SUPPLEMENT TO: "A NEW CLASS OF CHANGE POINT TEST STATISTICS OF RÉNYI TYPE"

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Abstract. This supplement contains the proofs of the results in the main paper, as well as multivariate generalizations of Theorem 2.1. Generalizations to Rényi type statistics defined with asymmetric trimming are also developed. We provide the details of the consistency of the variance estimators defined in Section 5 of the main paper.

Appendix A. Proofs of Main Results

A.1. **Proof of Theorem 2.1.** First we note that under H_0

$$\max_{t_T \le t \le T-t_T} \left| \frac{1}{t} \sum_{s=1}^{t} X_s - \frac{1}{T-t} \sum_{s=t+1}^{T} X_s \right| = \max(V_{T,1}, V_{T,2}), \tag{A.1}$$

where

$$V_{T,1} = \max_{t_T \le t \le T/2} \left| \frac{1}{t} \sum_{s=1}^{t} e_s - \frac{1}{T-t} \sum_{s=t+1}^{T} e_s \right| \quad \text{and} \quad V_{T,2} = \max_{T/2 < t \le T-t_T} \left| \frac{1}{t} \sum_{s=1}^{t} e_s - \frac{1}{T-t} \sum_{s=t+1}^{T} e_s \right|.$$

Lemma A.1. *If Assumptions 2.1–2.2 hold, then we have*

$$t_T^{1/2} V_{T,1} = t_T^{1/2} \max_{t_T \le t \le T/2} \left| \frac{1}{t} \sum_{s=1}^{t} e_s \right| + o_P(1) \tag{A.2}$$

and

$$t_T^{1/2} V_{T,2} = t_T^{1/2} \max_{T/2 < t \le T-t_T} \left| \frac{1}{T-t} \sum_{s=t+1}^{T} e_s \right| + o_P(1). \tag{A.3}$$

Proof. It is easy to see that

$$\left| V_{T,1} - \max_{t_T \le t \le T/2} \left| \frac{1}{t} \sum_{s=1}^{t} e_s \right| \right| \le \frac{2}{T} \max_{t_T \le t \le T/2} \left| \sum_{s=t+1}^{T} e_s \right|$$

and

$$\max_{t_T \le t \le T/2} \left| \sum_{s=t+1}^{T} e_s \right| \le \max_{t_T \le t \le T/2} \left| \sum_{s=1}^{t} e_s \right| + \left| \sum_{s=1}^{T/2} e_s \right| + \left| \sum_{s=T/2+1}^{T} e_s \right|.$$

By Assumption 2.2 we have

$$\max_{t_T \le t \le T/2} \left| \sum_{s=1}^{t} e_s \right| \le \max_{t_T \le t \le T/2} \left| \sum_{s=1}^{t} e_s - \sigma W_{T,1}(t) \right| + \sigma \max_{t_T \le u \le T/2} |W_{T,1}(u)|$$

and

$$\max_{1 \le t \le T/2} \left| \sum_{s=1}^{t} e_s - \sigma W_{T,1}(t) \right| = O_P(T^{\kappa}).$$

By the scale transformation of the Wiener process

$$\sup_{0 \le x \le T} |W_{T,1}(x)| \stackrel{\mathcal{D}}{=} T^{1/2} \sup_{0 \le u \le 1} |W(u)|.$$

Thus we get that

$$\max_{t_T \le t \le T/2} \left| \sum_{s=1}^{t} e_s \right| = O_P(T^{1/2}) \text{ and } \left| \sum_{s=1}^{T/2} e_s \right| = O_P(T^{1/2}).$$

Similarly,

$$\left| \sum_{s=T/2+1}^{T} e_s \right| = O_P(T^{1/2}).$$

Thus we conclude

$$\frac{1}{T} \max_{t_T \le t \le T/2} \left| \sum_{s=t+1}^{T} e_s \right| = O_P(T^{-1/2})$$

and therefore (A.2) follows Assumption 2.1. Similar arguments can be used to prove (A.3).

□

Lemma A.2. *If Assumptions 2.1–2.2 hold, then we have*

$$\left(t_T^{1/2} \max_{t_T \le t \le T/2} \left| \frac{1}{t} \sum_{s=1}^{t} e_s \right|, \quad t_T^{1/2} \max_{T/2 < t \le T - t_T} \left| \frac{1}{T-t} \sum_{s=t+1}^{T} e_s \right| \right) \xrightarrow{\mathcal{D}} \sigma \max(\xi_1, \xi_2),$$

where ξ_1 and ξ_2 are defined in Theorem 2.1.

Proof. It follows from Assumption 2.2 that

$$t_T^{1/2} \sup_{t_T \le x \le T/2} \frac{1}{\lfloor x \rfloor} \left| \sum_{s=1}^{\lfloor x \rfloor} e_s - \sigma W_{T,1}(x) \right| = O_P\left(t_T^{1/2} \sup_{t_T \le x \le T/2} \frac{1}{\lfloor x \rfloor} x^\kappa \right) = O_P(t_T^{\kappa - 1/2}) = o_P(1) \tag{A.4}$$

by Assumption 2.1 and similarly

$$t_T^{1/2} \sup_{T/2 < x \le T - t_T} \frac{1}{T - \lfloor x \rfloor} \left| \sum_{s=\lfloor x \rfloor + 1}^{T} e_s - \sigma W_{T,2}(T - x) \right| = o_P(1). \tag{A.5}$$

Since the Wiener processes $W_{T,1}$ and $W_{T,2}$ are independent for all T, the asymptotic independence in Lemma A.2 follows from (A.4) and (A.5). By symmetry, we need to show only that

$$t_T^{1/2} \sup_{t_T \le x \le T/2} \frac{1}{\lfloor x \rfloor} \left| W_{T,1}(x) \right| \xrightarrow{\mathcal{D}} \xi,$$

where ξ is defined above Theorem 2.1. By the scale transformation of the Wiener process

$$t_T^{1/2} \sup_{t_T \le x \le T/2} \frac{1}{x} \left| W_{T,1}(x) \right| = t_T^{1/2} \sup_{1 \le y \le T/(2t_T)} \left| \frac{1}{y t_T} W_{T,1}(y t_T) \right| \stackrel{\mathcal{D}}{=} \sup_{1 \le y \le T/(2t_T)} \left| \frac{W(y)}{y} \right|,$$

where W denotes a Wiener process. Since by Assumption 2.1, $T/t_T \to \infty$, elementary arguments give that

$$\sup_{1 \le y \le T/(2t_T)} \left| \frac{W(y)}{y} \right| \to \sup_{1 \le y < \infty} \left| \frac{W(y)}{y} \right| \quad \text{a.s.} \qquad \text{and} \qquad \sup_{1 \le y < \infty} \left| \frac{W(y)}{y} \right| \stackrel{\mathcal{D}}{=} \xi,$$

completing the proof of the lemma. □

Proof of Theorem 2.1. It follows immediately from Lemmas A.1 and A.2. □

A.2. **Proof of Theorems 3.1, 3.2, and 3.3.** *Proof of Theorem 3.1.* It follows from Górecki et al. (2016) that

$$T^{1/2} \| \hat{\boldsymbol{\beta}}_T - \boldsymbol{\beta}_0 \| = O_P(1),$$

where $\boldsymbol{\beta}_0$ denotes the common coefficient vector under H_0^*. Using the definition of the residuals $\hat{e}_t$ we get

$$\frac{1}{t}\sum_{s=1}^{t}\hat{e}_s-\frac{1}{T-t}\sum_{s=t+1}^{T}\hat{e}_s=\frac{1}{t}\sum_{s=1}^{t}e_s-\frac{1}{T-t}\sum_{s=t+1}^{T}e_s+\left(\frac{1}{t}\sum_{s=1}^{t}\mathbf{x}_s-\frac{1}{T-t}\sum_{s=t+1}^{T}\mathbf{x}_s\right)^{\top}(\boldsymbol{\beta}_0-\hat{\boldsymbol{\beta}}_T).$$

It follows from Assumptions 2.1, 2.2 and 3.1 that

$$\max_{t_T\le t\le T-t_T}\left\|\frac{1}{t}\sum_{s=1}^{t}\mathbf{x}_s-\frac{1}{T-t}\sum_{s=t+1}^{T}\mathbf{x}_s\right\|=o_P(1),$$

and therefore

$$t_T^{1/2}\max_{t_T\le t\le T-t_T}\left|\left(\frac{1}{t}\sum_{s=1}^{t}\mathbf{x}_s-\frac{1}{T-t}\sum_{s=t+1}^{T}\mathbf{x}_s\right)^{\top}(\boldsymbol{\beta}_0-\hat{\boldsymbol{\beta}}_T)\right|=o_P((t_T/T)^{1/2})=o_P(1).$$

□

Proof of Theorem 3.2. Górecki et al. (2016) proved that

$$T^{1/2}\|\hat{\boldsymbol{\theta}}_T-\boldsymbol{\theta}_0\|=O_P(1). \tag{A.6}$$

Hence using a two term Taylor expansion with Assumption 3.4 we conclude

$$\begin{aligned}&\frac{t_T^{1/2}}{\sigma}\max_{t_T\le t\le T-t_T}\left|\frac{1}{t}\sum_{s=1}^{t}\tilde{e}_s-\frac{1}{T-t}\sum_{s=t+1}^{T}\tilde{e}_s\right|\\&=\frac{t_T^{1/2}}{\sigma}\max_{t_T\le t\le T-t_T}\left|\frac{1}{t}\sum_{s=1}^{t}e_s-\frac{1}{T-t}\sum_{s=t+1}^{T}e_s+\left(\frac{1}{t}\sum_{s=1}^{t}\frac{\partial}{\partial\boldsymbol{\theta}}h(\mathbf{x}_s,\boldsymbol{\theta}_0)\right.\right.\\&\qquad\left.\left.-\frac{1}{T-t}\sum_{s=t+1}^{T}\frac{\partial}{\partial\boldsymbol{\theta}}h(\mathbf{x}_s,\boldsymbol{\theta}_0)\right)^{\top}(\boldsymbol{\theta}_0-\hat{\boldsymbol{\theta}}_T)\right|+O_P\left(t_T^{1/2}\frac{1}{T}\right).\end{aligned}$$

Using again Assumption 3.4 with (A.6) we get that

$$t_T^{1/2}\max_{t_T\le t\le T-t_T}\left|\left(\frac{1}{t}\sum_{s=1}^{t}\frac{\partial}{\partial\boldsymbol{\theta}}h(\mathbf{x}_s,\boldsymbol{\theta}_0)-\frac{1}{T-t}\sum_{s=t+1}^{T}\frac{\partial}{\partial\boldsymbol{\theta}}h(\mathbf{x}_s,\boldsymbol{\theta}_0)\right)^{\top}(\boldsymbol{\theta}_0-\hat{\boldsymbol{\theta}}_T)\right|=o_P(1).$$

The result now follows from Theorem 2.1. □

Proof of Theorem 3.3.

