Supplemental Material for Zhao et al., "A Novel Hybrid Combination Optimization Algorithm Based on Search Area Segmentation and Fast Fourier Transform," *Engineering Optimization*, 2018.

S1. Convergence analysis of HSAS/FFT

Convergence of the HSAS/FFT is proved in this part. The evolutionary process of HSAS/FFT can be regarded as a series of the stochastic sequence. The stochastic sequence is used to analyse the process of HSAS/FFT. And the convergence of the HSAS/FFT is proved by the criterion in (Peng and Xie 2012). Meanwhile, two theorems are presented to prove the convergence of HSAS/FFT.

Definition 1. $S = I^D$ is the search space, and $f: S \to I^+$ is fitness. I is the set of spaces divided by period that frequency corresponds to, a^* denotes the global optimal area. Then the processing of the divided area which is searched by HSAS/FFT can be described as $\{a \in I | f(a^*) = \min f(a)\}$.

Lemma 1. (Peng and Xie 2012) if a sequence is monotonous and no ascending as well as has lower bound, it must possess a limit.

Lemma 2. (Peng and Xie 2012) if a sequence is monotonous and no ascending as well as has lower bound, its subsequence must possess a limit.

Theorem 1. The direction of the population movement is monotonous, that is $f(A(n+1)) \leq f(A(n))$, thus the sequence $\{f(a^{(n)})\}$ is monotonous and no ascending as well as has lower bound.

Theorem 2. HSAS/FFT can converge to the subspace with global optimum with the probability of 1.

Theorem 1. Proof. According to the theory of HSAS/FFT, the direction of individuals toward optima is the maximum of gradient descent. If there is no gradient descent in the neighbourhood, individuals will stop moving. Therefore, the fitness of the population is not ascending that is $f(A(n+1)) \leq f(A(n))$. In consequence, there exists $f(a^{(n+1)}) \leq f(a^{(n)})$, where $a^{(n)}$ presents the minimum of n-th $(n = 1,2,\ldots,k)$ time iteration, k is the maximum number of iteration and sufficiently large, $f(a^{(n)})$ is the fitness of individual, $f(A(n))$ is the fitness of the population. It has $f(a^{(1)}) \geq f(a^{(2)}) \geq \cdots \geq f(a^{(n)}) \geq \cdots$, thus $\{f(a^{(n)})\}$ is monotonic sequence. Definition 1 shows that the optimum problems exist global optimum, and so mean of sampling points in the subspace $\{f(a^{(n)})\}$ is bounded. Therefore, $\{f(a^{(n)})\}$ is monotonic, no ascending and bounded sequence.

Theorem 2. Proof. According to Theorem 1, $\{f(a^{(n)})\}$ is monotonic, non-ascending and bounded sequence, and then $\{f(a^{(n)})\}$ must possess a limit with Lemmas 1 and 2. If $\lim_{n\to+\infty} f(a^{(n)}) = f(a^*)$ exists and a^* is the subspace with global optimum, and then $\{f(a^{(n)})\}$ is globally convergent. $\lim_{n\to+\infty} f(a^{(n)}) = f(a^*)$ is random event, as $\{f(a^{(n)})\}$ is stochastic sequence. So when $P(\lim_{n\to+\infty} f(a^{(n)}) = f(a^*)) = 1$ is true, and then the sequence of $\{f(a^{(n)})\}$ can converge with the probability of 1. It has been proved as follows.

For $\forall\varepsilon \geq 0$, there exists $T_\varepsilon = \{a \in D, f(a) - f(a^*) < \varepsilon\}$ and the monotonic, non-ascending and bounded sequence $\{f(a^{(n)})\}$ which is $f(a^{(1)}) \geq f(a^{(2)}) \geq \cdots \geq f(a^{(n)}) \geq \cdots$. Let $f(a^{(1)}) \geq f(a^{(2)}) \geq \cdots \geq f(a^{(n)}) \geq \cdots$ subtract $f(a^*)$, so $f(a^{(1)}) - f(a^*) \geq f(a^{(2)}) - f(a^*) \geq \cdots \geq f(a^{(n)}) - f(a^*) \geq \cdots$. Let $C = \{a^{(i)} \in T_\varepsilon, i \in (1,2,\ldots k)\}$ represent the iterated sequence which stuck into the neighborhood T_ε for $i-th$ times. Thus, there exists $C_1 \subseteq C_2 \subseteq$

$\cdots \subseteq C_i \subseteq \cdots$ for ε, thereby the inequality $P(C_1) \leq P(C_2) \leq \cdots \leq P(C_i) \leq \cdots$ is true. And as $0 \leq P(C_1) \leq 1$, $\lim_{i \to +\infty} P(C_i)$ is existent.

