On integers that are covering numbers of groups: Supplementary Material

Martino Garonzi, Luise-Charlotte Kappe, and Eric Swartz

The material presented here is a supplement to the results of the main paper [4]. It contains calculations of or bounds for the covering number of various monolithic groups with a degree of primitivity at most 129 that could not be dealt with using the methods of the main paper such as Algorithms KNS and GKS as well as known results.

We use the following notation in the tables. The notation $\mathcal{M}_i$ indicates a conjugacy class of maximal subgroups. Below the symbol $\mathcal{M}_i$, the number in parentheses indicates the number of conjugate subgroups in the class. The notation "$cl_{m,j}$" refers to a class of elements of order m; the "j" will be omitted when we are considering a single class of this order. If the $(cl_m, \mathcal{M}_i)$-entry of the table is n_k, then each subgroup of $\mathcal{M}_i$ contains n elements of class cl_m, and each element in cl_m is contained in k subgroups of $\mathcal{M}_i$. Instead of writing n_1, we will write n, P to indicate that the elements of cl_m are partitioned among the subgroups in $\mathcal{M}_i$. If the $(cl_m, \mathcal{M}_i)$-entry is written as n, then the information about how many subgroups in $\mathcal{M}_i$ contain a given element of cl_m is unimportant to the proof and is omitted. We observe that the smallest primitivity degree of each of the following subgroups is an index of one of its maximal subgroups, and hence this value appears as an index in the corresponding table (when such a table is provided).

One argument that is used repeatedly in the following propositions is the following, which we state here for emphasis: if there are c elements from class cl_j remaining to be covered and the $(cl_j, \mathcal{M}_i)$-entry of the table is n_k, then at least $\lceil c/n \rceil$ subgroups from $\mathcal{M}_i$ are needed to cover the c elements of class cl_j. In particular, if a class of maximal subgroups $\mathcal{M}_i$ has size m and the $(cl_j, \mathcal{M}_i)$-entry of the table is n_k, then at least m/k subgroups from $\mathcal{M}_i$ are needed to cover the elements of class cl_j.

Proposition A.1. *We have the following covering number values:*

(i) $\sigma(2^6 : \mathrm{O}^-(6,2)) = 67$;
(ii) $\sigma(2^6 : A_8) = 71$;
(iii) $\sigma((\mathrm{PSL}(2,7) \times \mathrm{PSL}(2,7)).4) = 498$;
(iv) $\sigma(\mathrm{PSL}(2,64).3) = 2080$;
(v) $\sigma(\mathrm{PSL}(2,8) \,\mathrm{wr}\, 2) = 586$;
(vi) $\sigma((A_6 \times A_6).4) = 1387$;
(vii) $\sigma(\mathrm{Sp}(8,2)) = 256$.

Proof. (i) Using [4, Algorithm GKS], we have that $\sigma(\mathrm{O}^-(6,2)) = 67$, and so $\sigma(2^6 : \mathrm{O}^-(6,2)) \leqslant 67$. Suppose, for the purpose of contradiction, that $\sigma(2^6 : \mathrm{O}^-(6,2)) < 67$, and let $\mathcal{B}$ be this smaller cover. By [4, Lemma 2.10], this means that all 64 conjugates of the point stabilizer in the primitive action on 64 points are contained in $\mathcal{B}$. By GAP [3], there is a class of elements of order 12 that are not contained in the point stabilizers. The most number of elements of this class that are contained in a maximal subgroup is 1920, and so at least an additional 18 subgroups are needed to cover this class. However, $64 + 18 > 67$, a contradiction to minimality. Therefore, the covering number of $2^6 : \mathrm{O}^-(6,2)$ is 67.

(ii) Since $\sigma(A_8) = 71$, if $\sigma(2^6 : A_8) < 71$, then by [4, Lemma 2.10], any minimal cover of $2^6 : A_8$ would have to contain all maximal subgroups isomorphic to A_8. However, by GAP, there are a total of 128 maximal subgroups isomorphic to A_8, a contradiction.

(iii) By GAP, there are four classes of maximal subgroups, and we have the following distribution of elements:

	$\mathcal{M}_1$ (1)	$\mathcal{M}_2$ (64)	$\mathcal{M}_3$ (441)	$\mathcal{M}_4$ (784)
cl_{24}	$4704, P$	0	0	0
cl_{16}	0	0	$16, P$	0
$cl_{12,1}$	0	294_2	0	$12, P$
$cl_{12,2}$	0	294_2	0	$12, P$

Table 1. Element distribution in $\mathrm{PSL}(2,7)^2.4$

The unique minimal normal subgroup is the only class containing elements from cl_{24}. Moreover, the elements of cl_{16} are partitioned among the 441 subgroups in $\mathcal{M}_3$, so these 441 subgroups are also contained in a minimal cover. Only the two classes $cl_{12,1}$ and $cl_{12,2}$ are left uncovered after including these 442 subgroups. Using [4, Algorithm KNS] and GUROBI [5] for the elements in these two classes, we find that the minimal cover of these two classes contains 56 subgroups. Therefore, the covering number of $(\mathrm{PSL}(2,7) \times \mathrm{PSL}(2,7)).4$ is 498.

