Appendix. A. Summary of the Notation

Table 8 Summary of the notation

$\mathcal{I}$	Set of patients
k	Health states $\{0,1\}$
t	Periods
T	Final period
N	Number of patients
β	Discounting factor
l	Type of appointments
o	Office appointments
v	Virtual appointments
$\mathcal{P}$	Disease progression matrix
$\mathcal{Q}_o$	Treatment matrix for OAs
$\mathcal{Q}_v$	Treatment matrix for VAs
p	Probability of staying in the controlled health state at the beginning of period $t+1$, if a patient is in the controlled health state at the beginning of period t
q_o	Probability of being in the controlled health state after an OA at the end of period t, if a patient is in the uncontrolled health state at the beginning of period t
q_v	Probability of being in the controlled health state after a VA at the end of period t, if a patient is in the uncontrolled health state at the beginning of period t
$\pi_{i,t}$	Probability of patient i being in the controlled health state at the beginning of period t
e^k	Diagnosis vector for health state k
$x^l_{i,t}$	1 if a patient $i \in \mathcal{I}$ is scheduled for an appointment type $l \in \{v, o\}$ in period t, 0 otherwise.
C^l	Capacity for appointment type l
$\mu^*_t(\pi_{1,t},\ldots,\pi_{N,t})$	The value function in period t
$\overline{\pi}_{i,t}$	Average of initial probabilities of patients being in the controlled health state
$\Delta_{\pi_{i,t}}$	Gap between maximum and minimum initial probabilities of patients being in the controlled health state
n_v	Minimum required number of VAs to substitute for one additional OA
n	Number of baseline OAs.

Appendix. B. Tables and Figures

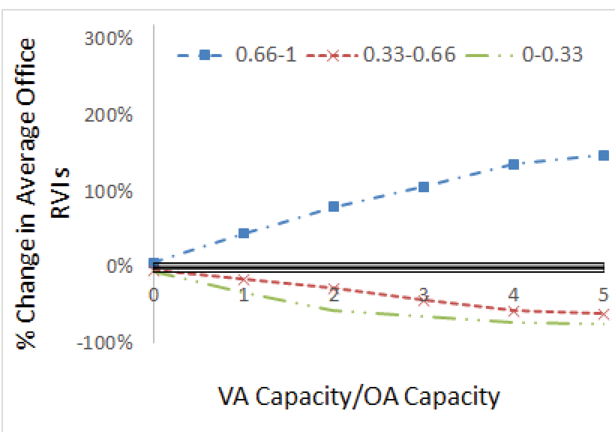
Table 9 summarizes the change in the number of scheduled OAs (the change in office RVIs) for individual patients when VAs are introduced when the disease progression is 0.4 (i.e., $p = 0.4$).

q_v/q_o	Percent of patients that the number of OAs decreases
0.0	13%
0.1	15%
0.5	19%
0.9	20%

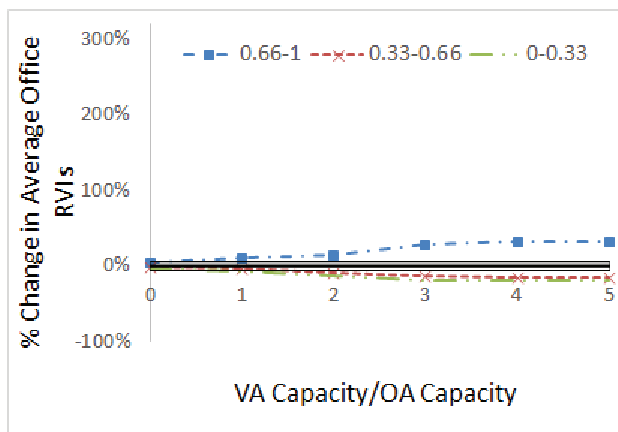
Table 9 **Percentage of patients where the number of scheduled OAs decrease when** $p = 0.4$

In Figure 7, the percent change in the average office RVIs for the Myopic Heuristic with respect to VA capacity/OA capacity is summarized for three patient groups.

