# Online Supplement for "A Bi-Criteria Multiple-Choice Secretary Problem" by "Ge Yu, Sheldon H Jacobson and Negar Kiyavash"

## 1. Proof of Proposition 1

We show that for any $0 < \lambda < 1$,

$$R_A(\Phi_\lambda) = \lambda R_A(\Phi_1) + (1-\lambda) R_A(\Phi_2) \text{ and } R_B(\Phi_\lambda) = \lambda R_B(\Phi_1) + (1-\lambda) R_B(\Phi_2), \tag{1}$$

where $\Phi_\lambda = \lambda\Phi_1 + (1-\lambda)\Phi_2$. Thus, $R_A(\Phi)$ and $R_B(\Phi)$ are both convex and concave functions of $\Phi$. Therefore, $R_A(\Phi)$ and $R_B(\Phi)$ are both affine functions of $\Phi$. In the following, we consider $R_A(\Phi_\lambda)$ first. The derivation for $R_B(\Phi_\lambda)$ follows the same arguments, with $\mathcal{A}(t)$ substituted by $\mathcal{B}(t)$.

Rewrite the objective function $R_A(\Phi_\lambda)$ by conditioning on the sequence of $\{(\mathcal{A}(t),\mathcal{B}(t))\}_{t=1}^T$ as

$$R_A(\Phi_\lambda) = \mathbb{E}[\sum_{t=1}^T X_t^{\Phi_\lambda}\mathcal{A}(t)] = \mathbb{E}_{\mathcal{A},\mathcal{B}}\left[\mathbb{E}_X[\sum_{t=1}^T X_t^{\Phi_\lambda}\mathcal{A}(t)|\{(\mathcal{A}(t),\mathcal{B}(t))\}_{t=1}^T]\right], \tag{2}$$

where the outer expectation is taken with respect to the joint distribution of $\{(\mathcal{A}(t),\mathcal{B}(t))\}_{t=1}^T$ and the inner expectation is taken with respect to $\{X_t^{\Phi_\lambda}\}_{t=1}^T$.

Consider $\mathbb{E}_X[\sum_{t=1}^T X_t^{\Phi_\lambda}\mathcal{A}(t)|\{(\mathcal{A}(t),\mathcal{B}(t))\}_{t=1}^T = \{(\alpha_t,\beta_t)\}_{t=1}^T]$. Then,

$$\begin{aligned}\mathbb{E}_X[\sum_{t=1}^T X_t^{\Phi_\lambda}\mathcal{A}(t)|\{(\mathcal{A}(t),\mathcal{B}(t))\}_{t=1}^T = \{(\alpha_t,\beta_t)\}_{t=1}^T] &= \sum_{t=1}^T \mathbb{E}_X[X_t^{\Phi_\lambda}|\{(\mathcal{A}(t),\mathcal{B}(t))\}_{t=1}^T = \{(\alpha_t,\beta_t)\}_{t=1}^T]\alpha_t \\ &= \sum_{t=1}^T \mathbb{P}(X_t^{\Phi_\lambda} = 1|\{(\mathcal{A}(t'),\mathcal{B}(t'))\}_{t'=1}^t = \{(\alpha_{t'},\beta_{t'})\}_{t'=1}^t)\alpha_t,\end{aligned} \tag{3}$$

where (3) follows from $X_t^{\Phi_\lambda}$ being a sequential candidate assignment with information of candidate attribute vectors till time $t$. From (8) and (10),

$$\mathbb{P}(X_t^{\Phi_\lambda}=1|\{(\mathcal{A}(t'),\mathcal{B}(t'))\}_{t'=1}^t=\{(\alpha_{t'},\beta_{t'})\}_{t'=1}^t)=\chi_t^{\Phi_\lambda}=\lambda\chi_t^{\Phi_1}+(1-\lambda)\chi_t^{\Phi_1},$$

for $t=1,2,\dots,T$, where $\{\chi_t^{\Phi_1}\}_{t=1}^T$, $\{\chi_t^{\Phi_2}\}_{t=1}^T$ and $\{\chi_t^{\Phi_\lambda}\}_{t=1}^T$ denote the realized profiles of policy $\Phi_1$, $\Phi_2$ and $\Phi_\lambda$ with respect to $\{(\alpha_t,\beta_t)\}_{t=1}^T$, respectively. Moreover, substituting $\Phi_\lambda$ with $\Phi_1$ and $\Phi_2$ in (3) respectively, we have

