Online Supplement for "Optimal Demand Shaping for a Dual-channel Retailer under Growing E-commerce Adoption" by Nevin Mutlu and Ebru K. Bish

Appendix A. Notation

Parameters:	
a:	Coefficient of imitation
λ:	Speed at which consumers adaptively learn
v:	Value that consumers place on the product
p:	Price of the product in both channels
k:	Service radius
c:	Retailer's unit purchasing cost
s:	Lost sales cost per unit ($s < c < p$)
l:	Delivery time for the online channel
T:	Length of the planning time horizon, indexed by $t = 1, 2, ..., T$
b:	Market size parameter
Decision Variables:	
α_t:	Type I service level for the store channel in period t where $t = 1, 2, ..., T$
ω:	The level of marketing efforts
Random Variables:	
M:	Market size ($M \sim U[0,b]$)
D:	Consumer index for unwillingness to wait for delivery in the online channel ($D \sim U[0,1]$)
R:	Consumer index for her distance to the store ($R \sim U[0,k]$)
X_t:	Bernoulli variable (X_t=1 if the consumer has adopted the online channel option by time t, X_t=0 otherwise, with $Pr(X_t = 1) = F_t(\omega)$)
Dependent (intermediary) variables:	
$U_{i,t}(d, r)$:	The utility that a consumer with indices (d, r) derives from channel $i \in \{s, o\}$ in period t where $t = 1, 2, ..., T$
$\beta(\alpha_t)$:	Type II service level for the store channel in period t where $t = 1, 2, ..., T$
ξ_t:	Predicted availability level for the store channel in period t where $t = 1, 2, ..., T$
$q_{i,t}$:	Order quantity for channel $i \in \{s, o\}$ in period t where $t = 1, 2, ..., T$
$Z_{i,t}$:	Random demand for option $i \in \{s, o, n\}$ in period t where $t = 1, 2, ..., T$
$I_{i,t}(\omega, \xi_t)$:	Proportion of total demand for option $i \in \{s, n\}$ in period t where $t = 1, 2, ..., T$
$I^y_{o,t}(\omega, \xi_t)$:	Proportion of primary ($y = 1$) and secondary ($y = 2$) demand for the online channel in period t where $t = 1, 2, ..., T$
$F_t(\omega)$:	Proportion of the population who have adopted e-commerce by time t where $t = 1, 2, ..., T$

Table 6: Notation

Appendix B. Proofs

Recall that $W \equiv \frac{k}{v-p}$, and $B \equiv \frac{v-p}{l}$. We also define, $Y_t \equiv \begin{cases} \frac{[1-BF_t(\omega)]\xi_t}{W}, & \text{if } \xi_t \le W \\ [1-BF_t(\omega)], & \text{otherwise.} \end{cases}$, for $t = 1, ..., T$.

Proof of Lemma 1. In order for no choice-based lost demand to occur (i.e. $(B_{n,t}|X_t = 0 \cup B_{n,t}|X_t = 1) = \emptyset$), both e-commerce adopters and non-adopters must receive positive utility from at least one channel. Since the only available channel for the e-commerce non-adopters is the store channel, it must hold that $TH_{r,t} = k \Leftrightarrow \xi_t > k/(v-p) = W$, i.e. all e-commerce non-adopters derive a positive utility from the store. This condition also implies that all e-commerce adopters derive a positive utility from the store, thus, completing the proof. ■

Proof of Lemma 2.
Proof of Part 1. The proof proceeds in two parts. First, we derive closed-form expressions for the choice probabilities, $Pr(B_{j,t}|X_t = 1)$, for $j \in \{o, c\}$, and $Pr(B_{i,t}|X_t = x)$, for $i \in \{s, n\}, x \in \{0, 1\}$; and the expressions in Table 2 for $I_{i,t}(\omega, \xi_t)$, for $i \in \{s, n\}$, $I^y_{o,t}(\omega, \xi_t)$, for $y \in \{1, 2\}$, follow directly from Eqn. (1). Next, we show that $I_{i,t}(\omega, \xi_t)$, for $i \in \{s, n\}$, $I^y_{o,t}(\omega, \xi_t)$, for $y \in \{1, 2\}$, are piece-wise continuous functions of ξ_t.

In the following, we consider the case where $TH_{d,t} < 1 \Leftrightarrow TH_{d,t} = (v-p)/l < 1$, and $TH_{r,t} < k \Leftrightarrow TH_{r,t} = \xi_t(v-p) < k$ (see Figure 2); other cases defined in Lemma 2 can be proven similarly (see Table 7). We start with the segmentation of e-commerce adopters. Since the y-intercept ($r = 0$) of the line $TH_{l,t}$ is $d(r=0) = \frac{(1-\xi_t)(v-p)}{l}$, we can write:

$$Pr(B_{o,t}|X_t=1) = \frac{1}{k}\left[k \cdot \frac{v-p}{l} - \frac{\xi_t(v-p)}{2}\left(\frac{v-p}{l} - \frac{(1-\xi_t)(v-p)}{l}\right)\right] = \frac{(v-p)[2k-\xi_t^2(v-p)]}{2kl}. \quad (8)$$

Similarly, it follows that:

$$Pr(B_{s,t}|X_t=1) = \frac{\xi_t(v-p)[2l-2(v-p)+\xi_t(v-p)]}{2kl}, \quad Pr(B_{n,t}|X_t=1) = 1 - \sum_{i\in\{s,o\}} Pr(B_{i,t}|X_t=1),$$
$$\text{and } Pr(B_{c,t}|X_t = 1) = \frac{v-p}{l} - Pr(B_{o,t}|X_t = 1). \quad (9)$$

Since the online channel is not an option for consumers with $X_t = 0$, consumers with $\{r : r < TH_{r,t}\}$ choose the store; otherwise, they choose to not make a purchase, leading to:

$$Pr(B_{s,t}|X_t = 0) = \frac{\xi_t(v-p)}{k} \text{ and } Pr(B_{n,t}|X_t = 0) = 1 - Pr(B_{s,t}|X_t = 0). \quad (10)$$

By substituting Eqn.s. (8)-(10), along with the equalities, $Pr(X_t = 1) = F_t(\omega)$ and $Pr(X_t = 0) = 1 - F_t(\omega)$, into Eqn. (1), we get:

$$I^1_{o,t}(\omega, \xi_t) = \left[1 - \frac{\xi_t^2}{2W}\right] BF_t(\omega), \quad I^2_{o,t}(\omega, \xi_t) = BF_t(\omega) - I^1_{o,t}(\omega, \xi_t), \quad I_{s,t}(\omega, \xi_t) = \frac{\xi_t}{W}\left[1 - \frac{(2-\xi_t)B}{2} F_t(\omega)\right],$$
$$\text{and } I_{n,t}(\omega, \xi_t) = 1 - \sum_{i\in\{s,o\}} I_{i,t}(\omega, \xi_t).$$

Next, we show that function $I^1_{o,t}(\omega, \xi_t)$ is continuous in ξ_t. (Continuity proofs for $I_{i,t}(\omega, \xi_t)$, for

$i \in \{s, n\}$, and $I^2_{o,t}(\omega, \xi_t)$ are similar.) From Table 1, we have that:

$$I^1_{o,t}(\omega, \xi_t) = \begin{cases} B\left[1 - \frac{\xi_t^2}{2W}\right] F_t(\omega), & \text{if } \xi_t < \frac{k}{v-p} \\ \left[(1-\xi_t)B + \frac{BW}{2}\right] F_t(\omega), & \text{otherwise.} \end{cases} \tag{11}$$

We have that $\lim_{\xi_t \to \frac{k}{v-p}-} I^1_{o,t}(\omega, \xi_t) = \lim_{\xi_t \to \frac{k}{v-p}+} I^1_{o,t}(\omega, \xi_t) = \left[B - \frac{BW}{2}\right] F_t(\omega)$. Hence, $I_{o,t}(\omega, \xi_t)$ is continuous in ξ_t.

