

# Supplementary Online Appendix

to

## Improved Nonparametric Bootstrap Tests of Lorenz Dominance

Zhenting Sun<sup>1</sup> and Brendan K. Beare<sup>2</sup>

<sup>1</sup>National School of Development, Peking University

<sup>2</sup>School of Economics, University of Sydney

July 4, 2019

This supplement to our paper “Improved nonparametric bootstrap tests of Lorenz dominance” is organized as follows. Section A contains proofs of Propositions 3.1, 3.2 and 3.5 appearing in Section 3 of the main paper. Section B contains the results of additional numerical simulations for the matched pairs sampling framework, complementing those given for the independent sampling framework in Section 4 of the main paper. References not given in the main paper are listed at the end of the supplement.

### A Proofs of Propositions 3.1, 3.2 and 3.5

For  $j = 1, 2$  and  $p, t \in (0, 1)$ , define

$$H_{j,p}(t) = \frac{1}{\mu_j} (L_j(p)Q_j(t) - Q_j(t \wedge p)).$$

Note that  $H_{j,p}(\cdot)$  is square-integrable under Assumption 2.1. The following lemma is used in the proofs of Propositions 3.1 and 3.5.

**Lemma A.1.** *Under Assumptions 2.1 and 2.2,  $\mathcal{L}_1$  and  $\mathcal{L}_2$  satisfy  $\text{Var}(\mathcal{L}_1(p)) = \text{Var}(H_{1,p}(U))$ ,  $\text{Var}(\mathcal{L}_2(p)) = \text{Var}(H_{2,p}(V))$ , and*

$$\text{Cov}(\mathcal{L}_1(p), \mathcal{L}_2(p)) = \text{Cov}(H_{1,p}(U), H_{2,p}(V))$$

for each  $p \in (0, 1)$ , where  $(U, V)$  is a pair of random variables whose joint CDF is the copula  $C$ .

*Proof of Lemma A.1.* In view of (2.5), the covariances between the Brownian bridges  $\mathcal{B}_1$  and  $\mathcal{B}_2$  satisfy

$$\text{Cov}(\mathcal{B}_1(s), \mathcal{B}_2(t)) = C(s, t) - st, \quad s, t \in [0, 1], \quad (\text{A.1})$$

where under Assumption 2.2(i) (independent sampling),  $C$  is the product copula, and under Assumption 2.2(ii) (matched pairs),  $C$  is the bivariate copula common to the pairs  $(X_i^1, X_i^2)$ . For  $j = 1, 2$  and  $p, t \in (0, 1)$ , define

$$h_{j,p}(t) = \frac{1}{\mu_j} (L_j(p) - \mathbb{1}(t \leq p)) Q'_j(t).$$

Note that almost sure integrability of  $h_{j,p}(\cdot) \mathcal{B}_j(\cdot)$  follows from the fact that  $n_j^{1/2}(\hat{Q}_j - Q_j) \rightsquigarrow -Q'_j \cdot \mathcal{B}_j$  in  $L^1(0, 1)$ , as shown by Kaji (2018). Note also that  $\mathcal{L}_j(p) = \int_0^1 h_{j,p}(t) \mathcal{B}_j(t) dt$ . Therefore, applying (A.1) and Fubini's theorem, we obtain

$$\text{Cov}(\mathcal{L}_1(p), \mathcal{L}_2(p)) = \int_0^1 \int_0^1 h_{1,p}(s) h_{2,p}(t) (C(s, t) - st) ds dt.$$

The function  $H_{j,p}$  is the antiderivative of  $h_{j,p}$ : it satisfies  $H_{j,p}(\cdot) = \int_0^\cdot h_{j,p} dt$ . A generalization of Hoeffding's lemma due to Lo (2017, Thm. 3.1) – see also Cuadras (2002) and Beare (2009) – thus implies that

$$\text{Cov}(H_{1,p}(U), H_{2,p}(V)) = \int_0^1 \int_0^1 h_{1,p}(s) h_{2,p}(t) (C(s, t) - st) ds dt.$$

This proves our claimed covariance formula. From this we obtain the claimed variance formulas for  $\mathcal{L}_1(p)$  and  $\mathcal{L}_2(p)$  by setting  $F_1 = F_2$  and setting  $C$  equal to the Fréchet-Hoeffding upper bound, so that  $H_{1,p}(U) = H_{2,p}(V)$  almost surely. (Note that the derivation of our covariance formula was valid for any copula  $C$ , including the Fréchet-Hoeffding upper bound.)  $\square$