$$\begin{aligned}&\mathbb{E}_X[\sum_{t=1}^T X_t^{\Phi_1}\mathcal{A}(t)|\{(\mathcal{A}(t),\mathcal{B}(t))\}_{t=1}^T=\{(\alpha_t,\beta_t)\}_{t=1}^T]\\ &=\sum_{t=1}^T\mathbb{P}(X_t^{\Phi_1}=1|\{(\mathcal{A}(t'),\mathcal{B}(t'))\}_{t'=1}^t=\{(\alpha_{t'},\beta_{t'})\}_{t'=1}^t)\alpha_t=\sum_{t=1}^T\chi_t^{\Phi_1}\alpha_t,\\ &\mathbb{E}_X[\sum_{t=1}^T X_t^{\Phi_2}\mathcal{A}(t)|\{(\mathcal{A}(t),\mathcal{B}(t))\}_{t=1}^T=\{(\alpha_t,\beta_t)\}_{t=1}^T]=\sum_{t=1}^T\chi_t^{\Phi_2}\alpha_t.\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathbb{E}_X[\sum_{t=1}^T X_t^{\Phi_\lambda}\mathcal{A}(t)|\{(\mathcal{A}(t),\mathcal{B}(t))\}_{t=1}^T=\{(\alpha_t,\beta_t)\}_{t=1}^T]=\sum_{t=1}^T\chi_t^{\Phi_\lambda}\alpha_t=\sum_{t=1}^T\left(\lambda\chi_t^{\Phi_1}+(1-\lambda)\chi_t^{\Phi_1}\right)\alpha_t\\ =&\lambda\mathbb{E}_X[\sum_{t=1}^T X_t^{\Phi_1}\mathcal{A}(t)|\{(\mathcal{A}(t),\mathcal{B}(t))\}_{t=1}^T=\{(\alpha_t,\beta_t)\}_{t=1}^T]\\ &+(1-\lambda)\mathbb{E}_X[\sum_{t=1}^T X_t^{\Phi_2}\mathcal{A}(t)|\{(\mathcal{A}(t),\mathcal{B}(t))\}_{t=1}^T=\{(\alpha_t,\beta_t)\}_{t=1}^T],\end{aligned}\tag{4}$$

which holds for any sequence of realized candidate attribute vectors $\{(\alpha_t,\beta_t)\}_{t=1}^T$. Substituting (4) into (2) leads to

$$\begin{aligned}R_A(\Phi_\lambda)=&\lambda\mathbb{E}_{\mathcal{A},\mathcal{B}}\left[\mathbb{E}_X[\sum_{t=1}^T X_t^{\Phi_1}\mathcal{A}(t)|\{(\mathcal{A}(t),\mathcal{B}(t))\}_{t=1}^T=\{(\alpha_t,\beta_t)\}_{t=1}^T]\right]\\ &+(1-\lambda)\mathbb{E}_{\mathcal{A},\mathcal{B}}\left[\mathbb{E}_X[\sum_{t=1}^T X_t^{\Phi_2}\mathcal{A}(t)|\{(\mathcal{A}(t),\mathcal{B}(t))\}_{t=1}^T=\{(\alpha_t,\beta_t)\}_{t=1}^T]\right]\\ =&\lambda R_A(\Phi_1)+(1-\lambda)R_A(\Phi_2),\end{aligned}$$

where the last equality follows from substituting $\Phi_\lambda$ with $\Phi_1$ and $\Phi_2$ in (2), respectively. Therefore, $R_A(\Phi)$ is an affine function of $\Phi$.

Following the same arguments with $\mathcal{A}(t)$ substituted by $\mathcal{B}(t)$, we have $R_B(\Phi)$ is an affine function of $\Phi$, which finishes the proof. $\square$

## 2. Proof of Theorem 2

First, we show every $\mathbb{M}$-optimal policy for WOSA-$\eta$ is a Pareto optimal policy for BMSP. Let $\Phi_1$ be an $\mathbb{M}$-optimal policy for WOSA-$\eta$, then $\Phi_1$ must satisfy one of the three conditions in Definition 2. If $\Phi_1$ satisfies Definition 2(a), then $\mathbf{w} > 0$, and hence, $\Phi_1$ is a Pareto optimal policy for BMSP by *Theorem 3.1.2* (Miettinen 1999, p. 78).

