ARTICLE TEMPLATE

Supplementary Materials for "Variable Screening for Ultrahigh Dimensional Censored Quantile Regression"

ARTICLE HISTORY
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In this document, we provide some useful lemmas and detailed proofs for Theorems 1-2. Subsequently, some additional simulation results are presented in Section 2.

1. Technical proofs

Lemma 1 (Hoeffding's inequality, [2]). *For independent random variables $X_1, \ldots, X_n$ which satisfy that $P(X_i \in [a_i, b_i]) = 1$ for $1 \leq i \leq n$, where a_i and b_i are some constants. Then for any $\epsilon > 0$, we have*

$$P\big(|\bar{X} - E(\bar{X})| \geq \epsilon\big) \leq 2\exp\left\{ - \frac{2\epsilon^2 n^2}{\sum_{i=1}^n (b_i - a_i)^2} \right\},$$

where $\bar{X} = \sum_{i=1}^n X_i/n$ and $E(\bar{X})$ is the expected value of $\bar{X}$.

Lemma 2. *Let $\mathcal{F}$ be a class of distribution functions whose support is the same as that of F, and let $\mathcal{T}$ be the support of T. For any $\epsilon > 0$, define $\mathcal{H}(\epsilon) = \{F^* \in \mathcal{F} : ||F^* - F||_\infty \equiv \sup_{t\in\mathcal{T}} |F^*(t) - F(t)| \leq \epsilon$ and $|Q_\tau(T^*) - Q_\tau(T)| \leq \epsilon\}$, where T^* follows the distribution F^*. Then*

$$\begin{aligned}
&\sup_{\mathcal{H}(\epsilon)} |I\{Y < Q_\tau(T^*)\} - I\{Y < Q_\tau(T)\}| \\
&\leq I\{Q_\tau(T) - \epsilon \leq T < Q_\tau(T) + \epsilon\} + 3I\{Q_\tau(T) - \epsilon \leq C < Q_\tau(T) + \epsilon\}.
\end{aligned}$$

Proof of Lemma 2. Notice that

$$\begin{aligned}
I\{Y < Q_\tau(T)\} = &I\{C \geq Q_\tau(T), T < Q_\tau(T)\} + I\{C < Q_\tau(T), T \leq C\} \\
&+ I\{C < Q_\tau(T), T > C\},
\end{aligned}$$

thus

$$\sup_{\mathcal{H}(\epsilon)} |I\{Y < Q_\tau(T^*)\} - I\{Y < Q_\tau(T)\}| \leq \sup_{\mathcal{H}(\epsilon)} (S_1 + S_2 + S_3),$$

where

$$
\begin{aligned}
S_1 &= \left| I\{C \geq Q_\tau(T^*), T < Q_\tau(T^*)\} - I\{C \geq Q_\tau(T), T < Q_\tau(T)\} \right|, \\
S_2 &= \left| I\{C < Q_\tau(T^*), T \leq C\} - I\{C \geq Q_\tau(T), T \leq C\} \right|, \\
S_3 &= \left| I\{C < Q_\tau(T^*), T > C\} - I\{C \geq Q_\tau(T), T > C\} \right|.
\end{aligned}
$$

Using the fact that $\sup_{t':|t-t'|\leq\epsilon} \left| I\{T < t'\} - I\{T < t\} \right| \leq I\{t - \epsilon \leq T < t + \epsilon\}$, for any $\epsilon > 0$, we have

$$
\begin{aligned}
\sup_{\mathcal{H}(\epsilon)} S_1 &\leq \sup_{\mathcal{H}(\epsilon)} \left| I\{C \geq Q_\tau(T^*)\} - I\{C \geq Q_\tau(T)\} \right| + \left| I\{T < Q_\tau(T^*)\} - I\{T < Q_\tau(T)\} \right| \\
&\leq I\{Q_\tau(T) - \epsilon \leq C < Q_\tau(T) + \epsilon\} + I\{Q_\tau(T) - \epsilon \leq T < Q_\tau(T) + \epsilon\}, \\
\sup_{\mathcal{H}(\epsilon)} S_2 &\leq I\{Q_\tau(T) - \epsilon \leq C < Q_\tau(T) + \epsilon\}, \\
\sup_{\mathcal{H}(\epsilon)} S_3 &\leq I\{Q_\tau(T) - \epsilon \leq C < Q_\tau(T) + \epsilon\}.
\end{aligned}
$$

