Supplemental Material: Time use of married couples: Bayesian approach

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ARTICLE HISTORY
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Appendix A. Derivation of Algorithms 1 and 2

A.1. Derivation of Algorithm 1

A.1.1. FCD of $\operatorname{vec} \boldsymbol{B}$

The FCD of $\operatorname{vec} \boldsymbol{B}$ is

$$p(\operatorname{vec} \boldsymbol{B} | \cdots) \propto \exp\Bigg[-\frac{1}{2}\Bigg\{\sum(\boldsymbol{z}_i - \boldsymbol{B}'\boldsymbol{x}_i)'\boldsymbol{V}^{-1}(\boldsymbol{z}_i - \boldsymbol{B}'\boldsymbol{x}_i) + \frac{1}{\lambda}(\operatorname{vec} \boldsymbol{B} - \operatorname{vec} \boldsymbol{B}_0)'(\operatorname{vec} \boldsymbol{B} - \operatorname{vec} \boldsymbol{B}_0)\Bigg\}\Bigg],$$

where

$$\begin{aligned}
&\sum(\boldsymbol{z}_i - \boldsymbol{B}'\boldsymbol{x}_i)'\boldsymbol{V}^{-1}(\boldsymbol{z}_i - \boldsymbol{B}'\boldsymbol{x}_i) + \frac{1}{\lambda}(\operatorname{vec} \boldsymbol{B} - \operatorname{vec} \boldsymbol{B}_0)'(\operatorname{vec} \boldsymbol{B} - \operatorname{vec} \boldsymbol{B}_0) \\
&= \sum \boldsymbol{x}_i'\boldsymbol{B}\boldsymbol{V}^{-1}\boldsymbol{B}_j'\boldsymbol{x}_i - 2\sum \boldsymbol{z}_i'\boldsymbol{V}^{-1}\boldsymbol{B}'\boldsymbol{x}_i \\
&\quad + \frac{1}{\lambda}(\operatorname{vec} \boldsymbol{B})' \operatorname{vec} \boldsymbol{B} - \frac{2}{\lambda}(\operatorname{vec} \boldsymbol{B})' \operatorname{vec} \boldsymbol{B}_0 + \cdots .
\end{aligned}$$

From [1, p.283, Exercise 10.20], we have

$$\begin{aligned}
\sum \boldsymbol{x}_i'\boldsymbol{B}\boldsymbol{V}^{-1}\boldsymbol{B}'\boldsymbol{x}_i &= (\operatorname{vec} \boldsymbol{B})'(\boldsymbol{V}^{-1} \otimes \boldsymbol{X}'\boldsymbol{X}) \operatorname{vec} \boldsymbol{B} \\
\sum \boldsymbol{z}_i'\boldsymbol{V}^{-1}\boldsymbol{B}'\boldsymbol{x}_i &= (\operatorname{vec} \boldsymbol{B})'(\boldsymbol{V}^{-1} \otimes \boldsymbol{I}) \operatorname{vec}(\boldsymbol{X}'\boldsymbol{Z}).
\end{aligned}$$

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Then, since

$$\begin{gathered}\sum \boldsymbol{x}_i' \boldsymbol{B} \boldsymbol{V}^{-1} \boldsymbol{B}' \boldsymbol{x}_i - 2 \sum \boldsymbol{z}_i' \boldsymbol{V}^{-1} \boldsymbol{B}' \boldsymbol{x}_i + \frac{1}{\lambda} (\operatorname{vec} \boldsymbol{B})' \operatorname{vec} \boldsymbol{B} \\ - \frac{2}{\lambda} (\operatorname{vec} \boldsymbol{B})' \operatorname{vec} \boldsymbol{B}_0 + \cdots \\ = (\operatorname{vec} \boldsymbol{B})' \left[\lambda^{-1} \boldsymbol{I} + (\boldsymbol{V}^{-1} \otimes \boldsymbol{X}' \boldsymbol{X}) \right] \operatorname{vec} \boldsymbol{B} \\ -2 (\operatorname{vec} \boldsymbol{B})' \left[\lambda^{-1} \operatorname{vec} \boldsymbol{B}_0 + (\boldsymbol{V}^{-1} \otimes \boldsymbol{I}) \operatorname{vec}(\boldsymbol{X}' \boldsymbol{Z}) \right] + \cdots ,\end{gathered}$$

the FCD of $\operatorname{vec} \boldsymbol{B}$ is

$$\operatorname{vec} \boldsymbol{B} | \cdots \sim \mathrm{N} \left(\boldsymbol{\beta}^*, \boldsymbol{\Omega}^* \right),$$

where

$$\begin{gathered}\boldsymbol{\Omega}^* = \left[\lambda^{-1} \boldsymbol{I} + (\boldsymbol{V}^{-1} \otimes \boldsymbol{X}' \boldsymbol{X}) \right]^{-1} \\ \boldsymbol{\beta}^* = \boldsymbol{\Omega}^* \left[\lambda^{-1} \operatorname{vec} \boldsymbol{B}_0 + (\boldsymbol{V}^{-1} \otimes \boldsymbol{I}) \operatorname{vec}(\boldsymbol{X}' \boldsymbol{Z}) \right].\end{gathered}$$