Proof of Part 2. First, notice that we are slightly abusing the notation since the time index t in Eqn. (2) stands for the time elapsed after e-commerce for the specific product becomes available to consumers (for a given set of ω and a), while the time index t in Eqn. (7) stands for the time period within the planning time horizon. Hence, in order to mimic the behavior of the diffusion curve given in Eqn. (2) for a given set of ω and a, we first provide the following transformation. Let $t^*(\omega)$ denote the diffusion time that the retailer's planning time horizon starts; that is, $F_{t^*(\omega)}(\omega) \equiv F_0$, where $\omega > 0$, as $\omega = 0$ implies that $F_t = 0, \forall t \geq 0$. Then, using Eqn. (2), the following must hold:

$$F_{t^*(\omega)}(\omega) = \frac{1 - e^{-(\omega+a)t^*(\omega)}}{1 + \frac{a}{\omega} e^{-(\omega+a)t^*(\omega)}} = F_0 \Rightarrow t^*(\omega) = -\frac{1}{\omega + a} \ln\left(\frac{\omega(1-F_0)}{\omega + aF_0}\right). \tag{12}$$

Then, using Eqn. (2), the adoption level at time period t within the planning horizon as a function of ω is as follows:

$$F_t(\omega) = \frac{1 - e^{-(\omega+a)(t^*(\omega)+t-1)}}{1 + \frac{a}{\omega} e^{-(\omega+a)(t^*(\omega)+t-1)}}, \quad \omega > 0,\ t = 1, ..., T. \tag{13}$$

Since Eqn. (13) is continuous and differentiable, it trivially follows that $I_{i,t}(\omega, \xi_t)$, $i \in \{s, n\}$, and $I^y_{o,t}(\omega, \xi_t)$, $y \in \{1, 2\}$ are continuous and differentiable in ω.

Table 7: Choice probabilities

	$TH_{d,t} = 1 \Leftrightarrow l \in (0, v-p]$	$TH_{d,t} < 1 \Leftrightarrow l \in (v-p, \infty)$
$TH_{r,t} < k \Leftrightarrow$ $\xi_t < W$	$Pr(B_{s,t} \mid X_t = 1) = \begin{cases} 0 & \text{if } \xi_t \leq 1 - \frac{l}{v-p} \\ \frac{[l-(1-\xi_t)(v-p)]^2}{2kl} & \text{otherwise} \end{cases}$	$Pr(B_{s,t} \mid X_t = 1) = \frac{\xi_t(v-p)[2l-2(v-p)+\xi_t(v-p)]}{2kl}$
	$Pr(B_{o,t} \mid X_t = 1) = 1 - Pr(B_{s,t} \mid X_t = 1)$	$Pr(B_{o,t} \mid X_t = 1) = \frac{(v-p)[2k-\xi_t^2(v-p)]}{2kl}$
	$Pr(B_{n,t} \mid X_t = 1) = 0$	$Pr(B_{n,t} \mid X_t = 1) = 1 - \sum_{i \in \{s,o\}} Pr(B_{i,t} \mid X_t = 1)$
	$Pr(B_{c,t} \mid X_t = 1) = 1 - Pr(B_{o,t} \mid X_t = 1)$	$Pr(B_{c,t} \mid X_t = 1) = \frac{v-p}{l} - Pr(B_{o,t} \mid X_t = 1)$
	$Pr(B_{s,t} \mid X_t = 0) = \frac{\xi_t(v-p)}{k}$	$Pr(B_{s,t} \mid X_t = 0) = \frac{\xi_t(v-p)}{k}$
	$Pr(B_{n,t} \mid X_t = 0) = 1 - Pr(B_{s,t} \mid X_t = 0)$	$Pr(B_{n,t} \mid X_t = 0) = 1 - Pr(B_{s,t} \mid X_t = 0)$
$TH_{r,t} = k \Leftrightarrow$ $\xi_t \geq W$	$Pr(B_{s,t} \mid X_t = 1) = \begin{cases} 0 & \text{if } \xi_t \leq 1 - \frac{l}{v-p} \\ \frac{[l-(1-\xi_t)(v-p)]^2}{2kl} & 1 - \frac{l}{v-p} < \xi_t \leq 1 - \frac{l-k}{v-p} \\ 1 - \frac{2(1-\xi_t)(v-p)+k}{2l} & \text{otherwise} \end{cases}$	$Pr(B_{s,t} \mid X_t = 1) = 1 - \frac{2(1-\xi_t)(v-p)+k}{2l}$
	$Pr(B_{o,t} \mid X_t = 1) = 1 - Pr(B_{o,t} \mid X_t = 1)$	$Pr(B_{o,t} \mid X_t = 1) = 1 - Pr(B_{s,t} \mid X_t = 1)$
	$Pr(B_{n,t} \mid X_t = 1) = 0$	$Pr(B_{n,t} \mid X_t = 1) = 0$
	$Pr(B_{c,t} \mid X_t = 1) = 1 - Pr(B_{o,t} \mid X_t = 1)$	$Pr(B_{c,t} \mid X_t = 1) = \frac{v-p}{l} - Pr(B_{o,t} \mid X_t = 1)$
	$Pr(B_{s,t} \mid X_t = 0) = 1$	$Pr(B_{s,t} \mid X_t = 0) = 1$
	$Pr(B_{n,t} \mid X_t = 0) = 0$	$Pr(B_{n,t} \mid X_t = 0) = 0$

Derivation of the Expected Profit Function. Recall that $M \sim U[0,b]$. Then, using Eqn.s (1), (3) and (4), we derive the following:

$$
\begin{aligned}
E[min(Z_{s,t}, q_{s,t})] &= \int_0^{q_{s,t}} z_{s,t} \frac{1}{bI_{s,t}(\omega,\xi_t)} d\, z_{s,t} + \int_{q_{s,t}}^{bI_{s,t}(\omega,\xi_t)} q_{s,t} \frac{1}{bI_{s,t}(\omega,\xi_t)} d\, z_{s,t} \\
&= \frac{1}{bI_{s,t}(\omega,\xi_t)} \left[\frac{q_{s,t}^2}{2} + q_{s,t}\Big(bI_{s,t}(\omega,\xi_t) - q_{s,t}\Big)\right] = \alpha_t bI_{s,t}(\omega,\xi_t)\left[1 - \frac{\alpha_t}{2}\right], \\
E[(Z_{s,t} - q_{s,t})^+] &= \int_{q_{s,t}}^{bI_{s,t}(\omega,\xi_t)} (z_{s,t} - q_{s,t}) \frac{1}{bI_{s,t}(\omega,\xi_t)} d\, z_{s,t} \\
&= \frac{1}{bI_{s,t}(\omega,\xi_t)} \left[\frac{b^2 I_{s,t}(\omega,\xi_t)^2 - q_{s,t}^2}{2} - q_{s,t}\Big(bI_{s,t}(\omega,\xi_t) - q_{s,t}\Big)\right] = \frac{bI_{s,t}(\omega,\xi_t)(1-\alpha_t)^2}{2}, \\
E[Z_{o,t}] &= \frac{bI_{o,t}^1(\omega,\xi_t)}{2} + \frac{I_{o,t}^2(\omega,\xi_t)}{I_{s,t}(\omega,\xi_t)} E[(Z_{s,t} - q_{s,t})^+] = \frac{bI_{o,t}^1(\omega,\xi_t)}{2} + \frac{bI_{o,t}^2(\omega,\xi_t)(1-\alpha_t)^2}{2}, \\
E[Z_{n,t}] &= \frac{bI_{n,t}(\omega,\xi_t)}{2} = \frac{b(1 - I_{s,t}(\omega,\xi_t) - I_{o,t}^1(\omega,\xi_t))}{2}.
\end{aligned}
\tag{14}
$$

Then, Eqn. (7) follows by substituting the terms derived in Eqn. (14), along with Eqn.s (1)-(4), into Eqn (6). This completes the proof. ■

Proof of Lemma 3. In the following, we consider the case where $TH_{d,t} < 1 \Leftrightarrow TH_{d,t} = (v-p)/l < 1$; the case where $TH_{d,t} = 1 \Leftrightarrow TH_{d,t} = (v-p)/l \geq 1$ can be proven similarly. Before we present the proof for Lemma 3, we derive several properties.

Property 1. *1.* $[I_{o,t}^1(\omega,\xi_t) + I_{o,t}^2(\omega,\xi_t)] = BF_t(\omega) \geq 0$ *which implies that* $\frac{\partial [I_{o,t}^1(\omega,\xi_t)+I_{o,t}^2(\omega,\xi_t)]}{\partial \xi_t} = 0$*;*

2. $[I_{s,t}(\omega,\xi_t) - I_{o,t}^2(\omega,\xi_t)] = Y_t > 0$*, and* $\frac{\partial Y_t}{\partial \xi_t} \geq 0$*;*

3. Given that $\xi_t = \lambda\beta(\alpha_{t-1}) + \lambda\xi_{t-1}$*, where* $\beta(\alpha_t) = \alpha_t(1 - ln(\alpha_t))$*, it holds that* $\frac{\partial \xi_t}{\partial \alpha_{t-i}} = -\lambda(1-\lambda)^{i-1}\ln\alpha_{t-i}$*, for* $t \leq T$*, and* $i \leq t-1$*.*

Proof of Property 1. Throughout the proof, we consider the range $\alpha_t \in (0,1]$, for $t = 1,2,...,T$, as $\ln\alpha_t$ is undefined at $\alpha_t = 0$.
Proof of Parts 1 and 2. The results directly follow from Table 1.

