The Scaled Uniform Model Revisited
Supplementary Online Materials

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This document provides a simple proof for Proposition 1. The result is a direct corollary of Equations (9) and (10) of Kagan and Malinovsky (2013). However, the following constructive proof explains why this happens.

Proposition 1. *Suppose that* $(T(X), A(X))$ *is a minimal sufficient statistic for* θ *and that* $A(X)$ *is ancillary. Let* $\hat{\theta}$ *be an estimator and* θ_0 *be a value in the parameter space* Θ *such that (i)* $E_{\theta_0}(\hat{\theta} \mid A) = \theta_0$ *almost surely , (ii)* $\mathrm{Var}_{\theta_0}(\hat{\theta}) < \infty$*, and (iii)* $\mathrm{Var}_{\theta_0}(\hat{\theta} \mid A)$ *is non-degenerate. Then* $\hat{\theta}$ *is not a global UMVUE.*

Proof. Suppose that A is ancillary with density f_A and that $E_{\theta_0}(\hat{\theta} \mid A) = \theta_0$ almost surely. Let $\hat{\theta}_c = c(A)\hat{\theta}$ be a new estimator, where $c(A)$ satisfies $E\{c(A)\} = 1$ and $E\{c^2(A)\} < \infty$. Then

$$E_{\theta_0}(\hat{\theta}_c) = E_{\theta_0}\{c(A)E_{\theta_0}(\hat{\theta} \mid A)\} = E_{\theta_0}\{c(A)\theta_0\} = \theta_0, \tag{1}$$

having variance

$$\begin{aligned} \mathrm{Var}_{\theta_0}(\hat{\theta}_c) &= E_{\theta_0}\{\mathrm{Var}_{\theta_0}(\hat{\theta}_c \mid A)\} + \mathrm{Var}_{\theta_0}\{E_{\theta_0}(\hat{\theta}_c \mid A)\} \\ &= E_{\theta_0}\{c(A)^2 \mathrm{Var}_{\theta_0}(\hat{\theta} \mid A)\} + \theta_0^2 \mathrm{Var}\{c(A)\}. \end{aligned} \tag{2}$$

It then follows that

$$\mathrm{Var}_{\theta_0}(\hat{\theta}_c) - \mathrm{Var}_{\theta_0}(\hat{\theta}) = E_{\theta_0}[\{c(A)^2 - 1\}\mathrm{Var}_{\theta_0}(\hat{\theta} \mid A)] + \theta_0^2 \mathrm{Var}\{c(A)\}. \tag{3}$$

If $\mathrm{Var}_{\theta_0}(\hat{\theta} \mid A)$ is non-degenrate, there exist constants $v_1 < v_2$ and non-null events $\mathcal{A}_1$ and $\mathcal{A}_2$ such that $\mathrm{Var}_{\theta_0}(\hat{\theta} \mid A) < v_1$ on $\mathcal{A}_1$ and $\mathrm{Var}_{\theta_0}(\hat{\theta} \mid A) > v_2$ on $\mathcal{A}_2$. For any $\epsilon > 0$ consider the estimator $\hat{\theta}_{c_\epsilon}$, where

$$c_\epsilon(A) = \begin{cases} 1 + \epsilon/P(\mathcal{A}_1) & \text{on } \mathcal{A}_1 \\ 1 - \epsilon/P(\mathcal{A}_2) & \text{on } \mathcal{A}_2 \\ 1 & \text{on } \overline{\mathcal{A}_1 \cup \mathcal{A}_2} \end{cases}$$

Write $V_a = \mathrm{Var}_{\theta_0}(\hat{\theta} \mid A = a)$. For $c = c_\epsilon$, the first term on the right-hand side of (3) becomes

$$\begin{aligned} &\int_{\mathcal{A}_1} \{\epsilon^2/P^2(\mathcal{A}_1) + 2\epsilon/P(\mathcal{A}_1)\} V_a f_A(a) da + \int_{\mathcal{A}_2} \{\epsilon^2/P^2(\mathcal{A}_2) - 2\epsilon/P(\mathcal{A}_2)\} V_a f_A(a) da \\ &< \{\epsilon^2/P^2(\mathcal{A}_1) + \epsilon^2/P^2(\mathcal{A}_2)\} \mathrm{Var}_{\theta_0}(\hat{\theta}) + 2\epsilon(v_1 - v_2), \end{aligned}$$

where the inequality follows from the definition of v_1 and v_2 and the fact that $\mathrm{Var}_{\theta_0}(\hat{\theta}) = \int_{\mathbb{R}} V_a f_A(a) da \geq \int_{\mathcal{A}_k} V_a f_A(a) da$ for $k = 1, 2$. The second term on the right-hand side of (3) is readily calculated as

$$\theta_0^2 \{\epsilon^2/P(\mathcal{A}_1) + \epsilon^2/P(\mathcal{A}_2)\},$$

so the difference of variances is bounded by

$$\mathrm{Var}_{\theta_0}(\hat{\theta}_{c_\epsilon}) - \mathrm{Var}_{\theta_0}(\hat{\theta}) < \epsilon^2 \left[\frac{\mathrm{Var}_{\theta_0}(\hat{\theta}) + \theta_0^2 P(\mathcal{A}_1)}{P^2(\mathcal{A}_1)} + \frac{\mathrm{Var}_{\theta_0}(\hat{\theta}) + \theta_0^2 P(\mathcal{A}_2)}{P^2(\mathcal{A}_2)} \right] + 2\epsilon(v_1 - v_2). \tag{4}$$

Since $v_1 - v_2 < 0$ and $\mathrm{Var}_{\theta_0}(\hat{\theta}) < \infty$, Equation (4) shows that there is a small enough ϵ such that $\mathrm{Var}_{\theta_0}(\hat{\theta}_{c_\epsilon}) - \mathrm{Var}_{\theta_0}(\hat{\theta}) < 0$, and thus $\hat{\theta}$ cannot be a UMVUE. Therefore, a necessary condition for a conditional unbiased estimator to be a global UMVUE is that, for all $\theta \in \Theta$, $\mathrm{Var}(\hat{\theta} \mid A)$ be almost surely constant.

References

[1] Kagan, A. M.,and Malinovsky, Y. (2013), "On the Nile problem by Sir Ronald Fisher," *Electronic Journal of Statistics*, 7, 1968-1982.