If $\Phi_1$ satisfies Definition 2(b), then $\mathbf{w} = (w_1, 0)$ with $w_1 > 0$ and $\max_{\Phi\in\Psi^\eta} R_w(\Phi) \Leftrightarrow \max_{\Phi\in\Psi^\eta} R_A(\Phi) = R_A(\Phi_1)$. If $\Phi_1$ is the unique policy that maximizes $R_A(\Phi)$ in $\Psi^\eta$, then $\Phi_1$ is a Pareto optimal policy for BMSP by *Theorem 3.1.3* (Miettinen 1999, p. 79). Otherwise, $\Phi_1$ is not unique and we prove $\Phi_1$ is Pareto optimal for BMSP by contradiction. Suppose $\Phi_1$ is not Pareto optimal, then there exists another policy $\Phi' \in \Psi^\eta$ such that $R_A(\Phi') \geq R_A(\Phi), R_B(\Phi') \geq R_B(\Phi)$ with at least one strict inequality. Since $R_A(\Phi_1) = \max_{\Phi\in\Psi^\eta} R_A(\Phi)$, then $R_A(\Phi') \leq R_A(\Phi_1)$, and hence, $R_A(\Phi') = R_A(\Phi_1), R_B(\Phi') > R_B(\Phi_1)$. However, this is a contradictory to $\mathbb{M}$-optimality of $\Phi_1$. Therefore, $\Phi_1$ is a Pareto optimal policy for BMSP.

If $\Phi_1$ satisfies Definition 2(c), similar arguments can be applied.

For the reverse direction, we show every Pareto optimal policy for BMSP is $\mathbb{M}$-optimal for WOSA-$\eta$. Let $\Phi_2 \in \Psi^{\eta+}$ be a mixed policy, which is Pareto optimal for BMSP indexed by $\eta$. Then, by Proposition 1, $\Psi^{\eta+}$ is convex and the objective functions $R_A(\Phi)$ and $R_B(\Phi)$ are affine functions of $\Phi$. Then from *Theorem 3.1.4* (Miettinen 1999, p. 79), there exists a non-negative weight vector $\mathbf{w}$ such that $\Phi_2$ maximizes $R_w(\Phi)$, which is the objective function of WOSA indexed by $\mathbf{w}$ defined in (6). From Section 4.3, optimal policies for WOSA-$\eta$ are all pure policies, and hence, $\Phi_2 \in \Psi^\eta$. We are left to show $\Phi_2$ satisfies one of the three conditions in Definition 2 to prove $\Phi_2$ is $\mathbb{M}$-optimal for WOSA-$\eta$.

If $\mathbf{w} > 0$, then Definition 2(a) is satisfied and $\Phi$ is $\mathbb{M}$-optimal for WOSA-$\eta$.

If $\mathbf{w} = (w_1, 0)$ with $w_1 > 0$, then $R_A(\Phi_2) = \max_{\Phi\in\Psi^\eta} R_A(\Phi)$. We prove $R_B(\Phi_2) = \max_{\Phi\in\Lambda_A^\eta} R_B(\Phi)$ with $\Lambda_A^\eta$ defined by Definition 2(b) by contradiction. Suppose $R_B(\Phi_2) \neq \max_{\Phi\in\Lambda_A^\eta} R_B(\Phi)$, then there exists $\Phi' \in \Lambda_A^\eta$ such that $R_B(\Phi') = \max_{\Phi\in\Lambda_A^\eta} R_B(\Phi)$. Therefore, $R_A(\Phi') = R_A(\Phi_2)$, $R_B(\Phi') >$

$R_B(\Phi_2)$, which is a contradictory to the Pareto optimality of $\Phi_2$. Therefore, $\Phi_2$ is $\mathbb{M}$-optimal for WOSA-$\eta$.

If $\mathbf{w} = (0, w_2)$ with $w_2 > 0$, similar arguments can be applied, which finishes the proof. $\square$

## 3. Proof of Proposition 2

The proof is based on induction on $T$. First, we consider the corresponding SSAP with almost-binary success rates for $T$ tasks. In this case, the SSAP optimal policy, denoted by $\Phi_B$, is a direct application of *Theorem 1* given by Derman et al. (1972).

THEOREM 1 (***Theorem 1*, Derman et al. (1972)**). *For the $t^{th}$ task arrival with the task value $\mathcal{C}_t$, there are $T-t+1$ workers available for $t = 1, 2, \ldots, T$. The thresholds for $\mathcal{C}_t$ are given by $-\infty = a_{0,t} \leq a_{1,t} \leq \ldots \leq a_{T-t+1,t} = +\infty$, obtained based on the recursive equations*

$$a_{i,t} = \int_{a_{i-1,t+1}}^{a_{i,t+1}} x dF_{\mathcal{C}}(x) + a_{i-1,t+1} F_{\mathcal{C}}(a_{i-1,t+1}) + a_{i,t+1}(1 - F_{\mathcal{C}}(a_{i,t+1})), \ i = 1, 2, \ldots, T-t. \tag{5}$$

*If the $t^{th}$ task value $\mathcal{C}_t \in (a_{i-1,t}, a_{i,t}]$, then the worker with the $i^{th}$ smallest success rate among the $T-t+1$ available workers will be assigned to the $t^{th}$ task under the optimal policy. Moreover, $a_{i,t}$ is the expected task value that will be assigned to the worker with $i^{th}$ smallest success rate among the $T-t$ available workers for $i = 1, 2, \ldots, T-t$.*