A.1.2. FCD of $\boldsymbol{V}$

Since the FCD of $\boldsymbol{V}$ is

$$\begin{gathered}p(\boldsymbol{V} | \cdots) \propto |\boldsymbol{V}|^{-\frac{1}{2}(m+n+d+1)} \\ \times \exp \left[-\frac{1}{2} \operatorname{tr} \left\{ \boldsymbol{M} + \sum (\boldsymbol{z}_i - \boldsymbol{B}' \boldsymbol{x}_i)(\boldsymbol{z}_i - \boldsymbol{B}' \boldsymbol{x}_i)' \right\} \boldsymbol{V}^{-1} \right],\end{gathered}$$

we have

$$\boldsymbol{V} | \cdots \sim \mathrm{IW} \left(m^*, \boldsymbol{M}^{*-1} \right),$$

where

$$m^* = m + n, \ \boldsymbol{M}^* = \boldsymbol{M} + \sum (\boldsymbol{z}_i - \boldsymbol{B}' \boldsymbol{x}_i)(\boldsymbol{z}_i - \boldsymbol{B}' \boldsymbol{x}_i)'.$$

A.1.3. FCD of $\boldsymbol{z}_{i\,\mathrm{o}}$

The FCD of $\boldsymbol{z}_{i\,\mathrm{o}}$ is

$$p(\boldsymbol{z}_{i\,\mathrm{o}} | \cdots) \propto \begin{cases} \mathrm{N}\Big(\boldsymbol{z}_{i\,\mathrm{o}} |\, \mathrm{E}(\boldsymbol{z}_{i\,\mathrm{o}} | \boldsymbol{z}_{i+}, \boldsymbol{B}, \boldsymbol{V}), \operatorname{var}(\boldsymbol{z}_{i\,\mathrm{o}} | \boldsymbol{z}_{i+}, \boldsymbol{B}, \boldsymbol{V}) \Big) & \\ & \text{if some elements of } \boldsymbol{y}_i \text{ are zero} \\ \mathrm{N}(\boldsymbol{B}' \boldsymbol{x}_i, \boldsymbol{V}) 1(\boldsymbol{z}_{i\,\mathrm{o}} \leq \boldsymbol{0}) & \text{if } \boldsymbol{y}_i = \boldsymbol{0}, \end{cases}$$

where

$$\begin{gathered}\mathrm{E}(\boldsymbol{z}_{i\,\mathrm{o}} | \boldsymbol{z}_{i+}, \boldsymbol{B}, \boldsymbol{V}) = \boldsymbol{B}_{i\,\mathrm{o}}' \boldsymbol{x}_i + \boldsymbol{V}_{i\,\mathrm{o}+} \boldsymbol{V}_{i++}^{-1} (\boldsymbol{z}_{i+} - \boldsymbol{B}_{i+}' \boldsymbol{x}_i) \\ \operatorname{var}(\boldsymbol{z}_{i\,\mathrm{o}} | \boldsymbol{z}_{i+}, \boldsymbol{B}, \boldsymbol{V}) = \boldsymbol{V}_{i\,\mathrm{o}\,\mathrm{o}} - \boldsymbol{V}_{i\,\mathrm{o}+} \boldsymbol{V}_{i++}^{-1} \boldsymbol{V}_{i+\mathrm{o}}.\end{gathered}$$

A.2. Derivation of Algorithm 2

A.2.1. FCD of $\operatorname{vec} \boldsymbol{B}_j$

The FCD of $\operatorname{vec} \boldsymbol{B}_1$ is

$$p(\operatorname{vec} \boldsymbol{B}_1|\cdots) \propto \exp\left[-\frac{1}{2}\left\{\sum \begin{pmatrix} \boldsymbol{z}_{i1} - \boldsymbol{B}_1'\boldsymbol{x}_{i1} \\ \boldsymbol{z}_{i2} - \boldsymbol{B}_2'\boldsymbol{x}_{i2} \end{pmatrix}' \boldsymbol{V}^{-1} \begin{pmatrix} \boldsymbol{z}_{i1} - \boldsymbol{B}_1'\boldsymbol{x}_{i1} \\ \boldsymbol{z}_{i2} - \boldsymbol{B}_2'\boldsymbol{x}_{i2} \end{pmatrix} \right.\right.$$
$$\left.\left. + \frac{1}{\lambda_1}(\operatorname{vec} \boldsymbol{B}_1 - \operatorname{vec} \boldsymbol{B}_{10})'(\operatorname{vec} \boldsymbol{B}_1 - \operatorname{vec} \boldsymbol{B}_{10})\right\}\right].$$