(iv) By GAP, we have the following distribution of elements:

	$\mathcal{M}_1$ (1)	$\mathcal{M}_2$ (65)	$\mathcal{M}_3$ (520)	$\mathcal{M}_4$ (2016)	$\mathcal{M}_5$ (2080)	$\mathcal{M}_6$ (4368)
cl_{63}	$12480, P$	384	0	0	6	0
cl_{15}	0	0	0	$26, P$	0	12
$cl_{9,1}$	0	2688_2	168	0	42	0
$cl_{9,2}$	0	2688_2	168	0	42	0

Table 2. Element distribution in $\mathrm{PSL}(2,64).3$

The classes cl_{15}, $cl_{9,1}$, and $cl_{9,2}$ are not contained in the minimal normal subgroup in $\mathcal{M}_1$. The elements of cl_{15} are partitioned among the 2016 subgroups of $\mathcal{M}_4$, and no subgroup contains more elements of cl_{15} than a subgroup in $\mathcal{M}_4$ does. Using [4, Algorithm KNS] and GUROBI, the minimal cover of $cl_{9,1}$ and $cl_{9,2}$ has size 64, and calculations in GAP show that a random choice of 64 subgroups from $\mathcal{M}_2$ (say, the first 64 in a given list, excluding the last) plus the 2016 aforementioned maximal subgroups from $\mathcal{M}_4$ are a cover. Therefore, the covering number of $\mathrm{PSL}(2,64).3$ is 2080.

(v) By GAP, we have the following distribution of elements in $\mathrm{PSL}(2,8) \operatorname{wr} 2$:

	$\mathcal{M}_1$ (1)	$\mathcal{M}_2$ (504)	$\mathcal{M}_3$ (81)	$\mathcal{M}_4$ (784)	$\mathcal{M}_5$ (1296)
cl_{63}	$8064, P$	0	0	0	0
cl_{18}	0	$56, P$	0	36	0
cl_4	0	0	$392, P$	162	98

Table 3. Element distribution in $\mathrm{PSL}(2,8) \operatorname{wr} 2$

By [4, Algorithm KNS] and GUROBI, the subgroups from $\mathcal{M}_2$ and $\mathcal{M}_3$ are a minimal cover of cl_{18} and cl_4. However, these two classes together are not a cover, whereas including the minimal normal subgroup from $\mathcal{M}_1$ with these is a cover. Therefore, the covering number of $\mathrm{PSL}(2,8) \operatorname{wr} 2$ is 586.

(vi) Here $(A_6 \times A_6).4$ is number 16 of the list AllPrimitiveGroups(NrMovedPoints,100) returned by GAP. A class of elements of order 40 is only contained in the minimal normal subgroup, and a class of elements of order 20 is only contained in the 1296 subgroups from another class. The only elements not covered by this class are four classes of elements of order 16. By [4, Algorithm KNS] and GUROBI, a minimal cover of these elements contains 90 subgroups. The result follows.

(vii) Using [4, Algorithm GKS], $\sigma(\mathrm{O}^+(8,2):2) = \sigma(\mathrm{O}^-(8,2):2) = 256$. By [4, Lemma 2.10], if $\sigma(\mathrm{Sp}(8,2)) < 256$, then all maximal subgroups isomorphic to either $\mathrm{O}^+(8,2):2$ or $\mathrm{O}^-(8,2):2$ are in such a minimal cover. However, there are $136+120 = 256$ such subgroups, so $\sigma(\mathrm{Sp}(8,2)) \geqslant 256$. On the other hand, calculations in GAP show that all 256 subgroups isomorphic to either $\mathrm{O}^+(8,2):2$ or $\mathrm{O}^-(8,2):2$ are a cover. The result follows. □

Proposition A.2. *We have the following lower and upper bounds for the indicated covering number values:*

(i) $447 \leqslant \sigma(A_7 \operatorname{wr} 2) \leqslant 667$;
(ii) $196 \leqslant \sigma(\mathrm{PSp}(4,4).2) \leqslant 222$;
(iii) $11859 \leqslant \sigma(HS:2) \leqslant 22375$;
(iv) $22746 \leqslant \sigma((A_{10} \times A_{10}).4) \leqslant 30377$;
(v) $344 \leqslant \sigma(\mathrm{PSU}(4,3)) \leqslant 442$;
(vi) $256 \leqslant \sigma(\mathrm{PSU}(4,3).2_1) \leqslant 554$, *where* $\mathrm{PSU}(4,3).2_1$ *is the group* $U(4,3).2_1$ *in the ATLAS* [2];
(vii) $183 \leqslant \sigma(\mathrm{PSU}(4,3).2_2) \leqslant 365$, *where* $\mathrm{PSU}(4,3).2_2$ *is the group* $U(4,3).2_2$ *in the ATLAS* [2];
(viii) $412 \leqslant \sigma(\mathrm{PSU}(4,3).2_3) \leqslant 554$, *where* $\mathrm{PSU}(4,3).2_3$ *is the group* $U(4,3).2_3$ *in the ATLAS* [2];
(ix) $242 \leqslant \sigma(\mathrm{PSL}(4,3).2) \leqslant 365$;
(x) $25706 \leqslant \sigma(\mathrm{O}^-(8,2)) \leqslant 26283$;
(xi) $204 \leqslant \mathrm{O}^+(8,2) \leqslant 765$;
(xii) $570 \leqslant \sigma(\mathrm{PSL}(2,11) \operatorname{wr} 2) \leqslant 926$.