It follows from Assumptions 3.1 and 3.6 that for all $\theta \in \Theta$,

$$\frac{1}{T}\sum_{t=1}^{T} m_t(\theta) \to Em_0(\theta), \quad a.s. \tag{A.7}$$

By the mean value theorem for all $\theta, \theta' \in \Theta$, we have that for some $\nu \in (\theta, \theta')$,

$$|m_t(\theta) - m_t(\theta')| = \left|\frac{\partial}{\partial\theta} m_t(\nu)\right| |\theta - \theta'| \leq M_t|\theta - \theta'|, \tag{A.8}$$

where according to Assumption 3.6 $EM_t < \infty$. Putting together (A.7) and (A.8) and using the assumption that Θ is compact, we get that

$$\sup_{\theta\in\Theta}\left|\frac{1}{T}\sum_{t=1}^{T} m_t(\theta) - Em_0(\theta)\right| \to 0, \ a.s.,$$

which yields that

$$\hat{\theta}_T \to \theta_0, \quad a.s.$$

Again by the mean value theorem we have that

$$\frac{1}{T}\sum_{t=1}^{T} m_t(\hat{\theta}_T) - \frac{1}{T}\sum_{t=1}^{T} m_t(\theta_0) = (\hat{\theta}_T - \theta_0)\frac{1}{T}\sum_{t=1}^{T}\frac{\partial}{\partial\theta} m_t(\nu_T),$$

where ν_T satisfies $|\nu_T - \theta_0| \leq |\hat{\theta}_T - \theta_0|$. By Assumption 3.6, we have that

$$\frac{1}{T}\sum_{t=1}^{T}\frac{\partial}{\partial\theta} m_t(\hat{\theta}_T) \to E\frac{\partial}{\partial\theta} m_t(\theta_0), \ a.s.$$

Hence by Assumption 3.7, we get that

$$|\hat{\theta}_T - \theta_0| = O_P(T^{-1/2}). \tag{A.9}$$

Applying Assumptions 2.1, 3.7 and (A.9), we get that

$$t_T^{1/2} \max_{t_T \le t \le T - t_T} \left| \frac{1}{t} \sum_{s=1}^{t} (m_t(\hat{\theta}_T) - m_t(\theta_0)) \right| = o_P(1), \tag{A.10}$$

$$t_T^{1/2} \max_{t_T \le t \le T - t_T} \left| \frac{1}{T-t} \sum_{s=t+1}^{T} (m_t(\hat{\theta}_T) - m_t(\theta_0)) \right| = o_P(1). \tag{A.11}$$

Indeed by the mean value theorem, we have that

$$\frac{1}{t} \sum_{s=1}^{t} (m_t(\hat{\theta}_T) - m_t(\theta_0)) = (\hat{\theta}_T - \theta_0) \frac{1}{t} \sum_{s=1}^{t} \frac{\partial}{\partial \theta} m_t(\nu_t),$$

with $\nu_t \in (\hat{\theta}_T, \theta_0)$, and (A.9) and (A.10) imply that

$$\max_{t_T \le t \le T - t_T} \left| \frac{1}{t} \sum_{s=1}^{t} \frac{\partial}{\partial \theta} m_t(\nu_t) \right| = O_P(1),$$

which establishes (A.10). (A.11) follows similarly. Now the result follows from (A.10), (A.11), Assumption 3.7, and Theorem 2.1.

A.3. **Proof of Theorems 4.1–4.3.** Since the change in mean occurs at time t^* we have that

$$T^{-1/2} \left(\sum_{s=1}^{t} X_s - \frac{t}{T} \sum_{s=1}^{T} X_s \right) = T^{-1/2} \left(\sum_{s=1}^{t} e_s - \frac{t}{T} \sum_{s=1}^{T} e_s \right) - z_{t,T}, \tag{A.12}$$

where

$$z_{t,T} = \begin{cases} T^{-3/2} t (T - t^*) \Delta & \text{if } 1 \le t \le t^*, \\ T^{-3/2} t^* (T - t) \Delta & \text{if } t^* < t \le T \end{cases}$$

and

$$t_T^{1/2} \left(\frac{1}{t} \sum_{s=1}^{t} X_s - \frac{1}{T-t} \sum_{s=t+1}^{T} X_s \right) = t_T^{1/2} \left(\frac{1}{t} \sum_{s=1}^{t} e_s - \frac{1}{T-t} \sum_{s=t+1}^{T} e_s \right) - v_{t,T}, \tag{A.13}$$

where

$$v_{t,T} = \begin{cases} t_T^{1/2} \dfrac{T-t^*}{T-t}\Delta, & \text{if } 1 \le t \le t^*, \\ t_T^{1/2} \dfrac{t^*}{t}\Delta, & \text{if } t^* < t \le T. \end{cases} \tag{A.14}$$

Proof of Theorem 4.1. It follows from (A.12) that

$$\max_{1\le t\le T} |z_{t,T}| - \max_{1\le t\le T} T^{-1/2} \left| \sum_{s=1}^{t} e_s - \frac{t}{T}\sum_{s=1}^{T} e_s \right|$$
$$\le A_T \le \max_{1\le t\le T} |z_{t,T}| + \max_{1\le t\le T} T^{-1/2} \left| \sum_{s=1}^{t} e_s + \frac{t}{T}\sum_{s=1}^{T} e_s \right|,$$

and therefore (4.1) from the main text and Condition 4.1 imply (4.2) of the main paper. Similarly,

$$\max_{t_T\le t\le T-t_T} |v_{t,T}| - \max_{t_T\le t\le T-t_T} t_T^{1/2} \left| \frac{1}{t}\sum_{s=1}^{t} e_s - \frac{1}{T-t}\sum_{s=t+1}^{T} e_s \right|$$
$$\le t_T^{1/2} D_T \le \max_{t_T\le t\le T-t_T} |v_{t,T}| + \max_{t_T\le t\le T-t_T} t_T^{1/2} \left| \frac{1}{t}\sum_{s=1}^{t} e_s - \frac{1}{T-t}\sum_{s=t+1}^{T} e_s \right|.$$

Hence (4.3) of the main paper follows from Theorem 2.1 and Condition 4.2. □

Lemma A.3. *If (4.4) from the main paper holds, then we have that*

$$\max_{1\le t\le t^*} \frac{1}{(t^*-t+1)^{1/2}} \left| \sum_{s=t}^{t^*} e_s \right| = O_P((\log T)^{1/\bar\nu}) \tag{A.15}$$

and

$$\max_{t^*<t\le T} \frac{1}{(t-t^*+1)^{1/2}} \left| \sum_{s=t^*}^{t} e_s \right| = O_P((\log T)^{1/\bar\nu}). \tag{A.16}$$

Furthermore, for all $x>0$ we have

$$P\left\{ \max_{t^*-A\le t\le t^*} \frac{1}{t^*-t+1} \sum_{s=t}^{t^*} e_s > xA^{-1/2} \right\} \le \frac{c_*}{x^{\bar\nu}} \tag{A.17}$$

and

$$P\left\{\max_{t^*+A\le t\le T}\frac{1}{t-t^*+1}\sum_{s=t^*}^{t}e_s > xA^{-1/2}\right\}\le\frac{c_*}{x^{\bar\nu}} \tag{A.18}$$

with some constant c_*.

Proof. With $\bar e_s = e_{t^*-s+1}, 1\le s\le t^*$ and $\bar e_s=0, s>t^*$ we have

$$\max_{1\le t\le t^*}\frac{1}{(t^*-t+1)^{1/2}}\left|\sum_{s=t}^{t^*}e_s\right| = \max_{1\le t\le t^*}\frac{1}{t^{1/2}}\left|\sum_{s=1}^{t}\bar e_s\right| \le \max_{1\le u\le \log t^*}\max_{e^{u-1}\le t\le e^u}\frac{1}{t^{1/2}}\left|\sum_{s=1}^{t}\bar e_s\right|$$
$$\le \max_{1\le u\le\log t^*}e^{-(u-1)/2}\max_{e^{u-1}\le t\le e^u}\left|\sum_{s=1}^{t}\bar e_s\right|.$$

By the the maximal inequality of Billingsley (1968) (cf. also Corollary 3.1 of Móricz et al. (1982)) we have

$$E\left(\max_{e^{u-1}\le t\le e^u}\left|\sum_{s=1}^{t}\bar e_s\right|\right)^{\bar\nu}\le c_1e^{u\bar\nu/2} \tag{A.19}$$

with some constant c_1. Hence Markov's inequality yields for all $x>0$ that

$$P\left\{\max_{1\le t\le t^*}\frac{1}{(t^*-t+1)^{1/2}}\left|\sum_{s=t}^{t^*}e_s\right|>x\right\}\le\frac{e^{\bar\nu/2}}{x^{\bar\nu}}\sum_{u=0}^{\log t^*}e^{-u\bar\nu}E\max_{e^{u-1}\le t\le e^u}\left(\sum_{s=1}^{t}\bar e_s\right)^{\bar\nu}\le\frac{c_2}{x^{\bar\nu}}\log T$$

with $c_2=c_1e^{\bar\nu/2}$, completing the proof of (A.15). Similar arguments yield (A.16). Following the proof of (A.15) we have

$$\max_{1\le t\le t^*-A}\frac{1}{t^*-t+1}\left|\sum_{s=t}^{t^*}e_s\right|\le\max_{\log A\le u\le\log t^*}e^{-(u-1)}\max_{e^{u-1}\le t\le e^u}\left|\sum_{s=1}^{t}\bar e_s\right|.$$

Combining (A.19) with Markov's inequality we have

$$P\left\{\max_{1\le t\le t^*-A}\frac{1}{t^*-t+1}\left|\sum_{s=t}^{t^*}e_s\right|>x\right\}\le\frac{e^{\bar\nu}}{x^{\bar\nu}}\sum_{u=\log A}^{\log t^*}e^{-u\bar\nu}E\max_{e^{u-1}\le t\le e^u}\left(\sum_{s=1}^{t}\bar e_s\right)^{\bar\nu}$$
$$\le\frac{c_1e^{\bar\nu}}{x^{\bar\nu}}\sum_{u=\log A}^{\infty}e^{-u\bar\nu/2}\le\frac{c_*}{x^{\bar\nu}A^{\bar\nu/2}}.$$

Hence the proof of (A.17) is complete and similar arguments give (A.18). ∎

Proof of Theorem 4.2. We can assume without loss of generality that $\Delta > 0$. It follows from Lemma A.3 that

$$\max_{1\le t<t^*}(t^*-t)^{-1/2}\left\{\left|\sum_{s=t}^{t^*}e_s\right|+\frac{t^*-t}{T^{3/2}}\left|\sum_{s=1}^{T}e_s\right|\right\}=O_P((\log T)^{1/\bar{\nu}}). \tag{A.20}$$

We note that there is a constant $c_1 > 0$ such that $z_{t^*,T}-z_{t,T}\ge c_1(t^*-t)\Delta T^{-1/2}$, if $1\le t\le t^*$, and therefore for all c_2 we have

$$\limsup_{T\to\infty}\max_{1\le t\le t^*-t_1}(T^{1/2}(z_{t,T}-z_{t^*,T})+c_2(t^*-t)^{1/2}(\log T)^{1/\bar{\nu}})=-\infty, \tag{A.21}$$

where $t_1=(\log T)/\Delta^2$. Next we observe that

$$\begin{aligned}\lim_{T\to\infty}P\Bigg\{&\max_{1\le t\le t^*-t_1}\left|\sum_{s=1}^{t}X_s-\frac{t}{T}\sum_{s=1}^{T}X_s\right|-\left|\sum_{s=1}^{t^*}X_s-\frac{t^*}{T}\sum_{s=1}^{T}X_s\right|\\ &=\max_{1\le t\le t^*-t_1}\left(\sum_{s=1}^{t}X_s-\frac{t}{T}\sum_{s=1}^{T}X_s\right)-\left(\sum_{s=1}^{t^*}X_s-\frac{t^*}{T}\sum_{s=1}^{T}X_s\right)\Bigg\}=1.\end{aligned} \tag{A.22}$$

Hence it follows from (A.20)–(A.22) that $t^*\Delta^2\to\infty$ implies

$$\limsup_{T\to\infty}P\left\{\max_{1\le t\le t^*-t_1}\left|\sum_{s=1}^{t}X_s-\frac{t}{T}\sum_{s=1}^{T}X_s\right|-\left|\sum_{s=1}^{t^*}X_s-\frac{t^*}{T}\sum_{s=1}^{T}X_s\right|>-x\right\}=0$$

for all $x > 0$.