We have

$$\begin{aligned}
&\sum \begin{pmatrix} \boldsymbol{z}_{i1} - \boldsymbol{B}_1'\boldsymbol{x}_{i1} \\ \boldsymbol{z}_{i2} - \boldsymbol{B}_2'\boldsymbol{x}_{i2} \end{pmatrix}' \begin{pmatrix} \boldsymbol{V}^{11} & \boldsymbol{V}^{12} \\ \boldsymbol{V}^{12\prime} & \boldsymbol{V}^{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{z}_{i1} - \boldsymbol{B}_1'\boldsymbol{x}_{i1} \\ \boldsymbol{z}_{i2} - \boldsymbol{B}_2'\boldsymbol{x}_{i2} \end{pmatrix} \\
&\quad + \frac{1}{\lambda_1}(\operatorname{vec} \boldsymbol{B}_1 - \operatorname{vec} \boldsymbol{B}_{10})'(\operatorname{vec} \boldsymbol{B}_j - \operatorname{vec} \boldsymbol{B}_{10}) \\
&= \sum \boldsymbol{x}_{i1}'\boldsymbol{B}_1\boldsymbol{V}^{11}\boldsymbol{B}_1'\boldsymbol{x}_{i1} - 2\sum \boldsymbol{z}_{1i}'\boldsymbol{V}^{11}\boldsymbol{B}_1'\boldsymbol{x}_{i1} - 2\sum (\boldsymbol{z}_{i2} - \boldsymbol{B}_2'\boldsymbol{x}_{i2})'\boldsymbol{V}^{12\prime}\boldsymbol{B}_1'\boldsymbol{x}_{i1} \\
&\quad + \frac{1}{\lambda_1}(\operatorname{vec} \boldsymbol{B}_1)' \operatorname{vec} \boldsymbol{B}_1 - \frac{2}{\lambda_1}(\operatorname{vec} \boldsymbol{B}_1)' \operatorname{vec} \boldsymbol{B}_{10} + \cdots .
\end{aligned}$$

From [1, p.283, Exercise 10.20], we have

$$\begin{aligned}
\sum \boldsymbol{x}_{i1}'\boldsymbol{B}_1\boldsymbol{V}^{11}\boldsymbol{B}_1'\boldsymbol{x}_{i1} &= (\operatorname{vec} \boldsymbol{B}_1)'(\boldsymbol{V}^{11} \otimes \boldsymbol{X}_1'\boldsymbol{X}_1) \operatorname{vec} \boldsymbol{B}_1 \\
\sum \boldsymbol{z}_{i1}'\boldsymbol{V}^{11}\boldsymbol{B}_1'\boldsymbol{x}_{i1} &= (\operatorname{vec} \boldsymbol{B}_1)'(\boldsymbol{V}^{11} \otimes \boldsymbol{I}) \operatorname{vec}(\boldsymbol{X}_1'\boldsymbol{Z}_1) \\
\sum (\boldsymbol{z}_{i2} - \boldsymbol{B}_2'\boldsymbol{x}_{i2})'\boldsymbol{V}^{12\prime}\boldsymbol{B}_1'\boldsymbol{x}_{i1} &= (\operatorname{vec} \boldsymbol{B}_1)'(\boldsymbol{V}^{12} \otimes \boldsymbol{I}) \operatorname{vec}[\boldsymbol{X}_1'(\boldsymbol{Z}_2 - \boldsymbol{X}_2\boldsymbol{B}_2)].
\end{aligned}$$

Then, since

$$\begin{aligned}
&\sum \boldsymbol{x}_{i1}'\boldsymbol{B}_1\boldsymbol{V}^{11}\boldsymbol{B}_1'\boldsymbol{x}_{i1} - 2\sum \boldsymbol{z}_{i1}'\boldsymbol{V}^{11}\boldsymbol{B}_1'\boldsymbol{x}_{i1} - 2\sum (\boldsymbol{z}_{i2} - \boldsymbol{B}_2'\boldsymbol{x}_{i2})'\boldsymbol{V}^{12\prime}\boldsymbol{B}_1'\boldsymbol{x}_{i1} \\
&\quad + \frac{1}{\lambda_1}(\operatorname{vec} \boldsymbol{B}_1)' \operatorname{vec} \boldsymbol{B}_1 - \frac{2}{\lambda_1}(\operatorname{vec} \boldsymbol{B}_1)' \operatorname{vec} \boldsymbol{B}_{10} + \cdots \\
&= (\operatorname{vec} \boldsymbol{B}_1)' \left[\lambda_1^{-1}\boldsymbol{I} + (\boldsymbol{V}^{11} \otimes \boldsymbol{X}_1'\boldsymbol{X}_1)\right] \operatorname{vec} \boldsymbol{B}_1 \\
&\quad -2(\operatorname{vec} \boldsymbol{B}_1)' \left[\lambda_1^{-1} \operatorname{vec} \boldsymbol{B}_{10} + (\boldsymbol{V}^{11} \otimes \boldsymbol{I}) \operatorname{vec}(\boldsymbol{X}_1'\boldsymbol{Z}_1)\right. \\
&\quad \left. + (\boldsymbol{V}^{12} \otimes \boldsymbol{I}) \operatorname{vec}[\boldsymbol{X}_1'(\boldsymbol{Z}_2 - \boldsymbol{X}_2\boldsymbol{B}_2)]\right] + \cdots ,
\end{aligned}$$

the FCD of $\operatorname{vec} \boldsymbol{B}_1$ is

$$\operatorname{vec} \boldsymbol{B}_1|\cdots \sim \mathrm{N}\left(\boldsymbol{\beta}_1^*, \boldsymbol{\Omega}_1^*\right),$$