Proof of Part 3. The result directly follows from the definition of ξ_t.

This completes the proof. ■

Proof of Lemma 3. Using Property 1, for $\lambda \in [0,1]$ it follows that:

$$
E[\pi_t(\omega,\alpha_t,\xi_t)] = -G(\omega) + bBF_t(\omega)\left[\frac{p+s}{2} - c\right] + bY_t\left[(p+s-c)\alpha_t - (p+s)\frac{\alpha_t^2}{2}\right] - \frac{bs}{2} \tag{15}
$$

$$
\Rightarrow \frac{\partial E[\pi_t(\omega,\alpha_t,\xi_t)]}{\partial \alpha_{t-i}} = -b\frac{\partial Y_t}{\partial \xi_t}\lambda(1-\lambda)^{i-1}\ln\alpha_{t-i}\left[(p+s-c)\alpha_t - (p+s)\frac{\alpha_t^2}{2}\right], \text{for } \ i < t \tag{16}
$$

$$
\frac{\partial E[\pi_t(\omega,\alpha_t,\xi_t)]}{\partial \alpha_t} = bY_t\left[p+s-c-(p+s)\alpha_t\right] \tag{17}
$$

$$\Rightarrow \frac{\partial^2 E[\pi_t(\omega,\alpha_t,\xi_t)]}{\partial \alpha_{t-i}^2} = -b\frac{\partial Y_t}{\partial \xi_t}\lambda(1-\lambda)^{i-1}\frac{1}{\alpha_{t-i}}\left[(p+s-c)\alpha_t - (p+s)\frac{\alpha_t^2}{2}\right], \text{for } \ i<t \tag{18}$$

$$\frac{\partial^2 E[\pi_t(\omega,\alpha_t,\xi_t)]}{\partial \alpha_{t-i}\alpha_{t-j}} = 0, \ \text{ for } \ i,j<t, i\neq j \tag{19}$$

$$\frac{\partial^2 E[\pi_t(\omega,\alpha,\xi_t)]}{\partial \alpha_{t-i}\alpha_t} = -b\frac{\partial Y_t}{\partial \xi_t}\lambda(1-\lambda)^{i-1}\ln\alpha_{t-i}\left[p+s-c-(p+s)\alpha_t\right], \text{for } \ i<t \tag{20}$$

$$\frac{\partial^2 E[\pi_t(\omega,\alpha_t,\xi_t)]}{\partial \alpha_t^2} = -bY_t\,(p+s) \tag{21}$$

Using the definition of Y_t, Property 1 and Eqn. (15), it follows that $E[\pi_t(\omega,\alpha_t,\xi_t)]$ is continuous in $\alpha_j \in (0,1]$, for $j \leq t$. Also, Eqn.s. (16) and (17) imply that $E[\pi_t(\omega,\alpha_t,\xi_t)]$ is strictly increasing in $\alpha_j \in (0, CF)$, for $j \leq t$. Let $A^- \equiv \{(\alpha_1,\alpha_2,...,\alpha_{t-1}) : CF\cdot\overrightarrow{1} \leq (\alpha_1,\alpha_2,...,\alpha_{t-1}) \leq \overrightarrow{1}, \xi_t < W\}$ and $A^+ \equiv \{(\alpha_1,\alpha_2,...,\alpha_{t-1}) : CF\cdot\overrightarrow{1} \leq (\alpha_1,\alpha_2,...,\alpha_{t-1}) \leq \overrightarrow{1}, \xi_t \geq W\}$, where $\xi_t = \lambda\beta(\alpha_{t-1}) + (1-\lambda)\xi_{t-1}$. Letting $T \equiv b\lambda\left[(p+s-c)\alpha_t - (p+s)\frac{\alpha_t^2}{2}\right]$, and $V \equiv b\lambda\left[p+s-c-(p+s)\alpha_t\right]$, the Hessian matrix for $-E[\pi_t(\omega,\alpha_t,\xi_t)]$ for A^- in this case, where the entries are in the order of $\alpha_1,\alpha_2,...,\alpha_t$, is as follows.

$$H^{A^-} = \frac{[1-BF_t(\omega)]}{W}\begin{bmatrix} \frac{T(1-\lambda)^{t-2}}{\alpha_1} & 0 & \cdots & 0 & 0 & V(1-\lambda)^{t-2}\ln\alpha_1 \\ 0 & \frac{T(1-\lambda)^{t-3}}{\alpha_2} & \cdots & 0 & 0 & V(1-\lambda)^{t-3}\ln\alpha_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{T(1-\lambda)^1}{\alpha_{t-2}} & 0 & V(1-\lambda)^1\ln\alpha_{t-2} \\ 0 & 0 & \cdots & 0 & \frac{T(1-\lambda)^0}{\alpha_{t-1}} & V(1-\lambda)^0\ln\alpha_{t-1} \\ V(1-\lambda)^{t-2}\ln\alpha_1 & V(1-\lambda)^{t-3}\ln\alpha_2 & \cdots & V(1-\lambda)^1\ln\alpha_{t-2} & V(1-\lambda)^0\ln\alpha_{t-1} & \xi_t(p+s) \end{bmatrix}$$

Using the definitions of T and V, it follows that $T > 0$ and $V \leq 0$, for $\alpha_t \in [CF,1]$. In order to prove that $E[\pi_t(\omega,\alpha_t,\xi_t)]$ is jointly concave in $\alpha_t \in [CF,1]$ and $(\alpha_1,\alpha_2,...,\alpha_{t-1}) \in A^-$, it is sufficient to show that H^{A^-} is positive semi-definite (PSD) in $\alpha_j \in [CF,1]$, for $j \leq t$. For that, we need to show that all principal minors of H^{A^-}, which we denote by D_i, are greater than or equal to zero for $\alpha_j \in [CF,1]$, for $j \leq t$. Clearly, $D_1 = \xi_t(p+s) \geq 0$. In order to show that H^{A^-} is PSD, we only need to show that $D_2 \geq 0$, then $D_j \geq 0$ for $j > 2$ can be proven by induction. In order to show that $D_2 \geq 0$ for $\alpha_j \in [CF,1]$, for $j \leq t$, we need to show that:

$$D_2 = \frac{T(1-\lambda)^0}{\alpha_{t-1}}\xi_t(p+s) - V^2(\ln\alpha_{t-1})^2 \geq 0. \tag{22}$$

It follows that D_2 is decreasing for $\alpha_t \in [CF,1]$; hence, for any given α_{t-1}, D_2 is minimized at $\alpha_t = 1$. Therefore, in order for D_2 to be positive at $\alpha_t = 1$, it must hold that:

$$\lambda\left[\frac{p+s}{2} - c\right]\frac{\xi_t(p+s)}{\alpha_{t-1}} - \lambda^2c^2(\ln\alpha_{t-1})^2 \geq 0 \Rightarrow \left[\frac{p+s}{2} - c\right]\frac{\xi_t(p+s)}{\alpha_{t-1}} - c^2(\ln\alpha_{t-1})^2 \geq 0$$

$$\Rightarrow f_1 = \left[\frac{p+s}{2} - c\right]\xi_t(p+s) - c^2(\ln\alpha_{t-1})^2\alpha_{t-1} \geq 0 \tag{23}$$

f_1 is increasing for $\alpha_{t-i} \in [CF,1]$, for $i < t$. Thus, it must hold that:

$$\left[\frac{p+s}{2} - c\right]CF(1-\ln CF) - c^2(\ln CF)^2 CF \geq 0 \Rightarrow \left[\frac{p+s}{2} - c\right](1-\ln CF) - c^2(\ln CF)^2 \geq 0 \tag{24}$$