Let $a_{i,t}$ denote the threshold values defined by (5) for $T$, with $i = 1, 2, \ldots, T-t$ and $t = 0, 1, \ldots, T-1$. Then, the first task assignment under policy $\Phi_B$ is

$$\tau_{j_1}^{\Phi_B} = \begin{cases} 1, & \text{if} \quad \mathcal{C}_1 > a_{T-\lfloor \Upsilon \rfloor, 1}, \\ \Upsilon - \lfloor \Upsilon \rfloor, & \text{if } a_{T-\lfloor \Upsilon \rfloor - 1, 1} < \mathcal{C}_1 \leq a_{T-\lfloor \Upsilon \rfloor, 1}, \\ 0, & \text{if} \quad \mathcal{C}_1 \leq a_{T-\lfloor \Upsilon \rfloor - 1, 1}. \end{cases} \tag{6}$$

Moreover, the optimal expected assignment reward is achieved under policy $\Phi_B$ as $\mathbb{E}[\sum_{t=2}^{T} \tau_{j_t}^{\Phi_B} \mathcal{C}_t] = \sum_{i=T-\lfloor \Upsilon \rfloor+1}^{T} a_{i,0} + a_{T-\lfloor \Upsilon \rfloor, 0}(\Upsilon - \lfloor \Upsilon \rfloor)$.

When $T = 1$, there is only one task to be assigned to one worker, and hence, $\Phi_B$ is trivially optimal for the SSAP with the fixed success rate sum.

Suppose Proposition 2 holds for $T' \leq T-1$. When there are $T$ tasks to be assigned, we need to show $\tau_{j_1}^{\Phi_B}$ is optimal for the first task assignment in the fixed success rate sum scenario, and the remaining $T-1$ task assignments are optimal under policy $\Phi_B$ by the induction assumption. Let $a'_{i,t}$ denote the threshold values defined by (5) for $T' = T-1$, with $i = 1,2,\ldots,T-t$ and $t = 0,1,\ldots,T-2$. We compute the optimal conditional expected reward for $T$ tasks assignments given the first task value,

$$\begin{aligned}\max_{\{\tau_{j_t}\}_{t=1}^{T}} \mathbb{E}[\sum_{t=1}^{T}\tau_{j_t}\mathcal{C}_t|\sum_{t=1}^{T}\tau_{j_t} = \Upsilon, \mathcal{C}_1 = x_1] &= \max_{\{\tau_{j_t}\}_{t=1}^{T}} \mathbb{E}[\tau_{j_1}x_1 + \sum_{t=2}^{T}\tau_{j_t}\mathcal{C}_t|\sum_{t=1}^{T}\tau_{j_t} = \Upsilon, \mathcal{C}_1 = x_1] \\ &= \max_{0\leq\tau_{j_1}\leq 1}(\tau_{j_1}x_1 + \max_{\{\tau_{j_t}\}_{t=2}^{T}} \mathbb{E}[\sum_{t=2}^{T}\tau_{j_t}\mathcal{C}_t|\sum_{t=2}^{T}\tau_{j_t} = \Upsilon - \tau_{j_1}]), \quad (7)\end{aligned}$$

where $x_1$ denotes the realized value of the first task.

The second term on the right-hand side of (7) is the optimal expected reward for $T-1$ task assignments with the success rate sum of $T-1$ workers as $\Upsilon - \tau_{j_1}$, and

$$\lfloor \Upsilon - \tau_{j_1} \rfloor = \begin{cases} \lfloor \Upsilon \rfloor - 1, & \text{if } \tau_{j_1} > \Upsilon - \lfloor \Upsilon \rfloor, \\ \lfloor \Upsilon \rfloor, & \text{if } \tau_{j_1} \leq \Upsilon - \lfloor \Upsilon \rfloor. \end{cases} \quad (8)$$

Then by the induction assumption, the maximum of the second term on the right-hand side of (7) is achieved under the SSAP optimal policy for the almost-binary success rate scenario with $T-1$ workers. Therefore,