The stochastic sequence is presented by:

$$\zeta_i = \begin{cases} 1, & i\text{th iteration drop into } \mathrm{T}_\varepsilon \\ 0, & i\text{th iteration do not drop into } \mathrm{T}_\varepsilon \end{cases} \quad i = 1,2,\cdots,k \tag{1}$$

and $C_i = \{\zeta_i = 1\}$. Let $P\{\zeta_i = 1\} = q_i$, $P\{\zeta_i = 0\} = 1 - q_i$. Therefore, $B_i = \frac{1}{i}\sum_{j=1}^{i} \zeta_j$, there exists:

$$E(B_i) = \frac{1}{i}\sum_{j=1}^{i} q_j\,, \quad i = 1,2,\cdots,k \tag{2}$$

$$D(B_i) = \frac{1}{i^2}\sum_{j=1}^{i} D(\zeta_i) = \frac{1}{i^2}\sum_{j=1}^{i} q_j\left(1 - q_j\right), \; i = 1,2,\cdots,k \tag{3}$$

where $E(B_i)$, $D(B_i)$ are the expected value and standard deviation of the sequence $B_i(i = 1,2,\cdots,k)$, respectively. And due to Chebyshev inequality, there exists:

$$P\{|B_i - E(B_i) < \varepsilon|\} \geq 1 - \frac{D(B_i)}{\varepsilon^2} \tag{4}$$

and $q_j(1 - q_j) \leq \frac{1}{4}$ is true in equation (9), therefore,

$$P\{|B_i - E(B_i)| < \varepsilon\} \geq 1 - \frac{1}{4i\varepsilon^2} \tag{5}$$

$$\lim_{i \to +\infty} P\{|B_i - E(B_i)| < \varepsilon\} = 1 \tag{6}$$

and because of $\zeta_i = iB_i - (i-1)B_{i-1}$, $i = 1,2,\cdots,k$, there exists:

$$\lim_{i \to +\infty} P\{|\zeta_i - E(\zeta_i)| < \varepsilon\} = 1 \tag{7}$$

The sequence of stochastic variables $\zeta_i \ (i = 1,2,\cdots,k)$ converges with the probability, so that the sequence of stochastic event $B_i \ (i = 1,2,\cdots,k)$ also converges with the probability, that is $\lim_{i\to+\infty} P\{B_i\} = 1$. Therefore, there exists:

$$\lim_{n\to+\infty} P\{|f(A(n)) - f(A^*)| < \varepsilon\} = 1 \tag{8}$$

For $\forall\varepsilon \geq 0$, when ε tend to very small, that is:

$$\lim_{\varepsilon\to 0}|f(a^{(n)}) - f(a^*)| = 0 \tag{9}$$

$$\lim_{\varepsilon\to 0} f(a^{(n)}) = f(a^*) \tag{10}$$

Thus,

$$\lim_{n\to+\infty} P\{f(a^{(n)}) = f(a^*)\} = 1 \tag{11}$$

Namely, the sequence $\{f(a^{(n)})\}$ can converge to the global optimum with the probability of 1.

S2. Description of CEC2017 benchmark set

To assess the performance of the proposed algorithm on single objective real-parameter numerical optimization, thirty benchmark functions on the CEC 2017 test suite are employed in the following experiments. A brief description of these functions with different characteristics which used to conduct the performance analysis of the proposed algorithms is listed in Table 1. $f_1 - f_3$ are unimodal functions. Simple Multimodal Functions consists of seven functions from $f_4 - f_{10}$ are multimodal functions with a lot of local optima, so it is easy to trap into local optima. Considering that in the real-world optimization problems, different subcomponents of the variables may have different properties, therefore functions from $f_{11} - f_{20}$ are proposed as the hybrid functions. The remaining functions of the test suite are composition functions. All these benchmark functions are evaluated as the minimization problems. More details about the definition of these functions can be found in the literature (Awad *et al.* 2016).

Table 1. Summary of the CEC 2017 test functions.