As a result, the conclusion of Lemma 2 holds. □

Lemma 3 (Lemma B.1, [1]). *Under Condition (C2), if $nh^2 \to \infty$ and $h \to 0$, then the local Kaplan-Meier estimator $\widehat{G}(t|x)$ satisfies:*
(1) $\sup_s \sup_x \left| \widehat{G}(s|x) - G(s|x) \right| = O\left(\{\log(n)/(nh)\}^{1/2} + h^2 \right)$ *a.s.;*
(2)

$$
\frac{1}{\widehat{G}(s|x)} - \frac{1}{G(s|x)} = \frac{1}{n} \sum_{j=1}^{n} \frac{B_{nj}(x)\xi(Y_j, \Delta_j, s, x)}{G^2(s|x)} + R_n(s, x),
$$

where

$$
\xi(Y_j, \Delta_j, s, x) = G(s|x) \left[-\int_0^{\min(Y_j, s)} \frac{dG(u|x)}{\{1 - F(u|x)\}G^2(u|x)} + \frac{I(Y_j \leq 0, \Delta_j = 0)}{\{1 - F(Y_j|x)\}G(Y_j|x)} \right],
$$

and $\sup_s \sup_x |R_n(s, x)| = O\left(\{\log(n)/(nh)\}^{3/4} + h^2 \right)$ *a.s., for $s \in [0, t_1]$ such that* $\inf_x \{1 - F(s|x)\} G(s|x) > 0$*;*
(3) $\sup_s \sup_x \left| \frac{1}{\widehat{G}(s|x)} - \frac{1}{G(s|x)} \right| = O\left(\{\log(n)/(nh)\}^{1/2} + h^2 \right)$ *a.s..*

Lemma 4 (Bernstein's inequality for U-statistics, [2]). *Let $U_n^2(g)$ denote the second-order U-statistics with kernel function $g(x_1, x_2)$ based on the independent random variables $X_1, \ldots, X_n$. Assume that the function g is bounded: $a < g < b$ for some finite constants a and b. If $E[g(X_i, X_j)] = 0$ when $i \neq j$, then for any $t > 0$, we have*

$$
P\left(|U^2(g)| > t \right) \leq 2 \exp\left\{ -\frac{2kt^2}{(b - a)^2} \right\},
$$

where k denotes the integer part of $n/2$.

Proof of Theorem 1. To start with, denote $\omega_k^* = \sqrt{\tau - \tau^2}\omega_k$, and $\widehat{\omega}_k^* = \sqrt{\tau - \tau^2}\widehat{\omega}_k$, for all $1 \le k \le p$, we have

$$P\left\{|\widehat{\omega}_k - \omega_k| \ge cn^{-\alpha}\right\} = P\left\{|\widehat{\omega}_k^* - \omega_k^*| \ge c^* n^{-\alpha}\right\},$$

where $c^* = \sqrt{\tau - \tau^2}c$. Note that

$$\begin{aligned}
|\widehat{\omega}_k^* - \omega_k^*| &= \tau\left|\frac{1}{n}\sum_{i=1}^n X_{ik} - EX_k\right| + \left|\frac{1}{n}\sum_{i=1}^n\left(\frac{1}{\widehat{G}(Y_i|X_{ik})} - \frac{1}{G(Y_i|X_{ik})}\right) I\{Y_i < \widehat{Q}_\tau(T)\}\Delta_i X_{ik}\right| \\
&+ \left|\frac{1}{n}\sum_{i=1}^n\left[I\{Y_i < \widehat{Q}_\tau(T)\} - I\{Y_i < Q_\tau(T)\}\right]\frac{\Delta_i}{G(Y_i|X_{ik})}X_{ik}\right| \\
&+ \left|\frac{1}{n}\sum_{i=1}^n\frac{\Delta_i}{G(Y_i|X_{ik})}I\{Y_i < Q_\tau(T)\}X_{ik} - E\left[\frac{\Delta}{G(Y|X_k)}I\{Y < Q_\tau(T)\}X_k\right]\right| \\
&\triangleq |I_1| + |I_2| + |I_3| + |I_4|.
\end{aligned} \tag{1}$$

For I_1, by Condition (C1) and Hoeffding's inequality in Lemma 1, we have

$$P\left(|I_1| \ge \frac{c^*}{4}n^{-\alpha}\right) \le 2\exp\left\{-\frac{(c^*)^2 n^{1-2\alpha}}{32\tau^2 M^2}\right\} = 2\exp\{-c_1 n^{1-2\alpha}\}, \tag{2}$$

where $c_1 = (c^*)^2/32\tau^2 M^2$.