$$\begin{aligned}\max_{\{\tau_{j_t}\}_{t=2}^{T}} \mathbb{E}[\sum_{t=2}^{T}\tau_{j_t}\mathcal{C}_t|\sum_{t=2}^{T}\tau_{j_t} = \Upsilon - \tau_{j_1}] &= \sum_{i=T-\lfloor\Upsilon-\tau_{j_1}\rfloor}^{T-1} a'_{i,0} + a'_{T-\lfloor\Upsilon-\tau_{j_1}\rfloor-1,0}(\Upsilon - \tau_{j_1} - \lfloor\Upsilon - \tau_{j_1}\rfloor) \\ &= \begin{cases} \sum_{i=T-\lfloor\Upsilon\rfloor+1}^{T-1} a_{i,1} + a_{T-\lfloor\Upsilon\rfloor,1}(\Upsilon - \tau_{j_1} - \lfloor\Upsilon\rfloor + 1), & \text{if } \tau_{j_1} > \Upsilon - \lfloor\Upsilon\rfloor, \\ \sum_{i=T-\lfloor\Upsilon\rfloor}^{T-1} a_{i,1} + a_{T-\lfloor\Upsilon\rfloor-1,1}(\Upsilon - \tau_{j_1} - \lfloor\Upsilon\rfloor), & \text{if } \tau_{j_1} \leq \Upsilon - \lfloor\Upsilon\rfloor, \end{cases}\end{aligned} \quad (9)$$

where the second equality follows from (8) and the recursive definitions of threshold values (i.e., $a'_{i,0} = a_{i,1}$ for $i = 1,2,\ldots,T-1$). Then, we substitute (9) into (7) and compute the optimal expected conditional reward (7) for two cases: (a) $\tau_{j_1} > \Upsilon - \lfloor\Upsilon\rfloor$ and (b) $\tau_{j_1} \leq \Upsilon - \lfloor\Upsilon\rfloor$, respectively, and then combine them together to obtain the optimal assignment for the first task. Note that (7) is an affine function of $\tau_{j_1}$ in both cases, and hence, the maximum of (7) is achieved at the boundary point of $\tau_{j_1}$. Therefore, in each case, we take derivative with respect to $\tau_{j_1}$ and if the derivative is positive, the maximum (supremum) of (7) is achieved when $\tau_{j_1}$ takes the maximum (supremum) value. Otherwise, the maximum (supremum) of (7) is achieved when $\tau_{j_1}$ takes the minimum (infimum) value. Therefore, for case (a), $\tau_{j_1} > \Upsilon - \lfloor \Upsilon \rfloor$,

$$\max_{\{\tau_{j_t}\}_{t=1}^T} \mathbb{E}[\sum_{t=1}^T \tau_{j_t} \mathcal{C}_t | \sum_{t=1}^T \tau_{j_t} = \Upsilon, \mathcal{C}_1 = x_1] = \begin{cases} \sum_{i=T-\lfloor \Upsilon \rfloor+1}^{T-1} a_{i,1} + x_1 + a_{T-\lfloor \Upsilon \rfloor,1}(\Upsilon - \lfloor \Upsilon \rfloor), & \text{if } x_1 > a_{T-\lfloor \Upsilon \rfloor,1}, \\ \sum_{i=T-\lfloor \Upsilon \rfloor+1}^{T-1} a_{i,1} + x_1(\Upsilon - \lfloor \Upsilon \rfloor) + a_{T-\lfloor \Upsilon \rfloor,1}, & \text{if } x_1 \leq a_{T-\lfloor \Upsilon \rfloor,1}, \end{cases} \tag{10}$$

where the second line is the supremum of the left-hand side and

$$\tau_{j_1} = \begin{cases} 1, & \text{if } x_1 > a_{T-\lfloor \Upsilon \rfloor,1}, \\ \Upsilon - \lfloor \Upsilon \rfloor, & \text{if } x_1 \leq a_{T-\lfloor \Upsilon \rfloor,1}, \end{cases}$$

with the second line being the infimum of the left-hand side.

For case (b), $\tau_{j_1} \leq \Upsilon - \lfloor \Upsilon \rfloor$,

$$\max_{\{\tau_{j_t}\}_{t=1}^T} \mathbb{E}[\sum_{t=1}^T \tau_{j_t} \mathcal{C}_t | \sum_{t=1}^T \tau_{j_t} = \Upsilon, \mathcal{C}_1 = x_1] = \begin{cases} \sum_{i=T-\lfloor \Upsilon \rfloor}^{T-1} a_{i,1} + x_1(\Upsilon - \lfloor \Upsilon \rfloor), & \text{if } x_1 > a_{T-\lfloor \Upsilon \rfloor-1,1}, \\ \sum_{i=T-\lfloor \Upsilon \rfloor}^{T-1} a_{i,1} + a_{T-\lfloor \Upsilon \rfloor-1,1}(\Upsilon - \lfloor \Upsilon \rfloor), & \text{if } x_1 \leq a_{T-\lfloor \Upsilon \rfloor-1,1}, \end{cases} \tag{11}$$

where

$$\tau_{j_1} = \begin{cases} \Upsilon - \lfloor \Upsilon \rfloor, & \text{if } x_1 > a_{T-\lfloor \Upsilon \rfloor-1,1}, \\ 0, & \text{if } x_1 \leq a_{T-\lfloor \Upsilon \rfloor-1,1}. \end{cases}$$