Proof. (i) Using GAP, we find the following distribution of elements:

	$\mathcal{M}_1$ (1)	$\mathcal{M}_2$ (2520)	$\mathcal{M}_3$ (2520)	$\mathcal{M}_4$ (1225)	$\mathcal{M}_5$ (441)	$\mathcal{M}_6$ (49)	$\mathcal{M}_7$ (225)	$\mathcal{M}_8$ (225)
cl_{14}	0	360	0	0	0	0	$4032, P$	$4032, P$
cl_{12}	0	0	420	432	2400_2	0	0	0
cl_3	$39200, P$	0	0	320	0	3200	0	0

Table 4. Element distribution in $A_7 \operatorname{wr} 2$

Either $\mathcal{M}_7$ or $\mathcal{M}_8$ along with the minimal normal subgroup in $\mathcal{M}_1$ constitute a minimal cover of cl_{14} and cl_{12}. For the lower bound, it takes at least 221 subgroups from $\mathcal{M}_5$ to cover the elements cl_{12}. The upper bound comes from [4, Algorithm GKS].

(ii) First, by GAP, we have the following distribution of elements:

	$\mathcal{M}_1$ (1)	$\mathcal{M}_2$ (85)	$\mathcal{M}_3$ (85)	$\mathcal{M}_4$ (120)	$\mathcal{M}_5$ (120)	$\mathcal{M}_6$ (136)	$\mathcal{M}_7$ (136)	$\mathcal{M}_8$ (1360)
cl_{10}	0	0	0	0	0	$1440, P$	$1440, P$	144
cl_8	0	$1440, P$	$1440, P$	2040_2	2040_2	0	0	0

Table 5. Element distribution in $\mathrm{PSp}(4,4).2$

The elements of cl_{10} are partioned among the subgroups in $\mathcal{M}_6$ and $\mathcal{M}_7$ in each class, and each of these classes contains 136 subgroups, so at least 136 subgroups are necessary to cover these elements. On the other hand, no maximal subgroup containing an element of cl_{10} contains an element from cl_8. The most number of elements from cl_8 in a single maximal subgroup is 2040, and each element of cl_8 is contained in exactly two of the 120 subgroups in each of $\mathcal{M}_4$ or $\mathcal{M}_5$. Hence it takes at least 120/2 subgroups to cover these elements, giving a lower bound of $136 + 60 = 196$. On the other hand, using GAP, it can be verified that the minimal normal subgroup in $\mathcal{M}_1$ together with $\mathcal{M}_2$ and $\mathcal{M}_6$ is a cover, giving the upper bound of 222.

(iii) By GAP, we have the following distribution of elements in $HS : 2$:

	$\mathcal{M}_1$ (1)	$\mathcal{M}_2$ (100)	$\mathcal{M}_3$ (1100)	$\mathcal{M}_4$ (1100)	$\mathcal{M}_5$ (3850)
cl_{11}	$8064000, P$	$80640, P$	0	0	0
cl_{30}	0	0	0	$2688, P$	0
$cl_{20,1}$	0	0	0	0	0
$cl_{20,2}$	0	0	0	0	0
cl_{10}	0	$88704, P$	8064	0	2304

Table 6. Element distribution in $HS : 2$

	$\mathcal{M}_6$ (4125)	$\mathcal{M}_7$ (5775)	$\mathcal{M}_8$ (15400)	$\mathcal{M}_9$ (22176)	$\mathcal{M}_{10}$ (36960)
cl_{11}	0	0	0	0	0
cl_{30}	0	0	0	0	80
$cl_{20,1}$	0	$768, P$	0	$200, P$	0
$cl_{20,2}$	0	0	$288, P$	400_2	$120, P$
cl_{10}	0	0	0	$400, P$	0

Table 7. Element distribution in $HS : 2$, cont.

