x,z_0)\frac{f^-(x|z_0) + f^+(x|z_0)}{2}dx + \int \left(w^+(x,z_0) - w^+(z_0)\right)^2 \frac{f^-(x|z_0) + f^+(x|z_0)}{2}dx.$$

With this preliminary we consider the difference $\mathcal{V}_{RDD} - \mathcal{V}_{no\,X}$, where for notational convenience we premultiply with the common scaling factor

$$\left(\frac{\bar{\mu}_2^2\ddot{\mu}_0 - 2\bar{\mu}_2\bar{\mu}_1\ddot{\mu}_1 + \bar{\mu}_1^2\ddot{\mu}_2}{\Gamma^2\tilde{\mu}^2 f(z_0) r_z}\right)^{-1}\left(\mathcal{V}_{RDD} - \mathcal{V}_{no\,X}\right) = \int \left(\frac{\sigma_W^{2+}(x,z_0)}{f^+(x|z_0)} + \frac{\sigma_W^{2-}(x,z_0)}{f^-(x|z_0)}\right)\frac{\left(f^+(x|z_0) + f^-(x|z_0)\right)^2}{4}dx$$
$$+ \frac{r_z}{4}\int \left\{w^+(x,z_0) - w^-(x,z_0)\right\}^2 \cdot \left(f^+(x|z_0) + f^-(x|z_0)\right) dx - \left(\sigma_W^{2+}(z_0) + \sigma_W^{2-}(z_0)\right)$$

$$= \int \left(\frac{\sigma_W^{2+}(x,z_0)}{f^+(x|z_0)} + \frac{\sigma_W^{2-}(x,z_0)}{f^-(x|z_0)}\right)\frac{\left(f^+(x|z_0) + f^-(x|z_0)\right)^2}{4}dx$$
$$+ \frac{r_z}{4}\int \left\{w^+(x,z_0) - w^-(x,z_0)\right\}^2 \cdot \left(f^+(x|z_0) + f^-(x|z_0)\right) dx$$
$$- \int \sigma_W^{2+}(x,z_0)\frac{f^-(x|z_0) + f^+(x|z_0)}{2}dx - \int \left(w^+(x,z_0) - w^+(z_0)\right)^2 \frac{f^-(x|z_0) + f^+(x|z_0)}{2}dx$$
$$- \int \sigma_W^{2-}(x,z_0)\frac{f^-(x|z_0) + f^+(x|z_0)}{2}dx - \int \left(w^-(x,z_0) - w^-(z_0)\right)^2 \frac{f^-(x|z_0) + f^+(x|z_0)}{2}dx.$$

Using the assumption that $f^+(x,z_0) = f^-(x,z_0)$ this expression simplifies to

$$= \int \left(\frac{r_z}{2} \left\{ w^+(x,z_0) - w^-(x,z_0) \right\}^2 - \left(w^+(x,z_0) - w^+(z_0) \right)^2 - \left(w^-(x,z_0) - w^-(z_0) \right)^2 \right) f(x|z_0) dx$$

$$= \int \Big(\frac{r_z}{2} \left\{ \left(w^+(x,z_0) - w^+(z_0) \right) - \left(w^-(x,z_0) - w^-(z_0) \right) + \left(w^+(z_0) - w^-(z_0) \right) \right\}^2 - \left(w^+(x,z_0) - w^+(z_0) \right)^2 - \left(w^-(x,z_0) - w^-(z_0) \right)^2 \Big) f(x|z_0) dx$$

$$= \int \Big(\frac{r_z}{2} \left\{ \left(w^+(x,z_0) - w^+(z_0) \right) - \left(w^-(x,z_0) - w^-(z_0) \right) + \left(w^+(z_0) - w^-(z_0) \right) \right\}^2 - \left(w^+(x,z_0) - w^+(z_0) \right)^2 - \left(w^-(x,z_0) - w^-(z_0) \right)^2 \Big) f(x|z_0) dx.$$

Now using that $w^+(z_0) - w^-(z_0) = 0$ we obtain

$$= \frac{r_z - 2}{2} \int \left\{ w^+(x,z_0) - w^+(z_0) \right\}^2 f(x|z_0) dx + \frac{r_z - 2}{2} \int \left\{ w^-(x,z_0) - w^-(z_0) \right\}^2 f(x|z_0) dx - r_z \int \left(w^+(x,z_0) - w^+(z_0) \right) \left(w^-(x,z_0) - w^-(z_0) \right) f(x|z_0) dx$$

$$= \frac{r_z - 2}{2} V^+ + \frac{r_z - 2}{2} V^- - r_z R \sqrt{V^+ V^-}, \tag{27}$$

where V^+ is defined as $V^+ = Var(w^+|z_0) = \int \left\{ w^+(x,z_0) - w^+(z_0) \right\}^2 f(x|z_0) dx$ and V^- analogously and C is defined as the covariance $\int \left(w^+(x,z_0) - w^+(z_0) \right) \left(w^-(x,z_0) - w^-(z_0) \right) f(x|z_0) dx$ and $R = \frac{C}{\sqrt{V^+V^-}}$ is the correlation coefficient. If $V^+ = V^- = 0$, then $\mathcal{V}_{RDD}$ and $\mathcal{V}_{no\,X}$ are identical. Otherwise, we consider conditions under which (27) is negative for *every* value of V^+ and V^-. A first observation is that (27) is *not* bounded from above if $r_z \geq 2$. Now, assume $r_z < 2$ in the following. It follows immediately that if $R \geq 0$, then (27) is negative.

Finally, consider the case when $r_z < 2$ and $R < 0$. We can write (27) as:

$$= V^- \left(\frac{r_z - 2}{2} \frac{V^+}{V^-} + \frac{r_z - 2}{2} - r_z R \sqrt{\frac{V^+}{V^-}} \right), \tag{28}$$

which has a global *maximum* at $\sqrt{V^+/V^-} = \frac{r_z R}{r_z - 2}$. Evaluating (28) at this maximum gives

$$V^- \left(\frac{r_z - 2}{2} \left(\frac{r_z R}{r_z - 2} \right)^2 + \frac{r_z - 2}{2} - \frac{r_z^2 R^2}{r_z - 2} \right)$$

$$= \frac{1}{2} \frac{V^-}{r_z - 2} \left(r_z^2 \left(1 - R^2 \right) - 4 r_z + 4 \right)$$

$$= \frac{1}{2} \frac{V^-}{r_z - 2} \left(1 - R^2 \right) \left(r_z - 2 \frac{1+R}{1-R^2} \right) \left(r_z - 2 \frac{1-R}{1-R^2} \right)$$

which is *negative* if $r_z < 2$ and

$$r_z < 2 \frac{1+R}{1-R^2}.$$

Finally, if $R = -1$, the expression (28) is negative only for $r_z < 1$.

References

SERFLING, R. (1980): *Approximation Theorems of Mathematical Statistics.* Wiley, New York.