Recall that $CF = \frac{p+s-c}{p+s}$, letting $\eta \equiv \frac{p+s}{c}$ and simplifying, we can rewrite Eqn. (24) as:

$$f_2 = \eta \left[\frac{\eta-2}{2}\right]\left(1 - \ln\frac{\eta-1}{\eta}\right) - \left(\ln\frac{\eta-1}{\eta}\right)^2 \geq 0 \tag{25}$$

f_2 is increasing in η, which implies that any η that is greater than the root of f_2 satisfies Eqn. (25), which assures that $D_2 \geq 0$. By numerically solving, we get: $f_2 = 0$ at $\eta = 2.206$. Thus, we prove that $D_2 \geq 0$ for $\eta \geq 2.206$, which implies that H^{A^-} is PSD for $\eta \geq 2.206$. Thus, $E[\pi_t(\omega, \alpha_t, \xi_t)]$ is concave in A^-. Next, the Hessian matrix for $-E[\pi_t(\omega, \xi_t)]$ for A^+ is:

$$H^{A^+} = [1 - BF_t(\omega)] \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & (p+s) \end{bmatrix}$$

which is PSD for $\forall \alpha_j \in (0,1], j \leq t$. Then, $E[\pi_t(\omega, \alpha_t, \xi_t)]$ is jointly concave in $\alpha_j \in [CF, 1]$, for $j \leq t$ since $E[\pi_t(\omega, \alpha_t, \xi_t)]$ is: 1) jointly concave in $\forall \alpha_j, j \leq t$ in both regions A^- and A^+, 2) continuous and differentiable in α_t, and 3) $\frac{\partial E[\pi_t(\omega,\alpha_t,\xi_t)]}{\partial \alpha_{t-i}} \geq 0, i < t$, for $(\alpha_1, \alpha_2, ..., \alpha_{t-1}) \in A^-$, and $\frac{\partial E[\pi_t(\omega,\alpha_t,\xi_t)]}{\partial \alpha_{t-i}} = 0, i < t$, for $(\alpha_1, \alpha_2, ..., \alpha_{t-1}) \in A^+$ (using Eqn. (16)).

Proof of Remark 1. Using Property 1, it follows that $\xi_t = \beta(\alpha_0)$ for $\lambda = 0$. Hence, $E[\pi_t(\omega, \alpha_t, \xi_t)]$ is independent of α_j, for $\forall j \neq t$. Then, Eqn.s. (17) and (21) imply that the FOC for $E[\pi_t(\omega, \alpha_t, \xi_t)]$, $\forall t$, for $\lambda = 0$, is satisfied at $\alpha_t = CF$ for $t \leq T$.

Proof of Theorem 1. In the following, we consider the case where $TH_{d,t} < 1 \Leftrightarrow TH_{d,t} = (v-p)/l < 1$; the case where $TH_{d,t} = 1 \Leftrightarrow TH_{d,t} = (v-p)/l \geq 1$ can be proven similarly. For a given $\omega > 0$, since ξ_t is a function of all $\alpha_i, i < t$, it follows that:

$$E[\Pi(\omega, \overrightarrow{\alpha})] = -G(\omega) + \sum_{t=1}^{T} E[\pi_t(\omega, \alpha_t, \xi_t)] \Rightarrow \frac{\partial E[\Pi(\omega, \overrightarrow{\alpha})]}{\partial \alpha_t} = \sum_{i=0}^{T-t} \frac{\partial E[\pi_{t+i}(\omega, \alpha_{t+i}, \xi_{t+i})]}{\partial \alpha_t}, \text{ for } t \leq T.$$

$$\Rightarrow \frac{\partial E[\Pi(\omega, \overrightarrow{\alpha})]}{\partial \alpha_t} = Y_t[p+s-c-(p+s)\alpha_t] - \lambda \ln \alpha_t \sum_{i=1}^{T-t} \frac{\partial Y_{t+i}}{\partial \xi_{t+i}} (1-\lambda)^{i-1} \left[(p+s-c)\alpha_{t+i} - (p+s)\frac{\alpha_{t+i}^2}{2}\right]. \tag{26}$$

Then using Eqn. (26), $\alpha_T^* = CF$. Let ξ_t^*, for $t \leq T$, denote ξ_t at optimality. Then, using Eqn.s. (16) and (17), it follows that the FOC for α_t, for $t \leq T$, is satisfied at $\alpha_t^* = CF$ if $\xi_t^* = \beta(CF) > W$; otherwise, there exists a vector, say, $\overrightarrow{\alpha}'' = (\alpha_1'', \alpha_2'', ..., \alpha_{T-1}'')$, that satisfies the FOC if $\xi_t''(\omega) \leq W, \forall t \leq T$, where $\xi_t''(\omega) = \lambda\beta(\alpha_{t-1}'') + (1-\lambda)\xi_{t-1}$, with $\xi_1'' = \beta(\alpha_0)$; thus, $(\alpha_1^*, \alpha_2^*, ..., \alpha_{T-1}^*) = \overrightarrow{\alpha}''$. If $\beta(CF) \leq W < \min_{1<t\leq T}\{\xi_t''(\omega)\}$, then $E[\Pi(\omega, \overrightarrow{\alpha})]$ is increasing in all directions in $(\alpha_1, \alpha_2, ..., \alpha_{T-1})$ when $\xi_t \leq W, \forall t \in [2,T]$; therefore, it follows that $(\alpha_1^*, \alpha_2^*, ..., \alpha_{T-1}^*) = \overrightarrow{\alpha}'''_{prior} \equiv \{(\alpha_1, \alpha_2, ..., \alpha_{T-1}) : \xi_t = W, \forall t \in [2,T]\}$. Consequently, if $\min_{1<t\leq T}\{\xi_t''(\omega)\} \leq W < \max_{1<t\leq T}\{\xi_t''(\omega)\}$, then $\alpha_t''' \leq \alpha_t^* \leq \alpha_t''$ since $E[\Pi(\omega, \overrightarrow{\alpha})]$ is decreasing at $\overrightarrow{\alpha}''$ in some directions while maximized in others. This completes the proof. ■

Proof of Theorem 2. In the following, we consider the case where $TH_{d,t} < 1 \Leftrightarrow TH_{d,t} = (v-p)/l <$

1, and $TH_{r,t} < k \Leftrightarrow TH_{r,t} = \xi_t(v-p) < k$ (see Figure 2); other cases can be proven similarly. Before we present the proof, we derive the following property.

Property 2. $\frac{\partial F_t(\omega)}{\partial \omega} < 0$ *for* $\omega > 0, t = 2, ..., T$.

Proof of Property 2. Using Eqns. (12) and (13), it follows that:

$$\begin{aligned}
F_t(\omega) &= \frac{\omega + aF_1 - \omega(1-F_1)e^{-(\omega+a)(t-1)}}{\omega + aF_1 + a(1-F_1)e^{-(\omega+a)(t-1)}} = 1 - \frac{(a+\omega)(1-F_1)e^{-(\omega+a)(t-1)}}{\omega + aF_1 + a(1-F_1)e^{-(\omega+a)(t-1)}} \\
&= 1 - \frac{(a+\omega)(1-F_1)}{(\omega+aF_1)e^{(\omega+a)(t-1)} + a(1-F_1)} \\
\Rightarrow \frac{\partial F_t(\omega)}{\partial \omega} &= (1-F_1)\frac{(\omega+aF_1)e^{(\omega+a)(t-1)} + a(1-F_1) - (a+\omega)e^{(\omega+a)(t-1)}(1+(\omega+aF_1)(t-1))}{\left((\omega+aF_1)e^{(\omega+a)(t-1)} + a(1-F_1)\right)^2} \\
&= (1-F_1)e^{(\omega+a)(t-1)}\frac{(\omega+aF_1) + a(1-F_1)e^{-(\omega+a)(t-1)} - (a+\omega)(1+(\omega+aF_1)(t-1))}{\left((\omega+aF_1)e^{(\omega+a)(t-1)} + a(1-F_1)\right)^2} \\
&= (1-F_1)e^{(\omega+a)(t-1)}\frac{a(1-F_1)(e^{-(\omega+a)(t-1)}-1) - (a+\omega)(\omega+a)(\omega+aF_1)(t-1)}{\left((\omega+aF_1)e^{(\omega+a)(t-1)} + a(1-F_1)\right)^2} < 0.
\end{aligned}$$

Proof of Theorem 2. Using Eqn.s. (3), (4), Property (2), and Theorem 3, when $\beta(CF) < W$, it holds that:

$$\begin{aligned}
q^*_{s,t}(\omega) = CF\, b\, I_{s,t}(\omega, \beta(CF)) \Rightarrow \frac{\partial q^*_{s,t}(\omega)}{\partial \omega} &= -b\, CF\, \frac{(2-\beta(CF))\,\beta(CF)\,B}{2W}\,\frac{\partial F_t(\omega)}{\partial \omega} < 0 \\
q^*_{o,t}(\omega) = b\, I^1_{o,t}(\omega, \beta(CF)) &+ (1-CF)\, b\, [BF_t(\omega) - I^1_{o,t}(\omega, \beta(CF))] \\
\Rightarrow \frac{\partial q^*_{o,t}(\omega)}{\partial \omega} = b\, CF\, B\left[1 - \frac{\beta(CF)^2}{2W}\right]\frac{\partial F_t(\omega)}{\partial \omega} &+ b\,(1-CF)B\frac{\partial F_t(\omega)}{\partial \omega} > 0 \\
\Rightarrow \frac{\partial q^*_{s,t}(\omega)}{\partial \omega} + \frac{\partial q^*_{o,t}(\omega)}{\partial \omega} &= bB(1 - CF\beta(CF)) > 0.
\end{aligned}$$