Now, using GAP, the subgroups in classes $\mathcal{M}_2$, $\mathcal{M}_4$, $\mathcal{M}_7$, and $\mathcal{M}_8$ form a cover, giving the upper bound. On the other hand, the elements of cl_{30} are covered by the 1100 maximal subgroups in $\mathcal{M}_4$. At least 5775 different subgroups are needed for $cl_{20,1}$, and the minimal normal subgroup in $\mathcal{M}_1$ is a minimal cover of a class of elements of order 11. At this point, at most 2442000 elements can possibly be covered from $cl_{20,2}$, being $120 \cdot 1100 + 5775 \cdot 400 = 2442000$. Since $15400 \cdot 288 - (120 \cdot 1100 + 5775 \cdot 400) = 1993200$, this leaves at least 1993200 elements still uncovered. The most elements of this class in any maximal subgroup is 400, which means at least an additional 4983 subgroups are required to cover these elements. Since $1100 + 5775 + 1 + 4983 = 11859$, the covering number is bounded below by 11859.

(iv) Using GAP, we obtain the following information about some classes of elements in $(A_{10} \times A_{10}).4$.

	$\mathcal{M}_1$ (1)	$\mathcal{M}_2$ (44100)	$\mathcal{M}_3$ (14400)	$\mathcal{M}_4$ (2025)
cl_{72}	$182891520000, P$	0	0	0
cl_{20}	0	0	0	0
cl_{28}	0	0	$3259200, P$	232243200
$cl_{24,1}$	0	2073600	12700800_2	0
$cl_{24,2}$	0	2073600	12700800_2	0

Table 8. Element distribution in $(A_{10} \times A_{10}).4$

	$\mathcal{M}_5$ (100)	$\mathcal{M}_6$ (893025)	$\mathcal{M}_7$ (15876)	$\mathcal{M}_8$ (6350400)
cl_{72}	0	0	0	0
cl_{20}	0	737280	$41472000, P$	103680
cl_{28}	$4702924800, P$	0	0	0
$cl_{24,1}$	0	307200	0	0
$cl_{24,2}$	0	307200	0	0

TABLE 9. Element distribution in $(A_{10} \times A_{10}).4$, cont.

Using [4, Algorithm GKS], the subgroups in classes $\mathcal{M}_1$, $\mathcal{M}_3$, $\mathcal{M}_5$ and $\mathcal{M}_7$ collectively form a cover, giving the upper bound. On the other hand, the information in Tables 8 and 9 shows the necessity of the subgroup in $\mathcal{M}_1$ to cover the elements in cl_{72}, and it takes at least 15876 additional subgroups to cover cl_{20}. At this point, since

$$14400 \cdot \frac{12700800}{2} - 15876 \cdot 307200 = 86568652800,$$

at least 86568652800 elements from each of $cl_{24,1}$ and $cl_{24,2}$ are still uncovered. Because

$$\frac{86568652800}{12700800} = 6816,$$

at least 6816 subgroups are still needed to cover the elements from these classes. Noting that

$$\left\lceil \frac{470292480000 - 6816 \cdot 32659200}{4702924800} \right\rceil = 53,$$

at least 53 more subgroups from $\mathcal{M}_5$ are needed to cover the elements of cl_{28}, giving a lower bound of 22746.

(v) Using GAP, we have the following information about elements of $\mathrm{PSU}(4,3)$.

	$\mathcal{M}_1$ (112)	$\mathcal{M}_2$ (126)	$\mathcal{M}_3$ (126)	$\mathcal{M}_4$ (162)	$\mathcal{M}_5$ (162)	$\mathcal{M}_6$ (280)	$\mathcal{M}_7$ (540)	$\mathcal{M}_8$ (567)
cl_7	0	0	0	$2880, P$	$2880, P$	0	$864, P$	0
$cl_{9,1}$	$1080, P$	2880_3	0	0	0	$432, P$	0	0
$cl_{9,2}$	$1080, P$	0	2880_3	0	0	$432, P$	0	0
cl_8	0	0	0	0	0	2916_2	1512_2	$720, P$

TABLE 10. Element distribution in $\mathrm{PSU}(4,3)$

	$\mathcal{M}_9$ (567)	$\mathcal{M}_{10}$ (1296)	$\mathcal{M}_{11}$ (1296)	$\mathcal{M}_{12}$ (1296)	$\mathcal{M}_{13}$ (1296)	$\mathcal{M}_{14}$ (2835)	$\mathcal{M}_{15}$ (4536)	$\mathcal{M}_{16}$ (4536)
cl_7	0	360	360	360	360	0	0	0
$cl_{9,1}$	0	0	0	0	0	0	0	0
$cl_{9,2}$	0	0	0	0	0	0	0	0
cl_8	$720, P$	0	0	0	0	144	180	180

TABLE 11. Element distribution in $\mathrm{PSU}(4,3)$, cont.