Let $t_{C,1}=C/\Delta^2$. We note that by (A.17) for all $C > 2$

$$\lim_{\alpha\to\infty}P\left\{\max_{t^*-t_1\le t\le t_{C,1}}\frac{1}{t^*-t+1}\left|\sum_{s=t}^{t^*}e_s\right|>\alpha t_{C,1}^{-1/2}\right\}=0.$$

It is easy to see that for all α and $c_1 > 0$

$$\limsup_{C\to\infty}\max_{t^*-t_1\le t\le t^*-t_{C,1}}\left(\alpha t_{C,1}^{1/2}(t^*-t)-c_1(t^*-t)\Delta\right)<0,$$

if $\Delta = O(1)$. (Similar argument can be used when $\Delta \to \infty$.) Similarly to (A.22) we have that

(A.23)

$$\lim_{C\to\infty} \liminf_{T\to\infty} P\Bigg\{ \max_{t^*-t_1\le t\le t^*-t_{C,1}} \left| \sum_{s=1}^{t} X_s - \frac{t}{T}\sum_{s=1}^{T} X_s \right| - \left| \sum_{s=1}^{t^*} X_s - \frac{t^*}{T}\sum_{s=1}^{T} X_s \right| \\ = \max_{t^*-t_1\le t\le t^*-t_{C,1}} \left(\sum_{s=1}^{t} X_s - \frac{t}{T}\sum_{s=1}^{T} X_s \right) - \left(\sum_{s=1}^{t^*} X_s - \frac{t^*}{T}\sum_{s=1}^{T} X_s \right) \Bigg\} = 1.$$

Thus we obtain that

$$\hat{t}_T = \mathrm{argmax}_{1\le t\le T}\left\{ \left| \sum_{s=1}^{t} X_s - \frac{t}{T}\sum_{s=1}^{T} X_s \right| \right\}$$

satisfies $|\hat{t}_T - t^*| I\{\hat{t}_T \le t^*\} = O_P(\Delta^{-2})$. Similarly to (A.24) we have

$$\max_{t^*<t\le T}(t-t^*)^{-1/2}\left\{ \left| \sum_{s=t^*}^{t} e_s \right| + \frac{t-t^*}{T^{3/2}}\left| \sum_{s=1}^{T} e_s \right| \right\} = O_P((\log T)^{1/\bar{\nu}}). \tag{A.24}$$

Elementary arguments give that

$$z_{t^*,T} - z_{t,T} \ge (t-t^*)\Delta T^{-3/2} t^*, \quad \text{if } \ t^* \le t \le T. \tag{A.25}$$

Let $t_2 = (T(\log T)/(\Delta t^*))^2$. It follows from (A.25) that

$$\limsup_{T\to\infty} \max_{t^*+t_2\le t\le T} (T^{1/2}(z_{t,T} - z_{t^*,T}) + c_3(t-t^*)^{1/2}(\log T)^{1/\bar{\nu}}) = -\infty \tag{A.26}$$

for all $c_3 > 0$. Putting together (A.24) and (A.26) we conclude

$$\limsup_{T\to\infty} P\left\{ \max_{t^*+t_2\le t\le T} T^{-1/2}\left| \sum_{s=1}^{t} X_s - \frac{t}{T}\sum_{s=1}^{T} X_s \right| - T^{-1/2}\left| \sum_{s=1}^{t^*} X_s - \frac{t^*}{T}\sum_{s=1}^{T} X_s \right| > -x \right\} = 0$$

for all $x > 0$. Let $t_{C,2} = (T/(\Delta t^*))^2$. Using (A.18) we conclude that for all $C > 0$

$$\lim_{\alpha\to\infty} \limsup_{T\to\infty} P\left\{ \max_{t^*+t_{C,2}\le t\le t^*+t_2} \frac{1}{t-t^*}\left| \sum_{s=t^*}^{t} e_s \right| > \alpha t_{C,2}^{1/2} \right\} = 0.$$

Elementary arguments give that for C sufficiently large

$$\limsup_{T\to\infty} \max_{t^*+t_2\le t\le T} (T^{1/2}(z_{t,T} - z_{t^*,T}) + \alpha(t-t^*)t_{C,2}^{-1/2}) = -\infty.$$

Similarly to (A.22) and (A.23) we have

$$\lim_{C\to\infty} \liminf_{T\to\infty} P\Bigg\{ \max_{t^*+t_{C,2}\le t\le t^*+t_2} \left|\sum_{s=1}^{t} X_s - \frac{t}{T}\sum_{s=1}^{T} X_s\right| - \left|\sum_{s=1}^{t^*} X_s - \frac{t^*}{T}\sum_{s=1}^{T} X_s\right|$$
$$= \max_{t^*+t_{C,2}\le t\le t^*+t_2} \left(\sum_{s=1}^{t} X_s - \frac{t}{T}\sum_{s=1}^{T} X_s\right) - \left(\sum_{s=1}^{t^*} X_s - \frac{t^*}{T}\sum_{s=1}^{T} X_s\right) \Bigg\} = 1.$$

Thus we conclude

$$|\hat{t}_T - t^*| I\{\hat{t}_T \ge t^*\} = O_P\left(\left(\frac{T}{\Delta t^*}\right)^2\right).$$

Hence

$$\lim_{C\to\infty} \liminf_{T\to\infty} P\left\{ \max_{1\le t\le T} \left|\sum_{s=1}^{t} X_s - \frac{t}{T}\sum_{s=1}^{T} X_s\right| = \max_{t^*-t_{C,1}\le t\le t^*+t_{C,2}} \left|\sum_{s=1}^{t} X_s - \frac{t}{T}\sum_{s=1}^{T} X_s\right| \right\} = 1.$$

Also,

$$\lim_{T\to\infty} P\left\{ \max_{t^*-t_{C,1}\le t\le t^*+t_{C,2}} \left|\sum_{s=1}^{t^*} e_s - \frac{t^*}{T}\sum_{s=1}^{T} e_s - T^{1/2} z_{t,T}\right| = -\sum_{s=1}^{t^*} e_s + \frac{t^*}{T}\sum_{s=1}^{T} e_s + T^{1/2} z_{t^*,T} \right\} = 1$$

for all $C > 0$. It follows from Assumption 2.2 and the stationarity of $e_s, -\infty < s < \infty$ that for all $C > 0$

$$\max_{t^*-t_{C,1}\le t\le t^*} \left|\sum_{s=t}^{t^*} e_s\right| = O_P\left(\frac{1}{\Delta}\right) \quad \text{and} \quad \max_{t^*\le t\le t^*+t_{C,2}} \left|\sum_{s=t^*}^{t} e_s\right| = O_P\left(\frac{T}{\Delta t^*}\right).$$

Using again Assumption 2.2 and $t^*/T \to 0$ we conclude

$$\frac{1}{\sqrt{t^*}}\left(\sum_{s=1}^{t^*} e_s - \frac{t^*}{T}\sum_{s=1}^{T} e_s\right) \quad \stackrel{\mathcal{D}}{\to} \quad \sigma\mathcal{N},$$

where $\mathcal{N}$ denotes a standard normal random variable. The result in Theorem 4.2(i) is now proven, since

$$\frac{1}{\sqrt{t^*}}\left(\frac{T}{\Delta t^*} + \frac{1}{\Delta}\right) \to 0.$$

We note that

$$\lim_{T\to\infty} P\left\{D_T = \sup_{t_T\le t\le T-t_T}\left(-\frac{1}{t}\sum_{s=1}^{t} e_s + \frac{1}{T-t}\sum_{s=t+1}^{T} e_s + t_T^{-1/2}|v_{t,T}|\right)\right\} = 1,$$

where $v_{t,T}$ is defined in A.14. We write

$$\sup_{t_T\le t\le T-t_T}\left(-\frac{1}{t}\sum_{s=1}^{t} e_s + \frac{1}{T-t}\sum_{s=t+1}^{T} e_s + t_T^{-1/2}|v_{t,T}|\right) - |\Delta| = \max\{G_{T,1}, G_{T,2}\},$$

where

$$G_{T,1} = \sup_{t_T\le t\le t^*}\left(-\frac{1}{t}\sum_{s=1}^{t} e_s + \frac{1}{T-t}\sum_{s=t+1}^{T} e_s + t_T^{-1/2}\bar{v}_{t,T}\right),$$

and

$$G_{T,2} = \sup_{t^*\le t\le T-t_T}\left(-\frac{1}{t}\sum_{s=1}^{t} e_s + \frac{1}{T-t}\sum_{s=t+1}^{T} e_s + t_T^{-1/2}\bar{v}_{t,T}\right),$$

with $\bar{v}_{t,T} = |v_{t,T}| - t_T^{1/2}|\Delta|$. It follows from the proof of Lemma A.1 that $t_T^{1/2}G_{T,1} = t_T^{1/2}G_{T,3} + o_P(1)$, where

$$G_{T,3} = \sup_{t_T\le t\le t^*}\left(-\frac{1}{t}\sum_{s=1}^{t} e_s + t_T^{-1/2}\bar{v}_{t,T}\right).$$