where

$$\begin{aligned}
\boldsymbol{\Omega}_1^* &= \left[\lambda_1^{-1}\boldsymbol{I} + (\boldsymbol{V}^{11} \otimes \boldsymbol{X}_1'\boldsymbol{X}_1)\right]^{-1} \\
\boldsymbol{\beta}_1^* &= \boldsymbol{\Omega}_1^* \left[\lambda_1^{-1} \operatorname{vec} \boldsymbol{B}_{10} + (\boldsymbol{V}^{11} \otimes \boldsymbol{I}) \operatorname{vec}(\boldsymbol{X}_1'\boldsymbol{Z}_1)\right. \\
&\quad \left. + (\boldsymbol{V}^{12} \otimes \boldsymbol{I}) \operatorname{vec}[\boldsymbol{X}_1'(\boldsymbol{Z}_2 - \boldsymbol{X}_2\boldsymbol{B}_2)]\right] .
\end{aligned}$$

Similarly, the FCD of $\operatorname{vec} \boldsymbol{B}_2$ is

$$\operatorname{vec} \boldsymbol{B}_2 | \cdots \sim \mathrm{N}\left(\boldsymbol{\beta}_2^*, \boldsymbol{\Omega}_2^*\right),$$

where

$$\begin{aligned}\boldsymbol{\Omega}_2^* &= \left[\lambda_2^{-1} \boldsymbol{I} + (\boldsymbol{V}^{22} \otimes \boldsymbol{X}_2' \boldsymbol{X}_2)\right]^{-1} \\ \boldsymbol{\beta}_2^* &= \boldsymbol{\Omega}_2^* \left[\lambda_2^{-1} \operatorname{vec} \boldsymbol{B}_{20} + (\boldsymbol{V}^{22} \otimes \boldsymbol{I}) \operatorname{vec}(\boldsymbol{X}_2' \boldsymbol{Z}_2)\right. \\ &\quad \left. + (\boldsymbol{V}^{12\prime} \otimes \boldsymbol{I}) \operatorname{vec}[\boldsymbol{X}_2'(\boldsymbol{Z}_1 - \boldsymbol{X}_1 \boldsymbol{B}_1)]\right].\end{aligned}$$

A.2.2. FCD of $\boldsymbol{V}$

Since the FCD of $\boldsymbol{V}$ is

$$\begin{aligned} p(\boldsymbol{V} | \cdots) &\propto |\boldsymbol{V}|^{-\frac{1}{2}(m+n+2d+1)} \\ &\times \exp\left[-\frac{1}{2} \operatorname{tr}\left\{\boldsymbol{M} + \sum \begin{pmatrix} \boldsymbol{z}_{i1} - \boldsymbol{B}_1' \boldsymbol{x}_{i1} \\ \boldsymbol{z}_{i2} - \boldsymbol{B}_2' \boldsymbol{x}_{i2} \end{pmatrix} \begin{pmatrix} \boldsymbol{z}_{i1} - \boldsymbol{B}_1' \boldsymbol{x}_{i1} \\ \boldsymbol{z}_{i2} - \boldsymbol{B}_2' \boldsymbol{x}_{i2} \end{pmatrix}' \right\} \boldsymbol{V}^{-1}\right],\end{aligned}$$

we have

$$\boldsymbol{V} | \cdots \sim \mathrm{IW}\left(m^*, \boldsymbol{M}^{*-1}\right),$$

where

$$m^* = m + n, \ \boldsymbol{M}^* = \boldsymbol{M} + \sum \begin{pmatrix} \boldsymbol{z}_{i1} - \boldsymbol{B}_1' \boldsymbol{x}_{i1} \\ \boldsymbol{z}_{i2} - \boldsymbol{B}_2' \boldsymbol{x}_{i2} \end{pmatrix} \begin{pmatrix} \boldsymbol{z}_{i1} - \boldsymbol{B}_1' \boldsymbol{x}_{i1} \\ \boldsymbol{z}_{i2} - \boldsymbol{B}_2' \boldsymbol{x}_{i2} \end{pmatrix}'.$$

A.2.3. FCD of $\boldsymbol{z}_{i\,\mathrm{o}}$

The FCD of $\boldsymbol{z}_{i\,\mathrm{o}}$ is

$$p(\boldsymbol{z}_{i\,\mathrm{o}} | \cdots) \propto \begin{cases} \mathrm{N}\Big(\boldsymbol{z}_{i\,\mathrm{o}} |\, \mathrm{E}(\boldsymbol{z}_{i\,\mathrm{o}} | \boldsymbol{z}_{i+}, \boldsymbol{B}_i, \boldsymbol{V}_i), \operatorname{var}(\boldsymbol{z}_{i\,\mathrm{o}} | \boldsymbol{z}_{i+}, \boldsymbol{B}_i, \boldsymbol{V}_i)\Big) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if some elements of } \boldsymbol{y}_i \text{ are zero} \\ \mathrm{N}(\boldsymbol{B}_i' \boldsymbol{x}_i, \boldsymbol{V}_i) 1(\boldsymbol{z}_{i\,\mathrm{o}} \leq \boldsymbol{0}) \quad \text{if } \boldsymbol{y}_i = \boldsymbol{0}, \end{cases}$$