First, [4, Algorithm GKS] shows that the 442 subgroups in $\mathcal{M}_4$ and $\mathcal{M}_6$ form a cover. On the other hand, the information in Tables 10 and 11 shows that at least 162 subgroups are needed to cover cl_7. Suppose that we use $162 - m$ subgroups from $\mathcal{M}_4$ and $\mathcal{M}_5$ and that we use m_7 subgroups from $\mathcal{M}_7$. This means we use $(162 - m) + m_7$ groups to cover cl_7. This implies that $864m_7 \geqslant 2880m$, that is, this implies that $m \leqslant 3m_7/10$, and so $162 + (m_7 - m) \geqslant 162 + 7m_7/10$. For each group that we use from class $\mathcal{M}_7$, potentially 1512 elements from cl_8 are covered. Since

$$\frac{408240 - 1512m_7}{2916} = 140 - \frac{14m_7}{27},$$

we still need at least $140 - 14m_7/27$ groups to cover cl_8. Noting that

$$\left(162 + \frac{7m_7}{10}\right) + \left(140 - \frac{14m_7}{27}\right) = 302 + \frac{49m_7}{270} \geqslant 302,$$

at least 302 subgroups are required to cover classes cl_7 and cl_8. Since $120960 - 140 \cdot 432 = 60480$, at the very least 60480 of the elements from each of $cl_{9,1}$ and $cl_{9,2}$ are still uncovered. Because $2 \cdot 60480/2880 = 42$, an additional 42 subgroups are needed, and hence at least 344 subgroups are needed to cover $\mathrm{PSU}(4,3)$.

(vi) $\mathrm{PSU}(4,3).2_1$ is the group $\mathrm{U}(4,3).2_1$ in the ATLAS [2]. Using GAP, we have the following information about elements of $\mathrm{PSU}(4,3).2_1$.

	$\mathcal{M}_1$ (1)	$\mathcal{M}_2$ (112)	$\mathcal{M}_3$ (126)	$\mathcal{M}_4$ (126)	$\mathcal{M}_5$ (162)	$\mathcal{M}_6$ (162)	$\mathcal{M}_7$ (280)
cl_{14}	0	0	0	0	$2880, P$	$2880, P$	0
cl_{10}	0	11664_2	$5184, P$	$5184, P$	0	0	0
cl_6	0	0	0	0	0	0	$108, P$

TABLE 12. Element distribution in $\mathrm{PSU}(4,3).2_1$

	$\mathcal{M}_8$ (540)	$\mathcal{M}_9$ (567)	$\mathcal{M}_{10}$ (567)	$\mathcal{M}_{11}$ (2835)	$\mathcal{M}_{12}$ (4536)	$\mathcal{M}_{13}$ (4536)
cl_{14}	$864, P$	0	0	0	0	0
cl_{10}	0	0	0	0	144	144
cl_6	560_{10}	0	0	96_9	0	0

TABLE 13. Element distribution in $\mathrm{PSU}(4,3).2_1$, cont.

First, using GAP, the subgroups in $\mathcal{M}_2$, $\mathcal{M}_5$, $\mathcal{M}_7$ are a cover, giving the upper bound of 554. On the other hand, examining Tables 12 and 13, we see that at least 54 subgroups are needed to cover cl_6. Supposing that 54 subgroups from $\mathcal{M}_8$ are used to cover cl_6, which would be optimal, we would then have covered $864 \cdot 54$ elements from cl_{14}. Since

$$\lceil 864(540 - 54)/2880 \rceil = 146,$$

at least 146 subgroups are still needed to cover cl_{14}. Finally, at least 56 subgroups are still needed to cover cl_{10}, since no subgroup that contains an element of cl_{14} or cl_6 contains an element of cl_{10}. Therefore, at least 256 subgroups are needed to cover $\mathrm{PSU}(4,3).2_1$.

(vii) $\mathrm{PSU}(4,3).2_2$ is the group $\mathrm{U}(4,3).2_2$ in the ATLAS [2]. Using GAP, we have the following information about elements of $\mathrm{PSU}(4,3).2_2$.

	$\mathcal{M}_1$ (1)	$\mathcal{M}_2$ (112)	$\mathcal{M}_3$ (126)	$\mathcal{M}_4$ (126)	$\mathcal{M}_5$ (280)	$\mathcal{M}_6$ (540)
cl_{10}	0	0	$5184, P$	$5184, P$	0	0
cl_{18}	0	$3240, P$	0	2880	1296	0
$cl_{12,1}$	0	$4860, P$	4320	0	0	0
$cl_{12,2}$	0	0	0	4320_2	972	2016
cl_8	0	0	6480_2	0	2916	1512
cl_7	$933120, P$	0	0	0	0	1728

TABLE 14. Element distribution in $\mathrm{PSU}(4,3).2_2$

	$\mathcal{M}_7$ (567)	$\mathcal{M}_8$ (567)	$\mathcal{M}_9$ (1296)	$\mathcal{M}_{10}$ (1296)	$\mathcal{M}_{11}$ (2835)
cl_{10}	0	2304	504	504	0
cl_{18}	0	0	0	0	0
$cl_{12,1}$	960	0	420	420	192
$cl_{12,2}$	0	0	0	0	96
cl_8	720	720	0	0	144
cl_7	0	0	$720, P$	$720, P$	0

TABLE 15. Element distribution in $\mathrm{PSU}(4,3).2_2$, cont.