For I_2, notice that

$$\begin{aligned}
I_2 &= \frac{1}{n}\sum_{i=1}^n\left(\frac{1}{\widehat{G}(Y_i|X_{ik})} - \frac{1}{G(Y_i|X_{ik})}\right)\left[I\{Y_i < \widehat{Q}_\tau(T)\} - I\{Y_i < Q_\tau(T)\}\right]\Delta_i X_{ik} \\
&+ \frac{1}{n}\sum_{i=1}^n\left(\frac{1}{\widehat{G}(Y_i|X_{ik})} - \frac{1}{G(Y_i|X_{ik})}\right) I\{Y_i < Q_\tau(T)\}\Delta_i X_{ik} \triangleq I_{21} + I_{22}.
\end{aligned} \tag{3}$$

We first deal with the term I_{21}. According to Lemma 3, we know that

$$\sup_s \sup_x \left|\frac{1}{\widehat{G}(s|x)} - \frac{1}{G(s|x)}\right| = O\left(\{\log n/(nh)\}^{1/2} + h^2\right) = o(1),$$

if $nh^2 \to \infty$, $h \to 0$. Thus for sufficiently large n, we have

$$P\left(\max_{1\le i\le n}\left|\frac{1}{\widehat{G}(Y_i|X_{ik})} - \frac{1}{G(Y_i|X_{ik})}\right| \ge 1\right) = 0.$$

Therefore, it follows that

$$
\begin{aligned}
&P\left(|I_{21}| \geq \frac{c^*}{8} n^{-\alpha}\right) \\
&\leq P\left(\max_{1\leq i\leq n}\left|\frac{1}{\widehat{G}(Y_i|X_{ik})} - \frac{1}{G(Y_i|X_{ik})}\right| \cdot \frac{1}{n}\sum_{i=1}^n \left|\left[I\{Y_i < \widehat{Q}_\tau(T)\} - I\{Y_i < Q_\tau(T)\}\right]\Delta_i X_{ik}\right| \geq \frac{c^*}{8} n^{-\alpha}\right) \\
&\leq P\left(\frac{1}{n}\sum_{i=1}^n \left|\left[I\{Y_i < \widehat{Q}_\tau(T)\} - I\{Y_i < Q_\tau(T)\}\right]\Delta_i X_{ik}\right| \geq \frac{c^*}{8} n^{-\alpha}\right) \\
&\leq P\left(\frac{1}{n}\sum_{i=1}^n \left|I\{Y_i < \widehat{Q}_\tau(T)\} - I\{Y_i < Q_\tau(T)\}\right| \geq c_2 n^{-\alpha}\right), \qquad (4)
\end{aligned}
$$

where $c_2 = c^*/8M$. Under Condition (C2) and (C3), we have $||\widehat{F}_{\text{KM}} - F|| = O(n^{-1/2}(\log n)^{1/2})$ and $|\widehat{F}^{-1}_{\text{KM}}(\tau) - Q_\tau(Y)| = O(n^{-1/2}(\log n)^{1/2})$ almost surely [3]. Let $c_3 = 1 + 16f(Q_\tau(T))$ and $\mu_1 = E[I\{Q_\tau(T) - c_2 n^{-\alpha} c_3^{-1} \leq T < Q_\tau(T) + c_2 n^{-\alpha} c_3^{-1}\}]$, where $f(t)$ is the distribution function of T. Applying Taylor's expansion, we can obtain that $\mu_1 = 2f(Q_\tau(T))c_2 n^{-\alpha} c_3^{-1} + [f'(Q^*_\tau(T)) - f'(Q^{**}_\tau(T))]c_2^2 n^{-2\alpha} c_3^{-2}/2$, where both $Q^*_\tau(T)$ and $Q^{**}_\tau(T)$ lie in $[Q_\tau(T) - c_2 n^{-\alpha} c_3^{-1}, Q_\tau(T) + c_2 n^{-\alpha} c_3^{-1}]$. Under Condition (C3), we can choose c_4 and n such that $|f'(Q^*_\tau(T)) - f'(Q^{**}_\tau(T))| \leq c_4$ and $2c_2 n^{-\alpha} c_4 \leq c_3$. Then we have

$$
\frac{c_2}{4} n^{-\alpha} - \mu_1 \geq 2f(Q_\tau(T))c_2 n^{-\alpha} c_3^{-1}.
$$