The assumptions of Theorem 4.2(ii) imply that

$$\max_{t_T\le t\le t^*}|\bar{v}_{t,T}| = O\left(\frac{t_T^{1/2}t^*|\Delta|}{T}\right) = o(1),$$

and therefore $t_T^{1/2}G_{T,3} = t_T^{1/2}\max_{t_T\le t\le t^*} -\frac{1}{t}\sum_{s=1}^{t} e_s + o_P(1)$. Using the proof of Theorem 2.1 we obtain that

$$t_T^{1/2}\max_{t_T\le t\le t^*} -\frac{1}{t}\sum_{s=1}^{t} e_s \xrightarrow{D} \sigma\sup_{0\le u\le 1} W(u),$$

where $W(u)$ is a standard Wiener process. Also, by Theorem 2.1, we have that

$$t_T^{1/2} \sup_{t^* \le t \le 2t^*} \left(-\frac{1}{t} \sum_{s=1}^{t} e_s + \frac{1}{T-t} \sum_{s=t+1}^{T} e_s \right) = o_P(1),$$

and

$$t_T^{1/2} \sup_{2t^* \le t \le T-t_T} \left(-\frac{1}{t} \sum_{s=1}^{t} e_s + \frac{1}{T-t} \sum_{s=t+1}^{T} e_s \right) = O_P(1). \tag{A.27}$$

Since $\bar{v}_{t,T} \le 0$ for all t, we have that

$$t_T^{1/2} \sup_{t^* \le t \le 2t^*} \left(-\frac{1}{t} \sum_{s=1}^{t} e_s + \frac{1}{T-t} \sum_{s=t+1}^{T} e_s + t^{-1/2} \bar{v}_{t,T} \right) = o_P(1). \tag{A.28}$$

Again due to the assumptions of Theorem 4.2(ii) we have that $\sup_{2t^* \le t \le T-t_T} \bar{v}_{t,T} \to -\infty$ as $T \to \infty$, and therefore by (A.27) we have

$$t_T^{1/2} \sup_{2t^* \le t \le T-t_T} \left(-\frac{1}{t} \sum_{s=1}^{t} e_s + \frac{1}{T-t} \sum_{s=t+1}^{T} e_s + t_T^{1/2} \bar{v}_{t,T} \right) \xrightarrow{P} -\infty,$$

as $T \to \infty$. Hence

$$P\left\{ t_T^{1/2} G_{T,2} = \sup_{t^* \le t \le 2t^*} t_T^{1/2} \left(-\frac{1}{t} \sum_{s=1}^{t} e_s + \frac{1}{T-t} \sum_{s=t+1}^{T} e_s + t_T^{1/2} \bar{v}_{t,T} \right) \right\} = 1.$$

It now follows from (A.28) that $t_T^{1/2} G_{T,2} = o_P(1)$, from which the result follows. □

Proof of Theorem 4.3. It follows from the definition of the residuals in equation (3.1) of the main text that

$$\frac{1}{t^*} \sum_{s=1}^{t^*} \hat{e}_s - \frac{1}{T-t^*} \sum_{s=t^*+1}^{T} \hat{e}_s$$

$$= \frac{1}{t^*} \sum_{s=1}^{t^*} e_s - \frac{1}{T-t^*} \sum_{s=t^*+1}^{T} e_s + \left(\frac{1}{t^*} \sum_{s=1}^{t^*} \mathbf{x}_s \right)^\top (\boldsymbol{\beta}_{(1)} - \hat{\boldsymbol{\beta}}_T) - \left(\frac{1}{T-t^*} \sum_{s=t^*+1}^{T} \mathbf{x}_s \right)^\top (\boldsymbol{\beta}_{(2)} - \hat{\boldsymbol{\beta}}_T)$$

and

$$t_T^{1/2} \left(\frac{1}{t^*} \sum_{s=1}^{t^*} e_s - \frac{1}{T-t^*} \sum_{s=t^*+1}^{T} e_s \right) = O_P(1)$$

by Theorem 3.1. Using Assumption 3.1 one can verify that

$$\hat{\boldsymbol{\beta}}_N = \frac{t^*}{T}\boldsymbol{\beta}_{(1)} + \frac{T-t^*}{T}\boldsymbol{\beta}_{(2)} + O_P\left(\frac{\sqrt{t^*}}{T} + \frac{\sqrt{T-t^*}}{T}\right).$$

Hence Assumption 3.1(ii) yields

$$\begin{aligned} &r_T^{1/2}\left\{\left(\frac{1}{t^*}\sum_{s=1}^{t^*}\mathbf{x}_s\right)^\top(\boldsymbol{\beta}_{(1)} - \hat{\boldsymbol{\beta}}_T) - \left(\frac{1}{T-t^*}\sum_{s=t^*+1}^{T}\mathbf{x}_s\right)^\top(\boldsymbol{\beta}_{(2)} - \hat{\boldsymbol{\beta}}_T)\right\} \\ &\quad = t_T^{1/2}\left\{\bar{\mathbf{x}}_0^\top(\boldsymbol{\beta}_{(1)} - \hat{\boldsymbol{\beta}}_T) - \bar{\mathbf{x}}_0^\top(\boldsymbol{\beta}_{(2)} - \hat{\boldsymbol{\beta}}_T)\right\}(1+o_P(1)), \end{aligned}$$

so the result follows from Assumption 4.1(ii). □

Appendix B. Multivariate generalization of Theorem 2.1

Consider a stationary sequence of random vectors $\mathbf{X}_t \in \mathbb{R}^d$ that satisfies which

$$\mathbf{X}_t = \boldsymbol{\mu} + \boldsymbol{\varepsilon}_t, \quad 1 \le t \le T, \tag{B.1}$$

where $E\boldsymbol{\varepsilon}_t = \mathbf{0}$, and

$$\Sigma = \lim_{n\to\infty}\frac{1}{n}\mathrm{cov}\left(\sum_{j=1}^n \boldsymbol{\varepsilon}_j, \sum_{j=1}^n \boldsymbol{\varepsilon}_j\right),$$

which we assume is well defined and invertible. We let $\|\cdot\|$ denote the Euclidean norm in $\mathbb{R}^d$. We consider the asymptotic properties of D_T defined by

$$D_T^2 = \sup_{t_T \le x \le T - t_T}\left(\frac{1}{\lfloor x\rfloor}\sum_{j=1}^{\lfloor x\rfloor}\boldsymbol{X}_j - \frac{1}{T-\lfloor x\rfloor}\sum_{j=\lfloor x\rfloor+1}^{T}\boldsymbol{X}_j\right)^\top \Sigma^{-1}\left(\frac{1}{\lfloor x\rfloor}\sum_{j=1}^{\lfloor x\rfloor}\boldsymbol{X}_j - \frac{1}{T-\lfloor x\rfloor}\sum_{j=\lfloor x\rfloor+1}^{T}\boldsymbol{X}_j\right),$$

i.e. D_T is the square root of the trimmed maximally selected quadratic form based the difference between the averages of $\boldsymbol{X}_t$.

Assumption B.1. $t_T \to \infty$ and $t_T/T \to 0$, *as* $T \to \infty$.

In order to establish the limit distribution of D_T under Assumption B.1, we require a rate in the weak convergence of the partial sum process of the $\boldsymbol{\varepsilon}_j$'s, which we quantify with the following assumption:

Assumption B.2. There are two independent sequences of standard $d-$dimensional Wiener processes $\{\mathbf{W}_{T,1}(x), 0 \le x \le T/2\}$ and $\{\mathbf{W}_{T,2}(x), 0 \le x \le T/2\}$ such that

$$\max_{1 \le x \le T/2} x^{-\kappa} \left\| \sum_{s=1}^{\lfloor x \rfloor} \boldsymbol{\varepsilon}_s - \Sigma^{1/2} \mathbf{W}_{T,1}(x) \right\| = O_P(1) \tag{B.2}$$

and

$$\max_{T/2 \le x \le T-1} (T-x)^{-\kappa} \left\| \sum_{s=\lfloor x \rfloor +1}^{T} \boldsymbol{\varepsilon}_s - \Sigma^{1/2} \mathbf{W}_{T,2}(T-x) \right\| = O_P(1) \tag{B.3}$$

with some $0 < \kappa < 1/2$.

Theorem B.1. *Under these assumptions, we have that*

$$t_T^{1/2} D_T \stackrel{\mathcal{D}}{\to} \max(\xi_1, \xi_2),$$

where ξ_1, ξ_2 are independent and each have the same distribution as

$$\xi \stackrel{D}{=} \sup_{0 \le x \le 1} \|\mathbf{W}(x)\|,$$

where $\mathbf{W}(x)$ is a standard d-dimensional Brownian motion.

B.1. *Proof of Theorem B.1.* Evidently under (B.1), the statistic D_T does not depend on $\boldsymbol{\mu}$, and hence we may assume without loss of generality that D_T is defined by

$$D_T^2 = \sup_{t_T \le x \le T - t_T} \left(\frac{1}{\lfloor x \rfloor} \sum_{j=1}^{\lfloor x \rfloor} \boldsymbol{\varepsilon}_j - \frac{1}{T - \lfloor x \rfloor} \sum_{j=\lfloor x \rfloor +1}^{T} \boldsymbol{\varepsilon}_j \right)^{\top} \Sigma^{-1} \left(\frac{1}{\lfloor x \rfloor} \sum_{j=1}^{\lfloor x \rfloor} \boldsymbol{\varepsilon}_j - \frac{1}{T - \lfloor x \rfloor} \sum_{j=\lfloor x \rfloor +1}^{T} \boldsymbol{\varepsilon}_j \right).$$

It follows then that

$$\sup_{t_T\le x\le T-t_T}\left(\frac{1}{\lfloor x\rfloor}\sum_{j=1}^{\lfloor x\rfloor}\boldsymbol{\varepsilon}_j-\frac{1}{T-\lfloor x\rfloor}\sum_{j=\lfloor x\rfloor+1}^{T}\boldsymbol{\varepsilon}_j\right)^{\top}\Sigma^{-1}\left(\frac{1}{\lfloor x\rfloor}\sum_{j=1}^{\lfloor x\rfloor}\boldsymbol{\varepsilon}_j-\frac{1}{T-\lfloor x\rfloor}\sum_{j=\lfloor x\rfloor+1}^{T}\boldsymbol{\varepsilon}_j\right) = \max(V_{T,1},V_{T,2}) \tag{B.4}$$

where

$$V_{T,1}=\sup_{t_T\le x\le T/2}\left(\frac{1}{\lfloor x\rfloor}\sum_{j=1}^{\lfloor x\rfloor}\boldsymbol{\varepsilon}_j-\frac{1}{T-\lfloor x\rfloor}\sum_{j=\lfloor x\rfloor+1}^{T}\boldsymbol{\varepsilon}_j\right)^{\top}\Sigma^{-1}\left(\frac{1}{\lfloor x\rfloor}\sum_{j=1}^{\lfloor x\rfloor}\boldsymbol{\varepsilon}_j-\frac{1}{T-\lfloor x\rfloor}\sum_{j=\lfloor x\rfloor+1}^{T}\boldsymbol{\varepsilon}_j\right),\quad\text{and}$$

$$V_{T,2}=\sup_{T/2\le x\le T-t_T}\left(\frac{1}{\lfloor x\rfloor}\sum_{j=1}^{\lfloor x\rfloor}\boldsymbol{\varepsilon}_j-\frac{1}{T-\lfloor x\rfloor}\sum_{j=\lfloor x\rfloor+1}^{T}\boldsymbol{\varepsilon}_j\right)^{\top}\Sigma^{-1}\left(\frac{1}{\lfloor x\rfloor}\sum_{j=1}^{\lfloor x\rfloor}\boldsymbol{\varepsilon}_j-\frac{1}{T-\lfloor x\rfloor}\sum_{j=\lfloor x\rfloor+1}^{T}\boldsymbol{\varepsilon}_j\right)$$