*Proof of Proposition 3.1.* The CDFs  $F_1$  and  $F_2$  are continuous under Assumption 2.1, so the pair of random variables  $(F_1(X^1), F_2(X^2))$  has joint CDF given by the copula  $C$ . It therefore follows from Lemma A.1 that

$$\text{Var}(\mathcal{L}_j(p)) = \text{Var}(H_{j,p}(F_j(X^j))) = \text{Var}\left(\frac{1}{\mu_j} (L_j(p) X^j - Q_j(p) \wedge X^j)\right)$$

for  $j = 1, 2$ , and

$$\begin{aligned}\text{Cov}(\mathcal{L}_1(p), \mathcal{L}_2(p)) &= \text{Cov}(H_{1,p}(F_1(X^1)), H_{2,p}(F_2(X^2))) \\ &= \text{Cov}\left(\frac{1}{\mu_1}(L_1(p)X^1 - Q_1(p) \wedge X^1), \frac{1}{\mu_2}(L_2(p)X^2 - Q_2(p) \wedge X^2)\right).\end{aligned}$$

The desired result follows easily.  $\square$

For  $\delta > 0$ , define  $B_\delta(\phi) = \{p \in [0, 1] : |\phi(p)| \leq \delta\}$ . The following lemma is used in the proof of Proposition 3.2.

**Lemma A.2.** *Suppose that Assumptions 2.1 and 2.2 are satisfied, and that  $\tau_n \rightarrow \infty$  and  $T_n^{-1/2}\tau_n \rightarrow 0$  as  $n \rightarrow \infty$ . If  $H_0$  is true, then for any  $\delta > 0$  we have*

$$\mathbb{P}\left(B_0(\phi) \subseteq \widehat{B_0(\phi)} \subseteq B_\delta(\phi)\right) \rightarrow 1.$$

*Proof of Lemma A.2.* We first show that  $\sup_{p \in [0, 1]} \hat{V}(p)$  is  $\mathcal{O}_{\text{a.s.}}(1)$ . Since  $\hat{L}_j(p)X_i^j$  and  $\hat{Q}_j(p) \wedge X_i^j$  are both nonnegative and  $|\hat{L}_j(p)| \leq 1$ , we have

$$|\hat{L}_j(p)X_i^j - \hat{Q}_j(p) \wedge X_i^j| \leq (\hat{L}_j(p)X_i^j) \vee (\hat{Q}_j(p) \wedge X_i^j) \leq X_i^j,$$

for  $j = 1, 2$  and  $p \in [0, 1]$ . Using this bound and the strong law of large numbers, it is simple to show that  $\tilde{V}$  satisfies

$$\sup_{p \in [0, 1]} \tilde{V}(p) \leq \frac{T_n}{\hat{\mu}_1^2 n_1^2} \sum_{i=1}^{n_1} (X_i^1)^2 + \frac{T_n}{\hat{\mu}_2^2 n_2^2} \sum_{i=1}^{n_2} (X_i^2)^2 \rightarrow \frac{1-\lambda}{\mu_1^2} \mathbb{E}(X^1)^2 + \frac{\lambda}{\mu_2^2} \mathbb{E}(X^2)^2$$

almost surely under Assumption 2.2(i) (independent sampling), or

$$\begin{aligned}\sup_{p \in [0, 1]} \tilde{V}(p) &\leq \frac{1}{2n} \sum_{i=1}^n \left( \frac{1}{\hat{\mu}_1} X_i^1 + \frac{1}{\hat{\mu}_2} X_i^2 \right)^2 \leq \frac{1}{\hat{\mu}_1^2 n} \sum_{i=1}^n (X_i^1)^2 + \frac{1}{\hat{\mu}_2^2 n} \sum_{i=1}^n (X_i^2)^2 \\ &\rightarrow \frac{1}{\mu_1^2} \mathbb{E}(X^1)^2 + \frac{1}{\mu_2^2} \mathbb{E}(X^2)^2\end{aligned}$$

almost surely under Assumption 2.2(ii) (matched pairs). Thus  $\sup_{p \in [0, 1]} \hat{V}(p)$  is  $\mathcal{O}_{\text{a.s.}}(1)$  as claimed.

Let  $\|\cdot\|$  denote the uniform norm on  $C[0, 1]$ . The set  $\widehat{B_0(\phi)}$  will contain  $B_0(\phi)$  if, for all  $p \in [0, 1]$ , we have  $|T_n^{1/2}(\hat{\phi}(p) - \phi(p))| \leq \tau_n \hat{V}(p)^{1/2}$ . Since  $\hat{V}(p) \geq \nu$ , this will certainly be true if  $\|T_n^{1/2}(\hat{\phi} - \phi)\| \leq \tau_n \nu^{1/2}$ . Therefore,

$$\mathbb{P}\left(B_0(\phi) \subseteq \widehat{B_0(\phi)}\right) \geq \mathbb{P}\left(\|T_n^{1/2}(\hat{\phi} - \phi)\| \leq \tau_n \nu^{1/2}\right) \rightarrow 1,$$

with the convergence to one following from the fact that  $\tau_n \nu^{1/2} \rightarrow \infty$  while  $T_n^{1/2}(\hat{\phi} - \phi) \rightsquigarrow \bar{\mathcal{L}}$  in  $C[0, 1]$  by Lemma 2.1.