On the event $\{|\widehat{Q}_\tau(T) - Q_\tau(T)| \leq c_2 n^{-\alpha} c_3^{-1}\}$, we further have

$$
\begin{aligned}
&P\left(\frac{1}{n}\sum_{i=1}^n I\left\{Q_\tau(T) - c_2 n^{-\alpha} c_3^{-1} \leq T_i < Q_\tau(T) + c_2 n^{-\alpha} c_3^{-1}\right\} \geq \frac{c_2}{4} n^{-\alpha}\right) \\
&\leq P\left(\frac{1}{n}\sum_{i=1}^n I\left\{Q_\tau(T) - c_2 n^{-\alpha} c_3^{-1} \leq T_i < Q_\tau(T) + c_2 n^{-\alpha} c_3^{-1}\right\} - \mu_1 \geq 2f(Q_\tau(T))c_2 n^{-\alpha} c_3^{-1}\right) \\
&\leq \exp\{-8f^2(Q_\tau(T))c_2^2 n^{1-2\alpha} c_3^{-2}\}. \qquad (5)
\end{aligned}
$$

Similarly on the event $\{|\widehat{Q}_\tau(T) - Q_\tau(T)| \leq c_2 n^{-\alpha} c_5^{-1}\}$, under Condition (C3), we have

$$
\begin{aligned}
&P\left(\frac{1}{n}\sum_{i=1}^n I\left\{Q_\tau(T) - c_2 n^{-\alpha} c_5^{-1} \leq C_i < Q_\tau(T) + c_2 n^{-\alpha} c_5^{-1}\right\} \geq \frac{c_2}{4} n^{-\alpha}\right) \\
&\leq P\left(\frac{1}{n}\sum_{i=1}^n I\left\{Q_\tau(T) - c_2 n^{-\alpha} c_5^{-1} \leq C_i < Q_\tau(T) + c_2 n^{-\alpha} c_5^{-1}\right\} - \mu_2 \geq 2g(Q_\tau(T))c_2 n^{-\alpha} c_5^{-1}\right) \\
&\leq \exp\{-8g^2(Q_\tau(T))c_2^2 n^{1-2\alpha} c_5^{-2}\}, \qquad (6)
\end{aligned}
$$

where $c_5 = 1 + 16g(Q_\tau(T))$, $\mu_2 = E[I\{Q_\tau(T) - c_2 n^{-\alpha} c_5^{-1} \leq C < Q_\tau(T) + c_2 n^{-\alpha} c_5^{-1}\}]$ and $g(t)$ is the distribution function of C. Combining (4), (5) and (6) together with

Lemma 2, we have

$$
\begin{aligned}
&P\left(|I_{21}| \geq \frac{c^*}{8}n^{-\alpha}\right) \leq P\left(\frac{1}{n}\sum_{i=1}^{n} I\left\{Q_\tau(T) - c_2 n^{-\alpha}c_3^{-1} \leq T_i < Q_\tau(T) + c_2 n^{-\alpha}c_3^{-1}\right\} \geq \frac{c_2}{4}n^{-\alpha}\right) \\
&+ P\left(\frac{1}{n}\sum_{i=1}^{n} I\left\{Q_\tau(T) - c_2 n^{-\alpha}c_5^{-1} \leq C_i < Q_\tau(T) + c_2 n^{-\alpha}c_5^{-1}\right\} \geq \frac{c_2}{4}n^{-\alpha}\right) \\
&\leq \exp\{-8f^2(Q_\tau(T))c_2^2 n^{1-2\alpha}c_3^{-2}\} + \exp\{-8g^2(Q_\tau(T))c_2^2 n^{1-2\alpha}c_5^{-2}\}. \qquad (7)
\end{aligned}
$$

Next, we will consider the term I_{22}. By Lemma 3, we know that I_{22} can be computed as

$$
\begin{aligned}
I_{22} &= \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{n}\sum_{j=1}^{n}\frac{B_{nj}(X_{ik})\xi(Y_j,\Delta_j,Y_i,X_{ik})}{G^2(Y_i|X_{ik})}\right) I\{Y_i < Q_\tau(T)\}\Delta_i X_{ik} \\
&+ \frac{1}{n}\sum_{i=1}^{n} R_n(Y_i,X_{ik})\Delta_i X_{ik} \triangleq J_1 + J_2,
\end{aligned}
$$

where ξ and R_n are defined in Lemma 3.