Combining cases (a) and (b) together by comparing (11) and (10) leads to the optimal first task assignment as

$$\tau_{j_1} = \begin{cases} 1, & \text{if } x_1 > a_{T-\lfloor \Upsilon \rfloor,1}, \\ \Upsilon - \lfloor \Upsilon \rfloor, & \text{if } a_{T-\lfloor \Upsilon \rfloor-1,1} < x_1 \leq a_{T-\lfloor \Upsilon \rfloor,1}, \\ 0, & \text{if } x_1 \leq a_{T-\lfloor \Upsilon \rfloor-1,1}, \end{cases}$$

which is the same as the SSAP optimal policy assignment $\tau_{j_1}^{\Phi_B}$ in the almost-binary success rate scenario given by (6). By the induction assumption, the optimal expected reward for assigning the remaining $T-1$ tasks to $T-1$ workers with a fixed success rate sum $\Upsilon - \tau_{j_1}$ is achieved under the SSAP optimal policy $\Phi_B$. This completes the proof. $\square$

## 4. Proof of Theorem 3

The proof is by induction on $T$, using similar techniques as in Derman et al. (1972) but specifically trimmed for WOSA with discrete attribute distributions.

Let $f(\eta,T)$ denote the optimal expected reward for $T$ candidate assignments with a selectee capacity $\eta$. Further, let $f(\eta,T|\gamma_1)$ denote the optimal conditional expected reward for $T$ candidate assignments with a selectee capacity $\eta$ given the first candidate combined attribute $\mathcal{G}(1)=\gamma_1$, then

$$f(\eta,T) = \max_{\Phi\in\Psi^\eta} \mathbb{E}[\sum_{t=1}^{T} X_t^\Phi \mathcal{G}(t)],$$
$$f(\eta,T|\gamma_1) = \max_{\Phi\in\Psi^\eta} \mathbb{E}[\sum_{t=1}^{T} X_t^\Phi \mathcal{G}(t)|\mathcal{G}(1)=\gamma_1].$$

When $T=1$, there is only one candidate to be assigned and $a_{0,1} = -\infty,\ a_{1,1} = +\infty$. Then, under policy ($\Phi1$), this candidate will be assigned to the selectee category if $\eta=1$ while to the non-selectee category if $\eta=0$. Therefore, policy ($\Phi1$) is trivially optimal for $T=1$. Moreover, the expected combined attribute value of this candidate is the expected value of $\mathcal{G}(1)$, and from (20),

$$a_{i,t} = \sum_{\gamma=G_1}^{G_L} \gamma p_{\mathcal{G}}(\gamma) = \mathbb{E}[\mathcal{G}(1)].$$

Therefore, Theorem 3 holds for $T=1$.

Suppose Theorem 3 holds for $T' \le T-1$. Then, policy ($\Phi1$) with threshold values defined by (20) maximizes the objective function $R_w(\Phi)$ (6) for $T'=T-1$ candidates. Let $\{a'_{i,0}\}_{i=1}^{T'}$ denote the threshold values in the initial stage for $T'$ candidates, which are the expected combined attribute values for $T'$ candidates to be assigned by the induction assumption. Let $\{a_{i,t}\}$ denote the threshold values defined by (20) for $T$ candidates, for $i=1,2,\dots,T-t$ and $t=0,1,\dots,T-1$. We show the first candidate assignment under policy ($\Phi1$) is optimal for $T$ candidates, and optimal assignments for the remaining $T-1$ candidates follow from the induction assumption. When there are $T$ candidates to be assigned, conditional on the combined attribute value of the first candidate,

$$f(\eta, T|\gamma_1) = \max_{X_1^\Phi \in \{0,1\}} (\gamma_1 X_1^\Phi + f(\eta - X_1^\Phi, T-1)). \tag{12}$$

Note that $f(\eta - X_1^\Phi, T-1)$ is the optimal expected reward for $T-1$ candidate assignments with a selectee capacity $\eta - X_1^\Phi$. Then, by the induction assumption, the optimal expected reward for $T-1$ candidate assignments is achieved under policy ($\Phi$1). Since $\{a'_{i,0}\}_{i=1}^{T'}$ are monotonically increasing by (19), and hence,

$$f(\eta - X_1^\Phi, T-1) = \sum_{i=(T-1)-(\eta - X_1^\Phi)+1}^{T-1} a'_{i,0}, \tag{13}$$

where the $(\eta - X_1)$ largest expected candidate combined attribute values are assigned to the selectee category to maximize (6). Substitute (13) into (12); the optimal policy assigns $X_1 = 1$ if

$$\gamma_1 + \sum_{i=(T-1)-(\eta-1)+1}^{T-1} a'_{i,0} > \sum_{i=(T-1)-\eta+1}^{T-1} a'_{i,0},$$