Class	No.	Functions	$F_i^* = F_i(x^*)$
Unimodal Functions	1	Shifted and Rotated Bent Cigar Function	100
	2	Shifted and Rotated Sum of Different Power Function	200
	3	Shifted and Rotated Zakharov Function	300
Simple Multimodal Functions	4	Shifted and Rotated Rosenbrock's Function	400
	5	Shifted and Rotated Rastrigin's Function	500
	6	Shifted and Rotated Expanded Scaffer's F6 Function	600
	7	Shifted and Rotated Lunacek Bi_Rastrigin Function	700
	8	Shifted and Rotated Non-Continuous Rastrigin's Function	800
	9	Shifted and Rotated Levy Function	900
	10	Shifted and Rotated Schwefel's Function	1000
Hybrid Functions	11	Hybrid Function 1 (*N*=3)	1100
	12	Hybrid Function 2 (*N*=3)	1200
	13	Hybrid Function 3 (*N*=3)	1300
	14	Hybrid Function 4 (*N*=4)	1400
	15	Hybrid Function 5 (*N*=4)	1500
	16	Hybrid Function 6 (*N*=4)	1600
	17	Hybrid Function 6 (*N*=5)	1700
	18	Hybrid Function 6 (*N*=5)	1800
	19	Hybrid Function 6 (*N*=5)	1900
	20	Hybrid Function 6 (*N*=6)	2000
Composition Functions	21	Composition Function 1 (*N*=3)	2100
	22	Composition Function 2 (*N*=3)	2200
	23	Composition Function 3 (*N*=4)	2300
	24	Composition Function 4 (*N*=4)	2400
	25	Composition Function 5 (*N*=5)	2500
	26	Composition Function 6 (*N*=5)	2600
	27	Composition Function 7 (*N*=6)	2700
	28	Composition Function 8 (*N*=6)	2800

	29	Composition Function 9 (N=3)	2900
	30	Composition Function 10 (N=3)	3000
Search Range: $[-100,100]^D$			

According to the guidelines requirements of CEC2017 benchmark competition, all experimental algorithms are performed. More specifically, when the solution found by the experimental algorithm is smaller than 10−8, the error is set to 0. The dimension number (D) of these test functions is set to 10, 30, 50, and 100, respectively. The maximum evaluated times is set to $D \times 10{,}000$ on each run. Each problem is evaluated at 51 times. Moreover, five statistical metrics are designed, such as Best, Worse, Median, Mean, and Standard deviation (Std.) (Awad *et al.* 2016). These metrics can be employed to evaluate the solving performance of these various algorithms.

S3. Parameters analysis

The parameter setting plays an important part in the performance of HSAS/FFT. In HSAS/FFT, there are five crucial parameters: reduction factor W_T, population size N, population convergence threshold C, reinitializing probability P_r, and number of samples S. To analyse the influence of each parameter in HSAS/FFT, the Taguchi method (Montgomery 2008) of design for the experiments is adopted. In the experiment, the CEC2017 benchmark functions (Awad *et al.* 2016) is adopted to calibrate the proposed algorithm. The various values of W_T, N, C, P_r and S are listed in Table 2. The parameter combinations and average error values yielded by HSAS/FFT are listed in Table 3. Table 4 lists each parameter's significance rank according to Table 3. Meanwhile, the trend of the parameters is described in Figure 1.

From Table 4 it is observed that W_T is the most significant one among them, which implies that W_T is important to HSAS/FFT. The large W_T can improve the solution accuracy. But too large W_T will always need oversize computational budget. N ranks the second place, which illustrates that it is also an important factor in HSAS/FFT. A small N would cause the population easily fall into the local optimum. A large N can lead the algorithm to search in the global space, but also will take a huge time to evaluate the population. C ranks the third place. A large C represents a strict criterion of population convergence. But it will slow down the evolutionary speed. From Figure 1, the change trend of P_r and S is relatively stable, which means it has slight effect on the performance of HSAS/FFT. A small P_r and S can improve the convergence rate. A large P_r and S can lead the algorithm search in the global space to improve the quality of the solution. According to the results of Taguchi method, the parameters in HSAS/FFT are suggested as follows: $W_T = 1/3$, $N = 10$, $C = 80\%$, $P_r = 0.05$ and $S = 20$.

Table 2. The parameters for different levels.