Noting that $J_2 = O\left(\{\log(n)/(nh)\}^{3/4} + h^2\right) = o(n^{-\alpha})$ almost surely under Condition (C5) and $B_{nj}(X_{ik})\xi(Y_j,\Delta_j,Y_i,X_{ik})G^{-2}(Y_i|X_{ik})I\{Y_i < Q_\tau(T)\}\Delta_i X_{ik}$ are independent bounded random variables under Condition (C1) and (C2), then we can rewrite J_1 as follows

$$
\begin{aligned}
J_1 &= \frac{1}{n^2}\sum_{i\neq j}\frac{B_{nj}(X_{ik})\xi(Y_j,\Delta_j,Y_i,X_{ik})I\{Y_i < Q_\tau(T)\}\Delta_i X_{ik}}{G^2(Y_i|X_{ik})} + O(n^{-1}) \\
&\triangleq \frac{n(n-1)}{n^2}\cdot\frac{1}{C_n^2}\sum_{i\neq j}\frac{1}{2}\psi_{ij} + O(n^{-1}),
\end{aligned}
$$

where ψ_{ij} is the symmetric kernel of U-statistic. Since $\xi(Y_j,\Delta_j,t,s)$ are random variables with mean zero and finite variances for any given t and s, by Bernstein's inequality in Lemma 4, for sufficiently large n, we have

$$
\begin{aligned}
&P\left(|I_{22}| \geq \frac{c^*}{8}n^{-\alpha}\right) \leq P\left(|J_1| \geq \frac{c^*}{16}n^{-\alpha}\right) + P\left(|J_2| \geq \frac{c^*}{16}n^{-\alpha}\right) \\
&\leq P\left(\left|\frac{2}{n(n-1)}\sum_{i\neq j}\frac{B_{nj}(X_{ik})\xi(Y_j,\Delta_j,Y_i,X_{ik})I\{Y_i < Q_\tau(T)\}\Delta_i X_{ik}}{G^2(Y_i|X_{ik})}\right| \geq \frac{c^*}{16}n^{-\alpha}\right) \\
&\leq 2\exp\{-c_6 n^{1-2\alpha}\}, \qquad (8)
\end{aligned}
$$

where c_6 is a positive constant. Combining (3), (5) and (8), we have

$$
\begin{aligned}
P\left(|I_2| \geq \frac{c^*}{4}n^{-\alpha}\right) &\leq P\left(|I_{21}| \geq \frac{c^*}{8}n^{-\alpha}\right) + P\left(|I_{22}| \geq \frac{c^*}{8}n^{-\alpha}\right) \\
&\leq 4\exp\{-c_7 n^{1-2\alpha}\}, \qquad (9)
\end{aligned}
$$

where $c_7 = \min\{8f^2(Q_\tau(T))c_2^2c_3^{-2}, 8g^2(Q_\tau(T))c_2^2c_5^{-2}, c_6\}$.

For I_3, by the same technique for the term I_{21}, taking $c_8 = \frac{c^*\delta}{4M}$, we can obtain that

$$
\begin{aligned}
&P\left(|I_3| \geq \frac{c^*}{4}n^{-\alpha}\right) \leq P\left(\frac{1}{n}\sum_{i=1}^{n}\left|I\{Y_i < \widehat{Q}_\tau(T)\} - I\{Y_i < Q_\tau(T)\}\right| \geq c_8 n^{-\alpha}\right) \\
&\leq P\left(\frac{1}{n}\sum_{i=1}^{n} I\left\{Q_\tau(T) - c_8 n^{-\alpha}c_3^{-1} \leq T_i < Q_\tau(T) + c_8 n^{-\alpha}c_3^{-1}\right\} \geq \frac{c_8}{4}n^{-\alpha}\right) \\
&+ P\left(\frac{1}{n}\sum_{i=1}^{n} I\left\{Q_\tau(T) - c_8 n^{-\alpha}c_5^{-1} \leq C_i < Q_\tau(T) + c_8 n^{-\alpha}c_5^{-1}\right\} \geq \frac{c_8}{4}n^{-\alpha}\right) \\
&\leq \exp\{-8f^2(Q_\tau(T))c_8^2 n^{1-2\alpha}c_3^{-2}\} + \exp\{-8g^2(Q_\tau(T))c_8^2 n^{1-2\alpha}c_5^{-2}\} \\
&\leq 2\exp\{-c_9 n^{1-2\alpha}\},
\end{aligned} \tag{10}
$$

where $c_9 = \min\{8f^2(Q_\tau(T))c_8^2c_3^{-2}, 8g^2(Q_\tau(T))c_2^2c_5^{-2}\}$.