This completes the proof. ■

Proof of Lemma 4. In the following, we consider the case where $TH_{d,t} < 1 \Leftrightarrow TH_{d,t} = (v-p)/l < 1$; the case where $TH_{d,t} = 1 \Leftrightarrow TH_{d,t} = (v-p)/l \geq 1$ can be proven similarly. From Eqn. (15), it follows that:

$$\frac{\partial E[\pi_t(\omega, \alpha_t, \xi_t)]}{\partial \omega} = -\frac{\partial G(\omega)}{\partial \omega} + bB\frac{\partial F_t(\omega)}{\partial \omega}\left[\frac{p+s}{2} - c\right] - b\frac{\partial Y_t}{\partial \omega}\left[(p+s-c)\alpha_t - (p+s)\frac{\alpha_t^2}{2}\right] \tag{27}$$

By assumption, $G(\omega)$ is increasing in ω. Then, using the definition of Y_t and Property 2, the proof follows. ■

Proof of Theorem 3. From Theorem 1, if $W < \beta(CF)$, then $\overrightarrow{\alpha}^* = CF \cdot \overrightarrow{1} \Rightarrow \xi^*_t = \beta(CF) > W, \forall t = 1, ..., T$. Since, $\frac{p+s}{2} - c\left[(p+s-c)\alpha^*_t - (p+s)\frac{(\alpha^*_t)^2}{2}\right] \leq 0$ at $\alpha^*_t = CF$, and $\partial\alpha^*_t/\partial\omega = 0$, using Lemma 4, the proof follows. ■

Proof of Lemma 5. Given that $\xi_t^P = \beta(\alpha_t) \Rightarrow \frac{\partial \xi_t^P}{\partial \alpha_t} = -\ln \alpha_t$, and using Property 1, it follows that:

$$E[\pi_t^P(\omega,\alpha_t,\xi_t)] = bBF_t(\omega)\left[\frac{p+s}{2}-c\right]+bY_t\left[(p+s-c)\alpha_t-(p+s)\frac{\alpha_t^2}{2}\right]-\frac{bs}{2} \tag{28}$$

$$\Rightarrow \frac{\partial E[\pi_t^P(\omega,\alpha_t,\xi_t)]}{\partial \alpha_t} = -b\frac{\partial Y_t}{\partial \xi_t}\ln\alpha_t\left[(p+s-c)\alpha_t-(p+s)\frac{\alpha_t^2}{2}\right]+bY_t\left[p+s-c-(p+s)\alpha_t\right] \tag{29}$$

$$\Rightarrow \frac{\partial^2 E[\pi_t^P(\omega,\alpha_t,\xi_t)]}{\partial \alpha_t^2} = -b\frac{\partial Y_t}{\partial \xi_t}\frac{1}{\alpha_t}\left[(p+s-c)\alpha_t-(p+s)\frac{\alpha_t^2}{2}\right]-2b\frac{\partial Y_t}{\partial \xi_t}\ln\alpha_t\left[p+s-c-(p+s)\alpha_t\right]-bY_t(p+s) \tag{30}$$

Using the definition of Y_t, Property 1 and Eqn. (28), it follows that $E[\pi_t^P(\omega,\alpha_t,\xi_t)]$ is continuous in $\alpha_t \in (0,1]$, for a given $\omega > 0$. Let $\alpha_{TH} \equiv \{\alpha_t : \xi_t = W\}$, where $\xi_t = \beta(\alpha_t)$. Then, there are three cases to consider: *Case 1.* If $0 < \alpha_{TH} \le CF \Leftrightarrow 0 < W \le \beta(CF)$, then $E[\pi_t(\omega,\alpha_t,\xi_t)]$ is strictly increasing in $\alpha_t \in (0, CF)$, and strictly concave and differentiable in $\alpha_t \in [CF,1]$ with a maximizer at $\alpha_t^{*P} = CF$. *Case 2.* If $CF < \alpha_{TH} \le 1 \Leftrightarrow \beta(CF) < W \le 1$, then $E[\pi_t^P(\omega,\alpha_t,\xi_t)]$ is strictly increasing in $\alpha_t \in (0,CF)$, strictly concave in $\alpha_t \in (CF,\alpha_{TH})$, and strictly concave decreasing in $\alpha_t \in (\alpha_{TH},1]$ with $\lim\limits_{\alpha_t \to \alpha_{TH}-} \partial E[\pi_t^P(\omega,\alpha_t,\xi_t)]/\partial\alpha_t > \lim\limits_{\alpha_t \to \alpha_{TH}+} \partial E[\pi_t^P(\omega,\alpha_t,\xi_t)]/\partial\alpha_t$ by using Eqn. (29), which implies that $E[\pi_t^P(\omega,\alpha_t,\xi_t)]$ is strictly concave in $\alpha_t \in [CF,1]$ with a maximizer at $\alpha_t^{*P} \in (CF,\alpha_{TH}]$. *Case 3.* If $\alpha_{TH} > 1 \Leftrightarrow W > 1$, then $E[\pi_t^P(\omega,\alpha_t,\xi_t)]$ is strictly increasing in $\alpha_t \in (0,CF)$ and strictly concave in $\alpha_t \in [CF,1]$ with a maximizer at $\alpha_t^{*P} \in (CF,1)$. Also, using Eqn. (29), it follows that the first derivative of $E[\pi_t^P(\omega,\alpha_t,\xi_t)]$ is independent of the adoption level $F_t(\omega)$; therefore, $\alpha_t^{*P} = \alpha^{*P}$ for $t \le T$. This completes the proof. ■

Proof of Theorem 4.
Proof of Part 1. It directly follows from Lemma 5 that, for a given $\omega > 0$, if $0 < \alpha_{TH} \le CF \Leftrightarrow 0 < W \le \beta(CF)$, then $\alpha^{*P} = CF$. If, on the other hand, $CF < \alpha_{TH} \le 1 \Leftrightarrow \beta(CF) < W \le 1$, then $\alpha^{*P} \in (CF,\alpha_{TH}]$; that is, the maximizer of $E[\pi_t^P(\omega,\alpha_t,\xi_t)]$ is also the maximizer of $E[\pi_t^P(\omega,\alpha_t,\xi_t)]$ when $\xi_t \le W$ (equivalently, $\alpha_t < \alpha_{TH}$). Let α' denote the point that sets the FOC of $E[\pi_t^P(\omega,\alpha_t,\xi_t)]$ when $\xi_t \le W$ equal to zero. It follows from Eqn. (29) that $\alpha' \in (0,1)$. Then, there are two cases to consider: *Case 1.* If $\alpha_{TH} < \alpha'$ (equivalently, $W < \beta(\alpha')$) then $\alpha^{*P} = \alpha_{TH}$, and *Case 2.* If $\alpha_{TH} \ge \alpha'$ (equivalently, $W \ge \beta(\alpha')$) then $\alpha^{*P} = \alpha'$.
Proof of Part 2. Proof follows similar to the proof of Theorem 2.
Proof of Part 3. Proof follows similar to the proof of Theorem 3.
This completes the proof. ■

Proof of Theorem 5. Before we present the proof for Theorem 5, we derive the following property.

Property 3. $\partial^2 E[\Pi(\omega,\overrightarrow{\alpha})]/\partial\alpha_t\partial\alpha_j \le 0$, *for* $\forall \alpha_t,\alpha_j \in (CF,1), j,t \le T$.

Proof of Property 3. Taking the derivative of Eqn. (26) with respect to α_j, for $j \le T$, it follows that $\partial^2 E[\Pi(\omega,\alpha_t,\xi_t)]/\partial\alpha_t\partial\alpha_j \le 0$ for $j, \forall t \le T$. This completes the proof. ■

Proof of Theorem 5.
Proof of Part 1. The proof follows from Theorems 1 and 4.
Proof of Part 2. The proof proceeds in three parts. First, we derive the expression for the upper bound $UB_{\alpha_t^*}(\lambda, CF)$ on $\alpha_t^*, t = 1,2,...,T-1$, for a given pair of (λ, CF) and for any $\omega > 0$. Next, we show that $UB_{\alpha_t^*}(\lambda, CF)$ is decreasing in t and increasing λ, and hence $UB(CF) \equiv UB_{\alpha_1^*}(1, CF)$ is an upper bound on $\overrightarrow{\alpha}^*$ for a given CF. Finally, we complete the proof by showing that $\alpha^{*P} \le UB_{\alpha^{*P}} \equiv \alpha' < UB(CF)$.