Lemma B.1. *If Assumptions B.1 and B.2 hold, then we have*

$$t_TV_{T,1}=t_T\sup_{t_T\le x\le T/2}\left(\frac{1}{\lfloor x\rfloor}\sum_{j=1}^{\lfloor x\rfloor}\boldsymbol{\varepsilon}_j\right)^{\top}\Sigma^{-1}\left(\frac{1}{\lfloor x\rfloor}\sum_{j=1}^{\lfloor x\rfloor}\boldsymbol{\varepsilon}_j\right)+o_P(1) \tag{B.5}$$

and

$$t_TV_{T,2}=t_T\sup_{T/2\le x\le T-t_T}\left(\frac{1}{T-\lfloor x\rfloor}\sum_{j=\lfloor x\rfloor+1}^{T}\boldsymbol{\varepsilon}_j\right)^{\top}\Sigma^{-1}\left(\frac{1}{T-\lfloor x\rfloor}\sum_{j=\lfloor x\rfloor+1}^{T}\boldsymbol{\varepsilon}_j\right)+o_P(1). \tag{B.6}$$

Proof. Let $V_{T,1}^*=\sup_{t_T\le x\le T/2}\left(\frac{1}{\lfloor x\rfloor}\sum_{j=1}^{\lfloor x\rfloor}\boldsymbol{\varepsilon}_j\right)^{\top}\Sigma^{-1}\left(\frac{1}{\lfloor x\rfloor}\sum_{j=1}^{\lfloor x\rfloor}\boldsymbol{\varepsilon}_j\right)$. It follows that

$$\begin{aligned}|V_{T,1}-V_{T,1}^*|&\le 2\sup_{t_T\le x\le T/2}\left|\left(\frac{1}{T-\lfloor x\rfloor}\sum_{j=\lfloor x\rfloor+1}^{T}\boldsymbol{\varepsilon}_j\right)^{\top}\Sigma^{-1}\left(\frac{1}{\lfloor x\rfloor}\sum_{j=1}^{\lfloor x\rfloor}\boldsymbol{\varepsilon}_j\right)\right|\\&\quad+\sup_{t_T\le x\le T/2}\left|\left(\frac{1}{T-\lfloor x\rfloor}\sum_{j=\lfloor x\rfloor+1}^{T}\boldsymbol{\varepsilon}_j\right)^{\top}\Sigma^{-1}\left(\frac{1}{T-\lfloor x\rfloor}\sum_{j=\lfloor x\rfloor+1}^{T}\boldsymbol{\varepsilon}_j\right)\right|\\&=:2E_1+E_2.\end{aligned}$$

The Cauchy-Schwarz inequality implies that

$$E_1 \leq \|\Sigma^{-1}\|_{op} \sup_{t_T \leq x \leq T/2} \left\| \frac{1}{T - \lfloor x \rfloor} \sum_{j=\lfloor x \rfloor+1}^{T} \boldsymbol{\varepsilon}_j \right\| \left\| \frac{1}{\lfloor x \rfloor} \sum_{j=1}^{\lfloor x \rfloor} \boldsymbol{\varepsilon}_j \right\| \tag{B.7}$$

$$\leq \|\Sigma^{-1}\|_{op} \sup_{t_T \leq x \leq T/2} \left\| \frac{1}{T - \lfloor x \rfloor} \sum_{j=\lfloor x \rfloor+1}^{T} \boldsymbol{\varepsilon}_j \right\| \sup_{t_T \leq x \leq T/2} \left\| \frac{1}{\lfloor x \rfloor} \sum_{j=1}^{\lfloor x \rfloor} \boldsymbol{\varepsilon}_j \right\|,$$

Where $\|\cdot\|_{op}$ denotes the operator norm of a matrix. We first aim to bound the second term on the right hand side of the last line. We have for this term that

$$\sup_{t_T \leq x \leq T/2} \left\| \frac{1}{T - \lfloor x \rfloor} \sum_{j=\lfloor x \rfloor+1}^{T} \boldsymbol{\varepsilon}_j \right\| \leq \frac{2}{T} \sup_{t_T \leq x \leq T/2} \left\| \sum_{j=\lfloor x \rfloor+1}^{T} \boldsymbol{\varepsilon}_j \right\|.$$

By the triangle inequality it follows that

$$\sup_{t_T \leq x \leq T/2} \left\| \sum_{j=\lfloor x \rfloor+1}^{T} \boldsymbol{\varepsilon}_j \right\| \leq \sup_{t_T \leq x \leq T/2} \left\| \sum_{j=1}^{\lfloor x \rfloor} \boldsymbol{\varepsilon}_j \right\| + \left\| \sum_{j=1}^{T/2} \boldsymbol{\varepsilon}_j \right\| + \left\| \sum_{j=T/2+1}^{T} \boldsymbol{\varepsilon}_j \right\|.$$

Again using the triangle inequality, we obtain that

$$\sup_{t_T \leq x \leq T/2} \left\| \sum_{j=1}^{\lfloor x \rfloor} \boldsymbol{\varepsilon}_j \right\| \leq \sup_{t_T \leq x \leq T/2} \left\| \sum_{j=1}^{\lfloor x \rfloor} \boldsymbol{\varepsilon}_j - \Sigma^{1/2} \mathbf{W}_{T,2}(x) \right\| + \sup_{t_T \leq x \leq T/2} \left\| \Sigma^{1/2} \mathbf{W}_{T,2}(x) \right\|.$$

By Assumption B.1,

$$\sup_{t_T \leq x \leq T/2} \left\| \sum_{j=1}^{\lfloor x \rfloor} \boldsymbol{\varepsilon}_j - \Sigma^{1/2} \mathbf{W}_{T,2}(x) \right\| = O_P(T^{\kappa}),$$

and by the scale transformation of the Brownian motion,

$$\sup_{t_T \leq x \leq T/2} \left\| \Sigma^{1/2} \mathbf{W}_{T,2}(x) \right\| = O_P(T^{1/2}).$$

It follows similarly from Assumption B.1 that

$$\left\| \sum_{j=1}^{T/2} \boldsymbol{\varepsilon}_j \right\| = O_P(T^{1/2}), \text{ and } \left\| \sum_{j=T/2+1}^{T} \boldsymbol{\varepsilon}_j \right\| = O_P(T^{1/2}),$$

and therefore $\sup_{t_T \le x \le T/2} \left\| \frac{1}{T - \lfloor x \rfloor} \sum_{j=\lfloor x \rfloor + 1}^{T} \boldsymbol{\varepsilon}_j \right\| = O_P(T^{-1/2})$. We now turn to bounding the third term on the last line of (B.7). We have for this term by applying Assumption B.1, the triangle inequality, and the scale transformation of the Brownian motion that

$$\begin{aligned}
\sup_{t_T \le x \le T/2} \left\| \frac{1}{\lfloor x \rfloor} \sum_{j=1}^{\lfloor x \rfloor} \boldsymbol{\varepsilon}_j \right\| &\le \sup_{t_T \le x \le T/2} \frac{1}{\lfloor x \rfloor} \left\| \sum_{j=1}^{\lfloor x \rfloor} \boldsymbol{\varepsilon}_j - \Sigma^{1/2} \mathbf{W}_{T,2}(x) \right\| + \sup_{t_T \le x \le T/2} \frac{1}{\lfloor x \rfloor} \left\| \Sigma^{1/2} \mathbf{W}_{T,2}(x) \right\| \\
&\le \frac{1}{t_T^{1-\kappa}} \sup_{t_T \le x \le T/2} \frac{1}{\lfloor x \rfloor^{\kappa}} \left\| \sum_{j=1}^{\lfloor x \rfloor} \boldsymbol{\varepsilon}_j - \Sigma^{1/2} \mathbf{W}_{T,2}(x) \right\| + \max_{1 \le y \le T/(2t_T)} \frac{1}{t_T^{1/2}} \left\| \Sigma^{1/2} \frac{\mathbf{W}_{T,2}(y)}{y} \right\| \\
&= O_P\left(\frac{1}{t_T^{1-\kappa}} + \frac{1}{t_T^{1/2}} \right) = O_P(t_T^{-1/2}).
\end{aligned}$$

It now follows that $t_T E_1 = O_P((t_T/T)^{1/2}) = o_P(1)$. One may obtain by similar means that $t_T E_2 = o_P(1)$, from which (B.5) follows. The proof of (B.6) follows similar lines, and so we omit the details.

□

Lemma B.2. *If Assumptions B.1 and B.2 hold, then we have*

$$\begin{aligned}
\max\Bigg(& t_T \sup_{t_T \le x \le T/2} \left(\frac{1}{\lfloor x \rfloor} \sum_{j=1}^{\lfloor x \rfloor} \boldsymbol{\varepsilon}_j \right)^{\top} \Sigma^{-1} \left(\frac{1}{\lfloor x \rfloor} \sum_{j=1}^{\lfloor x \rfloor} \boldsymbol{\varepsilon}_j \right), \\
& t_T \sup_{T/2 \le x \le T - t_T} \left(\frac{1}{T - \lfloor x \rfloor} \sum_{j=\lfloor x \rfloor + 1}^{T} \boldsymbol{\varepsilon}_j \right)^{\top} \Sigma^{-1} \left(\frac{1}{T - \lfloor x \rfloor} \sum_{j=\lfloor x \rfloor + 1}^{T} \boldsymbol{\varepsilon}_j \right) \Bigg) \xrightarrow{\mathcal{D}} \sigma \max(\xi_1, \xi_2),
\end{aligned}$$

where ξ_1 and ξ_2 are defined in Theorem B.1.