Figure 7 **Change in the value of average office RVIs for three patient groups with respect to VA Capacity/OA Capacity for Model TD when p=0.9**



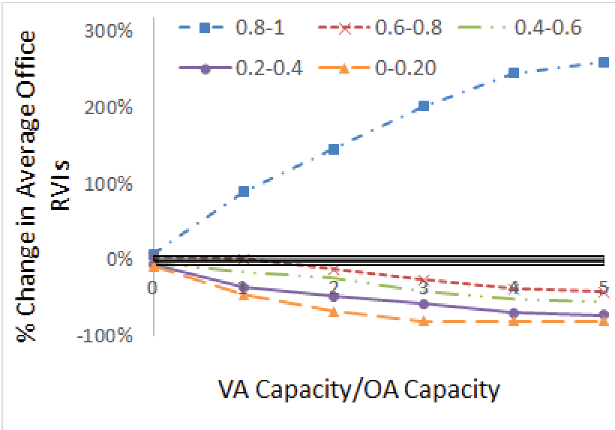
(a) Model TD when q_v/q_o=0.5



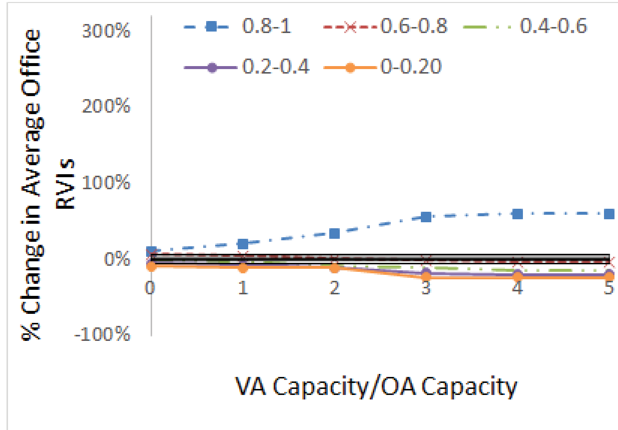
(b) Model TD when q_v/q_o=0.9

In Figure 8, the percent change in the average office RVIs for the Myopic Heuristic with respect to VA capacity/OA capacity is summarized for five patient groups.

Figure 8 **Change in the value of average office RVIs for five patient groups with respect to VA Capacity/OA Capacity for Model TD when p=0.9**



(a) Model TD when q_v/q_o=0.5



(b) Model TD when q_v/q_o=0.9

Appendix. C. Proofs

Table 10: An Overview/Road-map of Proofs

Proofs	The Purpose	Technical Results Used
Theorem 1	To characterize the optimal scheduling policy for OAs and VAs in Models TD and T	Proposition 1 Item 1, Proposition 1 Item 2, and Lemma 11
Proposition 1 Item 1	To characterize the optimal scheduling policy for OAs when the VA scheduling decisions are fixed in Models TD and T	Lemma 1, Lemma 2, Lemma 3, and Lemma 4
Lemma 1	To show $u_t^*(\pi_{1,t}, ..., \pi_{N,t})$ is nondecreasing in $\pi_{i,t}$ in Model TD.	
Lemma 2	To show $u_t^*(\pi_{1,t}, ..., \pi_{N,t})$ is componentwise convex in each $\pi_{i,t}$ in Model TD.	
Lemma 3	To show $u_t^*(\pi_{1,t}, ..., \pi_{N,t})$ is nondecreasing in $\pi_{i,t}$ in Model T.	
Lemma 4	To show $u_t^*(\pi_{1,t}, ..., \pi_{N,t})$ is componentwise convex in $\pi_{i,t}$ in Model T.	
Proposition 1 Item 2	To characterize the optimal scheduling policy for VAs when the OA scheduling decisions are fixed in Models TD and T	Lemma 5, Lemma 6, Lemma 7, Lemma 8,Lemma 9, and Lemma 10
Lemma 5	An intermediate finding to show that Proposition 1 Item 2 holds in Model TD	
Lemma 6	An intermediate finding to show that Proposition 1 Item 2 holds in Model TD	
Lemma 7	An intermediate finding to show that Proposition 1 Item 2 holds in Model TD	
Lemma 8	An intermediate finding to show that Proposition 1 Item 2 holds in Model T	
Lemma 9	An intermediate finding to show that Proposition 1 Item 2 holds in Model T	
Lemma 10	An intermediate finding to show that Proposition 1 Item 2 holds in Model T	
Lemma 11	To characterize the optimal scheduling policy for OAs and VAs when the non-scheduled patients are fixed in Models TD and T	
Theorem 2	To characterize the optimal scheduling policy for OAs in Model D	Proposition 1 Item 1 and Lemma 11
Corollary 1	To characterize the optimal scheduling policy for VAs in Model D	Lemma 12 and Lemma 13
Lemma 12	An intermediate finding to show that Corollary 1 holds in Model TD	
Lemma 13	An intermediate finding to show that Corollary 1 holds in Model T	
Theorem 3 Item 1	To show the value of VAs under perfect OA treatment and under capacity conditions	Lemma 14
Lemma 14	An intermediate finding to show that Theorem 3 Item 1 holds in Model D	
Theorem 3 Item 2	To show the value of VAs under perfect OA treatment and under capacity conditions	
Theorem 3 Item 3	To show the value of VAs under imperfect OA treatment	