Parameters	Levels			
	1	2	3	4
W_T	1/10	1/5	1/3	1/2
N	5	10	50	100
C	60%	70%	80%	90%
P_r	0.005	0.01	0.05	0.1
S	10	20	50	100

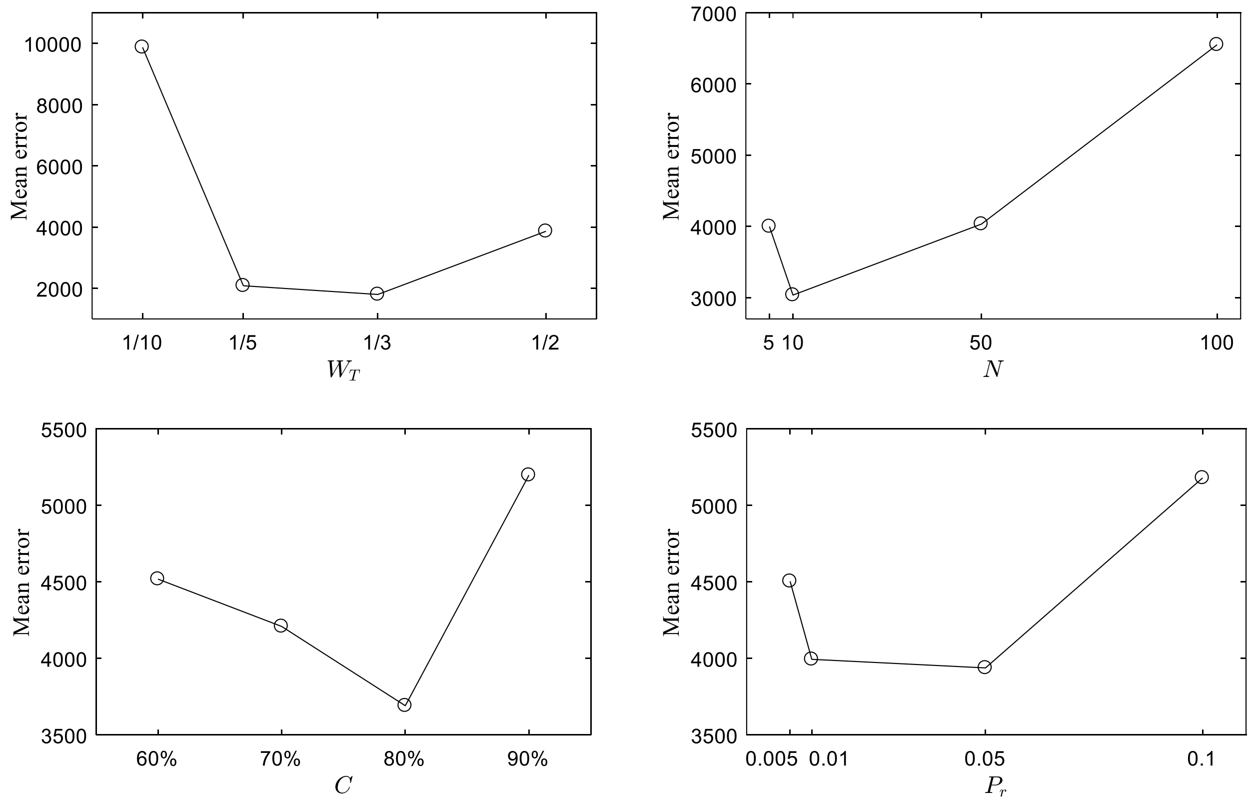
Table 3. Parameter combinations and average error values.

No.	Parameter combinations					Average error values
	W_T	N	C	P_r	S	
1	1/10	5	60%	0.005	10	9703.61
2	1/10	10	70%	0.010	20	7395.42
3	1/10	50	80%	0.050	50	8137.62
4	1/10	100	90%	0.100	100	14248.33
5	1/5	5	70%	0.050	100	3456.47
6	1/5	10	60%	0.100	50	3200.08
7	1/5	50	90%	0.005	20	3881.52
8	1/5	100	80%	0.010	10	4909.01
9	1/3	5	80%	0.100	20	953.01
10	1/3	10	90%	0.050	10	781.77

11	1/3	50	60%	0.010	100	1793.55
12	1/3	100	70%	0.005	50	3668.54
13	1/2	5	90%	0.010	50	1877.22
14	1/2	10	80%	0.005	100	763.18
15	1/2	50	70%	0.100	10	2314.92
16	1/2	100	60%	0.050	20	3375.24

Table 4. Response table for means.