Factoring out $p+s$ in Eqn. (26), we get:

$$\frac{\partial E[\Pi(\omega,\overrightarrow{\alpha})]}{\partial \alpha_t} = (p+s)Y_t[CF-\alpha_t] - (p+s)\lambda \ln \alpha_t \sum_{i=1}^{T-t} \frac{\partial Y_{t+i}}{\partial \xi_{t+i}}(1-\lambda)^{i-1}\left[CF\alpha_{t+i} - \frac{\alpha_{t+i}^2}{2}\right] \tag{31}$$

For a given ω, when $\xi_t^* \leq W, \forall t = 2,...,T$; that is $\overrightarrow{\alpha}^*(\omega) = \overrightarrow{\alpha}''(\omega)$, it follows that $\partial E[\Pi(\omega,\overrightarrow{\alpha})]/\partial\alpha_t$ is decreasing in α_j, for $j \leq T$ (using Property 3), and increasing in $\frac{[1-BF_{t+i}(\omega)]}{[1-BF_t(\omega)]}$, for $i \in [1, T-t]$ (from Eqn. (31)). Then, $UB_{\alpha_t''}(\lambda, CF)$ sets Eqn (31) equal to zero when $\alpha_j = CF, \forall j \leq T, j \neq t$, i.e. at its lower bound, and $\frac{[1-BF_{t+i}]}{[1-BF_t]} = 1$, i.e. at its maximum, which is obtained by setting $F_t(\omega) = F_{t+i}(\omega)$, for $i \in [1, T-t]$ (Hence, $UB_{\alpha_t^*}(\lambda, CF)$ is independent of B, ω and a.).

When $\alpha_j = CF$, for $\forall j \leq T, j \neq t$, and $F_t = F_{t+i}$, for $i \in [1, T-t]$, the first term in Eqn. (31) is the same for all $\partial E[\Pi(\omega,\overrightarrow{\alpha})]/\partial\alpha_t$; however, the second term is a summation over all $\alpha_{t+i}, i \in [1, T-t]$. Thus, for a larger t, the second term in Eqn. (31) is less positive; hence, $UB_{\alpha_t''}(\lambda, CF)$ is decresing in t. Also, when $\alpha_j = CF$, for $\forall j \leq T, j \neq t$, Eqn. (31) is decreasing in α_t and increasing in λ. Thus, $UB_{\alpha_t''}(\lambda, CF)$, $t < T$, is increasing in λ. Hence, $UB(CF) \equiv UB_{\alpha_1^*}(1, CF)$ is an upper bound on $\overrightarrow{\alpha}^*$.

Setting $\lambda = 1$, $\alpha_j = CF, \forall j \leq T, j \neq 1$, and $F_1 = F_{1+i}$, for $i = 1,...,T-1$, in Eqn. (31), the FOC for α_1 follows:

$$\beta(CF)\left[CF - UB(CF)\right] - \ln\left(UB(CF)\right)\left[\frac{CF^2}{2}\right] = 0. \tag{32}$$

When $\beta(\alpha') \leq W$; that is $\alpha^{*P} = \alpha'$, at optimality it must hold that (using Eqn. 29):

$$\beta(\alpha')[CF-\alpha'] - \ln\alpha'\left[CF\alpha' - \frac{\alpha'^2}{2}\right] = 0. \tag{33}$$

That is, using Eqn.s. (32) and (33),

$$\beta(CF)\left[CF - UB(CF)\right] - \ln\left(UB(CF)\right)\left[\frac{CF^2}{2}\right] = \beta(\alpha')[CF-\alpha'] - \ln\alpha'\left[CF\alpha' - \frac{\alpha'^2}{2}\right] \tag{34}$$

We prove that $\alpha^{*P} \leq UB_{\alpha^{*P}} \equiv \alpha' < UB(CF)$ by contradiction. Assume that $\alpha' \geq UB(CF) > \beta(CF)$, then using Eqn. (34), it must hold that:

$$\beta(CF)\left[CF - UB(CF)\right] - \beta(\alpha')[CF-\alpha'] = \ln\left(UB(CF)\right)\left[\frac{CF^2}{2}\right] - \ln\alpha'\left[CF\alpha' - \frac{\alpha'^2}{2}\right] > 0. \tag{35}$$

In addition, $\ln\left(UB(CF)\right) < \ln\alpha' < 0$ and $\frac{CF^2}{2} > CF\alpha' - \frac{\alpha'^2}{2} > 0 \Rightarrow \ln\left(UB(CF)\right)\left[\frac{CF^2}{2}\right] - \ln\alpha'\left[CF\alpha' - \frac{\alpha'^2}{2}\right] < 0$, which poses a contradiction with Eqn. (35). Hence, it must hold that $\alpha^{*P} \leq UB_{\alpha^{*P}} \equiv \alpha' < UB(CF)$, which implies that $|\alpha^{*P} - \alpha_t^*| < UB(CF) - CF$. This completes the proof. ■

Appendix C. Justification of Assumptions

In this section, we provide justifications for some of the modeling assumptions:

- **The online (store) channel has 100% (limited) availability.** The assumption of 100% availability for the online channel is not unreasonable given the current online retailing practices, and does not modify the important tradeoffs that we aim to model in this paper. Indeed, it is common for online retailers to target extremely high levels of online inventory availability. For example, the Journal of Commerce (2016) reports that "US retailers reducing inventory, but e-commerce stock balance still tricky," one reason being that "the goal of lowering inventory (and related costs) can clash with the desire to sell more products." Such pressure is inevitably higher in the online market due to the fierce competition between an ever-increasing number of retailers, and as a result, it is not uncommon for online channels to hold more inventory than their store counter-parts. This is also in line with our own experience with industry partners, including a leading global retailer in the sporting goods sector, which strategically targets a minimum of 95% availability in its online channel, and Bol.com, the leading Dutch online retailer, which targets an online availability of at least 96% for the majority of its product offerings. In reality, both companies provide higher availability (see Kiemeneij 2017). Second, modeling limited availability in the store channel is of main importance, because this has a major impact on the consumer's channel choice. Due to the time and energy consumers invest in their shopping trips: "Consumers would not patronize a firm without some form of assurance that they can find what they are looking for" (Su and Zhang 2009). While consumers incur a travel/time cost for visiting the store before product availability is actually observed, they can easily check the availability in the online channel on their smartphones, tablets, etc. at a negligible cost before finalizing their purchasing decision. Given that the focus of our paper is on the shaping of consumer demand, this simplifying assumption preserves the essential aspects of the problem, and is in line with the literature (see Chen et al. 2008, Gao and Su 2016, 2017 with similar assumptions).

- **Market size is stochastic, exogenous, and identical across the different time periods.** Retailers commonly face stochasticity in market size. Hence, they may execute their inventory planning based on a probability distribution of total demand, which relies on historical data and/or expert opinion. Since the focus of this paper is on tactical planning, it is not unreasonable to assume that this distribution can be used for the planning of inventory over the entire planning horizon. In other words, the operational aspect of continuously updating the demand distribution in each period based on recently observed data is beyond the scope of this work. The exogeneity in total market size is also a common assumption in the multi-channel retailing literature, as the main goal in such studies is to study the impact of multiple sales channels on the channel migration behavior of existing consumers, who would shop through a single channel in the absence of alternatives (e.g. Cattani et al. 2006, Chen et al. 2008, Yoo and Lee 2011, and Gao and Su 2016).

- **Diffusion of e-commerce adoption depends on the retailer's marketing efforts and the word of mouth effect.** In reality, the diffusion of e-commerce adoption may depend on a variety of other factors such as the online delivery lead time and consumer's distance to the store as well as product availability, customer service quality and convenience of returns. However, accounting for all these factors would lead to a significant increase in the parameter space, leading to a much less tractable problem. We consider a subset of the aforementioned factors in our discrete choice model, as explained in Section 3.2.

- **The retailer's business setting is determined by the exogenous parameters k and l.** The retailer's store service radius, k, is a strategic decision, as it strategically places its store to serve a neighborhood of consumers. In addition, the delivery time, l, for the online channel can be highly constrained by the current supply chain infrastructure and/or contracts with third party logistics providers. Since the retailer's strategic decisions and contracts with external parties are outside the scope of this paper, we characterize the retailer's business setting in terms of parameters k and l.