Using GAP, we see that the subgroups in $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$, and $\mathcal{M}_4$ constitute a cover, demonstrating the upper bound. On the other hand, Tables 14 and 15 show that at least 126 subgroups are needed for cl_{10}. Assume that m_3 subgroups from $\mathcal{M}_3$, m_4 subgroups from $\mathcal{M}_4$, m_8 subgroups from $\mathcal{M}_8$, m_9 subgroups from $\mathcal{M}_9$, and m_{10} subgroups from $\mathcal{M}_{10}$ are used in the cover; thus $m := m_3 + m_4 + m_8 + m_9 + m_{10} \geqslant 126$. Since

$$\frac{112 \cdot 3240 - 2880m_4}{3240} = 112 - \frac{8m_4}{9},$$

at least $112 - 8m_4/9$ subgroups are still needed to cover elements from cl_{18}, and, since

$$\frac{112 \cdot 4860 - 4320m_3 - 420(m_9 + m_{10})}{4860} \geqslant \frac{112 \cdot 4860 - 4320(m - m_4)}{4860} = 112 - \frac{8(m - m_4)}{9},$$

at least $112 - 8(m - m_4)/9$ subgroups are still needed to cover elements from $cl_{12,1}$. Now, it is possible that the elements from classes cl_{18} and $cl_{12,1}$ are covered simultaneously by subgroups from class $\mathcal{M}_2$, so we consider two different cases. Suppose first that $m_4 \geqslant 63$. Then, at least

$$m + (112 - 8(m - m_4)/9) = \frac{m}{9} + 112 + \frac{8m_4}{9} \geqslant 14 + 112 + 56 = 182$$

subgroups are still needed to cover classes cl_{10} and $cl_{12,1}$. On the other hand, suppose $m_4 < 63$. Then, the number of subgroups needed to cover classes cl_{10} and cl_{18} is

$$m + \left(112 - \frac{8m_4}{9}\right) > 126 + 112 - 56 = 182.$$

In any case, we see that at least 182 subgroups are needed to cover classes cl_{10}, cl_{18}, and $cl_{12,1}$, collectively. The class cl_7 has not yet been covered (unless 1296 subgroups from classes $\mathcal{M}_9$ and $\mathcal{M}_{10}$ have been used), so the subgroup in $\mathcal{M}_1$ is still needed. This gives the lower bound of 183.

(viii) $\mathrm{PSU}(4,3).2_3$ is the group $\mathrm{U}(4,3).2_3$ in the ATLAS [2]. Using GAP, we note the following distribution of elements in $\mathrm{PSU}(4,3).2_3$.

	$\mathcal{M}_1$ (1)	$\mathcal{M}_2$ (112)	$\mathcal{M}_3$ (162)	$\mathcal{M}_4$ (162)	$\mathcal{M}_5$ (280)	$\mathcal{M}_6$ (540)
cl_{24}	0	0	0	0	$972, P$	0
cl_{10}	0	0	8064_2	0	0	0
cl_8	0	14580_2	0	10080	0	0

TABLE 16. Element distribution in $\mathrm{PSU}(4,3).2_3$

	$\mathcal{M}_7$ (2835)	$\mathcal{M}_8$ (4536)	$\mathcal{M}_9$ (4536)	$\mathcal{M}_{10}$ (45366)	$\mathcal{M}_{11}$ (8505)
cl_{24}	96	0	0	0	0
cl_{10}	0	144	144	144	0
cl_8	288	0	0	360	96

TABLE 17. Element distribution in $\mathrm{PSU}(4,3).2_3$, cont.

Using GAP, we see that the subgroups in $\mathcal{M}_2$, $\mathcal{M}_3$, and $\mathcal{M}_5$ constitute a cover, giving the upper bound. On the other hand, Tables 16 and 17 show that it takes at least 280 subgroups to cover the elements in cl_{24}. No maximal subgroup that contains elements from cl_{24} contains elements of cl_{10}, so it takes at least $162/2$ subgroups to cover these elements. Finally, because

$$\left\lceil \frac{112 \cdot \frac{14580}{2} - 288 \cdot 280}{14580} \right\rceil = 51,$$

it takes at least an additional 51 subgroups to cover the elements of cl_8. Hence it takes at least 412 groups to cover these three classes. The result follows.

(ix) The upper bound comes from using [4, Algorithm GKS]. On the other hand, using GAP, we have the following distribution of elements in $\mathrm{PSL}(4,3).2$.