For I_4, by Condition (C1) and (C2) together with Hoeffding's inequality, we have

$$
P\left(|I_4| \geq \frac{c^*}{4}n^{-\alpha}\right) \leq 2\exp\left\{-\frac{\delta^2 c^{*2} n^{1-2\alpha}}{32}M^2\right\} \leq 2\exp\{-c_{10}n^{1-2\alpha}\}, \tag{11}
$$

where $c_{10} = \frac{(\delta c^*)^2 M^2}{32}$. Therefore, the equations (1)-(2) and (9)-(11) together can imply that

$$
P\left\{\max_{1\leq k\leq p}|\widehat{\omega}_k - \omega_k| \geq cn^{-\alpha}\right\} \leq \sum_{k=1}^{p} P\left(|\widehat{\omega}_k^* - \omega_k^*| \geq c^* n^{-\alpha}\right) \leq O\left(p\exp\{\eta n^{1-2\alpha}\}\right),
$$

where η is a positive constant depending on c_1, c_7, c_9 and c_{10}.

As for the second part of Theorem 1, if $\mathcal{A}_\tau$ is not the subset of $\widehat{\mathcal{A}}_\tau$, we obtain that there exist some $k \in \mathcal{A}_\tau$ such that $|\widehat{\omega}_k| < c_0 n^{-\alpha} \leq cn^{-\alpha}$ for some $k \in \mathcal{A}_\tau$. Combining with the assumption that $\min_{k\in\mathcal{A}_\tau}|\omega_k| \geq 2cn^{-\alpha}$, it follows that

$$
\{\mathcal{A}_\tau \not\subseteq \widehat{\mathcal{A}}_\tau\} \subseteq \{\max_{k\in\mathcal{A}_\tau}|\widehat{\omega}_k - \omega_k| > cn^{-\alpha}\}.
$$

Therefore,

$$
P\left\{\mathcal{A}_\tau \subset \widehat{\mathcal{A}}_\tau\right\} \geq 1 - O\left(s_n\exp\{-\eta n^{1-2\alpha}\}\right),
$$

where s_n is the cardinality of $\mathcal{A}_\tau$. Here we complete the proof of Theorem 1. □

Proof of Theorem 2. By Condition (C6), we have $\min_{k\in\mathcal{A}_\tau}|\omega_k| > \max_{k\in\mathcal{A}_\tau^c}|\omega_k|$ holds uniformly in n. Then for large n, there exists some $\delta \in (0,1]$ such that $\min_{k\in\mathcal{A}_\tau}|\omega_k| - \max_{k\in\mathcal{A}_\tau^c}|\omega_k| \geq \delta$, because $0 \leq |\omega_k| \leq 1$ for $1 \leq k \leq p$. Therefore, we can obtain

$$P\big(\min_{k\in\mathcal{A}_\tau}|\widehat{\omega}_k|\le\max_{k\in\mathcal{A}_\tau^c}|\widehat{\omega}_k|\big)\le P\big(\min_{k\in\mathcal{A}_\tau}|\widehat{\omega}_k|+\delta\le\max_{k\in\mathcal{A}_\tau^c}|\widehat{\omega}_k|+\min_{k\in\mathcal{A}_\tau}|\omega_k|-\max_{k\in\mathcal{A}_\tau^c}|\omega_k|\big)$$
$$\le P\big(\max_{k\in\mathcal{A}_\tau^c}|\widehat{\omega}_k-\omega_k|+\max_{k\in\mathcal{A}_\tau}|\widehat{\omega}_k-\omega_k|\ge\delta\big)\le\sum_{k=1}^{p}P\big(|\widehat{\omega}_k-\omega_k|\ge\frac{\delta}{2}\big).$$

Similarly, we denote $\omega_k^*=\sqrt{\tau-\tau^2}\omega_k$ and $\widehat{\omega}_k^*=\sqrt{\tau-\tau^2}\widehat{\omega}_k$. According to (1), we similarly decompose $|\widehat{\omega}_k^*-\omega_k^*|$ as

$$\begin{aligned}|\widehat{\omega}_k^*-\omega_k^*|=\tau&\left|\frac{1}{n}\sum_{i=1}^{n}X_{ik}-EX_k\right|+\left|\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\widehat{G}(Y_i|X_{ik})}-\frac{1}{G(Y_i|X_{ik})}\right)I\{Y_i<\widehat{Q}_\tau(T)\}\Delta_iX_{ik}\right|\\&+\left|\frac{1}{n}\sum_{i=1}^{n}\left[I\{Y_i<\widehat{Q}_\tau(T)\}-I\{Y_i<Q_\tau(T)\}\right]\frac{\Delta_i}{G(Y_i|X_{ik})}X_{ik}\right|\\&+\left|\frac{1}{n}\sum_{i=1}^{n}\frac{\Delta_i}{G(Y_i|X_{ik})}I\{Y_i<Q_\tau(T)\}X_{ik}-E\left[\frac{\Delta}{G(Y|X_k)}I\{Y<Q_\tau(T)\}X_k\right]\right|\\&\triangleq|I_1|+|I_2|+|I_3|+|I_4|.\end{aligned}$$