Levels	W_T	N	C	P_r	S
1	9871	3998	4518	4504	4427
2	2083	3035	4209	3994	3901
3	1799	4032	3691	3938	4221
4	3862	6550	5197	5179	5065
Delta	8072	3515	1507	1241	1164
Rank	1	2	3	4	5



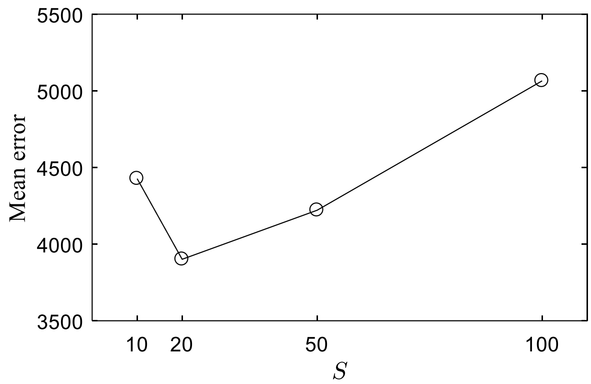


Figure 1. The trend of parameters of the HSAS/FFT.

S4. Statistical results of the HSAS/FFT

In this section, the proposed algorithm is run independently 51 times, and five statistical metrics are calculated, such as Best, Worse, Median, Mean, and Std. (Awad *et al.* 2016). These metrics can be used to analyse the performance of the proposed algorithm. The maximum number of objective function evaluations is $D \times 10{,}000$. Table 5 and Table 6 show the computational results of the proposed algorithm in solving the CEC2017 competition on real-parameter optimization problems for dimension 10, 30, 50, and 100. Each column shows Best, Worse, Median, Mean, and Std. of the error value between the true optimal value and the best fitness values found by the algorithm in each run. Max and Min represent the maximum and minimum fitness values of the algorithm over the 51 runs, respectively. Median denotes the median of the fitness values over the 51 runs. Mean represents the average value of the result fitness values over the 51 runs. Std. stands for standard deviation. It is worth noting that the error values smaller than $1.00\mathrm{E}-08$ has been set as zero.

From the results obtained in Table 5 for dimension 10, and 30, the HSAS/FFT could perform good results in unimodal problems from f_1 to f_3. While for multimodal function, the optimal solution is still available on f_4, f_6, f_9 in dimension 10, and available on f_6, f_9 in dimension 30.

For f_5, f_7, f_8, the error value is still smaller than $1.00\mathrm{E}-01$ in dimension 10. While for all the hybrid functions, the error value is still small than $1.00\mathrm{E}-01$ on f_{15}, f_{16}, f_{18}, f_{19} and f_{20} for dimension 10. As for the remaining composition functions, the best error value is larger than $1.00\mathrm{E}+02$ except f_{21}, f_{22}, f_{24} for dimension 10.

Similarly, Table 6 for dimension 50, and 100 presents the statistical results. From the comparisons regarding the unimodal problems, the proposed algorithm performs good results on f_1 for dimension 50 and 100. For multimodal functions, the error value of the proposed algorithm is smaller than $1.00\mathrm{E}-01$ on f_6, f_9. While for all the hybrid functions, the error value is still small than $1.00\mathrm{E}+03$ except f_{12}, f_{16}, f_{17} for dimension 50 and small than $1.00\mathrm{E}+04$ except f_{12}, f_{16}, f_{17}, f_{20} for dimension 100. As for the remaining composition functions, the best error value is still less than $1.00\mathrm{E}+04$ except f_{30} in dimension 50.

In summary, the proposed algorithm has obtained good results on the most of functions from low dimension to high dimension. Compared with solving unimodal functions, the proposed algorithm is difficult to solve multimodal functions, hybrid functions, and composition functions.

Table 5. The statistical results of HSAS/FFT on the CEC2017 benchmarks for $D = 10, 30$ dimensions.