- **Consumers are homogeneous in their learning speed and predicted store availability levels.** Our model assumes that all consumers learn at the same speed. Even though it may be desirable to introduce heterogeneity of the learning speed, this would highly complicate the problem. The assumption that consumers are homogeneous in their learning speed captures the essential component of the problem, and, thus, is commonly used in the literature that utilizes adaptive learning within multi-period settings, see Section 2. Also, our learning model implies that consumers are homogeneous in their predicted store availability (i.e. ξ_t is the same for all consumers). This is not an unreasonable assumption, as today's consumers can easily share information with each other through word of mouth, social media, etc. to gather information about the past availability levels in order to form a common prediction of the current availability level (β_t) of the company (Bernstein and Federgruen 2004). Of course it would be interesting to study the setting in which consumers construct individual availability predictions based on their own history of stock-outs. However, this approach would require tracking each individual's stock availability experience over the planning horizon (either 0% or 100%), resulting in a substantially more complicated model. Further, as Liu and Van Ryzin (2011) emphasizes, "assuming customers only form expectations based on their individual purchase experience is equally unrealistic. It would suggest, for example, that we could only know whether a car maker's automobiles are overstocked if we attempt to buy one." Hence, especially for products that are not purchased repeatedly, updating of the predicted availability levels based on individual history is the opposite extreme. Even though the reality lies somewhere in between, our stylized model (which assumes that consumers have the same availability expectation, similar to Dana and Petruzzi 2001, Liu and Van Ryzin 2011, Gao and Su 2016, among others) captures the essence of the problem, and applies to a wide selection of products that are purchased repeatedly or not, and to consumers who have previously purchased the product through the store channel or not.

- **The unit purchasing cost is uniform across channels.** In reality, there may be a multitude of other costs related to each channel that the retailer operates, such as the cost of operating a physical store (e.g. rent, labor cost, utilities), the cost of operating an online channel (e.g. web hosting, IT resources, cyber security), fulfillment cost associated with an online channel (e.g. rent for the warehouse, labor cost for order picking/packaging, utilities, cost of delivery to consumer), fulfillment cost associated with a store channel (e.g. rent for the distribution center, labor cost, utilities, cost of delivery to store). Such costs would be highly dependent on the business models that retailers adopt for their operations. Even though most of these costs can be easily incorporated into our model, such an approach would make our parameter space quite large. Therefore, in order to improve the exposition of the paper and to keep our analysis general, we normalize certain costs, and focus on those that are most relevant within the scope of our paper. This allows us to create a parsimonious model that incorporates the essential cost components of the problem that we study and to derive analytical insights related to the retailer's tactical decisions, which are our main focus.

Appendix D. Additional Analyses

1. An example of the retailer's optimal service levels for a given ω

Example 1 depicts the change in optimal service levels with service radius, k, for a given ω.

Example 1. *Let* $a = 0.0065$, $\omega = 0.0003$, $\lambda = 0.4$, $v = 29.99$, $p = 24.99$, $c = 10.75$, $s = 0$, $l = 3$, $T = 4$, *and* $\alpha_0 = CF = 0.57$. *Figure 4 depicts the optimal service levels as a function of service radius, k, for e-commerce adoption level vector* $\overrightarrow{F} = (F_1, F_2, F_3, F_4) = (0.0299, 0.0304, 0.0309, 0.0314)$.

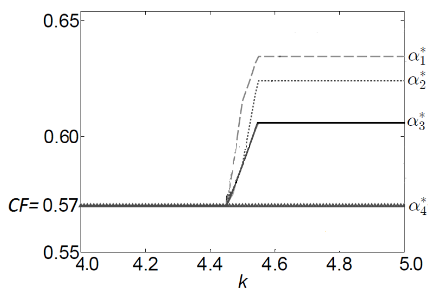

Figure 4: $\overrightarrow{\alpha}^*$ as a function of k.

2. Sensitivity analysis with respect to parameters k, l, and v

A sensitivity analysis was conducted to observe the change in the optimal solution with respect to low, medium and high values of k, l, and v parameters for each product; namely, $l = 2, 6, 10$, $k = 15, 22, 29$, and $v = p+1, p+5, p+9$ when $\lambda = 0.5$ (all other parameters have been kept the same as in Section 6). Table 8 shows a summary of the results obtained for 3^3 experiments (i.e. problems with different parameter values) per product. That is, after computing the optimal solution for 3^3 problems, we report the mean, minimum, and maximum (optimal) service levels, level of marketing efforts, and profit, which are represented by α_y^*, ω_y^*, and $E[\Pi_y^*]$, for $y \in \{\text{mean, min, max}\}$, respectively.

Table 8: Summary of the sensitivity analysis..

Product	ω^*_{mean} $[\omega^*_{min}, \omega^*_{max}]$	α^*_{mean} $[\alpha^*_{min}, \alpha^*_{max}]$	$E[\Pi^*_{mean}]$ $\left[E[\Pi^*_{min}], E[\Pi^*_{max}]\right]$
Candle	0.0797 [0, 0.1675]	0.625 [0.570, 0.640]	3,866 [554, 8,716]
Wallet	0.0690 [0, 0.1730]	0.606 [0.550, 0.621]	6,951 [1,107, 15,296]
Auto navigator	0.2139 [0, 0.3797]	0.631 [0.580, 0.650]	25,834 [4,495, 45,546]

Below, we present a subset of our results (for the candle example as a representative product; other products show similar patterns). Columns 1-3 show the results for the case with $v = p+5$ and $k = 22$ with varying l, columns 4-6 show the results for the case with $v = p+5$ and $l = 10$ with varying k, and columns 7-9 show the results for the case with $l = 10$ and $k = 22$ with varying v. The results show a profit increase as delivery lead time gets smaller, as this allows more consumers to patronize the retailer. Not surprisingly, the same trend prevails for the store radius. Similarly, the retailer's profit increases as the value that consumers place on the product increases. When consumers have a higher valuation of the product, they are willing to accomodate higher inconvenience costs. This lowers choice-based lost demand. Some findings, which have been described in Section 6, also show in Table 9. For example, as inventory service levels increase, we see a decrease in the level of e-commerce marekting efforts due to the cost trade-off between the sets of two decisions.

Table 9: A subset of sensitivity results for the candle example.

	$l = 2$	$l = 6$	$l = 10$	$k = 15$	$k = 22$	$k = 29$	$v = p + 1$	$v = p + 5$	$v = p + 9$
ω^*	0.1462	0.1285	0.0873	0.0553	0.0873	0.1018	0.0363	0.1018	0.1006
α_1^*	0.628	0.632	0.637	0.638	0.637	0.636	0.640	0.636	0.633
α_2^*	0.622	0.625	0.629	0.630	0.629	0.628	0.631	0.628	0.626
α_3^*	0.608	0.610	0.612	0.612	0.612	0.612	0.613	0.612	0.610
α_4^*	0.570	0.570	0.570	0.570	0.570	0.570	0.570	0.570	0.570
$E[\Pi^*]$	4,088	3,897	3,577	4,977	3,577	2,871	554	2,871	4,878

3. Sensitivity analysis with respect to parameter T

A sensitivity analysis was conducted to demonstrate the impact of the length of the planning horizon, T, on the retailer's optimal decisions. Table 10 presents the summary of results for the candle example for different values of T when $\lambda = 0.5$ and $\lambda = 1$ (all other parameters have been kept the same as in Section 6). Our results show that the optimal service levels continue to follow a monotone decreasing pattern in t, i.e. $\alpha_1^* > \alpha_2^* > \alpha_3^* > \alpha_4^*$, for smaller λ (i.e. $\lambda = 0.5$), regardless of the length of the planning horizon. However, the magnitude of the service levels tends to increase as the planning horizon gets longer, since this strategy allows the retailer to increase consumers' service level expectations early in the planning horizon to attract demand. The findings for the case with larger λ (i.e. $\lambda = 1$) show that the optimal service levels continue to follow a non-monotonic pattern, regardless of the length of the planning horizon. However, in this setting, we consistently see the pattern that towards the end of the planning horizon, the optimal service levels get quite close to that of the perfect information case – with the exception of the last period, where the optimal service level will always equal to CF.

Table 10: Retailer's optimal decisions for different planning horizons.