Let  $\varepsilon_n = T_n^{-1/2} \tau_n \sup_{p \in [0, 1]} \hat{V}(p)$ . Since  $\sup_{p \in [0, 1]} \hat{V}(p)$  is  $\mathcal{O}_{\text{a.s.}}(1)$ , we have  $\varepsilon_n \rightarrow 0$  almost surely. The set  $B_\delta(\phi)$  will contain  $\widehat{B_0(\phi)}$  if  $|\phi(p)| \leq \delta$  whenever  $|T_n^{1/2} \hat{\phi}(p)| \leq \tau_n \hat{V}(p)^{1/2}$ . If  $\varepsilon_n < \delta$ , then this occurs if  $\hat{\phi}$  is everywhere within  $\delta - \varepsilon_n$  of  $\phi$ . Since  $\varepsilon_n \rightarrow 0$  almost surely, for sufficiently large  $n$  we therefore have

$$\mathbb{P}\left(\widehat{B_0(\phi)} \subseteq B_\delta(\phi)\right) \geq \mathbb{P}\left(\|T_n^{1/2}(\hat{\phi} - \phi)\| \leq T_n^{1/2}(\delta - \varepsilon_n)\right) \rightarrow 1,$$

with the convergence to one following from the fact that  $T_n^{1/2}(\delta - \varepsilon_n) \rightarrow \infty$  almost surely while  $T_n^{1/2}(\hat{\phi} - \phi) \rightsquigarrow \bar{\mathcal{L}}$  in  $C[0, 1]$  by Lemma 2.1.  $\square$

*Proof of Proposition 3.2.* It is easy to see that our estimated functionals satisfy the Lipschitz conditions

$$|\widehat{\mathcal{S}'_\phi}(h_1) - \widehat{\mathcal{S}'_\phi}(h_2)| \leq \|h_1 - h_2\| \quad \text{and} \quad |\widehat{\mathcal{I}'_\phi}(h_1) - \widehat{\mathcal{I}'_\phi}(h_2)| \leq \|h_1 - h_2\|$$

for  $h_1, h_2 \in C[0, 1]$ . Therefore, Lemma S.3.6 of Fang and Santos (2019) implies that a sufficient condition for  $\widehat{\mathcal{S}'_\phi}$  and  $\widehat{\mathcal{I}'_\phi}$  to satisfy their Assumption 4 is that, for any  $\epsilon > 0$ ,

$$\mathbb{P}\left(\left|\widehat{\mathcal{S}'_\phi}(h) - \mathcal{S}'_\phi(h)\right| > \epsilon\right) \rightarrow 0 \quad \text{and} \quad \mathbb{P}\left(\left|\widehat{\mathcal{I}'_\phi}(h) - \mathcal{I}'_\phi(h)\right| > \epsilon\right) \rightarrow 0 \quad (\text{A.2})$$

for each  $h \in C[0, 1]$ . Moreover, since  $n^{1/2}(\hat{\phi}^* - \hat{\phi})$  is a Borel measurable map into the separable space  $C[0, 1]$ , Assumption 4 of Fang and Santos (2019) is equivalent to our Assumption 3.1; see their Remark 3.3. Thus we need only verify (A.2).

To verify the first part of (A.2), fix  $h \in C[0, 1]$  and  $\epsilon > 0$ , and choose  $\delta > 0$  small enough that  $|h(p) - h(q)| < \epsilon$  whenever  $|p - q| < \delta$ . Next choose  $\eta > 0$  small enough that  $|p - q| < \delta$  for some  $q \in B_0(\phi)$  whenever  $p \in B_\eta(\phi)$ . Observe that if

$$B_0(\phi) \subseteq \widehat{B_0(\phi)} \subseteq B_\eta(\phi),$$

then it must be the case that

$$\left|\widehat{\mathcal{S}'_\phi}(h) - \mathcal{S}'_\phi(h)\right| \leq \sup_{p \in B_\eta(\phi)} h(p) - \sup_{p \in B_0(\phi)} h(p) \leq \epsilon.$$

The first part of (A.2) now follows from Lemma A.2.

To verify the second part of (A.2), fix  $h \in C[0, 1]$  and  $\epsilon > 0$ , and choose  $\delta > 0$  small

enough that

$$\|h\| \int_0^1 \mathbb{1}(0 < |\phi(p)| \leq \delta) dp \leq \epsilon,$$

which is possible by the dominated convergence theorem. Observe that if

$$B_0(\phi) \subseteq \widehat{B_0(\phi)} \subseteq B_\delta(\phi),$$

then it must be the case that

$$\left| \widehat{\mathcal{I}'_\phi}(h) - \mathcal{I}'_\phi(h) \right| \leq \|h\| \int_0^1 \mathbb{1}(0 < |\phi(p)| \leq \delta) dp \leq \epsilon.$$

The second part of (A.2) now follows from Lemma A.2.  $\square$

The following lemma is used in the proof of Proposition 3.5.