	$\mathcal{M}_1$ (1)	$\mathcal{M}_2$ (117)	$\mathcal{M}_3$ (117)	$\mathcal{M}_4$ (130)	$\mathcal{M}_5$ (520)	$\mathcal{M}_6$ (1080)	$\mathcal{M}_7$ (2106)	$\mathcal{M}_8$ (8424)	$\mathcal{M}_9$ (10530)
$cl_{12,1}$	0	0	$4320, P$	$3888, P$	0	0	240	0	48
$cl_{12,2}$	0	$4320, P$	0	$3888, P$	0	0	240	0	48
cl_6	0	0	0	10368_2	$1296, P$	1872	0	240	64
cl_{20}	$606528, P$	0	0	0	0	0	288	0	0
cl_8	0	$6480, P$	$6480, P$	0	2916_2	1404	0	180	0
$cl_{10,1}$	0	10368_2	0	0	0	0	288	0	0
$cl_{10,2}$	0	0	10368_2	0	0	0	288	0	0

TABLE 18. Element distribution in $\mathrm{PSL}(4,3).2$

Assume that $117 - m_{2,3}$ subgroups are used from classes $\mathcal{M}_2$ and $\mathcal{M}_3$ to cover cl_8. In this case, an additional m subgroups from classes $\mathcal{M}_5$, $\mathcal{M}_6$, and $\mathcal{M}_8$ are needed to cover cl_8. Now, $6480 m_{2,3} \leqslant 2916m$, and so $117 - m_{2,3} + m \geqslant 117 + 11m/20$. At this point, we have covered at most $1872m$ elements of cl_6, and so there are $673920 - 1872m$ elements still to cover. Since

$$\frac{673920 - 1872m}{10368} = 65 - \frac{13m}{72},$$

at least $65 - 13m/72$ subgroups are needed to cover the remaining elements of cl_6. So far, we have used at least 182 subgroups, since

$$\left(117 + \frac{11m}{20}\right) + \left(65 - \frac{13m}{72}\right) = 182 + \frac{133m}{360} \geqslant 182.$$

None of the subgroups that contain elements from cl_6 or cl_8 contain elements from cl_{20}, so including the subgroup from $\mathcal{M}_1$ means at least 183 subgroups are needed to cover cl_6, cl_8, and cl_{20}. Of the elements in $cl_{12,1}$ and $cl_{12,2}$, we have covered at most $4320 \cdot 117 + 3888 \cdot 65$, which leaves at least 252720 still uncovered. This means at least an additional $\lceil 252720/4320 \rceil$ more subgroups are needed, and, since $\lceil 252720/4320 \rceil = 59$, we have a lower bound of 242 subgroups.

(x) The upper bound comes from [4, Algorithm GKS]. Using GAP, we have the following distribution of elements in $\mathrm{O}^-(8,2)$.

	$\mathcal{M}_1$ (119)	$\mathcal{M}_2$ (136)	$\mathcal{M}_3$ (765)	$\mathcal{M}_4$ (1071)	$\mathcal{M}_5$ (1632)	$\mathcal{M}_6$ (24192)	$\mathcal{M}_7$ (45696)	$\mathcal{M}_8$ (1175040)
cl_{17}	0	0	0	0	0	$480, P$	0	0
cl_{30}	0	0	0	$6144, P$	0	0	144	0
cl_{21}	0	0	24576_2	0	5760	0	0	0
cl_9	368640_2	$161280, P$	0	0	0	0	0	0
cl_{15}	0	96768_3	0	0	5376	0	96	0

TABLE 19. Element distribution in $\mathrm{O}^-(8,2)$

It is clear from Table 19 that at least $24192 + 1071$ subgroups are needed to cover cl_{17} and cl_{30}. No maximal subgroups that contain elements in cl_{17} or cl_{30} contain elements in cl_{21}, so at least another $\lceil 765/2 \rceil$ are needed. Finally, no subgroup that contains elements in cl_{17}, cl_{30}, or cl_{21} contains elements in cl_9, which takes at least $\lceil 119/2 \rceil$ additional subgroups, giving a lower bound of 25706.

(xi) [4, Algorithm GKS] shows that the covering number of $\mathrm{O}^+(8,2)$ is at most 765. Using GAP, we have the following element distribution.

	$\mathcal{M}_1$ (120)	$\mathcal{M}_2$ (120)	$\mathcal{M}_3$ (120)	$\mathcal{M}_4$ (135)	$\mathcal{M}_5$ (135)	$\mathcal{M}_6$ (135)	$\mathcal{M}_7$ (960)	$\mathcal{M}_8$ (960)
$cl_{15,1}$	0	0	$96768, P$	172032_2	0	0	0	0
$cl_{15,2}$	$96768, P$	0	0	0	0	172032_2	24192	0
$cl_{15,3}$	0	$96768, P$	0	0	172032_2	0	0	24192

TABLE 20. Element distribution in $\mathrm{O}^+(8,2)$

	$\mathcal{M}_9$ (960)	$\mathcal{M}_{10}$ (1120)	$\mathcal{M}_{11}$ (1120)	$\mathcal{M}_{12}$ (1120)	$\mathcal{M}_{13}$ (1575)	$\mathcal{M}_{14}$ (11200)	$\mathcal{M}_{15}$ (12096)	$\mathcal{M}_{16}$ (12096))	$\mathcal{M}_{17}$ (12096))
$cl_{15,1}$	24192	0	10368	0	0	0	0	960	0
$cl_{15,2}$	0	0	0	10368	0	0	0	0	960
$cl_{15,3}$	0	10368	0	0	0	0	960	0	0

TABLE 21. Element distribution in $\mathrm{O}^+(8,2)$, cont.