where

$$\begin{aligned}\mathrm{E}(\boldsymbol{z}_{i\,\mathrm{o}} | \boldsymbol{z}_{i+}, \boldsymbol{B}_i, \boldsymbol{V}_i) &= \begin{pmatrix} \boldsymbol{B}_{i1\,\mathrm{o}}' \boldsymbol{x}_{i1} \\ \boldsymbol{B}_{i2\,\mathrm{o}}' \boldsymbol{x}_{i2} \end{pmatrix} + \boldsymbol{V}_{ij\,\mathrm{o}+} \boldsymbol{V}_{ij++}^{-1} \left(\boldsymbol{z}_{i+} - \begin{pmatrix} \boldsymbol{B}_{i1+}' \boldsymbol{x}_{i1} \\ \boldsymbol{B}_{i2+}' \boldsymbol{x}_{i2} \end{pmatrix}\right) \\ \operatorname{var}(\boldsymbol{z}_{i\,\mathrm{o}} | \boldsymbol{z}_{i+}, \boldsymbol{B}_i, \boldsymbol{V}_i) &= \boldsymbol{V}_{i\,\mathrm{oo}} - \boldsymbol{V}_{i\,\mathrm{o}+} \boldsymbol{V}_{i++}^{-1} \boldsymbol{V}_{i+\mathrm{o}}.\end{aligned}$$

Appendix B. Marginal Likelihood Functions and the Bayes Factor

B.1. Two Independent Compositional Data

In model $\mathcal{M}_1$, $\boldsymbol{\theta} = \{\boldsymbol{B}_1, \boldsymbol{B}_2, \boldsymbol{V}_1, \boldsymbol{V}_2\}$. The log marginal likelihood of model $\mathcal{M}_1$ at a given value of parameters, $\hat{\boldsymbol{\theta}} = \{\hat{\boldsymbol{B}}_1, \hat{\boldsymbol{B}}_2, \hat{\boldsymbol{V}}_1, \hat{\boldsymbol{V}}_2\}$, can be written as

$$\log m(\boldsymbol{y}|\mathcal{M}_1) = \log p(\boldsymbol{y}|\hat{\boldsymbol{B}}_1, \hat{\boldsymbol{B}}_2, \hat{\boldsymbol{V}}_1, \hat{\boldsymbol{V}}_2) + \log p(\hat{\boldsymbol{B}}_1) + \log p(\hat{\boldsymbol{B}}_2) \tag{B1}$$

Under model $\mathcal{M}_1$, we have

$$p(\boldsymbol{y}|\hat{\boldsymbol{B}}_1, \hat{\boldsymbol{B}}_2, \hat{\boldsymbol{V}}_1, \hat{\boldsymbol{V}}_2) = \prod_{d=1}^{2} p(\boldsymbol{y}_j|\hat{\boldsymbol{B}}_j, \hat{\boldsymbol{V}}_j)$$
$$p(\hat{\boldsymbol{B}}_1, \hat{\boldsymbol{B}}_2, \hat{\boldsymbol{V}}_1, \hat{\boldsymbol{V}}_2|\boldsymbol{y}) = \prod_{d=1}^{2} p(\hat{\boldsymbol{B}}_j, \hat{\boldsymbol{V}}_j|\boldsymbol{y}_j),$$

where

$$p(\hat{\boldsymbol{B}}_j, \hat{\boldsymbol{V}}_j|\boldsymbol{y}_j) = p(\hat{\boldsymbol{B}}_j|\boldsymbol{y}_j)p(\hat{\boldsymbol{V}}_j|\hat{\boldsymbol{B}}_j, \boldsymbol{y}_j),\ j = 1, 2.$$

Utilizing the latent variable $\boldsymbol{z}_j$, we can write

$$p(\hat{\boldsymbol{B}}_j|\boldsymbol{y}_j) = \int p(\hat{\boldsymbol{B}}_j|\boldsymbol{V}_j, \boldsymbol{z}_j, \boldsymbol{y}_j)p(\boldsymbol{V}_j, \boldsymbol{z}_j|\boldsymbol{y}_j)d\boldsymbol{V}_j d\boldsymbol{z}_j \tag{B2}$$
$$p(\hat{\boldsymbol{V}}_j|\hat{\boldsymbol{B}}_j, \boldsymbol{y}_j) = \int p(\hat{\boldsymbol{V}}_j|\hat{\boldsymbol{B}}_j, \boldsymbol{z}_j, \boldsymbol{y}_j)p(\boldsymbol{z}_j|\hat{\boldsymbol{B}}_j, \boldsymbol{y}_j)d\boldsymbol{z}_j. \tag{B3}$$

Using the sth draws of $(\boldsymbol{V}_j, \boldsymbol{z}_j)$ from their FCDs, we estimate $p(\hat{\boldsymbol{B}}_j|\boldsymbol{y})$ by

$$\hat{p}(\hat{\boldsymbol{B}}_j|\boldsymbol{y}_j) = \frac{1}{S}\sum_{s=1}^{S} p(\hat{\boldsymbol{B}}_j|\boldsymbol{V}_j^{(s)}, \boldsymbol{z}_j^{(s)}, \boldsymbol{y}_j). \tag{B4}$$

