Supplementary Material

1 A partition of the total responsive variance based on model (6)

Based on model (6), a partition of the total responsive variance V_y is given by

$$\begin{aligned} V_y &= V_{\tilde{\eta}} + V_A + V_B + V_{A\times B} + 2\mathrm{Cov}(\tilde{\eta}, T_{A\times B}) + 2\mathrm{Cov}(T_A, T_B) \\ &\quad +2\mathrm{Cov}(T_A, T_{A\times B}) + 2\mathrm{Cov}(T_B, T_{A\times B}) + V_e \end{aligned}$$

where

$$\begin{aligned} &V_{\tilde{\eta}} = \mathrm{Var}(\tilde{\eta}) = (1-k_2)\sigma^2_{D'}, \quad V_e = \sigma^2_e \\ &V_A = \mathrm{Var}(T_A) = \sum_{i=1}^{a} p_{i\cdot}(\alpha_i - \bar{\alpha}_\cdot)^2 + k_1(1-k_2)\sigma^2_{\alpha B} \\ &V_B = \mathrm{Var}(T_B) = k_2\sigma^2_{D'} \\ &V_{A\times B} = \mathrm{Var}(T_{A\times B}) = (k_1k_2 + 4\bar{d}^*_{\cdot\cdot})\sigma^2_{\alpha B} \\ &\mathrm{Cov}(\tilde{\eta}, T_{A\times B}) = \mathrm{Cov}(T_A, T_B) = \bar{d}^*_{\cdot\cdot}\sigma^2_{\alpha B} \\ &\mathrm{Cov}(T_A, T_{A\times B}) = \mathrm{Cov}(T_B, T_{A\times B}) = -2\bar{d}^*_{\cdot\cdot}\sigma^2_{\alpha B} \end{aligned}$$

where $k_1 = 1-\sum_{i=1}^a p^2_{i\cdot}$, $k_2 = 1-\sum_{j=1}^b p^2_{\cdot j}$, $\sigma^2_{D'} = \sigma^2_B + (1-k_1)\sigma^2_{\alpha B}$ and $\bar{d}^*_{\cdot\cdot} = \sum_{i=1}^a \sum_{j=1}^b p_{i\cdot}p_{\cdot j}d_{ij}$. When the distributions of factors A and B are independent (i.e., $\bar{d}^*_{\cdot\cdot} = 0$), there is an orthogonal partition of the total responsive variance

$$V_y = V_{\tilde{\eta}} + V_A + V_B + V_{A\times B} + V_e$$

2 A re-parameterization of the cell means $\{\bar{y}_{ij\cdot}\}$

Based on model (8), the cell means $\{\bar{y}_{ij\cdot}\}$ can be re-parameterized through the following

$$\bar{y}_{ij\cdot} = \tilde{\eta} + \tilde{\tau}_i + \tilde{D}_j + \widetilde{(\tau D)}_{ij} \tag{1}$$

where $\tilde{\tau}_i = \tau''_i - \bar{\tau}''_\cdot$, $\tilde{D}_j = D'_j - \bar{D}'_\cdot$ and $\widetilde{(\tau D)}_{ij} = (\alpha B)_{ij} - \overline{(\alpha B)}_{i\cdot} - \overline{(\alpha B)}_{\cdot j} + \overline{(\alpha B)}_{\cdot\cdot}$ for $i = 1, \cdots, a$ and $j = 1, \cdots, b$. Under the intrinsic constrains $\sum_{i=1}^a \hat{p}_{i\cdot}\tilde{\tau}_i = 0$, $\sum_{j=1}^b \hat{p}_{\cdot j}\tilde{D}_j = 0$ and $\sum_{i=1}^a \hat{p}_{i\cdot}\widetilde{(\tau D)}_{ij} = \sum_{j=1}^b \hat{p}_{\cdot j}\widetilde{(\tau D)}_{ij} = 0$, we have $\bar{y}^*_{i\cdot} = \sum_{j=1}^b \hat{p}_{\cdot j}\bar{y}^*_{ij\cdot} = \tilde{\eta} + \tilde{\tau}_i$, $\bar{y}^*_{\cdot j} = \sum_{i=1}^a \hat{p}_{i\cdot}\bar{y}^*_{ij\cdot} = \tilde{\eta} + \tilde{D}_j$ and $\bar{y}^*_{\cdot\cdot} = \sum_{i=1}^a \hat{p}_{i\cdot}\bar{y}^*_{i\cdot} = \sum_{i=1}^a (\tilde{\eta} + \tilde{\tau}_i)\hat{p}_{i\cdot} = \tilde{\eta}$. Therefore, $\tilde{\tau}_i = \bar{y}^*_{i\cdot} - \bar{y}^*_{\cdot\cdot}$ and $\tilde{D}_j = \bar{y}^*_{\cdot j} - \bar{y}^*_{\cdot\cdot}$. Meanwhile, $\widetilde{(\tau D)}_{ij} = \bar{y}_{ij\cdot} - \tilde{\eta} - \tilde{\tau}_i - \tilde{D}_j = \bar{y}_{ij\cdot} - \bar{y}^*_{i\cdot} - \bar{y}^*_{\cdot j} + \bar{y}^*_{\cdot\cdot}$, for $i = 1, \cdots, a$ and $j = 1, \cdots, b$.