It is clear from Tables 20 and 21 that no maximal subgroup contains elements from more than one of the classes $cl_{15,1}$, $cl_{15,2}$, or $cl_{15,3}$. A minimal cover for each class consists of at least $\lceil 135/2 \rceil$ subgroups, and so at least 204 subgroups are needed in any cover.

(xii) The upper bound comes from [4, Algorithm GKS]. Using GAP, we have the following distribution of elements.

	$\mathcal{M}_1$ (1)	$\mathcal{M}_2$ (660)	$\mathcal{M}_3$ (660)	$\mathcal{M}_4$ (121)	$\mathcal{M}_5$ (121)	$\mathcal{M}_6$ (144)	$\mathcal{M}_7$ (3025)
cl_{12}	0	0	220_2	0	0	0	24
cl_{22}	0	$60, P$	0	0	0	$275, P$	0
$cl_{6,1}$	0	$220, P$	0	1200_2	1200_2	0	24
$cl_{6,2}$	$220, P$	0	0	0	0	0	4

TABLE 22. Element distribution in $\mathrm{PSL}(2,11)\,\mathrm{wr}\,2$

Examining Table 22, it is clear that at least $660/2 + 144$ subgroups are needed to cover cl_{12} and cl_{22}, since these elements lie in disjoint classes of maximal subgroups. Since

$$\left\lceil \frac{145200 - 144 \cdot 220}{1200} \right\rceil = 95,$$

at least 95 more subgroups needed for $cl_{6,1}$. Finally, not all elements from $cl_{6,2}$ are covered, and so the subgroup from $\mathcal{M}_1$ is needed, giving the lower bound of 570. □

PROPOSITION A.3. *We have the following lower bounds for the indicated covering number values:*

(i) $\sigma(\mathrm{PSL}(5,3)) \geqslant 393030144$*;*
(ii) $\sigma((A_{11} \times A_{11}).4) \geqslant 213444$*;*
(iii) $\sigma(\mathrm{PSL}(7,2)) \geqslant 184308203520$.

PROOF. (i) A Sylow 11-subgroup of PSL(5,3) has order 121 and is cyclic. Using GAP (and/or [1, Tables 8.18-8.19]), there are 8 classes of maximal subgroups, and only one has order divisible by 121 (and hence is the only maximal subgroup containing an element of order 121). A maximal subgroup in this class is isomorphic to $121 : 5$, and the index of one of these groups in G is 393030144. The result follows.

(ii) By [4, Algorithm GKS] (or, more accurately, one iteration of the loop in [4, Algorithm GKS]), there exists a class cl_{60} of elements of order 60 that are distributed as follows.

	$\mathcal{M}_1$ (1)	$\mathcal{M}_2$ (213444)	$\mathcal{M}_3$ (108900)	$\mathcal{M}_4$ (27225)	$\mathcal{M}_5$ (3025)	$\mathcal{M}_6$ (121)	$\mathcal{M}_7$ (131681894400)
cl_{60}	0	124416000	0	0	0	0	0

TABLE 23. Element distribution in $(A_{11} \times A_{11}).4$

The elements in the class cl_{60} are partitioned among the subgroups in $\mathcal{M}_2$, so at least 213444 subgroups are needed.

(iii) This follows immediately from considering the elements of order $2^7 - 1$ and the result of Kantor [6] that only field extension subgroups contain a Singer cycle. □

References

[1] J. N. Bray, D. F. Holt, and C. M. Roney-Dougal. *The maximal subgroups of the low-dimensional finite classical groups.* London Mathematical Society Lecture Note Series, Cambridge University Press, 2013.

[2] J .H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson. *Atlas of Finite Groups* Oxford University Press, 1985.

[3] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.7.5*, 2014.

[4] M. Garonzi, L.-C. Kappe, and E. Swartz. On integers that are covering numbers of groups.

[5] Gurobi Optimizer Reference Manual, Gurobi Optimization, Inc., 2014, (http://www.gurobi.com).

[6] W. M. Kantor. Linear groups containing a Singer cycle. *J. Algebra*, 62:232–234, 1980.

(Garonzi) Departamento de Matemática, Universidade de Brasília, Brasília, DF 70910-900, Brasil
Email address: `mgaronzi@gmail.com`

(Kappe) Department of Mathematical Sciences, State University of New York at Binghamton, Binghamton, NY 13902-6000, USA
Email address: `menger@math.binghamton.edu`

(Swartz) Department of Mathematics, College of William & Mary, P.O. Box 8795, Williamsburg, VA 23187-8795, USA
Email address: `easwartz@wm.edu`