Func.	10D					30D				
	Best	Worst	Median	Mean	Std.	Best	Worst	Median	Mean	Std.
F1	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
F2	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
F3	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
F4	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	2.13E-03	3.13E-03	5.86E+01	2.67E-03	4.59E-03
F5	0.00E+00	1.99E+00	9.95E-01	1.09E+00	6.96E-01	5.07E+00	8.66E+00	6.96E+00	5.66E+00	5.51E+00
F6	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
F7	1.05E+01	1.23E+01	1.19E+01	1.16E+01	5.66E-01	6.61E+02	8.33E+02	3.73E+01	7.32E+02	2.54E+02
F8	9.95E-01	2.98E+00	1.99E+00	1.59E+00	6.60E-01	2.60E+00	3.93E+00	6.96E+00	3.37E+00	7.29E+00
F9	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
F10	3.48E+00	2.45E+02	1.04E+01	4.47E+01	7.47E+01	1.22E+03	2.30E+03	1.43E+03	1.88E+03	1.15E+02
F11	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	5.07E+00	9.87E+00	3.48E+00	5.34E+00	1.89E+00
F12	0.00E+00	1.35E+02	6.56E+01	5.22E+01	6.17E+01	1.69E+02	4.66E+02	7.17E+02	2.70E+02	1.87E+01
F13	0.00E+00	5.39E+00	4.84E+00	3.58E+00	2.15E+00	2.03E+02	2.11E+02	1.88E+01	2.09E+02	2.04E+01
F14	0.00E+00	2.00E+01	0.00E+00	2.10E+00	5.97E+00	3.26E+00	6.10E+00	2.20E+01	4.20E+00	2.05E+00
F15	4.16E-03	5.00E-01	4.12E-01	2.67E-01	1.99E-01	1.84E+00	8.12E+00	2.59E+00	3.69E+00	7.95E-01
F16	1.09E-03	1.46E+00	6.05E-01	6.13E-01	5.57E-01	1.97E+00	9.12E+00	2.19E+01	4.01E+00	6.44E+00
F17	1.97E-02	2.03E+01	2.11E-01	2.16E+00	6.06E+00	2.04E+02	3.23E+02	3.27E+01	2.76E+02	8.09E+01
F18	4.23E-04	5.00E-01	3.31E-01	2.64E-01	1.93E-01	8.98E-01	2.52E+01	2.15E+01	2.32E+01	1.17E+00
F19	0.00E+00	3.92E-02	1.97E-02	1.47E-02	1.18E-02	2.06E+01	3.54E+01	5.79E+00	2.38E+01	1.64E+01
F20	0.00E+00	3.12E-01	3.12E-01	1.56E-01	1.56E-01	2.71E+01	6.82E+01	2.75E+01	3.08E+01	4.41E+00
F21	1.00E+02	2.04E+02	2.03E+02	1.82E+02	4.12E+01	3.98E+02	8.27E+02	2.08E+02	4.53E+02	3.65E+02
F22	1.00E+02	1.00E+02	1.00E+02	1.00E+02	0.00E+00	1.00E+02	1.09E+02	1.00E+02	1.03E+02	5.51E-02
F23	3.00E+02	3.05E+02	3.00E+02	3.01E+02	1.92E+00	3.52E+02	3.73E+02	3.55E+02	3.62E+02	5.78E+01
F24	1.00E+02	3.32E+02	3.30E+02	3.07E+02	6.91E+01	4.35E+02	4.60E+02	4.30E+02	4.52E+02	9.98E+01
F25	3.98E+02	4.46E+02	4.21E+02	4.16E+02	2.25E+01	3.76E+02	3.81E+02	3.87E+02	3.79E+02	1.86E+01
F26	3.00E+02	3.00E+02	3.00E+02	3.00E+02	0.00E+00	8.45E+02	9.48E+02	9.89E+02	9.45E+02	1.46E+01
F27	3.89E+02	3.90E+02	3.90E+02	3.89E+02	2.05E-01	5.00E+02	5.00E+02	5.08E+02	5.00E+02	1.18E-03
F28	3.00E+02	6.12E+02	5.84E+02	4.47E+02	1.48E+02	3.00E+03	3.00E+03	3.00E+02	3.00E+03	2.68E-03
F29	2.32E+02	2.40E+02	2.34E+02	2.34E+02	2.63E+00	4.19E+02	4.82E+02	4.28E+02	4.75E+02	2.65E+02
F30	3.95E+02	4.64E+02	3.95E+02	4.04E+02	2.07E+01	4.23E+02	5.51E+02	1.97E+03	4.43E+02	1.26E+01

Table 6. The statistical results of HSAS/FFT on the CEC2017 benchmarks for $D = 50, 100$ dimensions.