3 Conditional expectations of the sums of squares

Based on UP model (1), we have

$$\begin{cases} \bar{y}^*_{i\cdot} = \mu + \alpha_i + \bar{B}^*_{\cdot} + \overline{(\alpha B)}^*_{i\cdot} + \bar{e}^*_{i\cdot} \\ \bar{y}^*_{\cdot j} = \mu + \bar{\alpha}^*_{\cdot} + B_j + \overline{(\alpha B)}^*_{\cdot j} + \bar{e}^*_{\cdot j} \\ \bar{y}^*_{\cdot\cdot} = \mu + \bar{\alpha}^*_{\cdot} + \bar{B}^*_{\cdot} + \overline{(\alpha B)}^*_{\cdot\cdot} + \bar{e}^*_{\cdot\cdot} \end{cases}$$

where $\bar{\alpha}^*_{\cdot} = \sum_{i=1}^a \hat{p}_{i\cdot}\alpha_i$, $\bar{B}^*_{\cdot} = \sum_{j=1}^b \hat{p}_{\cdot j}B_j$, $\overline{(\alpha B)}^*_{i\cdot} = \sum_{j=1}^b \hat{p}_{\cdot j}(\alpha B)_{ij}$, $\overline{(\alpha B)}^*_{\cdot j} = \sum_{i=1}^a \hat{p}_{i\cdot}(\alpha B)_{ij}$, $\overline{(\alpha B)}^*_{\cdot\cdot} = \sum_{i=1}^a\sum_{j=1}^b \hat{p}_{i\cdot}\hat{p}_{\cdot j}(\alpha B)_{ij}$, $\bar{e}^*_{i\cdot} = \sum_{j=1}^b \hat{p}_{\cdot j}\bar{e}_{ij\cdot}$, $\bar{e}^*_{\cdot j} = \sum_{i=1}^a \hat{p}_{i\cdot}\bar{e}_{ij\cdot}$ and $\bar{e}^*_{\cdot\cdot} = \sum_{i=1}^a\sum_{j=1}^b \hat{p}_{i\cdot}\hat{p}_{\cdot j}\bar{e}_{ij\cdot}$. Given the fixed cell counts $\{n_{ij}\}$'s, we can show that

$$\begin{cases} \mathrm{E}(SSA^*|\{n_{ij}\}) = n_{\cdot\cdot}[\sum_{i=1}^a \hat{p}_{i\cdot}(\alpha_i - \bar{\alpha}_{\cdot})^2 + \hat{k}_1(1-\hat{k}_2)\sigma^2_{\alpha B}] + df_A \cdot \sigma^2_e \\ \mathrm{E}(SSB^*|\{n_{ij}\}) = n_{\cdot\cdot}\hat{k}_2(\sigma^2_B + \hat{k}_1\sigma^2_{\alpha B}) + df_B \cdot \sigma^2_e \\ \mathrm{E}(SSAB^*|\{n_{ij}\}) = n_{\cdot\cdot}(\hat{k}_1\hat{k}_2 + 4\hat{\bar{d}}^*_{\cdot\cdot})\sigma^2_{\alpha B} + df_{AB} \cdot \sigma^2_e \\ \mathrm{E}(2\Delta_{A\cdot B}|\{n_{ij}\}) = 2n_{\cdot\cdot}\hat{\bar{d}}^*_{\cdot\cdot}\sigma^2_{\alpha B} + df_{A\cdot B} \cdot \sigma^2_e \\ \mathrm{E}(2\Delta_{A\cdot AB}|\{n_{ij}\}) = -4n_{\cdot\cdot}\hat{\bar{d}}^*_{\cdot\cdot}\sigma^2_{\alpha B} + df_{A\cdot AB} \cdot \sigma^2_e \\ \mathrm{E}(2\Delta_{B\cdot AB}|\{n_{ij}\}) = -4n_{\cdot\cdot}\hat{\bar{d}}^*_{\cdot\cdot}\sigma^2_{\alpha B} + df_{B\cdot AB} \cdot \sigma^2_e \\ \mathrm{E}(SSE|\{n_{ij}\}) = (n_{\cdot\cdot} - ab)\sigma^2_e \end{cases}$$