The evaluation of (B3) needs additional S iterations from the following FCDs:

$$p(\boldsymbol{V}_j|\hat{\boldsymbol{B}}_j, \boldsymbol{z}_j, \boldsymbol{y}_j) \text{ and } p(\boldsymbol{z}_j|\hat{\boldsymbol{B}}_j, \boldsymbol{V}_j, \boldsymbol{y}_j).$$

Using this additional Gibbs sampling, we estimate $p(\hat{\boldsymbol{V}}_j|\hat{\boldsymbol{B}}_j, \boldsymbol{y}_j)$ by

$$\hat{p}(\hat{\boldsymbol{V}}_j|\hat{\boldsymbol{B}}_j, \boldsymbol{y}_j) = \frac{1}{S}\sum_{s=1}^{S} p(\hat{\boldsymbol{V}}_j|\hat{\boldsymbol{B}}_j, \boldsymbol{z}_j^{(s)}, \boldsymbol{y}_j), \tag{B5}$$

where

$$p(\hat{\boldsymbol{V}}_j|\hat{\boldsymbol{B}}_j, \boldsymbol{z}_j^{(s)}, \boldsymbol{y}_j) = \left(2^{\frac{m_j^* d}{2}} \pi^{\frac{1}{4}d(d-1)} \prod_{l=1}^{d} \Gamma\left(\frac{m_j^*+1-l}{2}\right)\right)^{-1}$$
$$\times |\boldsymbol{M}_j^*|^{\frac{m_j}{2}} |\hat{\boldsymbol{V}}_j|^{-\frac{1}{2}(m_j^*+d+1)} \exp\left(-\frac{1}{2} \operatorname{tr} \boldsymbol{M}_j^* \hat{\boldsymbol{V}}_j^{-1}\right)$$
$$m_j^* = m_j + n,\ \boldsymbol{M}_j^* = \boldsymbol{M}_j + \sum (\boldsymbol{z}_{ij}^{(s)} - \hat{\boldsymbol{B}}_j' \boldsymbol{x}_{ij})(\boldsymbol{z}_{ij}^{(s)} - \hat{\boldsymbol{B}}_j' \boldsymbol{x}_{ij})'.$$

By substituting (B4) and (B5) into (B1), we have the following estimate of $\log m(\boldsymbol{y}|\mathcal{M}_1)$:

$$\log \hat{m}(\boldsymbol{y}|\mathcal{M}_1) = \sum_{j=1}^{2} \Big[\log p(\boldsymbol{y}_j|\hat{\boldsymbol{B}}_j, \hat{\boldsymbol{V}}_j) + \log p(\hat{\boldsymbol{B}}_j) + \log p(\hat{\boldsymbol{V}}_j)$$
$$- \log \hat{p}(\hat{\boldsymbol{B}}_j|\boldsymbol{y}_j) - \log \hat{p}(\hat{\boldsymbol{V}}_j|\hat{\boldsymbol{B}}_j, \boldsymbol{y}_j)\Big].$$

B.2. Two Dependent Compositional Data

In model $\mathcal{M}_2$, $\boldsymbol{\theta} = \{\boldsymbol{B}_1, \boldsymbol{B}_2, \boldsymbol{V}\}$. The log marginal likelihood of model $\mathcal{M}_2$ at a given value of parameters, $\hat{\boldsymbol{\theta}} = \{\hat{\boldsymbol{B}}_1, \hat{\boldsymbol{B}}_2, \hat{\boldsymbol{V}}\}$, can be written as

$$\log m(\boldsymbol{y}|\mathcal{M}_2) = \log p(\boldsymbol{y}|\hat{\boldsymbol{B}}_1, \hat{\boldsymbol{B}}_2, \hat{\boldsymbol{V}}) + \log p(\hat{\boldsymbol{B}}_1) + \log p(\hat{\boldsymbol{B}}_2)$$
$$+ \log p(\hat{\boldsymbol{V}}) - \log p(\hat{\boldsymbol{B}}_1, \hat{\boldsymbol{B}}_2, \hat{\boldsymbol{V}}|\boldsymbol{y}), \tag{B6}$$

where

$$p(\hat{\boldsymbol{B}}_1, \hat{\boldsymbol{B}}_2, \hat{\boldsymbol{V}}|\boldsymbol{y}) = p(\hat{\boldsymbol{B}}_1|\boldsymbol{y}) p(\hat{\boldsymbol{B}}_2|\hat{\boldsymbol{B}}_1, \boldsymbol{y}) p(\hat{\boldsymbol{V}}|\hat{\boldsymbol{B}}_1, \hat{\boldsymbol{B}}_2, \boldsymbol{y}).$$