or equivalently,

$$\gamma_1 > a'_{(T-1)-\eta+1,0} = a'_{T-\eta,0} = a_{T-\eta,1}, \tag{14}$$

where the last equality follows from the recursive definitions of threshold values (20), $a_{i,1} = a'_{i,0}$ for $i = 1, 2, \ldots, T-1$ (the first stage for $T$ candidates $\{a_{i,1}\}_{i=1}^{T-1}$ are equal to the threshold values in the initial stage for $T-1$ candidates). Therefore, the optimal first candidate assignment given by (14) is the same as that given by policy ($\Phi$1). By the induction assumption, the remaining $T-1$ candidates can be assigned under policy ($\Phi$1) to maximize the objective function $R_w(\Phi)$ (6). Therefore, policy ($\Phi$1) is optimal for $T$ candidates.

Next, we compute the expected combined attribute values for the $T$ candidates to be assigned. By the monotonicity of the threshold values and the induction assumption, $a_{i,1} = a'_{i,0}, i = 1, 2, \ldots, T-1$ is the $i^{th}$ smallest expected combined attribute value for the $T-1$ candidates to be assigned. Let $\hat{\mathcal{G}}_T^{(i)}$

(random variable) denote the $i^{th}$ smallest combined attribute value for $T$ candidates, $i=1,2,\dots,T$. Conditioning on the value of $\mathcal{G}(1)$ leads to

$$\begin{aligned}
\mathbb{E}[\hat{\mathcal{G}}_T^{(i)}] =& \mathbb{E}[\mathbb{E}[\hat{\mathcal{G}}_T^{(i)}|\mathcal{G}(1)]] \\
=& \mathbb{E}[\mathcal{G}(1)|a_{i-1,1}<\mathcal{G}(1)\le a_{i,1}]\mathbb{P}(a_{i-1,1}<\mathcal{G}(1)\le a_{i,1}) + \mathbb{E}[\hat{\mathcal{G}}_{T-1}^{(i-1)}|\mathcal{G}(1)\le a_{i-1,1}]\mathbb{P}(\mathcal{G}(1)\le a_{i-1,1}) \\
&+ \mathbb{E}[\hat{\mathcal{G}}_{T-1}^{(i)}|\mathcal{G}(1)>a_{i,1}]\mathbb{P}(\mathcal{G}(1)>a_{i,1}) \\
=& \left(\sum_{\gamma'=g^u_{i-1,1}}^{g^l_{i,1}} \gamma' p_{\mathcal{G}}(\gamma')\right) + a_{i-1,1}F_{\mathcal{G}}(a_{i-1,1}) + a_{i,1}(1-F_{\mathcal{G}}(a_{i,1})) \\
=& a_{i,0},
\end{aligned}$$

with $g^l_{i,1}$ and $g^u_{i-1,1}$ given by (21). Therefore, the threshold values in the initial stage $\{a_{i,0}\}_{i=1}^T$ are the expected combined attribute values for the $T$ candidates to be assigned. This completes the proof. □

## 5. Proof of Proposition 3

The proof is based on induction on $T$. When $T=1$, there is only one candidate to be assigned, and hence, $i=1,\ t=0$. The expected value of $\hat{\mathcal{A}}_1^{(1)}$ is just the expectation of $\mathcal{A}(t)$. From Proposition 3, only $b_{1,0}$ is defined by (24) when $T=1$, which is given by

$$b_{1,0} = \sum_{\gamma'=G_1}^{G_L} \mathbb{E}[\mathcal{A}(t)|\mathcal{G}(t)=\gamma']p_{\mathcal{G}}(\gamma') = \mathbb{E}[\mathcal{A}(t)]. \tag{15}$$

Therefore, Proposition 3 holds for $T=1$.

Suppose Proposition 3 holds for $T'\le T-1$ and $\{b'_{i,t}\}$ are defined by (24) for $T-1$, for $i=1,2,\dots,T-1-t$ and $t=0,1,\dots,T-2$. Then we show that (24) holds for $T$ with $\{b_{i,t}\}$, for $i=1,2,\dots,T-t$ and $t=0,1,\dots,T-1$. From the recursive definitions of $\{a_{i,t}\}$ and (24), $\{b_{i,t}\}$ are the same as $\{b'_{i,t-1}\}$ for $t=1,2,\dots,T-1$. We are left with $\{b_{i,0}\},\ i=1,2,\dots,T$, which need to be verified as the expected value of $\hat{\mathcal{A}}_T^{(i)}$ for $T$ candidates. Since the threshold values $\{a_{i,t}\}$ are monotonically increasing with respect to $i$, then conditioning on the combined attribute value of the first candidate $\mathcal{G}(1)$ leads to