**Lemma A.3.** *Under Assumptions 2.1 and 2.2, the variance of  $\bar{\mathcal{L}}(p)$  is strictly positive for all  $p \in (0, 1)$ .*

*Proof.* Under Assumption 2.2(i) (independent sampling)  $\mathcal{L}_1(p)$  and  $\mathcal{L}_2(p)$  are independent. Therefore, since each of them has strictly positive variance for  $p \in (0, 1)$  by Lemma A.1, their weighted difference  $\bar{\mathcal{L}}(p)$  trivially also has strictly positive variance. It remains to establish that  $\bar{\mathcal{L}}(p)$  has strictly positive variance under Assumption 2.2(ii) (matched pairs). Since  $C$  has maximal correlation strictly less than one, we must have

$$\text{Cov}(H_{1,p}(U), H_{2,p}(V)) < \sqrt{\text{Var}(H_{1,p}(U))\text{Var}(H_{2,p}(V))}. \quad (\text{A.3})$$

We thus deduce from Lemma A.1 that

$$\text{Cov}(\mathcal{L}_1(p), \mathcal{L}_2(p)) < \sqrt{\text{Var}(\mathcal{L}_1(p))\text{Var}(\mathcal{L}_2(p))}, \quad (\text{A.4})$$

meaning that the correlation between  $\mathcal{L}_1(p)$  and  $\mathcal{L}_2(p)$  must be strictly less than one. The weighted difference  $\bar{\mathcal{L}}(p)$  therefore cannot have zero variance.  $\square$

*Proof of Proposition 3.5.* We first observe that since  $\bar{\mathcal{L}}$  is Gaussian and the directional derivatives  $\mathcal{S}'_\phi$  and  $\mathcal{I}'_\phi$  are continuous and convex, Theorem 11.1 of Davydov et al. (1998) can be used to show that the CDFs of  $\mathcal{S}'_\phi(\bar{\mathcal{L}})$  and  $\mathcal{I}'_\phi(\bar{\mathcal{L}})$  are continuous everywhere except perhaps at zero, and that if either CDF assigns probability less than one to zero, then it is strictly increasing on  $(0, \infty)$ . Thus if either CDF is not continuous and strictly increasing at its  $1 - \alpha$  quantile, then it must assign probability of at least  $1 - \alpha$  to zero.

To demonstrate claim (a), observe that if the set  $\Psi(\phi)$  includes some point  $p_0 \notin \{0, 1\}$ , then

$$\mathbb{P}(\mathcal{S}'_\phi(\bar{\mathcal{L}}) > 0) \geq \mathbb{P}(\bar{\mathcal{L}}(p_0) > 0) = \frac{1}{2},$$

with the final equality following from Lemma A.3 and the fact that  $\bar{\mathcal{L}}(p_0)$  is a centered Gaussian random variable. Thus the CDF of  $\mathcal{S}'_\phi(\bar{\mathcal{L}})$  can assign probability of no greater than one half to zero. Since  $1 - \alpha > 1/2$ , we conclude that the CDF must be continuous and strictly increasing at its  $1 - \alpha$  quantile. On the other hand, if  $\Psi(\phi)$  does not include any point  $p_0 \notin \{0, 1\}$ , then clearly  $\mathcal{S}'_\phi(\bar{\mathcal{L}})$  is degenerate at zero.

To demonstrate claim (b), suppose that  $\mathcal{I}'_\phi(\bar{\mathcal{L}})$  is not degenerate at zero. Since we have assumed  $H_0$  to be satisfied, we have  $B_+(\phi) = \emptyset$ , and so  $B_0(\phi)$  must be a set of positive measure. Thus the Lebesgue density theorem ensures the existence of  $p_0 \in B_0 \cap (0, 1)$  such that the set  $(p_0 - \epsilon, p_0 + \epsilon) \cap B_0$  has positive measure for all  $\epsilon > 0$ . Since  $\bar{\mathcal{L}}(p)$  is continuous in  $p$ , if  $\bar{\mathcal{L}}(p_0) > 0$  then we must have  $\bar{\mathcal{L}} > 0$  on  $(p_0 - \epsilon, p_0 + \epsilon)$  for some  $\epsilon > 0$ , implying that

$$\mathcal{I}'_\phi(\bar{\mathcal{L}}) \geq \int_{(p_0 - \epsilon, p_0 + \epsilon) \cap B_0} \bar{\mathcal{L}}(p) dp > 0.$$

Thus we have

$$\mathbb{P}(\mathcal{I}'_\phi(\bar{\mathcal{L}}) > 0) \geq \mathbb{P}(\bar{\mathcal{L}}(p_0) > 0) = \frac{1}{2},$$

with the final equality following from Lemma A.3 and the fact that  $\bar{\mathcal{L}}(p_0)$  is a centered Gaussian random variable. Thus the CDF of  $\mathcal{I}'_\phi(\bar{\mathcal{L}})$  can assign probability of no greater than one half to zero, and since  $1 - \alpha > 1/2$ , we conclude that the CDF must be continuous and strictly increasing at its  $1 - \alpha$  quantile.  $\square$

## B Further numerical simulations

The numerical simulations reported in Section 4 pertained to the independent sampling framework. Here we report analogous simulations for the matched pairs sampling framework. The simulation design is the same as described in Section 4.1, except that dependence between pairs was induced by linking the random variables  $X_i^1$  and  $X_i^2$  with a Gaussian copula with parameter  $\rho = 0.25, 0.5, 0.75$ . In Tables B.1, B.2 and B.3 we report results analogous to those reported in Table 4.1, and in Figure B.1 we report results analogous to those reported in Figure 4.2. Qualitatively, the results for the matched pairs sampling framework are similar to those for the independent sampling framework.