where $\hat{k}_1 = 1-\sum_{i=1}^a \hat{p}^2_{i\cdot}$, $\hat{k}_2 = 1-\sum_{j=1}^b \hat{p}^2_{\cdot j}$, $\hat{\bar{d}}^*_{\cdot\cdot} = \sum_{i=1}^a\sum_{j=1}^b \hat{p}_{i\cdot}\hat{p}_{\cdot j}\hat{d}_{ij}$, $df_A = \sum_{i=1}^a\sum_{j=1}^b \hat{p}_{i\cdot}(1-\hat{p}_{i\cdot})\hat{p}^2_{\cdot j}/\hat{p}_{ij}$, $df_B = \sum_{i=1}^a\sum_{j=1}^b \hat{p}_{\cdot j}(1-\hat{p}_{\cdot j})\hat{p}^2_{i\cdot}/\hat{p}_{ij}$, $df_{AB} = (a-2)(b-2) + \sum_{i=1}^a\sum_{j=1}^b \hat{p}_{i\cdot}\hat{p}_{\cdot j}(\hat{p}_{i\cdot} + \hat{p}_{\cdot j} - 3\hat{p}_{i\cdot}\hat{p}_{\cdot j})/\hat{p}_{ij}$, $df_{A\cdot B} = 2\sum_{i=1}^a\sum_{j=1}^b \hat{d}_{ij}\hat{p}_{i\cdot}p_{\cdot j}/\hat{p}_{ij}$, $df_{A\cdot AB} = 2\sum_{i=1}^a\sum_{j=1}^b \hat{d}_{ij}\hat{p}_{\cdot j}(1-2\hat{p}_{i\cdot})/\hat{p}_{ij}$ and $df_{B\cdot AB} = 2\sum_{i=1}^a\sum_{j=1}^b \hat{d}_{ij}\hat{p}_{i\cdot}(1-2\hat{p}_{\cdot j})/\hat{p}_{ij}$. In addition, we can show that

$$\mathrm{E}(SST^*|\{n_{ij}\}) = n_{\cdot\cdot}\sum_{i=1}^a \hat{p}_{i\cdot}(\alpha_i - \bar{\alpha}_{\cdot})^2 + n_{\cdot\cdot}\hat{k}_2\sigma^2_B + n_{\cdot\cdot}[1-(1-\hat{k}_1)(1-\hat{k}_2) - 2\hat{\bar{d}}^*_{\cdot\cdot}]\sigma^2_{\alpha B} + df_{SST^*}\cdot\sigma^2_e$$

where $df_{SST^*} = (n_{\cdot\cdot} - 2) + \sum_{i=1}^a\sum_{j=1}^b \hat{p}^2_{i\cdot}\hat{p}^2_{\cdot j}/\hat{p}_{ij}$, and $\mathrm{E}[(\bar{y}^*_{\cdot\cdot} - \bar{y}_{\cdots})^2] = \sum_{i=1}^a\sum_{j=1}^b \hat{D}^2_{ij}(\sigma^2_{\alpha B} + \sigma^2_e/n_{ij})$.