Theorem 1 characterizes the optimal scheduling policy for OAs and VAs in Models TD and T and it is presented as follows:

Proof of Theorem 1 Suppose there are N patients and Condition 1 and Condition 2 in Section 4 are met. Reindex the patients by their information vectors such that $\pi_{1,t} \leq \pi_{2,t} \leq ... \leq \pi_{N,t}$. Then the optimal policy is to schedule patients $1, 2, ..., \boldsymbol{C^o}$ for OAs and patients $\boldsymbol{C^o}+1, \boldsymbol{C^o}+2, ..., \boldsymbol{C^o}+\boldsymbol{C^v}$ for VAs in a period $t \in \{1, 2, ..., T-1\}$.

Proposition 1 Item 1 and Item 2 and Lemma 11 describe the steps of the proof of Theorem 1 in order. When these steps are performed in order by fixing the previous step, the optimal policy becomes to schedule patients $1, 2, ..., \boldsymbol{C^o}$ for OAs and patients $\boldsymbol{C^o}+1, \boldsymbol{C^o}+2, ..., \boldsymbol{C^o}+\boldsymbol{C^v}$ for VAs in any period $t \in \{1, 2, ..., T-1\}$.

As a simplified illustrative example, consider that there are N patients and without loss of generality assume that there 3 VA and 4 OA capacities. At the first step, according to the Proposition 1 Item 1, reindex the patients by their information vectors such that $\pi_{1,t} \leq \pi_{2,t} \leq ... \leq \pi_{N,t}$ and arbitrarily schedule patients for VAs, such as Patients 1,3 and N. Then, among the remaining patients, the sickest ones, Patients 2, 4, 5 and 6 should be scheduled for OAs according to the optimal policy. At the second step, according to the Proposition 1 Item 2, fix patients who are scheduled for OAs at the first step. Patients 2, 4, 5 and 6 are fixed and among the remaining patients, the sickest patients, Patients 1,3 and 7 are scheduled for VAs according to the optimal policy defined through Proposition 1 Item 2. At the last step let fix the patients who are not scheduled for any type of appointment at the first two steps. Patients 8, 9, 10, ..., N are fixed. Among the remaining patients, Patients 1, 2, 3 and 4 (the sickest patients) are scheduled for OAs and patients 5, 6 and 7 are scheduled for VAs through Lemma 11. Thus, by the argument in the beginning of the proof, the sickest patients should be scheduled for OAs and the next sickest patients should be scheduled for VAs.

We next present the proof of Proposition 1 to characterize the optimal scheduling policy for OAs when the VA scheduling decisions are fixed in Models TD and T.

■

Proof of Proposition 1 Item 1 Suppose there are N patients and Condition 1 and Condition 2 in Section 4 are met. Consider any arbitrary allocation of $\boldsymbol{C^v}$ patients to VAs capacity in a period $t \in \{1, 2, ..., T-1\}$. Reindex the remaining patients by their information vectors such that $\pi_{1,t} \leq \pi_{2,t} \leq ... \leq \pi_{N-\boldsymbol{C^v},t}$. Then the optimal policy is to schedule patients $1, 2, ..., \boldsymbol{C^o}$ for OAs in a period $t \in \{1, 2, ..., T-1\}$. Note that this proof would be the same for both T and TD models as the scheduling decisions for VAs are fixed.