		$\beta$							
$\alpha$	$\tau_n$	1	2	3	4	5	6	7	8
$\mathcal{F} = \mathcal{S}$	2	1	<b>12.6</b>	<b>12.5</b>	<b>12.5</b>	<b>12.5</b>	<b>12.5</b>	<b>12.5</b>	<b>12.4</b>
		2	5.1	4.9	4.9	4.8	4.8	4.8	4.8
		3	5.1	4.9	4.9	4.8	4.8	4.8	4.8
		4	5.1	4.9	4.9	4.8	4.8	4.8	4.8
		$\infty$	5.1	4.9	4.9	4.8	4.8	4.8	4.8
	3	1	<b>10.3</b>	<b>10.8</b>	<b>10.8</b>	<b>10.9</b>	<b>10.8</b>	<b>10.8</b>	<b>10.7</b>
		2	4.8	4.6	4.6	4.5	4.6	4.6	4.7
		3	4.8	4.6	4.5	4.5	4.6	4.6	4.6
		4	4.8	4.6	4.5	4.5	4.6	4.6	4.6
		$\infty$	4.8	4.6	4.5	4.5	4.6	4.6	4.6
	4	1	<b>8.8</b>	<b>9.5</b>	<b>9.9</b>	<b>10.0</b>	<b>10.1</b>	<b>10.1</b>	<b>10.3</b>
		2	4.8	4.7	4.7	4.7	4.7	4.7	4.7
		3	4.7	4.7	4.7	4.7	4.7	4.7	4.6
		4	4.7	4.7	4.7	4.7	4.7	4.7	4.6
		$\infty$	4.7	4.7	4.7	4.7	4.7	4.7	4.6
	5	1	<b>9.4</b>	<b>9.3</b>	<b>9.3</b>	<b>9.6</b>	<b>9.7</b>	<b>9.8</b>	<b>10.0</b>
		2	4.9	4.6	4.7	4.7	4.7	4.6	4.7
		3	4.8	4.5	4.6	4.7	4.6	4.6	4.6
		4	4.8	4.5	4.6	4.7	4.6	4.6	4.6
		$\infty$	4.8	4.5	4.6	4.7	4.6	4.6	4.6
$\mathcal{F} = \mathcal{I}$	2	1	<b>18.3</b>	<b>17.9</b>	<b>17.8</b>	<b>17.9</b>	<b>17.8</b>	<b>17.9</b>	<b>17.9</b>
		2	<b>6.0</b>	<b>5.7</b>	<b>5.7</b>	<b>5.6</b>	<b>5.6</b>	5.5	<b>5.5</b>
		3	5.5	5.4	5.3	5.4	5.4	5.3	5.3
		4	5.5	5.4	5.3	5.4	5.4	5.3	5.3
		$\infty$	5.5	5.4	5.3	5.4	5.4	5.3	5.3
	3	1	<b>17.4</b>	<b>17.6</b>	<b>17.3</b>	<b>17.5</b>	<b>17.6</b>	<b>17.6</b>	<b>17.6</b>
		2	<b>5.9</b>	<b>5.5</b>	<b>5.5</b>	<b>5.4</b>	<b>5.4</b>	<b>5.3</b>	<b>5.3</b>
		3	5.2	5.0	4.9	4.9	4.9	4.9	4.9
		4	5.2	4.9	4.9	4.9	4.9	4.9	4.9
		$\infty$	5.2	4.9	4.9	4.9	4.9	4.9	4.9
	4	1	<b>16.3</b>	<b>16.7</b>	<b>16.6</b>	<b>16.7</b>	<b>16.8</b>	<b>16.7</b>	<b>16.8</b>
		2	<b>5.8</b>	<b>5.7</b>	<b>5.7</b>	<b>5.7</b>	<b>5.7</b>	<b>5.7</b>	<b>5.7</b>
		3	4.9	5.1	5.0	5.1	5.1	5.2	5.2
		4	4.9	5.1	5.0	5.1	5.1	5.2	5.2
		$\infty$	4.9	5.1	5.0	5.1	5.1	5.2	5.2
	5	1	<b>16.7</b>	<b>16.5</b>	<b>16.8</b>	<b>16.9</b>	<b>17.0</b>	<b>17.0</b>	<b>17.0</b>
		2	<b>5.9</b>	<b>5.8</b>	<b>5.8</b>	<b>5.8</b>	<b>5.8</b>	<b>5.7</b>	<b>5.6</b>
		3	5.1	5.1	5.0	5.1	5.2	5.2	5.1
		4	5.1	5.1	5.0	5.1	5.1	5.1	5.1
		$\infty$	5.1	5.1	5.0	5.1	5.1	5.1	5.1

**Table B.1:** Null rejection rates with  $X^1 \sim X^2 \sim \text{dP}(\alpha, \beta)$  and  $n = 2000$  matched pairs linked by a Gaussian copula with correlation parameter  $\rho = 0.25$ . Rejection rates are in bold when they exceed the corresponding rate obtained with  $\tau_n = \infty$  by more than 0.1 percentage point.