Utilizing the latent variable $\boldsymbol{z}$, we can write

$$p(\hat{\boldsymbol{B}}_1|\boldsymbol{y}) = \int p(\hat{\boldsymbol{B}}_1|\boldsymbol{B}_2, \boldsymbol{V}, \boldsymbol{z}, \boldsymbol{y}) p(\boldsymbol{B}_2, \boldsymbol{V}, \boldsymbol{z}|\boldsymbol{y}) d\boldsymbol{B}_2 d\boldsymbol{V} d\boldsymbol{z} \tag{B7}$$
$$p(\hat{\boldsymbol{B}}_2|\hat{\boldsymbol{B}}_1, \boldsymbol{y}) = \int p(\hat{\boldsymbol{B}}_2|\hat{\boldsymbol{B}}_1, \boldsymbol{V}, \boldsymbol{z}, \boldsymbol{y}) p(\boldsymbol{V}, \boldsymbol{z}|\hat{\boldsymbol{B}}_1, \boldsymbol{y}) d\boldsymbol{V} d\boldsymbol{z} \tag{B8}$$
$$p(\hat{\boldsymbol{V}}|\hat{\boldsymbol{B}}_1, \hat{\boldsymbol{B}}_2, \boldsymbol{y}) = \int p(\hat{\boldsymbol{V}}|\hat{\boldsymbol{B}}_1, \hat{\boldsymbol{B}}_2, \boldsymbol{z}, \boldsymbol{y}) p(\boldsymbol{z}|\hat{\boldsymbol{B}}_1, \hat{\boldsymbol{B}}_2, \boldsymbol{y}) d\boldsymbol{z}. \tag{B9}$$

Using the sth draw of $(\boldsymbol{B}_2, \boldsymbol{V}, \boldsymbol{z})$ from their FCDs, we estimate $p(\hat{\boldsymbol{B}}_1|\boldsymbol{y})$ by

$$\hat{p}(\hat{\boldsymbol{B}}_1|\boldsymbol{y}) = \frac{1}{S} \sum_{s=1}^{S} p(\hat{\boldsymbol{B}}_1|\boldsymbol{B}_2^{(s)}, \boldsymbol{V}^{(s)}, \boldsymbol{z}^{(s)}, \boldsymbol{y}). \tag{B10}$$

The evaluation of (B8) needs additional S iterations from the following FCDs:

$$p(\boldsymbol{B}_2|\hat{\boldsymbol{B}}_1, \boldsymbol{V}, \boldsymbol{z}, \boldsymbol{y}),\ p(\boldsymbol{V}|\hat{\boldsymbol{B}}_1, \boldsymbol{B}_2, \boldsymbol{z}, \boldsymbol{y}) \text{ and } p(\boldsymbol{z}|\hat{\boldsymbol{B}}_1, \boldsymbol{B}_2, \boldsymbol{V}, \boldsymbol{y}).$$

Using this additional Gibbs sampling, we estimate $p(\hat{\boldsymbol{B}}_2|\hat{\boldsymbol{B}}_1, \boldsymbol{y})$ by

$$\hat{p}(\hat{\boldsymbol{B}}_2|\hat{\boldsymbol{B}}_1, \boldsymbol{y}) = \frac{1}{S}\sum_{s=1}^{S} p(\hat{\boldsymbol{B}}_2|\hat{\boldsymbol{B}}_1, \boldsymbol{V}^{(s)}, \boldsymbol{z}^{(s)}, \boldsymbol{y}). \tag{B11}$$

The evaluation of (B9) needs additional S iterations from the following FCDs:

$$p(\boldsymbol{V}|\hat{\boldsymbol{B}}_1, \hat{\boldsymbol{B}}_2, \boldsymbol{z}, \boldsymbol{y}) \text{ and } p(\boldsymbol{z}|\hat{\boldsymbol{B}}_1, \hat{\boldsymbol{B}}_2, \boldsymbol{V}, \boldsymbol{y}).$$

Using this additional Gibbs sampling, we estimate $p(\hat{\boldsymbol{V}}|\hat{\boldsymbol{B}}_1, \hat{\boldsymbol{B}}_2, \boldsymbol{y})$ by

$$\hat{p}(\hat{\boldsymbol{V}}|\hat{\boldsymbol{B}}_1, \hat{\boldsymbol{B}}_2, \boldsymbol{y}) = \frac{1}{S}\sum_{s=1}^{S} p(\hat{\boldsymbol{V}}|\hat{\boldsymbol{B}}_1, \hat{\boldsymbol{B}}_2, \boldsymbol{z}^{(s)}, \boldsymbol{y}), \tag{B12}$$

where

$$\begin{aligned} p(\hat{\boldsymbol{V}}|\hat{\boldsymbol{B}}_1, \hat{\boldsymbol{B}}_2, \boldsymbol{z}^{(s)}, \boldsymbol{y}) &= \left(2^{m^* d}\pi^{\frac{1}{2}d(2d-1)}\prod_{l=1}^{2d}\Gamma\left(\tfrac{m^*+1-l}{2}\right)\right)^{-1} \\ &\times |\boldsymbol{M}^*|^{\frac{m^*}{2}}|\hat{\boldsymbol{V}}|^{-\frac{1}{2}(m^*+2d+1)}\exp\left(-\frac{1}{2}\operatorname{tr}\boldsymbol{M}^*\hat{\boldsymbol{V}}^{-1}\right) \\ m^* = m+n,\ \boldsymbol{M}^* &= \boldsymbol{M} + \sum \begin{pmatrix} \boldsymbol{z}_{i1}^{(s)} - \hat{\boldsymbol{B}}_1'\boldsymbol{x}_{i1} \\ \boldsymbol{z}_{i2}^{(s)} - \hat{\boldsymbol{B}}_2'\boldsymbol{x}_{i2} \end{pmatrix}\begin{pmatrix} \boldsymbol{z}_{i1}^{(s)} - \hat{\boldsymbol{B}}_1'\boldsymbol{x}_{i1} \\ \boldsymbol{z}_{i2}^{(s)} - \hat{\boldsymbol{B}}_2'\boldsymbol{x}_{i2} \end{pmatrix}'. \end{aligned}$$