For a scheduled Patient i for appointment type l, if the diagnosed health state is k, after one period of disease progression, the information vector is $\boldsymbol{\pi_{i,t+1}} = \boldsymbol{e}^k \boldsymbol{\mathcal{Q}}_l \boldsymbol{\mathcal{P}}$, which we denote by $\gamma_{i,t}^l = [\gamma_0^l, \gamma_1^l]$. γ_0^l represents the realization of controlled health state after appointment type l, while γ_1^l denotes the realization of uncontrolled health state after appointment type l. For unscheduled Patient j after one period of disease progression, the information vector is $\boldsymbol{\pi_{j,t+1}} = \boldsymbol{\pi}_{j,t}\boldsymbol{\mathcal{P}}$, which we denote by $z(\boldsymbol{\pi_{j,t}})$.

Without loss of generality, **let us define** two policies;

Policy 1: Patients $\{1, 3, ..., C_{o+1}\}$ are scheduled for OAs and patients $\{2, C_{o+2}, ..., C_{N-\boldsymbol{C^v}}\}$ are not scheduled (NS) in period t.

Policy 2: Patients $\{2,3,...,C_{o+1}\}$ are scheduled for OAs and patients $\{1,C_{o+2},...,C_{N-\boldsymbol{C^v}}\}$ are not scheduled (NS) in period t.

We define $\boldsymbol{\mu}'_{t+1}$ which represents the information vector of all patients except Patient 1 and Patient 2 in period $t+1$.

$$\boldsymbol{\mu}'_{t+1} = \big(\gamma^o_{3,t},...,\gamma^o_{\boldsymbol{C^o}+1,t}, z(\pi_{\boldsymbol{C^o}+2,t}), z(\pi_{\boldsymbol{C^o}+3,t}),...,z(\pi_{N-\boldsymbol{C^v},t})\big) \tag{7}$$

If optimal policy is followed from period $t+1$ to the end of the horizon, we define the value of Policy 1 as follows:

$$\begin{aligned} u_t\big(\pi_{1,t},...,\pi_{N,t};1\in OA,2\in \mathrm{NS}\big) = &\sum_{i=1}^{N}\pi_{i,t} + \beta\mathbb{E}_{\boldsymbol{\mu}'_{t+1}}\big[\pi_{1,t}u^*_{t+1}(\gamma^o_0, z(\pi_{2,t}),\boldsymbol{\mu}'_{t+1}) \\ &+ (1-\pi_{1,t})u^*_{t+1}(\gamma^o_1, z(\pi_{2,t}),\boldsymbol{\mu}'_{t+1})\big] \qquad (8) \\ = &\sum_{i=1}^{N}\pi_{i,t} + \beta\mathbb{E}_{\boldsymbol{\mu}'_{t+1}}\big[u^*_{t+1}(p, z(\pi_{2,t}),\boldsymbol{\mu}'_{t+1})\big] \qquad (9) \end{aligned}$$

where $\gamma^o_0 = \gamma^o_1 = p$ since office appointments provide perfect treatment according to Condition 1.

Similarly, if patient 2 is scheduled instead of patient 1 in period t and the optimal policy is followed from period $t+1$ until the end of the horizon, the value of Policy 2 can be expressed as follows:

$$\begin{aligned} u_t\big(\pi_{1,t},...,\pi_{N,t};2\in OA,1\in \mathrm{NS}\big) = &\sum_{i=1}^{N}\pi_{i,t} + \beta\mathbb{E}_{\boldsymbol{\mu}'_{t+1}}\big[\pi_{2,t}u^*_{t+1}(z(\pi_{1,t}),\gamma^o_0,\boldsymbol{\mu}'_{t+1}) \\ &(1-\pi_{2,t})u^*_{t+1}(z(\pi_{1,t}),\gamma^o_1,\boldsymbol{\mu}'_{t+1})\big] \qquad (10) \\ = &\sum_{i=1}^{N}\pi_{i,t} + \beta\mathbb{E}_{\boldsymbol{\mu}'_{t+1}}\big[u^*_{t+1}(z(\pi_{1,t}),p,\boldsymbol{\mu}'_{t+1})\big] \qquad (11) \end{aligned}$$

Now we show that $u_t\big(\pi_{1,t},...,\pi_{N,t};1\in OA,2\in \mathrm{NS}\big) \le u_t\big(\pi_{1,t},...,\pi_{N,t};2\in OA,1\in \mathrm{NS}\big)$ by showing that for any given realization $\boldsymbol{\mu}'_{t+1}$ the following inequality holds,

$$u^*_{t+1}(p, z(\pi_{2,t}),\boldsymbol{\mu}'_{t+1}) \le u^*_{t+1}(z(\pi_{1,t}),p,\boldsymbol{\mu}'_{t+1}) \tag{12}$$

As the value function is nondecreasing componentwise convex function for both model TD and model T according to the Lemmas 1, 2, 3, and 4, this inequality holds if and only if $z(\pi_{2,t}) \le z(\pi_{1,t}) \to \pi_{2,t}\cdot p \le \pi_{1,t}\cdot p$.