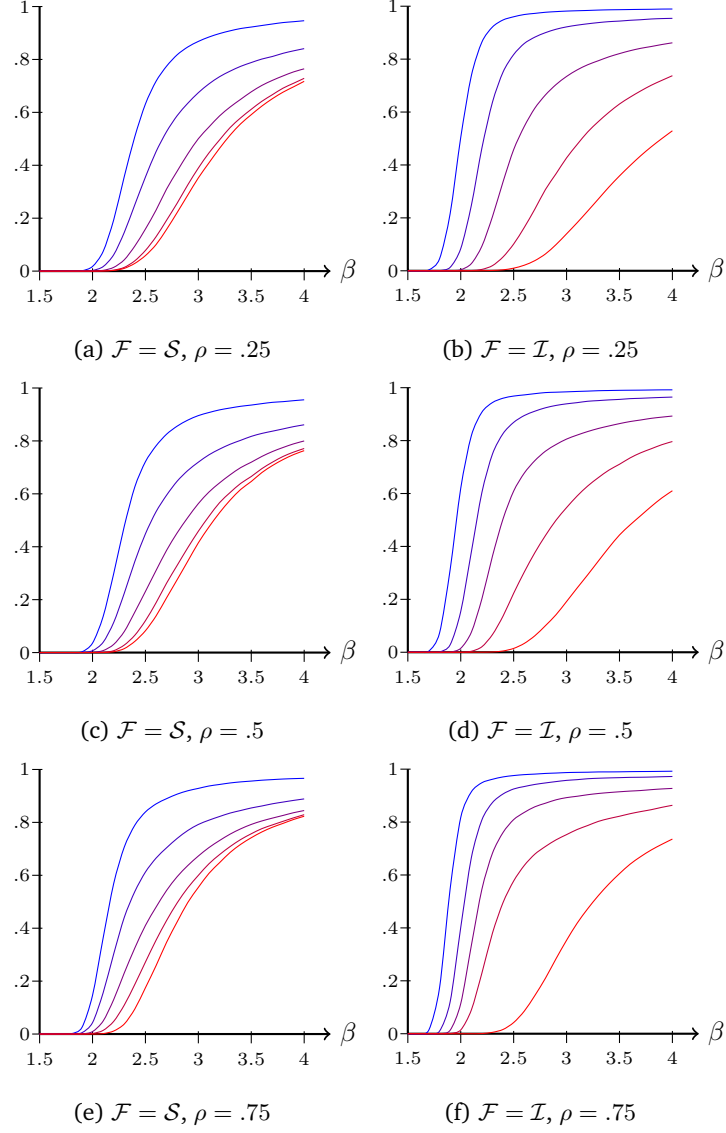
$\alpha$		$\beta$							
$\alpha$	$\tau_n$	1	2	3	4	5	6	7	8
$\mathcal{F} = \mathcal{S}$	1	<b>12.9</b>	<b>12.8</b>	<b>12.7</b>	<b>12.6</b>	<b>12.6</b>	<b>12.7</b>	<b>12.7</b>	<b>12.7</b>
	2	5.7	5.5	5.4	5.4	5.4	5.4	5.4	5.4
	3	5.7	5.5	5.4	5.4	5.4	5.4	5.4	5.4
	4	5.7	5.5	5.4	5.4	5.4	5.4	5.4	5.4
	$\infty$	5.7	5.5	5.4	5.4	5.4	5.4	5.4	5.4
	1	<b>10.3</b>	<b>10.4</b>	<b>10.4</b>	<b>10.4</b>	<b>10.4</b>	<b>10.5</b>	<b>10.5</b>	<b>10.4</b>
	2	4.8	4.7	4.6	4.7	4.8	4.7	4.7	4.7
	3	4.7	4.7	4.6	4.7	4.8	4.7	4.7	4.7
	4	4.7	4.7	4.6	4.7	4.8	4.7	4.7	4.7
	$\infty$	4.7	4.7	4.6	4.7	4.8	4.7	4.7	4.7
	1	<b>9.2</b>	<b>9.3</b>	<b>9.4</b>	<b>9.6</b>	<b>9.7</b>	<b>9.7</b>	<b>9.7</b>	<b>9.7</b>
	2	5.0	4.9	4.9	4.8	4.8	4.8	4.8	4.8
	3	5.0	4.9	4.9	4.8	4.8	4.8	4.8	4.8
	4	5.0	4.9	4.9	4.8	4.8	4.8	4.8	4.8
	$\infty$	5.0	4.9	4.9	4.8	4.8	4.8	4.8	4.8
	1	<b>8.5</b>	<b>8.8</b>	<b>9.0</b>	<b>9.3</b>	<b>9.4</b>	<b>9.5</b>	<b>9.5</b>	<b>9.5</b>
	2	4.8	4.8	4.7	4.7	4.7	4.8	4.8	4.7
	3	4.7	4.7	4.7	4.7	4.7	4.7	4.7	4.7
	4	4.7	4.7	4.7	4.7	4.7	4.7	4.7	4.7
	$\infty$	4.7	4.7	4.7	4.7	4.7	4.7	4.7	4.7
$\mathcal{F} = \mathcal{I}$	1	<b>18.9</b>	<b>18.7</b>	<b>18.8</b>	<b>18.7</b>	<b>18.7</b>	<b>18.7</b>	<b>18.6</b>	<b>18.6</b>