Proof. We first aim to show that

$$\begin{aligned}
t_T \Bigg| & \sup_{t_T \le x \le T/2} \left(\frac{1}{\lfloor x \rfloor} \sum_{j=1}^{\lfloor x \rfloor} \boldsymbol{\varepsilon}_j \right)^{\top} \Sigma^{-1} \left(\frac{1}{\lfloor x \rfloor} \sum_{j=1}^{\lfloor x \rfloor} \boldsymbol{\varepsilon}_j \right) \\
& - \sup_{t_T \le x \le T/2} \left(\frac{\Sigma^{1/2}}{\lfloor x \rfloor} \mathbf{W}_{T,1}(x) \right)^{\top} \Sigma^{-1} \left(\frac{\Sigma^{1/2}}{\lfloor x \rfloor} \mathbf{W}_{T,1}(x) \right) \Bigg| = o_P(1), \tag{B.8}
\end{aligned}$$

and

$$\text{(B.9)} \qquad t_T \Bigg| \sup_{T/2 \le x \le T - t_T} \left(\frac{1}{T - \lfloor x \rfloor} \sum_{j=\lfloor x \rfloor + 1}^{T} \boldsymbol{\varepsilon}_j \right)^{\top} \Sigma^{-1} \left(\frac{1}{T - \lfloor x \rfloor} \sum_{j=\lfloor x \rfloor + 1}^{T} \boldsymbol{\varepsilon}_j \right) - \sup_{T/2 \le x \le T - t_T} \left(\frac{\Sigma^{1/2}}{\lfloor x \rfloor} \mathbf{W}_{T,2}(x) \right)^{\top} \Sigma^{-1} \left(\frac{\Sigma^{1/2}}{\lfloor x \rfloor} \mathbf{W}_{T,2}(x) \right) \Bigg| = o_P(1).$$

Towards establishing (B.8), we note that the left hand side is bounded above by

$$t_T \sup_{t_T \le x \le T/2} \left| \left(\frac{1}{\lfloor x \rfloor} \sum_{j=1}^{\lfloor x \rfloor} \boldsymbol{\varepsilon}_j \right)^{\top} \Sigma^{-1} \left(\frac{1}{\lfloor x \rfloor} \sum_{j=1}^{\lfloor x \rfloor} \boldsymbol{\varepsilon}_j \right) - \left(\frac{\Sigma^{1/2}}{\lfloor x \rfloor} \mathbf{W}_{T,1}(x) \right)^{\top} \Sigma^{-1} \left(\frac{\Sigma^{1/2}}{\lfloor x \rfloor} \mathbf{W}_{T,1}(x) \right) \right|,$$

which by the triangle inequality is less than or equal to

$$t_T \sup_{t_T \le x \le T/2} \left| \left(\frac{1}{\lfloor x \rfloor} \sum_{j=1}^{\lfloor x \rfloor} \boldsymbol{\varepsilon}_j - \frac{\Sigma^{1/2}}{\lfloor x \rfloor} \mathbf{W}_{T,1}(x) \right)^{\top} \Sigma^{-1} \left(\frac{1}{\lfloor x \rfloor} \sum_{j=1}^{\lfloor x \rfloor} \boldsymbol{\varepsilon}_j \right) \right| + t_T \sup_{t_T \le x \le T/2} \left| \left(\frac{1}{\lfloor x \rfloor} \sum_{j=1}^{\lfloor x \rfloor} \boldsymbol{\varepsilon}_j \right)^{\top} \Sigma^{-1} \left(\frac{1}{\lfloor x \rfloor} \sum_{j=1}^{\lfloor x \rfloor} \boldsymbol{\varepsilon}_j - \frac{\Sigma^{1/2}}{\lfloor x \rfloor} \mathbf{W}_{T,1}(x) \right) \right| =: G_1 + G_2.$$

For the first summand we have by the Cauchy-Schwarz inequality that

$$t_T \sup_{t_T \le x \le T/2} \left| \left(\frac{1}{\lfloor x \rfloor} \sum_{j=1}^{\lfloor x \rfloor} \boldsymbol{\varepsilon}_j - \frac{\Sigma^{1/2}}{\lfloor x \rfloor} \mathbf{W}_{T,1}(x) \right)^{\top} \Sigma^{-1} \left(\frac{1}{\lfloor x \rfloor} \sum_{j=1}^{\lfloor x \rfloor} \boldsymbol{\varepsilon}_j \right) \right| \le t_T \|\Sigma^{-1}\|_{op} \sup_{t_T \le x \le T/2} \left\| \frac{1}{\lfloor x \rfloor} \sum_{j=1}^{\lfloor x \rfloor} \boldsymbol{\varepsilon}_j - \frac{\Sigma^{1/2}}{\lfloor x \rfloor} \mathbf{W}_{T,1}(x) \right\| \sup_{t_T \le x \le T/2} \left\| \frac{1}{\lfloor x \rfloor} \sum_{j=1}^{\lfloor x \rfloor} \boldsymbol{\varepsilon}_j \right\|.$$

It now follows as in the proof of Lemma B.1 that

$$\sup_{t_T \le x \le T/2} \left\| \frac{1}{\lfloor x \rfloor} \sum_{j=1}^{\lfloor x \rfloor} \boldsymbol{\varepsilon}_j - \frac{\Sigma^{1/2}}{\lfloor x \rfloor} \mathbf{W}_{T,1}(x) \right\| = O_P(t_T^{1-\kappa}), \text{ and } \sup_{t_T \le x \le T/2} \left\| \frac{1}{\lfloor x \rfloor} \sum_{j=1}^{\lfloor x \rfloor} \boldsymbol{\varepsilon}_j \right\| = O_P(t_T^{-1/2}),$$

from which we obtain that $G_1 = O_P(t_T^{\kappa-1/2}) = o_P(1)$. A parallel argument shows that $G_2 = o_P(1)$, which implies (B.8), and by the same argument one can establish (B.9). Now we aim to show that

$$t_T \sup_{t_T \le x \le T/2} \left(\frac{\Sigma^{1/2}}{\lfloor x \rfloor} \mathbf{W}_{T,1}(x) \right)^\top \Sigma^{-1} \left(\frac{\Sigma^{1/2}}{\lfloor x \rfloor} \mathbf{W}_{T,1}(x) \right) \xrightarrow{D} \xi_1, \quad T \to \infty.$$

By simple matrix algebra and the scale transformation of the Brownian motion, we have that

$$\begin{aligned} t_T \sup_{t_T \le x \le T/2} &\left(\frac{\Sigma^{1/2}}{\lfloor x \rfloor} \mathbf{W}_{T,1}(x) \right)^\top \Sigma^{-1} \left(\frac{\Sigma^{1/2}}{\lfloor x \rfloor} \mathbf{W}_{T,1}(x) \right) \\ &= t_T \sup_{t_T \le x \le T/2} \left\| \frac{\mathbf{W}_{T,1}(x)}{\lfloor x \rfloor} \right\|^2 \\ &= t_T \max_{1 \le y \le T/(2t_T)} \left\| \frac{\mathbf{W}_{T,1}(t_T y)}{t_T y} \right\|^2 \\ &= \max_{1 \le y \le T/(2t_T)} \left\| \frac{\mathbf{W}_{T,1}(y)}{y} \right\|^2 \xrightarrow{D} \sup_{1 \le y \infty} \left\| \frac{\mathbf{W}_{0,1}(y)}{y} \right\|^2, \end{aligned}$$

as $T \to \infty$. A simple calculation shows that $\{\mathbf{W}_{0,1}(y)/y \ : y \in [1,\infty)\} \stackrel{D}{=} \{\mathbf{W}_{0,1}(1/y) \ : y \in [1,\infty)\}$, and so we have that

$$t_T \sup_{t_T \le x \le T/2} \left(\frac{\Sigma^{1/2}}{\lfloor x \rfloor} \mathbf{W}_{T,1}(x) \right)^\top \Sigma^{-1} \left(\frac{\Sigma^{1/2}}{\lfloor x \rfloor} \mathbf{W}_{T,1}(x) \right) \xrightarrow{D} \xi_1.$$

A similar argument gives that

$$t_T \sup_{T/2 \le x \le T - t_T} \left(\frac{\Sigma^{1/2}}{\lfloor x \rfloor} \mathbf{W}_{T,2}(x) \right)^\top \Sigma^{-1} \left(\frac{\Sigma^{1/2}}{\lfloor x \rfloor} \mathbf{W}_{T,2}(x) \right) \xrightarrow{D} \xi_2,$$

where ξ_1 and ξ_2 are independent, which proves the result.

□

Proof of Theorem B.1. It follows immediately from Lemmas B.1 and B.2. □

Appendix C. Asymmetric Trimming

One can obtain similar results as Theorem 2.1 in the case when D_T is defined with asymmetric trimming.

The asymptotic distribution of $t_T^{1/2} D_T$ is established by means of the following, somewhat more general, result. Let s_T be a sequence satisfying the following assumption.

Assumption C.1. $s_T \to \infty$ and $s_T/T \to \infty$ *as* $T \to \infty$.

Let

$$r_T = \min(t_T, s_T) \tag{C.1}$$

and define

$$\lim_{T\to\infty} \frac{r_T}{t_T} = \gamma_1 \quad \text{and} \quad \lim_{T\to\infty} \frac{r_T}{s_T} = \gamma_2. \tag{C.2}$$

The limit distribution will be expressed in terms of the random variable

$$\xi = \sup_{0\le u\le 1} |W(u)|, \tag{C.3}$$

where $\{W(u), 0 \le u \le 1\}$ denotes a Wiener process.

Theorem C.1. *If H_0, the conditions of Theorem 2.1, Assumption C.1 hold, then, as $T \to \infty$, we have*

$$\frac{r_T^{1/2}}{\sigma} \max_{t_T \le t \le T - s_T} \left| \frac{1}{t}\sum_{s=1}^{t} X_s - \frac{1}{T-t}\sum_{s=t+1}^{T} X_s \right| \xrightarrow{\mathcal{D}} \max(\gamma_1^{1/2}\xi_1, \gamma_2^{1/2}\xi_2),$$

where ξ_1, ξ_2 are independent and each have the same distribution as ξ defined in (C.3).

We can also formulate our results in terms of more generally weighted CUSUM processes. Let

$$F_T(\tau) = \frac{1}{\sigma} T^{-\tau} r_T^{\tau-1/2} \sup_{t_T/T \le u \le 1 - s_T/T} (u(1-u))^{-\tau} \left| \sum_{s=1}^{\lfloor Tu \rfloor} X_s - \frac{\lfloor Tu \rfloor}{T} \sum_{s=1}^{T} X_s \right|,$$

where τ satisfies $1/2 < \tau < \infty$. The limit distribution of $F_T(\tau)$ will be given in terms of the distribution of the random variable

$$\xi(\tau) = \sup_{0 \le u \le 1} u^{\tau-1}|W(u)|. \tag{C.4}$$

We note that by the law of iterated logarithm at zero for the Wiener process, the random variable $\xi(\tau)$ is finite a.s. for all $1/2 < \tau < \infty$.

Theorem C.2. *If H_0 and Assumptions B.1–C.1 hold, and $1/2 < \tau < \infty$, then, as $T \to \infty$, we have*

$$F_T(\tau) \;\stackrel{\mathcal{D}}{\to}\; \max(\gamma_1^{\tau-1/2}\xi_1(\tau), \gamma_2^{\tau-1/2}\xi_2(\tau)),$$

where $\xi_1(\tau), \xi_2(\tau)$ are independent and they have the same distribution as $\xi(\tau)$.