Func.	50D					100D				
	Best	Worst	Median	Mean	Std.	Best	Worst	Median	Mean	Std.
F1	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
F2	0.00E+00	1.00E+00	0.00E+00	2.00E-01	4.00E-01	7.00E+00	9.47E+11	4.72E+04	9.47E+10	2.84E+11
F3	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	7.99E-07	1.97E-05	7.90E-06	7.70E-06	5.80E-06
F4	3.75E+00	1.42E+02	1.26E+02	8.07E+01	5.43E+01	1.92E+02	2.27E+02	1.99E+02	2.02E+02	1.10E+01
F5	8.95E+00	1.39E+01	1.19E+01	1.13E+01	1.62E+00	2.09E+01	3.59E+01	2.61E+01	2.65E+01	3.75E+00
F6	0.00E+00	1.92E-07	2.40E-08	4.79E-08	7.43E-08	2.19E-05	5.58E-03	1.42E-03	1.33E-03	1.61E-03
F7	5.98E+02	7.69E+02	7.37E+02	7.33E+02	3.07E+01	9.22E+02	9.36E+02	9.33E+02	9.30E+02	4.58E+01
F8	6.96E+00	1.39E+01	1.09E+01	1.06E+01	1.94E+00	2.12E+01	3.16E+01	2.57E+01	2.55E+01	3.10E+00
F9	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	5.44E-01	0.00E+00	1.09E-01	1.98E-01
F10	2.41E+03	3.71E+03	3.24E+03	3.14E+03	3.66E+02	7.14E+03	1.02E+04	9.50E+03	9.07E+03	8.37E+02
F11	2.72E+01	4.11E+01	3.32E+01	3.35E+01	4.01E+00	1.34E+02	3.09E+02	2.43E+02	2.29E+02	4.84E+01
F12	1.65E+03	2.77E+03	1.91E+03	1.97E+03	3.59E+02	1.13E+04	3.89E+04	2.52E+04	2.13E+04	8.55E+03
F13	1.65E+01	1.16E+02	4.87E+01	4.79E+01	2.67E+01	1.57E+02	2.96E+02	2.34E+02	2.21E+02	4.12E+01
F14	2.30E+01	3.02E+01	2.58E+01	2.59E+01	1.94E+00	1.07E+02	2.01E+02	1.28E+02	1.39E+02	2.71E+01
F15	2.50E+01	3.28E+01	2.80E+01	2.75E+01	2.40E+00	1.94E+02	3.77E+02	3.02E+02	2.83E+02	5.88E+01
F16	1.29E+02	6.50E+02	4.13E+02	3.74E+02	1.48E+02	5.08E+02	1.87E+03	1.63E+03	1.45E+03	3.69E+02
F17	7.73E+01	4.40E+02	2.51E+02	2.39E+02	1.15E+02	7.26E+02	1.48E+03	1.05E+03	1.05E+03	2.30E+02
F18	2.47E+01	6.41E+01	2.84E+01	3.14E+01	1.13E+01	1.93E+02	3.05E+02	2.51E+02	2.44E+02	3.82E+01
F19	1.27E+01	1.81E+01	1.67E+01	1.60E+01	1.50E+00	1.46E+02	2.11E+02	1.63E+02	1.69E+02	2.28E+01
F20	3.70E+01	2.75E+02	8.50E+01	9.40E+01	6.26E+01	4.23E+02	1.45E+03	1.10E+03	1.06E+03	3.13E+02
F21	2.09E+02	2.22E+02	2.15E+02	2.15E+02	3.14E+00	2.45E+02	2.61E+02	2.55E+02	2.54E+02	4.86E+00
F22	1.00E+02	3.79E+03	3.12E+03	2.03E+03	1.59E+03	8.24E+03	1.09E+04	9.71E+03	9.40E+03	8.78E+02
F23	4.29E+02	4.42E+02	4.37E+02	4.36E+02	3.93E+00	5.58E+02	5.94E+02	5.75E+02	5.75E+02	9.83E+00
F24	5.10E+02	5.19E+02	5.13E+02	5.14E+02	2.60E+00	9.08E+02	9.35E+02	9.25E+02	9.21E+02	8.61E+00
F25	4.80E+02	4.80E+02	4.80E+02	4.80E+02	2.42E-02	6.58E+02	7.77E+02	7.68E+02	7.42E+02	3.83E+01
F26	1.13E+03	1.36E+03	1.23E+03	1.22E+03	6.42E+01	3.32E+03	3.60E+03	3.44E+03	3.42E+03	7.70E+01
F27	5.18E+02	5.75E+02	5.34E+02	5.38E+02	1.72E+01	5.91E+02	6.55E+02	6.29E+02	6.26E+02	1.89E+01
F28	4.59E+02	4.59E+02	4.59E+02	4.59E+02	0.00E+00	4.78E+02	5.76E+02	5.26E+02	5.29E+02	2.71E+01
F29	3.09E+02	3.54E+02	3.39E+02	3.34E+02	1.48E+01	8.47E+02	1.81E+03	1.32E+03	1.26E+03	2.65E+02
F30	5.79E+05	8.62E+05	6.20E+05	6.35E+05	7.93E+04	2.15E+03	2.57E+03	2.34E+03	2.34E+03	1.07E+02

S5. Comparison of HSAS/FFT with some state-of-the-art algorithms under the CEC2017 evaluation criteria

The CEC2017 evaluation criteria (Awad *et al.* 2016) is adopted to analyse and compare the HSAS/FFT, jSO, RB-IPOP-CMA-ES, PPSO, and DE. The evaluation method for each algorithm is based on a score of 100. For detailed evaluation criteria, please refer to (Awad *et al.* 2016).