$$\mathbb{E}[\hat{\mathcal{A}}_T^{(i)}] = \mathbb{E}_{\mathcal{G}}[\mathbb{E}_{\mathcal{A}}[\hat{\mathcal{A}}_T^{(i)}|\mathcal{G}(1)]]$$

$$
\begin{aligned}
&= \mathbb{E}_{\mathcal{A}}[\hat{\mathcal{A}}_T^{(i)} | a_{i-1,1} < \mathcal{G}(1) \le a_{i,1}] \mathbb{P}(a_{i-1,1} < \mathcal{G}(1) \le a_{i,1}) + \mathbb{E}_{\mathcal{A}}[\hat{\mathcal{A}}_T^{(i)} | \mathcal{G}(1) \le a_{i-1,1}] \mathbb{P}(\mathcal{G}(1) \le a_{i-1,1}) \\
&\quad + \mathbb{E}_{\mathcal{A}}[\hat{\mathcal{A}}_T^{(i)} | \mathcal{G}(1) > a_{i,1}] \mathbb{P}(\mathcal{G}(1) > a_{i,1}) \\
&\overset{(a)}{=} \mathbb{E}_{\mathcal{G}}\left[\mathbb{E}_{\mathcal{A}}[\mathcal{A}(1)|\mathcal{G}(1)] | a_{i-1,1} < \mathcal{G}(1) \le a_{i,1}\right] \mathbb{P}(a_{i-1,1} < \mathcal{G}(1) \le a_{i,1}) + b_{i-1,1}\mathbb{P}(\mathcal{G}(1) \le a_{i-1,1}) + b_{i,1}\mathbb{P}(\mathcal{G}(1) > a_{i,1}) \\
&\overset{(b)}{=} \left( \sum_{\gamma' = g_{i-1,1}^u}^{g_{i,1}^l} \mathbb{E}[\mathcal{A}(t) | \mathcal{G}(t) = \gamma'] \right) + b_{i-1,1} F_{\mathcal{G}}(a_{i-1,1}) + b_{i,1}(1 - F_{\mathcal{G}}(a_{i,1})) \\
&= b_{i,0},
\end{aligned}
$$

where: equality (a) follows from the induction assumption; and equality (b) follows from that $\mathcal{A}(t)$ and $\mathcal{G}(t)$ are both IID. Therefore, $b_{i,0}$ defined by (24) is the expected value of $\hat{\mathcal{A}}_T^{(i)}$ for $i = 1, 2, \ldots, T$, which completes the proof. $\square$

## 6. A Short Proof of the Relationship between the Weight Vectors and the Achievement Ratios

We want to prove that if $w_1 = 1$ is fixed, then $\delta_A^\eta(\mathbf{w})$ decreases with $w_2$ while $\delta_B^\eta(\mathbf{w})$ increases with $w_2$. It is sufficient to prove that if $w_1 = 1$ is fixed, then $R_A(\Phi 1)$ decreases with $w_2$ while $R_B(\Phi 1)$ increases with $w_2$. Let $\mathbf{w}^1 = (1, w_2^1)$ and $\mathbf{w}^2 = (1, w_2^2)$ be two non-negative weight vectors, with $w_2^1 > w_2^2$. Let $\Phi_1$ and $\Phi_2$ denote the optimal SSAP optimal policy for WOSA indexed by $\mathbf{w}_1$ and $\mathbf{w}_2$, respectively. Therefore,

$$
\begin{aligned}
R_A(\Phi_1) + w_2^1 R_B(\Phi_1) \ge R_A(\Phi_2) + w_2^1 R_B(\Phi_2), \\
R_A(\Phi_2) + w_2^2 R_B(\Phi_2) \ge R_A(\Phi_1) + w_2^2 R_B(\Phi_1).
\end{aligned}
$$

Rearranging the terms leads to

$$
w_2^1(R_B(\Phi_1) - R_B(\Phi_2)) \ge R_A(\Phi_2) - R_A(\Phi_1) \ge w_2^2(R_B(\Phi_1) - R_B(\Phi_2)).
$$

Since $w_2^1 > w_2^2$, then $R_B(\Phi_1) - R_B(\Phi_2) \ge 0$, which completes the proof.

Following similar arguments, it can be proved that if $w_2 = 1$ is fixed, then $\delta_A^\eta(\mathbf{w})$ increases with $w_1$ while $\delta_B^\eta(\mathbf{w})$ decreases with $w_1$.

## References

Derman, C., G. J. Lieberman, S. M. Ross. 1972. A sequential stochastic assignment problem. *Management Science* **18**(7) 349–355.

Miettinen, K. 1999. *Nonlinear multiobjective optimization*, vol. 12. Springer Science & Business Media, New York, the United States.