We can also formulate our results in terms of more generally weighted CUSUM processes. Let

$$F_T(\tau) = \frac{1}{\sigma} T^{-\tau} r_T^{\tau-1/2} \sup_{t_T/T \le u \le 1 - s_T/T} (u(1-u))^{-\tau} \left| \sum_{s=1}^{\lfloor Tu \rfloor} X_s - \frac{\lfloor Tu \rfloor}{T} \sum_{s=1}^{T} X_s \right|,$$

where τ satisfies $1/2 < \tau < \infty$. The limit distribution of $F_T(\tau)$ will be given in terms of the distribution of the random variable

$$\xi(\tau) = \sup_{0 \le u \le 1} u^{\tau-1}|W(u)|. \tag{C.5}$$

We note that by the law of iterated logarithm at zero for the Wiener process, the random variable $\xi(\tau)$ is finite a.s. for all $1/2 < \tau < \infty$.

Theorem C.3. *If H_0 and Assumptions B.1–C.1 hold, and $1/2 < \tau < \infty$, then, as $T \to \infty$, we have*

$$F_T(\tau) \;\stackrel{\mathcal{D}}{\to}\; \max(\gamma_1^{\tau-1/2}\xi_1(\tau), \gamma_2^{\tau-1/2}\xi_2(\tau)),$$

where $\xi_1(\tau), \xi_2(\tau)$ are independent and they have the same distribution as $\xi(\tau)$.

To prove Theorem C.3, we write under H_0 that

$$\sup_{t_T/T\le u\le 1-t_T/T}(u(1-u))^{-\tau}\left|\sum_{s=1}^{\lfloor Tu\rfloor}X_s-\frac{\lfloor Tu\rfloor}{T}\sum_{s=1}^{T}X_s\right|=\max(U_{T,1},U_{T,2}), \tag{C.6}$$

where

$$U_{T,1}=\sup_{t_T/T\le u\le 1/2}(u(1-u))^{-\tau}\left|\sum_{s=1}^{\lfloor Tu\rfloor}e_s-\frac{\lfloor Tu\rfloor}{T}\sum_{s=1}^{T}e_s\right|,$$

and

$$U_{T,2}=\sup_{1/2<u<1-s_T/T}(u(1-u))^{-\tau}\left|\sum_{s=1}^{\lfloor Tu\rfloor}e_s-\frac{\lfloor Tu\rfloor}{T}\sum_{s=1}^{T}e_s\right|.$$

Lemma C.1. *If Assumptions B.1–B.2 hold, and $1/2<\tau<\infty$, then we have that*

$$T^{-\tau}t_T^{\tau-1/2}U_{T,1}=T^{-\tau}t_T^{\tau-1/2}\sup_{t_T/T\le u\le 1/2}(u(1-u))^{-\tau}\left|\sum_{s=1}^{\lfloor Tu\rfloor}e_s\right|+o_P(1) \tag{C.7}$$

and

$$T^{-\tau}t_T^{\tau-1/2}U_{T,2}=\sup_{1/2<u<1-s_T/T}(u(1-u))^{-\tau}\left|\sum_{s=\lfloor Tu\rfloor+1}^{T}e_s\right|+o_P(1). \tag{C.8}$$

Proof. Let

$$U_{T,1,1}=\left|\sup_{t_T/T\le u\le 1/2}(u(1-u))^{-\tau}\left|\sum_{s=1}^{\lfloor Tu\rfloor}e_s\right|-U_{T,1}\right|,$$

then we get

$$U_{T,1,1}\le\sup_{t_T/T\le u\le 1/2}(u(1-u))^{-\tau}\left|\frac{\lfloor Tu\rfloor}{T}\sum_{s=1}^{T}e_s\right|\le 2^{\tau}\left|\sum_{s=1}^{T}e_s\right|\sup_{t_T/T\le u\le 1/2}u^{1-\tau}.$$

We showed in the proof of Lemma B.1 that

$$\left|\sum_{s=1}^{T}e_s\right|=O_P(T^{1/2}) \tag{C.9}$$

and therefore

$$U_{T,1,1}=\begin{cases}O_P(T^{1/2}(t_T/T)^{1-\tau}), & \text{if } 1\le\tau<\infty\\[2ex] O_P(T^{1/2}), & \text{if } 1/2<\tau<1.\end{cases}$$

Since by Assumption B.1 we have

$$T^{-\tau} t_T^{\tau-1/2} T^{1/2} (t_T/T)^{1-\tau} = (t_T/T)^{1/2} \to 0, \quad \text{if } 1 \le \tau < \infty$$

and

$$T^{-\tau} t_T^{\tau-1/2} T^{1/2} = (t_T/T)^{\tau-1/2} \to 0, \quad \text{if } 1/2 < \tau < 1$$

implying

$$T^{-\tau} t_T^{\tau-1/2} U_{T,1,1} = o_P(1).$$

Hence (C.7) is proven and the same arguments give (C.8). □

Lemma C.2. *If Assumptions B.1, B.2 and* $1/2 < \tau < \infty$ *hold and*

$$\bar{t}_T/t_T \to \infty, \tag{C.10}$$

then we have that

$$t_T^{\tau-1/2} \sup_{t_T \le x \le \bar{t}_T} x^{-\tau} \left| \sum_{s=1}^{x} e_s \right| \xrightarrow{\mathcal{D}} \sigma \xi(\tau).$$

Proof. By Assumption B.2 we have

$$t_T^{\tau-1/2} \sup_{t_T \le x \le \bar{t}_T} x^{-\tau} \left| \sum_{s=1}^{x} e_s - \sigma W_{T,1}(x) \right| = O_P(1) t_T^{\tau-1/2} \sup_{t_T \le x \le \bar{t}_T} x^{\kappa-\tau} = o_P(1).$$

By the scale transformation of the Wiener process we have

$$t_T^{\tau-1/2} \sup_{t_T \le x \le \bar{t}_T} x^{-\tau} |W_{T,1}(x)| \stackrel{\mathcal{D}}{=} \sup_{1 \le y \le \bar{t}_T/t_T} y^{-\tau} |W(y)|,$$

where W is a Wiener process. It is easy to see that as $T \to \infty$

$$\sup_{1 \le y \le \bar{t}_T/t_T} y^{-\tau} |W(y)| \;\to\; \sup_{1 \le y < \infty} y^{-\tau} |W(y)| \text{ a.s. and } \sup_{1 \le y < \infty} x^{-\tau} |W(x)| \stackrel{\mathcal{D}}{=} \xi(\tau),$$

completing the proof of the lemma. □

Lemma C.3. *We assume that Assumptions B.1–B.2 are satisfied and $1/2 < \tau < \infty$. If* (C.10) *holds,*

$$\bar{t}_T/T \to 0, \;\; t_T/\bar{s}_T \;\to\; 0 \quad \text{and} \quad \bar{s}_T/T \to 0, \tag{C.11}$$

then we have

$$\frac{t_T^{\tau-1/2}}{T^\tau} \sup_{t_T/T \le u \le 1/2} (u(1-u))^{-\tau} \left| \sum_{s=1}^{\lfloor Tu \rfloor} e_s \right| = \frac{t_T^{\tau-1/2}}{T^\tau} \sup_{t_T/T \le u \le \bar{t}_T/T} u^{-\tau} \left| \sum_{s=1}^{\lfloor Tu \rfloor} e_s \right| + o_P(1) \tag{C.12}$$

and

$$\begin{aligned} T^{-\tau} t_T^{\tau-1/2} &\sup_{1/2 < u \le 1 - t_T/T} (u(1-u))^{-\tau} \left| \sum_{s=1}^{\lfloor Tu \rfloor} e_s \right| \\ &= T^{-\tau} t_T^{\tau-1/2} \sup_{1-\bar{s}_T/T < u \le 1 - s_T/T} (1-u)^{-\tau} \left| \sum_{s=\lfloor Tu \rfloor+1}^{T} e_s \right| + o_P(1). \end{aligned} \tag{C.13}$$

Proof. It follows from Lemma C.2 that

$$T^{-\tau} t_T^{\tau-1/2} \sup_{\bar{t}_T/T \le u \le 1/2} (u(1-u))^{-\tau} \left| \sum_{s=1}^{\lfloor Tu \rfloor} e_s \right| = O_P((t_T/\bar{t}_T)^{\tau-1/2}) = o_P(1)$$

and by the mean value theorem

$$\begin{aligned} T^{-\tau} t_T^{\tau-1/2} \sup_{t_T/T \le u \le \bar{t}_T/T} \left| (u(1-u))^{-\tau} - u^{-\tau} \right| \left| \sum_{s=1}^{\lfloor Tu \rfloor} e_s \right| &= O(1) T^{-\tau} t_T^{\tau-1/2} \sup_{t_T/T \le u \le \bar{t}_T/T} u^{1-\tau} \left| \sum_{s=1}^{\lfloor Tu \rfloor} e_s \right| \\ &\le O(1/\bar{t}_T) T^{-\tau} t_T^{\tau-1/2} \sup_{t_T/T \le u \le \bar{t}_T/T} u^{-\tau} \left| \sum_{s=1}^{\lfloor Tu \rfloor} e_s \right| \\ &= o_P(1). \end{aligned}$$

The result in (C.13) can be proven along the lines of that of (C.12) and therefore the proof is omitted. □

Lemma C.4. *If Assumptions B.1–B.2,* (C.11) *and* (C.10) *hold, then we have that*

$$\left(T^{-\tau} t_T^{\tau-1/2} \sup_{t_T/T \le u \le \bar{t}_T/T} u^{-\tau} \left| \sum_{s=1}^{\lfloor Tu \rfloor} e_s \right|, \quad T^{-\tau} t_T^{\tau-1/2} \sup_{1-\bar{s}_T/T < u \le 1 - s_T/T} (1-u)^{-\tau} \left| \sum_{s=\lfloor Tu \rfloor+1}^{T} e_s \right| \right)$$
$$\xrightarrow{\mathcal{D}} \sigma \max(\xi_1(\tau), \xi_2(\tau)),$$

where $\xi_1(\tau)$ and $\xi_2(\tau)$ are defined in Theorem C.3.

Proof. It follows from Assumption B.2 that

$$T^{-\tau} t_T^{\tau-1/2} \sup_{t_T/T \le u \le \bar{t}_T/T} u^{-\tau} \left| \sum_{s=1}^{\lfloor Tu \rfloor} e_s \right| = t_T^{\tau-1/2} \sup_{t_T \le x \le \bar{t}_T} x^{-\tau} \sigma |W_{T,1}(x)| + o_P(1)$$

and

$$\begin{aligned} T^{-\tau} t_T^{\tau-1/2} \sup_{1-\bar{s}_T/T < u \le 1-s_T/T} (1-u)^{-\tau} \left| \sum_{s=\lfloor Tu \rfloor+1}^{T} e_s \right| \\ = t_T^{\tau-1/2} \sup_{T-\bar{s}_T < x \le T - s_T} (T-x)^{-\tau} \sigma |W_{T,2}(x)| \\ = o_P(1). \end{aligned}$$

The asymptotic independence now follows from the independence of the Wiener processes and the asymptotic distribution is an immediate consequence of Lemma C.2. □

Proof of Theorem C.3. The result is an immediate consequence of Lemmas C.1–C.4. □

Appendix D. Estimation of the long run variance: Proof of Theorems 5.1 and consistency of estimators defined in (5.4)

In this section we provide justification of the Theorem 5.1, and establish the consistency of the estimators defined in (5.4). We begin with Theorem 5.1.