F2 has been excluded because it shows unstable behaviour especially for higher dimensions, and significant performance variations for the same algorithm implemented in Matlab, C. The final ranking result is shown in Table 7. HSAS/FFT is ranked the second of the comparative algorithms according to the results in Table 7.

Table 7. Ranking of HSAS/FFT, jSO, RB-IPOP-CMA-ES, PPSO, and DE obtained through the evaluation criteria in CEC2017.

Algorithms	Score1	Score2	Final Score	Rank
HSAS/FFT	4.7253E+01	3.2479E+01	7.9732E+01	2
jSO	5.0000E+01	5.0000E+01	1.0000E+02	1
RB-IPOP-CMA-ES	3.8162E+00	3.2386E+01	3.6203E+01	4
PPSO	3.9494E+00	3.1405E+01	3.5354E+01	5
DE	1.5719E+01	3.1579E+01	4.7298E+01	3

In summary, the statistical analysis of the results obtained by the algorithms in the comparative study showed that the jSO is really the best because it outperformed the results of all the other significantly by solving the CEC2017 benchmark functions of all observed dimensions. Even though HSAS/FFT cannot get the best results on some test problems, it can get the suboptimal results compared with all experimental algorithms. The proposed HSAS/FFT is superior to RB-IPOP-CMA-ES, PPSO, and DE on the most of benchmark problems with a different dimension. Compared with other methods, HSAS/FFT can get the best results on the unimodal and hybrid functions with the different dimensions. On the rest of test problems, the proposed algorithm still keeps the stable solving performance. Meanwhile, the convergence rate of HSAS/FFT is much better than other algorithms in most of the functions in the early stage of the search, which benefited from fast Fourier transform and search space segmentation. More specifically, the correct reduction of search space accelerates the convergence process.

S6. HSAS/FFT time complexity

This subsection deal with an analysis of the HSAS/FFT time complexity as defined in (Awad *et al.* 2016). The dunning time obtained by evaluating the benchmark function f_{18}, is compared with a running time of the test program presented in Algorithm 3. The experiments are run in Windows Server 2012 R2 under the hardware environment of Intel Core i7-2760QM CPU @2.40GHz processor and 8.0 GB of RAM. The proposed algorithm is implemented using C++ programming language.

Algorithm 3. The test program.

Initialize: $x = 0.55$
Start
  For $k = 1{:}D$
    $x = x + x;$
    $x = x/2\,;\ x = x \times x;\ x = \sqrt{x};$
    $x = \log(x)\,;\ x = \exp(x)\,;$
    $x = x/(x+2))$
  End for
End

The computing time of the test program is denoted as $T0$. Variable $T1$ is the time required for evaluating the benchmark function f_{18} and variable $T2$ the time of HSAS/FFT execution for function f_{18} within $200{,}000$ evaluations for each dimension. Variable $\widehat{T2}$ is an average of $T2$ values obtained in five independent runs. Table 8 shows the algorithm complexities relationship with dimension. The computational complexity of the algorithm HSAS/FFT is reflected by the measured and calculated variables $T0$, $T1$, $\widehat{T2}$ and $(\widehat{T2} - T1)/T0$ for each of the observed dimension $D = \{10, 30, 50, 100\}$. Obviously, this calculation is independent of the computing system and programming language in which the measured algorithm is implemented. According to Table 8, it is easy to conclude that more computational cost is required with the increasing number of dimensions for the benchmark functions.

Table 8. The computational complexity of the algorithm HSAS/FFT (all times are in seconds).

D	$T0$	$T1$	$\widehat{T2}$	$(\widehat{T2} - T1)/T0$
10	0.147522	0.270703	2.536710	15.360468
30		0.914563	6.916988	40.688338
50		1.995140	11.897618	67.125432
100		6.741860	27.265260	139.120945

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