	λ=0.5					λ=1				
T	4	6	8	10	12	4	6	8	10	12
w^*	0.0553	0.0677	0.0757	0.0807	0.0835	0.0542	0.0670	0.0753	0.0803	0.0832
α_1^*	0.638	0.643	0.643	0.643	0.643	0.646	0.646	0.646	0.646	0.645
α_2^*	0.630	0.640	0.642	0.642	0.642	0.644	0.643	0.643	0.643	0.643
α_3^*	0.612	0.636	0.641	0.642	0.642	0.645	0.644	0.643	0.643	0.643
α_4^*	0.570	0.629	0.639	0.641	0.642	0.570	0.644	0.643	0.643	0.643
α_5^*	N/A	0.612	0.636	0.641	0.642	N/A	0.645	0.644	0.643	0.643
α_6^*	N/A	0.570	0.629	0.639	0.641	N/A	0.570	0.644	0.644	0.643
α_7^*	N/A	N/A	0.612	0.636	0.641	N/A	N/A	0.645	0.644	0.644
α_8^*	N/A	N/A	0.570	0.629	0.639	N/A	N/A	0.570	0.644	0.644
α_9^*	N/A	N/A	N/A	0.612	0.636	N/A	N/A	N/A	0.645	0.644
α_{10}^*	N/A	N/A	N/A	0.570	0.629	N/A	N/A	N/A	0.570	0.644
α_{11}^*	N/A	N/A	N/A	N/A	0.612	N/A	N/A	N/A	N/A	0.645
α_{12}^*	N/A	N/A	N/A	N/A	0.570	N/A	N/A	N/A	N/A	0.570
$E[\Pi^*]$	4,975	7,571	10,224	12,926	15,668	5,010	7,609	10,263	12,965	15,707

In order to evaluate the optimal solution in longer planning horizons, we solved the problem for T = 20, 40, 60, 80, 100, and we observed a similar behavior. That is, in the long term when the planning horizon approaches to an infinite horizon case, we observe that the optimal service levels approach to that of the perfect information case. For instance, when we solve the problem for T

= 100, we observe that the optimal service levels from period 42 to 98 are identical to the optimal service level of the perfect information case with the accuracy of four decimal places ($\alpha^{*P} = 0.6453$). Note that the service level in the last period will always equal CF (and the service level set in the period before may be slightly higher due to the drop in service in the last period – in our case these numbers correspond to $\alpha^*_{99} = 0.6464$ and $\alpha^*_{100} = 0.5700$).

4. Intermediary results with respect to Table 3

In this section, we report the values of some of the intermediary variables at the optimal solution of the cases provided in Table 3. Specifically, for each λ value, we report the e-commerce adoption levels, F_t, the expected sales in the store channel, $E[min(Z_{s,t}, q_{s,t})]$, and the expected sales in the online channel, $E[Z_{o,t}]$, throughout the planning horizon, i.e. $t = 1, 2, 3, 4$ (the entries in brackets represents the respective values in each period for a planning horizon of four periods). Additionally, for the online sales, we present 1) the amount of sales due to lost sales recovery (i.e. number of consumers who switch to online channel as a result of a stock-out in the store), $\frac{I^2_{o,t}(\omega,\xi_t)}{I_{s,t}(\omega,\xi_t)} E[(Z_{s,t} - q_{s,t})^+]$, and 2) amount of sales due to market expansion through opening an online channel, $I^3_{o,t}(\omega, \xi_t) * b/2$, where $I^3_{o,t}(\omega, \xi_t) = (1 - \xi_t/W)BF_t$.

Table 11: Intermediary results with respect to Table 3.

λ	Adoption levels	Store sales	Online sales	Lost sales recovery	Market expansion
(a) Adaptive learning (limited information) setting with various values of λ					
0.00	[0.09, 0.14, 0.19, 0.24]	[118, 116, 115, 113]	[19, 31, 43, 54]	[0.53, 0.85, 1.17, 1.47]	[15, 25, 34, 42]
0.25	[0.09, 0.14, 0.19, 0.24]	[124, 122, 119, 114]	[19, 31, 43, 54]	[0.41, 0.72, 1.07, 1.51]	[15, 25, 34, 42]
0.50	[0.09, 0.14, 0.19, 0.24]	[126, 126, 123, 116]	[19, 31, 42, 54]	[0.37, 0.66, 1.00, 1.55]	[15, 24, 33, 42]
0.75	[0.09, 0.14, 0.19, 0.24]	[127, 128, 126, 117]	[19, 31, 42, 53]	[0.36, 0.63, 0.94, 1.58]	[15, 24, 33, 42]
1.00	[0.09, 0.14, 0.19, 0.24]	[127, 130, 128, 118]	[19, 31, 42, 53]	[0.36, 0.64, 0.87, 1.60]	[15, 24, 33, 42]
(b) Perfect information setting					
	[0.08, 0.13, 0.18, 0.23]	[132, 130, 129, 127]	[18, 29, 40, 50]	[0.37, 0.60, 0.83, 1.04]	[14, 23, 32, 40]

Expectedly, our results show that the volume of the store sales depend not only on the predicted product availability but also on the e-commerce adoption level. For example, when $\lambda = 0$, the optimal store service levels all equal CF; hence, the predicted store availability levels are constant throughout the planning horizon. As a result, when the e-commerce adoption, which is the only dynamic component in the problem, increases over time, the store sales decline, and online sales increase. The majority of the online sales can be attributed to the market expansion in this example, while amount of online sales due to lost sales recovery (i.e. availability-based substitution) is minimal. (It is also worth to mention that the number of consumers who would have shopped in the store channel in the absence of the online channel – which constitute the remainder of the online sales, but not explicitly shown in the above table due to space limitations – is also minimal.) On the other hand, when λ starts to increase, we see that availability expectations also start to take effect. For example, when $\lambda = 1$, we see that the store sales start slightly low at first due to relatively low availability expectation (i.e. $\xi_1 = \beta(CF)$). As store service levels increase, however, we see that store sales slightly increase, then start to drop due to e-commerce adoption growth. Similar to the smaller λ levels, we see that e-commerce sales increase over time. Note that the e-commerce adoption levels are very similar across different λ values. Although this may seem surprising at first, it is not unexpected, as the optimal marketing levels are very close to each other across different λ values (They differ at the third decimal place, which has very little impact on the overall adoption curve). Therefore, the difference in sales numbers can be mainly attributed to the changes in the service levels in Table 11. This is clearly demonstrated in the perfect information case: Constant (and high) availability levels in the store results in more sales in the store channel. The combination of high service levels with lower marketing efforts results in a decrease in the online sales and market expansion.

5. Analysis of the setting under constant e-commerce adoption levels

In this section, we analyze the setting in which the growth in e-commerce adoption is ignored. In other words, the e-commerce adoption levels throughout the planning horizon are constant, i.e. $a = 0, \omega = 0$. For this purpose, we use the candle example to construct Table 12, which is structured in a similar way to Table 3 to facilitate comparison of results under growing vs. constant adoption levels. Table 12 depicts the retailer's optimal service level policy for different values of λ. The results are intuitive. Specifically, in the perfect information case, the optimal service levels under constant e-commerce adoption levels equal to that of the setting with evolving e-commerce levels. This is not surprising, as the former is a special case of the latter setting. However, it is important to note that the total inventory ordered throughout the planning horizon decreases. This is because the market expansion that the online channel provides is limited (i.e. lower sales) when e-commerce adoption does not show any growth over time. In the adaptive learning case with constant e-commerce adoption, we observe that the optimal service levels follow the same pattern as in the growing adoption setting. That is, while they follow a decreasing pattern in t when λ is relatively small, they follow a non-monotone pattern when λ is large. However, we see that the service levels are slightly higher than that of the growing adoption setting. This is not surprising, as this case represents the setting when the retailer does not exert any e-commerce marketing effort. Hence, this enables the retailer to invest more in her inventory levels.

Table 12: Retailer's optimal decisions and resulting expected profit for the (a) adaptive learning, and (b) perfect information settings, with $F_t = 0.03$, for $t = 1, 2, 3, 4$.

(a) Adaptive learning (limited information) setting with various values of λ

λ	α_1^*	α_2^*	α_3^*	α_4^*	q_{total}^*	$E[\Pi^*]$	$\Delta(\%)$
0.00	0.570	0.570	0.570	0.570	727	4,854	2.33
0.25	0.620	0.609	0.594	0.570	765	4,874	1.93
0.50	0.641	0.631	0.613	0.570	791	4,903	1.35
0.75	0.647	0.643	0.631	0.570	807	4,925	0.91
1.00	0.648	0.645	0.646	0.570	817	4,941	0.58

(b) Perfect information setting

	α_1^{*P}	α_2^{*P}	α_3^{*P}	α_4^{*P}	q_{total}^{*P}	$E[\Pi^{*P}]$	
	0.645	0.645	0.645	0.645	850	4,970	

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