**Online Supplement for “****Higher-Order Normal Approximation Approach for Highly Reliable System Assessment” by “Zhaohui Li, Dan Yu, Jian Liu, Qingpei Hu”**

This supplementary material is organized as below. Appendix A contains the necessary proofs for theorem 3.1 (appendix A1) and theorem 3.2 (appendix A2). Appendix B provides the characteristic quantities which are indispensable when calculating the R-WCF expansion. Appendix C proposes the characteristic quantities for Weibull model.

**Appendix A.1: Proof of Theorem 3.1**

To prove Theorem3.1, we first define the concept of the inseparable cut set.

**Definition A.1**: Suppose random variables satisfy . The joint cumulant of is

where means ’s k-set partition.

The k-th order cumulant of is defined by with .

**Definition A.2**: Suppose is a collection of arrays, which can be listed as:

(A.1)

A cut set, is called inseparable if, for any proper subset U of , cannot be represented by the combined rows of (A.1)

The proof of Theorem 2.1 requires a lemma and a theorem proposed by Brillinger (1980).

**Theorem A.1** (Brillinger 1980).Denote , then

, (A.2)

where means sum by all inseparable cut .

**Lemma A.1.** Denote , and suppose . Then, .

Proof: For any inseparable cut satisfying , based on Theorem A.1, the terms in (A.2) corresponding to and denoted by

have the order . □

**Lemma A.2**. Suppose is an inseparable cut of . Denote as the set in which we delete all rows satisfying . Then is an inseparable cut of .

**Lemma A.3.** Suppose is an inseparable cut of . Then, is an inseparable cut of . Here .

Lemma A.2 and A.3 directly follow via proof by contradiction and Theorem A.1.

**Lemma A.4:** Suppose , is an inseparable cut of . If , then .

**Proof:** The conclusion is trivial with. Suppose the lemma is true with . We prove that it is true for . Suppose for there exists a cut satisfying the condition of Lemma A.4, . We may set with Lemma A.3. Denote as the set of all rows deleted, satisfying. Using Lemma A.2, is an inseparable cut of . Because , there exists a subset satisfying .=1. Suppose. Together with condition , , elements in are rows in with one column. Delete from , and denote as a subset of  and . Then, is an inseparable cut. Hence, the contradiction. □

**Proof of Theorem 3.1**: Suppose and , if . It can be inferred from Lemma A.1.

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If , then for any inseparable cut of denoted by , it follows from lemma A.4 that . Based on theorem A.1, the order of every term is at most . □

**Appendix A.2 Proof of Theorem 3.2**.

Let denote a vector of components’ reliability. As shown in Section 3.1 the derivation of (4), we only need to prove ’s cumulants satisfying (4), then

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Recall the system structure function:

where are point estimations of the system reliability and component reliability. It immediately follows the assumption that are independent. From Theorems 3.1 and 3.2, as well as the Taylor expansion of  over , the cumulant of satisfying the first cumulant of is , and the k-th cumulant is . Because is a smooth function, the Taylor expansion of over can be expressed by:

where denotes the vector . It is easy to prove that has the same cumulant property as ; i.e., the first cumulant of is , and the k-th cumulant is .

Coefficients are computed from (8), where . The polynomial satisfies .

**Appendix B: Character quantities for log-location-scale family**

The calculation of character quantities for an arbitrary system consisting of components with a log location scale class is given here. Recall equation (6),

The derivation requires the cumulant property of , retaining only the terms of order of . We omit the subscript *i* for different components because there is no difference in computation. Similarly, we have:

The only undetermined components in (7) are , :

**Appendix C: Character quantities for Weibull distribution.**

The moments of a standard extreme value distribution are obtained by numerical calculation:

where is the Euler constant. We omit subscript *i*, which denotes the coordinate of components and

Similar to Appendix B, let , in which denote , respectively. Deviations of are easily obtained through any software with symbolic computation or through direct calculation. Then