4 Asymptotic unbiasedness of the ANOVA estimators of the factor variances and covariances

Assume that the variances $V(y_{ijk})$ are finite, and $n_{ij}|n_{\cdot\cdot} \sim Mult(n_{\cdot\cdot}, \{p_{ij}, i = 1, \cdots, a, j = 1, \cdots, b\})$ with $n_{ij}/n_{\cdot\cdot} \to p_{ij}$ almost surely as $n_{\cdot\cdot} \to \infty$, for $i = 1, \cdots, a$ and $j = 1, \cdots, b$. Then, we have $\hat{p}_{i\cdot} = \frac{n_{i\cdot}}{n_{\cdot\cdot}} \to p_{i\cdot}$, $\hat{p}_{\cdot j} = \frac{n_{\cdot j}}{n_{\cdot\cdot}} \to p_{\cdot j}$, and $\hat{d}_{ij} = \hat{p}_{ij} - \hat{p}_{i\cdot}\hat{p}_{\cdot j} \to d_{ij}$ almost surely. Therefore, by the dominated convergence theorem, we have

$$\begin{cases} \mathrm{E}(\widehat{V}_A) = \mathrm{E}[\mathrm{E}(\widehat{V}_A|\{n_{ij}\})] \to V_A = \sum_{i=1}^a p_{i\cdot}(\alpha_i - \bar{\alpha}_{\cdot})^2 + k_1(1-k_2)\sigma^2_{\alpha B} \\ \mathrm{E}(\widehat{V}_B) = \mathrm{E}[\mathrm{E}(\widehat{V}_B|\{n_{ij}\})] \to V_B = k_2\sigma^2_{D'} \\ \mathrm{E}(\widehat{V}_{A\times B}) = \mathrm{E}[\mathrm{E}(\widehat{V}_{A\times B}|\{n_{ij}\})] \to V_{A\times B} = (k_1k_2 + 4\bar{d}^*_{\cdot\cdot})\sigma^2_{\alpha B} \\ \mathrm{E}[\widehat{\mathrm{Cov}}(\eta^*, T_B|\{n_{ij}\})] = \mathrm{E}[\widehat{\mathrm{Cov}}(T_A, T_B|\{n_{ij}\})] \to \bar{d}^*_{\cdot\cdot}\sigma^2_{\alpha B} \\ \mathrm{E}[\widehat{\mathrm{Cov}}(T_A, T_{A\times B})] = \mathrm{E}[\widehat{\mathrm{Cov}}(T_B, T_{A\times B})] \to -2\bar{d}^*_{\cdot\cdot}\sigma^2_{\alpha B} \end{cases}$$

as $n_{..} \to \infty$.

5 A proof that SSE is independent of SSA^*, SSB^* and $SSAB^*$

Note that $\bar{y}_{ij\cdot} = \sum_{k=1}^{n_{ij}} y_{ijk}/n_{ij} = \mu + \alpha_i + B_j + (\alpha B)_{ij} + \bar{e}_{ij\cdot}$, where $\bar{e}_{ij\cdot} = \sum_{k=1}^{n_{ij}} e_{ijk}/n_{ij}$. Based on UP model (1), by adopting the same strategy as used in Searle (1987), we can show that $\mathrm{Cov}(\bar{y}^*_{i'\cdot} - \bar{y}^*_{\cdot\cdot}, y_{ijk} - \bar{y}_{ij\cdot}) = 0$, $\mathrm{Cov}(\bar{y}^*_{\cdot j'} - \bar{y}^*_{\cdot\cdot}, y_{ijk} - \bar{y}_{ij\cdot}) = 0$ and $\mathrm{Cov}(\bar{y}_{i'j'\cdot} - \bar{y}^*_{i'\cdot} - \bar{y}^*_{\cdot j'} + \bar{y}^*_{\cdot\cdot}, y_{ijk} - \bar{y}_{ij\cdot}) = 0$ for any $i', i = 1, \cdots, a,\ j, j' = 1, \cdots, b$ and $k = 1, \cdots, n_{ij}$. Thus, under the normality assumption, $\{y_{ijk} - \bar{y}_{ij\cdot}, i = 1, \cdots, a, j = 1, \cdots, b, k = 1, \cdots, n_{ij}\}$ are independent of $\{\bar{y}^*_{i\cdot} - \bar{y}^*_{\cdot\cdot}, i = 1, \cdots, a\}$, $\{\bar{y}^*_{\cdot j} - \bar{y}^*_{\cdot\cdot}, j = 1, \cdots, b\}$ and $\{\bar{y}_{ij\cdot} - \bar{y}^*_{i\cdot} - \bar{y}^*_{\cdot j} + \bar{y}^*_{\cdot\cdot}, i = 1, \cdots, a, j = 1, \cdots, b\}$. Therefore, SSE is independent of SSA^*, SSB^* and $SSAB^*$.

References

Searle, S. R. (1987). *Linear Models for Unbalanced Data.* John Wiley & Sons, Inc.