	2	<b>6.3</b>	<b>6.2</b>	<b>6.1</b>	<b>6.0</b>	<b>6.0</b>	<b>6.0</b>	5.9	5.9
	3	6.0	5.9	5.8	5.8	5.8	5.8	5.8	5.8
	4	6.0	5.9	5.8	5.8	5.8	5.8	5.8	5.8
	$\infty$	6.0	5.9	5.8	5.8	5.8	5.8	5.8	5.8
	1	<b>17.2</b>	<b>17.2</b>	<b>17.1</b>	<b>17.1</b>	<b>17.1</b>	<b>17.0</b>	<b>17.0</b>	<b>17.0</b>
	2	<b>6.1</b>	<b>5.9</b>	<b>5.9</b>	<b>5.9</b>	<b>5.6</b>	<b>5.5</b>	<b>5.5</b>	<b>5.5</b>
	3	5.2	5.3	5.3	5.2	5.1	5.1	5.0	5.1
	4	5.2	5.3	5.3	5.2	5.1	5.1	5.0	5.1
	$\infty$	5.2	5.3	5.3	5.2	5.1	5.1	5.0	5.1
	1	<b>16.7</b>	<b>16.7</b>	<b>16.8</b>	<b>16.7</b>	<b>16.7</b>	<b>16.6</b>	<b>16.7</b>	<b>16.6</b>
	2	<b>6.2</b>	<b>6.2</b>	<b>5.9</b>	<b>6.0</b>	<b>5.9</b>	<b>5.7</b>	<b>5.8</b>	<b>5.7</b>
	3	5.3	5.3	5.1	5.1	5.2	5.2	5.2	5.1
	4	5.2	5.3	5.1	5.1	5.2	5.2	5.2	5.1
	$\infty$	5.2	5.3	5.1	5.1	5.2	5.2	5.2	5.1
	1	<b>16.0</b>	<b>16.5</b>	<b>16.1</b>	<b>16.0</b>	<b>16.2</b>	<b>16.2</b>	<b>16.2</b>	<b>16.3</b>
	2	<b>5.9</b>	<b>5.9</b>	<b>5.7</b>	<b>5.8</b>	<b>5.6</b>	<b>5.8</b>	<b>5.8</b>	<b>5.8</b>
	3	5.0	4.9	4.9	5.0	5.0	5.0	5.1	5.1
	4	5.0	4.9	4.9	5.0	5.0	5.0	5.1	5.1
	$\infty$	5.0	4.9	4.9	5.0	5.0	5.0	5.1	5.1