By the same technique as the proof of Theorem 1, there exists some constant $a>0$ such that

$$P\left(|\widehat{\omega}_k^*-\omega_k^*|\ge\frac{\delta'}{2}\right)\le O\left(\exp(-an)\right),$$

where $\delta'=\delta\sqrt{\tau^2-\tau}$. For $p=o\big\{\exp(an)\big\}$, we can obtain

$$P\Big(\liminf_{n\to\infty}\{\min_{k\in\mathcal{A}_\tau}|\widehat{\omega}_k|-\max_{k\in\mathcal{A}_\tau^c}|\widehat{\omega}_k|\}\le 0\Big)\le\lim_{n\to\infty}P\Big(\min_{k\in\mathcal{A}_\tau}|\widehat{\omega}_k|\le\max_{k\in\mathcal{A}_\tau^c}|\widehat{\omega}_k|\Big)=0.$$

That is

$$P\Big(\liminf_{n\to\infty}\{\min_{k\in\mathcal{A}_\tau}|\widehat{\omega}_k|-\max_{k\in\mathcal{A}_\tau^c}|\widehat{\omega}_k|\}>0\Big)=1.$$

□

2. Additional simulation results

In the Example 1 of the simulation studies, we further set $n=100$ and $p=1000,2000,3000$ to examine the performance of these method under different dimension. Table 1 and 2 summarize the corresponding results about the quantiles of the minimum model size $\mathcal{M}$ and the selection proportion $\mathcal{P}_k$ and $\mathcal{P}_a$ for a given model size d, respectively.

Table 1. The quantiles of the minimum model size $\mathcal{M}$ for Example 1 with two different random errors

		Case (1): $\epsilon \sim N(0,1)$					Case (2): $\epsilon \sim t(1)$				
τ	Method	5%	25%	50%	75%	95%	5%	25%	50%	75%	95%
						Scenario 1: $(n = 100, p = 1000)$					
0.3	QaSISc	8.0	20.0	42.0	76.0	182.1	16.0	42.0	75.5	153.5	521.3
	Q-SISc	5.0	5.0	5.0	5.0	12.0	5.0	5.0	6.0	10.0	57.0
	QC-SISc	5.0	5.0	5.0	5.0	7.0	5.0	5.0	5.0	7.0	34.1
0.5	QaSISc	11.0	23.0	45.5	86.5	345.7	17.0	43.8	96.5	179.3	712.1
	Q-SISc	5.0	5.0	5.0	5.0	6.0	5.0	5.0	5.0	7.0	23.2
	QC-SISc	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	6.0	16.1
0.7	QaSISc	22.8	68.3	139.0	279.0	721.6	37.9	96.5	227.0	529.5	940.2
	Q-SISc	5.0	5.0	5.0	5.0	10.0	5.0	5.0	6.0	10.0	73.2
	QC-SISc	5.0	5.0	5.0	5.0	7.0	5.0	5.0	5.0	7.0	33.3
						Scenario 2: $(n = 100, p = 2000)$					
0.3	QaSISc	14.0	34.0	70.0	127.8	431.3	44.0	110.8	192.0	303.8	689.1
	Q-SISc	5.0	5.0	5.0	5.0	12.0	5.0	5.0	7.0	16.3	179.1
	QC-SISc	5.0	5.0	5.0	5.0	6.1	5.0	5.0	6.0	10.0	100.4
0.5	QaSISc	20.0	48.8	122.5	219.0	513.3	30.0	99.5	186.5	376.8	1230.1
	Q-SISc	5.0	5.0	5.0	5.0	7.0	5.0	5.0	6.0	11.0	49.2
	QC-SISc	5.0	5.0	5.0	5.0	6.0	5.0	5.0	5.0	7.0	20.0
0.7	QaSISc	34.0	89.8	234.0	554.3	1440.8	70.0	188.0	487.5	1078.5	1878.1
	Q-SISc	5.0	5.0	5.0	5.0	7.0	5.0	5.0	7.0	14.3	93.8
	QC-SISc	5.0	5.0	5.0	5.0	6.1	5.0	5.0	5.0	9.0	50.2
						Scenario 3: $(n = 100, p = 3000)$					
0.3	QaSISc	14.0	47.0	92.0	186.8	480.2	55.0	141.3	268.0	434.0	960.1
	Q-SISc	5.0	5.0	5.0	5.0	8.0	5.0	5.0	7.0	21.0	226.4
	QC-SISc	5.0	5.0	5.0	5.0	7.0	5.0	5.0	6.0	13.3	124.6
0.5	QaSISc	21.0	49.8	129.0	280.5	626.7	49.9	144.5	279.5	512.3	1718.6
	Q-SISc	5.0	5.0	5.0	5.0	8.1	5.0	5.0	6.0	11.0	70.2
	QC-SISc	5.0	5.0	5.0	5.0	6.0	5.0	5.0	5.0	7.0	36.4
0.7	QaSISc	46.0	128.3	312.5	727.0	2091.0	77.5	241.0	601.0	1742.8	2669.5
	Q-SISc	5.0	5.0	5.0	5.0	8.1	5.0	5.0	8.0	26.0	217.4
	QC-SISc	5.0	5.0	5.0	5.0	7.0	5.0	5.0	6.0	13.0	97.8