Proof of Theorem 5.1. Elementary arguments show that under H_0 we have

$$T\hat{\sigma}_{T,t}^2 = \sum_{s=1}^{T} e_s^2 - t\bar{e}_t^2 - (T-t)\tilde{e}_{T-t}^2, \quad \text{with } \bar{e}_t = \frac{1}{t}\sum_{s=1}^{t} e_s \text{ and } \tilde{e}_{T-t} = \frac{1}{T-t}\sum_{s=t+1}^{T} e_s. \tag{D.1}$$

The approximations in Assumption B.2 with the Darling–Erdős (1956) yields

$$\max_{1\le t\le T} t|\bar{e}_t^2| = \max_{1\le t\le T}\left(\frac{1}{t^{1/2}}\sum_{s=1} e_s\right)^2 = O_P(\log\log T) \tag{D.2}$$

and

$$\max_{1\le t<T}\left|(T-t)\tilde{e}_{T-t}^2\right| = O_P(\log\log T). \tag{D.3}$$

By the ergodic theorem,

$$\frac{1}{T}\sum_{s=1}^{T} e_s^2 \;\to\; \sigma^2 \quad a.s., \tag{D.4}$$

completing the proof of (5.1). Observing that (D.1) holds true under H_A when $t = t^*$, we get immediately (5.2) from (D.2)–(D.4). □

We now turn to (5.4) from the main paper, in which we use the following assumption:

Assumption D.1. The sequence $e_s -\infty < s < \infty$ is a Bernoulli shift, i.e. there is measurable function f such that $e_s = f(\varepsilon_s, \varepsilon_{s-1}, \ldots)$, where $\varepsilon_s, -\infty < s < \infty$ are independent and identically distributed random variables in some measurable space. In addition,

$$Ee_0 = 0,\; E|e_0|^\nu < \infty \;\text{ with some }\; \nu > 2,$$

and

$$\left(E\left|e_{s,m} - e_s\right|^\nu\right)^{1/\nu} = O(m^{-\gamma}) \;\text{ with some }\; \gamma > 1,$$

where $e_{s,m} = f(\varepsilon_s, \varepsilon_{s-1}, \ldots, \varepsilon_{s-m}, \varepsilon^*_{s,m,s-m-1}, \varepsilon^*_{s,m,s-m-2}, \ldots)$ and $\varepsilon^*_{i,j,\ell}$ are independent and identically distributed copies of ε_0.

Theorem D.1. *If Assumption D.1 holds with $\nu > 4$ moments, and Assumptions 5.1, 5.2 hold, then the estimator $\hat{\sigma}^2_{T,t}$ defined by (5.4) satisfies (5.1) and (5.2).*

Lemma D.1. *Suppose Assumption D.1 is satisfied with $\nu > 4$, and Assumptions 5.1, 5.2 hold.*

(i) If H_0 holds, then we have

$$\max_{1\le t\le T}\left|\frac{1}{T-1}\sum_{s=1}^{T}e_s^2-\hat{\gamma}_{\ell,t}\right|=o_P(1)$$

and

$$\max_{1\le t\le T}\sum_{\ell=1}^{T-1}K\left(\frac{\ell}{h}\right)\left|\frac{1}{T-\ell}\sum_{s=1}^{T-\ell}e_se_{s+\ell}-\hat{\gamma}_{\ell,t}\right|=o_P(1).$$

(ii) If H_A holds, then we have

$$\left|\frac{1}{T}\sum_{s=1}^{T}e_s^2-\hat{\gamma}_{\ell,t^*}\right|=o_P(1)$$

and

$$\sum_{\ell=1}^{T-1}K\left(\frac{\ell}{h}\right)\left|\frac{1}{T-\ell}\sum_{s=1}^{T-\ell}e_se_{s+\ell}-\hat{\gamma}_{\ell,t^*}\right|=o_P(1).$$

Proof. First we note that Assumption B.2 and (4.4) hold under the conditions of the Lemmas (cf. Aue et al. (2014)). Under H_0 we have for all $0\le\ell\le t<T-\ell$ and T

(D.5)

$$\begin{aligned}(T-\ell)\hat{\gamma}_{\ell,t}&=\sum_{s=1}^{T-\ell}e_se_{s+\ell}-\bar{e}_t\sum_{s=1}^{t-\ell}e_{s+\ell}-\bar{e}_t\sum_{s=1}^{t-\ell}e_s+(t-\ell)\bar{e}_t^2-\bar{e}_t\sum_{s=t-\ell+1}^{t}e_{s+\ell}\\&\quad-\tilde{e}_{T-t}\sum_{s=t-\ell+1}^{t}e_s+\ell\bar{e}_t\tilde{e}_{T-t}-\tilde{e}_{T-t}\sum_{s=t+1}^{T-\ell}e_s-\tilde{e}_{T-t}\sum_{s=t+1}^{T-\ell}e_{s+\ell}+(T-\ell-t+1)\tilde{e}_{T-t}^2\\&=\sum_{s=1}^{T-\ell}e_se_{s+\ell}+p_{\ell,t,1}+\ldots+p_{\ell,t,9}.\end{aligned}$$

(We used $\sum_{\emptyset}=0$.) We have similar expressions for $1\le t<\ell$ and $T-\ell\le t\le T$.

By the maximal inequality of Móricz et al. (1982) we have for all $x > 0$ that

$$
\begin{aligned}
P\left\{\max_{\ell+1\le t\le T}\frac{1}{t^{1/2}}\left|\sum_{s=\ell+1}^{t} e_s\right| > x\right\} &\le P\left\{\max_{1\le i\le \log T} e^{-(i-1)/2}\max_{e^{i-1}\le t\le e^s}\left|\sum_{s=\ell+1}^{t} e_s\right| > x\right\}\\
&\le x^{-\bar\nu}\sum_{i=1}^{\log T} e^{-(i-1)\bar\nu/2} E\max_{e^{i-1}\le t\le e^s}\left|\sum_{s=\ell+1}^{t} e_s\right|^{\bar\nu}\\
&\le x^{-\bar\nu}\sum_{i=1}^{\log T} e^{-(i-1)\bar\nu/2} c_1 e^{i\bar\nu/2}\\
&\le c_2 x^{-\bar\nu}\log T,
\end{aligned}
$$

with some constants c_1 and c_2. Hence, there is constant c_3 such that for all ℓ and T

$$
E\max_{\ell+1\le t\le T}\frac{1}{t^{1/2}}\left|\sum_{s=\ell+1}^{t} e_s\right| \le c_3(\log T)^{1/\bar\nu}. \tag{D.6}
$$

We showed that Assumption B.2 yields

$$
\max_{1\le t\le T}\frac{1}{t^{1/2}}\left|\sum_{s=1}^{t} e_s\right| = O_P((\log\log T)^{1/2}). \tag{D.7}
$$

Using (D.7) we conclude

$$
\begin{aligned}
&\max_{1\le t\le T}\sum_{\ell=1}^{T-1} K(\ell/h)\left|\bar e_t\sum_{s=1}^{t-\ell} e_{s+\ell}\right|\\
&\quad\le \max_{1\le t\le T}\frac{1}{t^{1/2}}\left|\sum_{s=1}^{t} e_s\right| \max_{1\le t\le T}\sum_{\ell=1}^{T-1} K(\ell/h)\left|\frac{1}{t^{1/2}}\sum_{s=1}^{t-\ell} e_{s+\ell}\right|\\
&\quad= O_P((\log\log T)^{1/2})\sum_{\ell=1}^{T-1} K(\ell/h)\max_{\ell+1\le t\le T}\frac{1}{t^{1/2}}\left|\sum_{s=\ell+1}^{t} e_s\right|\\
&\quad= O_P(h(\log T)^{1/\bar\nu}(\log\log T)^{1/2}),
\end{aligned}
$$

since by Assumption 5.1 and (D.6) we have via Markov's inequality

$$
\sum_{\ell=1}^{T-1} K(\ell/h)\max_{\ell+1\le t\le T}\frac{1}{t^{1/2}}\left|\sum_{s=\ell+1}^{t} e_s\right| = O_P(h(\log T)^{1/\bar\nu}).
$$

Thus we have

$$\max_{1\le t\le T}\sum_{\ell=1}^{T-1}K(\ell/h)|p_{\ell,t,1}| = O_P(h(\log T)^{1/\bar\nu}(\log\log T)^{1/2}). \tag{D.8}$$

Similar arguments give

$$\max_{1\le t\le T}\sum_{\ell=1}^{T-1}K(\ell/h)|p_{\ell,t,i}| = O_P(h(\log T)^{1/\bar\nu}(\log\log T)^{1/2}), \quad i=2,4,5,7,8 \tag{D.9}$$

and

$$\max_{1\le t\le T}\sum_{\ell=1}^{T-1}K(\ell/h)|p_{\ell,t,i}| = O_P(h\log\log T), \quad i=3,6,9. \tag{D.10}$$

It follows from (D.5) and (D.8)–(D.10) that

$$\sum_{\ell=1}^{T-1}K(\ell/h)\frac{1}{T-\ell}\max_{\ell\le t<T-\ell}\left|\sum_{s=1}^{T-\ell}e_se_{s+\ell}-\hat\gamma_{\ell,t}\right| = O_P\left(\frac{h}{T}(\log T)^{1/\bar\nu}(\log\log T)^{1/2}\right). \tag{D.11}$$

Following the arguments leading to (D.11) one can verify that

$$\sum_{\ell=1}^{T-1}K(\ell/h)\frac{1}{T-\ell}\max_{0\le t<\ell}\left|\sum_{s=1}^{T-\ell}e_se_{s+\ell}-\hat\gamma_{\ell,t}\right| = O_P\left(\frac{h}{T}(\log T)^{1/\bar\nu}(\log\log T)^{1/2}\right) \tag{D.12}$$

and

$$\sum_{\ell=1}^{T}K(\ell/h)\frac{1}{T-\ell}\max_{T-\ell\le t\le T}\left|\sum_{s=1}^{T-\ell}e_se_{s+\ell}-\hat\gamma_{\ell,t}\right| = O_P\left(\frac{h}{T}(\log T)^{1/\bar\nu}(\log\log T)^{1/2}\right). \tag{D.13}$$

The result of Lemma D.1(i) is an immediate consequence of (D.11)–(D.13).

The same arguments give the proof of Lemma D.1(ii). □

Proof of Theorem D.1. The assumptions of the Theorem are stronger than those in Theorem 1 of Liu and Wu (2010),

$$\frac{1}{T}\sum_{t=1}^{T}e_t^2+2\sum_{\ell=1}^{T-1}K\left(\frac{\ell}{h}\right)\frac{1}{T-\ell}\sum_{s=1}^{T-\ell}e_se_{s+\ell}\xrightarrow{P}\sigma^2.$$

Hence (5.1) and (5.1) follow immediately from Lemma D.1. □

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