**Table B.2:** Null rejection rates with  $X^1 \sim X^2 \sim \text{dP}(\alpha, \beta)$  and  $n = 2000$  matched pairs linked by a Gaussian copula with correlation parameter  $\rho = 0.50$ . Rejection rates are in bold when they exceed the corresponding rate obtained with  $\tau_n = \infty$  by more than 0.1 percentage point.



		$\beta$							
$\alpha$	$\tau_n$	1	2	3	4	5	6	7	8
$\mathcal{F} = \mathcal{S}$	1	1	<b>13.2</b>	<b>13.0</b>	<b>12.9</b>	<b>12.9</b>	<b>12.9</b>	<b>12.9</b>	<b>12.9</b>
		2	5.6	5.5	5.4	5.4	5.3	5.3	5.3
		3	5.5	5.5	5.4	5.3	5.3	5.3	5.3
		4	5.5	5.5	5.4	5.3	5.3	5.3	5.3
		$\infty$	5.5	5.5	5.4	5.3	5.3	5.3	5.3
	2	1	<b>10.1</b>	<b>10.0</b>	<b>10.0</b>	<b>10.1</b>	<b>10.1</b>	<b>10.2</b>	<b>10.1</b>
		2	4.8	4.7	4.7	4.6	4.6	4.6	4.6
		3	4.8	4.7	4.7	4.6	4.6	4.6	4.6
		4	4.8	4.7	4.7	4.6	4.6	4.6	4.6
		$\infty$	4.8	4.7	4.7	4.6	4.6	4.6	4.6
	3	1	<b>9.0</b>	<b>9.5</b>	<b>9.8</b>	<b>9.9</b>	<b>10.0</b>	<b>10.1</b>	<b>10.0</b>
		2	4.9	4.9	5.0	5.0	4.9	5.0	4.9
		3	4.9	4.9	4.9	5.0	4.9	4.9	4.9
		4	4.9	4.9	4.9	5.0	4.9	4.9	4.9
		$\infty$	4.9	4.9	4.9	5.0	4.9	4.9	4.9
	4	1	<b>8.3</b>	<b>8.4</b>	<b>8.6</b>	<b>8.8</b>	<b>9.0</b>	<b>9.2</b>	<b>9.3</b>
		2	4.8	4.7	4.7	4.7	4.6	4.7	4.8
		3	4.7	4.6	4.7	4.6	4.6	4.7	4.8
		4	4.7	4.6	4.7	4.6	4.6	4.7	4.8
		$\infty$	4.7	4.6	4.7	4.6	4.6	4.7	4.8
$\mathcal{F} = \mathcal{I}$	1	1	<b>19.3</b>	<b>19.0</b>	<b>18.9</b>	<b>18.8</b>	<b>18.9</b>	<b>18.8</b>	<b>18.8</b>
		2	<b>6.6</b>	<b>6.2</b>	<b>6.0</b>	<b>5.9</b>	<b>5.8</b>	<b>5.8</b>	<b>5.7</b>
		3	6.1	5.8	5.6	5.6	5.6	5.5	5.5
		4	6.1	5.8	5.6	5.6	5.6	5.5	5.5
		$\infty$	6.1	5.8	5.6	5.6	5.6	5.5	5.4
	2	1	<b>17.0</b>	<b>16.9</b>	<b>17.3</b>	<b>17.3</b>	<b>17.3</b>	<b>17.2</b>	<b>17.2</b>
		2	<b>6.3</b>	<b>5.8</b>	<b>5.7</b>	<b>5.6</b>	<b>5.4</b>	<b>5.4</b>	<b>5.4</b>
		3	5.4	5.0	5.0	5.0	4.9	5.0	5.0
		4	5.3	5.0	5.0	5.0	4.9	5.0	5.0
		$\infty$	5.3	5.0	5.0	5.0	4.9	5.0	5.0
	3	1	<b>17.1</b>	<b>17.5</b>	<b>17.3</b>	<b>17.3</b>	<b>17.3</b>	<b>17.6</b>	<b>17.5</b>
		2	<b>6.5</b>	<b>6.5</b>	<b>6.3</b>	<b>6.2</b>	<b>6.2</b>	<b>6.1</b>	<b>6.1</b>
		3	5.4	5.5	5.5	5.5	5.3	5.3	5.3
		4	5.4	5.5	5.5	5.4	5.3	5.3	5.3
		$\infty$	5.4	5.5	5.5	5.4	5.3	5.3	5.3
	4	1	<b>16.2</b>	<b>16.3</b>	<b>16.5</b>	<b>16.6</b>	<b>16.7</b>	<b>16.4</b>	<b>16.4</b>
		2	<b>6.0</b>	<b>5.9</b>	<b>5.9</b>	<b>5.9</b>	<b>5.9</b>	<b>5.8</b>	<b>5.7</b>
		3	5.0	4.9	4.9	5.0	5.0	5.0	5.0
		4	5.0	4.9	4.9	5.0	5.0	5.0	5.0
		$\infty$	5.0	4.9	4.9	5.0	5.0	5.0	5.0

**Table B.3:** Null rejection rates with  $X^1 \sim X^2 \sim \text{dP}(\alpha, \beta)$  and  $n = 2000$  matched pairs linked by a Gaussian copula with correlation parameter  $\rho = 0.75$ . Rejection rates are in bold when they exceed the corresponding rate obtained with  $\tau_n = \infty$  by more than 0.1 percentage point.



**Figure B.1:** Power with  $X^1 \sim \text{dP}(3, 1.5)$  and  $X^2_{(\beta)} \sim \text{dP}(2.1, \beta)$  as a function of the parameter  $\beta$ . Going from top to bottom in each panel, the five power curves correspond to our test with  $\tau_n = 1, 2, 3, 4$ , and the test of Barrett et al. (2014). Samples are  $n = 2000$  matched pairs linked by a Gaussian copula with correlation parameter  $\rho$ .

## Additional references

- BEARE, B. K. (2009). A generalization of Hoeffding's lemma, and a new class of covariance inequalities. *Statistics and Probability Letters*, 79(5):637–642.
- CUADRAS, C. M. (2002). On the covariance between functions. *Journal of Multivariate Analysis*, 81(1):19–27.
- DAVYDOV, Y. A., LIFSHITS, M. A. AND SMORODINA, N. V. (1998). *Local Properties of Distributions of Stochastic Functionals*. American Mathematical Society, Providence.
- LO, A. (2017). Functional generalizations of Hoeffding's covariance lemma and a formula for Kendall's tau. *Statistics and Probability Letters*, 122:218–226.