By substituting (B10), (B11), and (B12) into (B6), we have the following estimate of $\log m(\boldsymbol{y}|\mathcal{M}_2)$:

$$\begin{aligned} \log \hat{m}(\boldsymbol{y}|\mathcal{M}_2) &= \log p(\boldsymbol{y}|\hat{\boldsymbol{B}}_1, \hat{\boldsymbol{B}}_2, \hat{\boldsymbol{V}}) + \log p(\hat{\boldsymbol{B}}_1) + \log p(\hat{\boldsymbol{B}}_2) \\ &+ \log p(\hat{\boldsymbol{V}}) - \log \hat{p}(\hat{\boldsymbol{B}}_1|\boldsymbol{y}) - \log \hat{p}(\hat{\boldsymbol{B}}_2|\hat{\boldsymbol{B}}_1, \boldsymbol{y}) - \log \hat{p}(\hat{\boldsymbol{V}}|\hat{\boldsymbol{B}}_1, \hat{\boldsymbol{B}}_2, \boldsymbol{y}). \end{aligned}$$

Appendix C. Average Partial Effect (APE) and Probability of the Positive APE

As [2, p.320] show, the posterior expected values $\mathrm{E}(Y_{ik}|\text{data}, \boldsymbol{x}_i)$ can be calculated from the MCMC results. We generate $\tilde{\boldsymbol{Y}}_i = (\tilde{Y}_{i1}, \ldots, \tilde{Y}_{i,D-1}, \tilde{Y}_{iD})'$ for S iterations from the predictive distribution of $\boldsymbol{Y}_i$, and calculate

$$\mathrm{E}(Y_{ik}|\text{data}, \boldsymbol{x}_i) \approx \frac{1}{S}\sum_{s=1}^{S}\tilde{Y}_{ik}^{(s)}. \tag{C1}$$

We apply this method to estimate the partial effects. Without loss of generality, we show the partial effect of the pth explanatory variable x_p. Suppose x_p is a continuous variable and increases by one unit. Denoting $\boldsymbol{x} = (x_1, \ldots, x_p)'$ and $\boldsymbol{x}^\star = (x_1, \ldots, x_p +$

$1)'$, the partial effect of the ith individual in the sth simulation is as follows:

$$\left(\tilde{\boldsymbol{Y}}_i^{(s)}|\text{data}, \boldsymbol{x}_i^\star\right) - \left(\tilde{\boldsymbol{Y}}_i^{(s)}|\text{data}, \boldsymbol{x}_i\right).$$

Following [3, p.592], we define the following average partial effect (APE) for the sth simulation:

$$APE(p)^{(s)} = \frac{1}{n}\sum_{i=1}^{n}\left[\left(\tilde{\boldsymbol{Y}}_i^{(s)}|\text{data}, \boldsymbol{x}_i^\star\right) - \left(\tilde{\boldsymbol{Y}}_i^{(s)}|\text{data}, \boldsymbol{x}_i\right)\right], \tag{C2}$$

where p denotes the pth explanatory variable.

If the pth explanatory variable is represented in logarithm, we calculate the partial effect using (C2), where $\boldsymbol{x} = (x_1, \ldots, x_{p-1}, \log x_p)'$ and $\boldsymbol{x}^\star = (x_1, \ldots, x_{p-1}, \log(x_p + 1))'$. Further, if the pth variable is a dummy variable, the partial effect can be obtained using (C2), where $\boldsymbol{x} = (x_1, \ldots, x_{p-1}, 0)'$ and $\boldsymbol{x}^\star = (x_1, \ldots, x_{p-1}, 1)'$.

The probability that $APE(p)$ is positive as follows. As [2, p.320] show, the posterior expected values $\mathrm{E}(Y_{ik}|\text{data}, \boldsymbol{x}_i)$ can be calculated from the MCMC results. We generate $\tilde{\boldsymbol{Y}}_i = (\tilde{Y}_{i1}, \ldots, \tilde{Y}_{i,D-1}, \tilde{Y}_{iD})'$ for S iterations from the predictive distribution of $\boldsymbol{Y}_i$, and calculate

$$\Pr(APE(p) > 0) \approx \frac{1}{S}\sum_{s=1}^{S}\left[APE(p)^{(s)} > 0\right]. \tag{C3}$$

References

[1] K.M. Abadir and J.R. Magnus, *Matrix Algebra*, Cambridge University Press, Cambridge, 2005.

[2] T. J. Leininger, A. E. Gelfand, J. M. Allen, and J. A. Silander Jr, *Spatial regression modeling for compositional data with many zeros.* J. Agric. Biol. Environ. Stat. 18 (2013), pp. 314–334.

[3] J. M. Wooldridge, *Introductory Econometrics: A Modern Approach, 6th ed.* Cengage Learning, Boston, 2016.