Table 2. The proportions $\mathcal{P}_k$ corresponding to $X_1 - X_5$ and $\mathcal{P}_a$ for Example 1 with a given model size $d = [n/\log(n)]$

		Case (1): $\epsilon \sim N(0,1)$						Case (2): $\epsilon \sim t(1)$					
τ	Method	$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_3$	$\mathcal{P}_4$	$\mathcal{P}_5$	$\mathcal{P}_a$	$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_3$	$\mathcal{P}_4$	$\mathcal{P}_5$	$\mathcal{P}_a$
							Scenario 1: $(n = 100, p = 1000)$						
0.3	QaSISc	0.91	0.94	0.88	0.68	0.40	0.28	0.62	0.67	0.64	0.44	0.26	0.09
	Q-SISc	1.00	1.00	1.00	1.00	0.99	0.99	1.00	1.00	1.00	0.99	0.88	0.87
	QC-SISc	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.99	0.92	0.91
0.5	QaSISc	0.86	0.91	0.86	0.68	0.39	0.25	0.61	0.69	0.64	0.44	0.30	0.10
	Q-SISc	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.95	0.95
	QC-SISc	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.95	0.96
0.7	QaSISc	0.63	0.74	0.63	0.50	0.30	0.05	0.45	0.48	0.55	0.32	0.22	0.04
	Q-SISc	1.00	1.00	1.00	1.00	0.98	0.98	1.00	1.00	1.00	0.96	0.87	0.87
	QC-SISc	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.97	0.94	0.92
							Scenario 2: $(n = 100, p = 2000)$						
0.3	QaSISc	0.77	0.92	0.81	0.59	0.33	0.14	0.44	0.50	0.46	0.30	0.19	0.02
	Q-SISc	1.00	1.00	1.00	1.00	0.98	0.98	0.99	1.00	0.99	0.94	0.82	0.79
	QC-SISc	1.00	1.00	1.00	1.00	0.99	0.99	1.00	1.00	1.00	0.97	0.87	0.86
0.5	QaSISc	0.69	0.84	0.73	0.46	0.25	0.09	0.45	0.57	0.40	0.29	0.22	0.03
	Q-SISc	1.00	1.00	1.00	1.00	0.99	0.99	1.00	1.00	1.00	1.00	0.89	0.89
	QC-SISc	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.96	0.96
0.7	QaSISc	0.54	0.59	0.54	0.38	0.23	0.02	0.30	0.41	0.30	0.22	0.15	0.00
	Q-SISc	1.00	1.00	1.00	1.00	0.99	0.99	1.00	1.00	1.00	0.97	0.82	0.81
	QC-SISc	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.99	0.87	0.87
							Scenario 3: $(n = 100, p = 3000)$						
0.3	QaSISc	0.70	0.85	0.78	0.56	0.33	0.14	0.41	0.46	0.36	0.24	0.14	0.01
	Q-SISc	1.00	1.00	1.00	1.00	0.99	0.99	0.99	1.00	0.99	0.94	0.97	0.77
	QC-SISc	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.97	0.85	0.84
0.5	QaSISc	0.64	0.75	0.65	0.46	0.26	0.06	0.35	0.47	0.36	0.23	0.15	0.01
	Q-SISc	1.00	1.00	1.00	1.00	0.99	0.99	1.00	1.00	1.00	0.98	0.84	0.83
	QC-SISc	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.99	0.93	0.93
0.7	QaSISc	0.41	0.59	0.49	0.33	0.14	0.02	0.28	0.28	0.32	0.26	0.13	0.02
	Q-SISc	1.00	1.00	1.00	1.00	0.98	0.98	0.98	0.98	0.98	0.92	0.76	0.72
	QC-SISc	1.00	1.00	1.00	1.00	0.99	0.99	0.98	1.00	0.99	0.96	0.84	0.83

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