Therefore by the argument in the beginning of the proof, every patient in the scheduled group must have a smaller information value than patients in the unscheduled group; i.e., to schedule patients with the smallest information is the optimal policy when Condition 1 is met.

■

Next, we show that $u^*_t(\pi_{1,t},...,\pi_{N,t})$ is nondecreasing in $\pi_{i,t}$ for Model TD thorugh Lemma 1.

LEMMA 1. *Suppose there are N patients. For all $i\in\{1,2,...,N\}$ and for all $t\in\{1,2,...,T\}$, $u^*_t(\pi_{1,t},...,\pi_{N,t})$ is nondecreasing in $\pi_{i,t}$ for Model TD.*

Proof:

Consider two systems, where the information for all patients at the beginning of Period t is denoted by $u^*_t(\pi_{1,t},...,\pi_{j,t},...,\pi_{N,t})$ and $u^*_t(\pi_{1,t},...,\pi'_{j,t},...\pi_{N,t})$, respectively such that $\pi_{j,t} \le \pi'_{j,t}$. We prove

$u^*_t(\pi_{1,t},...,\pi_{j,t},...,\pi_{N,t}) \leq u^*_t(\pi_{1,t},...,\pi_{j,t},...,\pi_{N,t})$ using induction. From the following equality $u_T(\pi_{1,T},...,\pi_{N,T}) = \sum_{i=1}^{N}\pi_{i,T}$, it can be shown that the value function is nondecreasing in each $\pi_{i,T}$. Now assume $u_{t+1}(\pi_{1,t+1},\pi_{2,t+1},...,\pi_{N,t+1})$ is nondecreasing in $\pi_{i,t+1}$, and let consider the following cases to show that $u_t(\pi_{1,t},\pi_{2,t},...,\pi_{N,t})$ is also nondecreasing in all $\pi_{i,t}$.

Case(1) Patient j with information $\pi_{j,t}$ is scheduled for an OA while patient k is scheduled for a VA with information $\pi_{k,t}$

We define $\boldsymbol{\mu}'_{j,t+1}$ to represent the information of all patients except for patient j and patient k in period $t+1$.

$$\boldsymbol{\mu}'_{j,t+1} = \big(\gamma^o_{1,t},...,\gamma^o_{j-1,t},\gamma^o_{j+1,t},...,\gamma^o_{\boldsymbol{C^o},t},\gamma^v_{\boldsymbol{C^o}+1,t},...,\gamma^v_{k-1,t},\gamma^v_{k+1,t},...,\gamma^v_{\boldsymbol{C^o}+\boldsymbol{C^v},t}$$
$$,z(\pi_{\boldsymbol{C^o}+\boldsymbol{C^v}+1,t}),...,z(\pi_{N,t})\big) \tag{13}$$

Then, the value function in period t can be represented as follows:

$$u^*_t(\pi_{1,t},..,\pi_{j,t},..,\pi_{k,t},..,\pi_{N,t}) = \sum_{i=1}^{N}\pi_{i,t} + \beta\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[\pi_{j,t}\pi_{k,t}u^*_{t+1}(\gamma^o_0,\gamma^v_0,\boldsymbol{\mu}'_{j,t+1}) + (1-\pi_{j,t})\pi_{k,t}u^*_{t+1}(\gamma^o_1,\gamma^v_0,\boldsymbol{\mu}'_{j,t+1})$$
$$+\pi_{j,t}(1-\pi_{k,t})u^*_{t+1}(\gamma^o_0,\gamma^v_1,\boldsymbol{\mu}'_{j,t+1}) + (1-\pi_{j,t})(1-\pi_{k,t})u^*_{t+1}(\gamma^o_1,\gamma^v_1,\boldsymbol{\mu}'_{j,t+1})\big] \tag{14}$$

For the system with state $(\pi_{1,t},..,\pi'_{j,t},..,\pi_{k,t},..,\pi_{N,t})$ the optimal future discounted value function is no less than the future discounted value function following any feasible policy. Thus;

$$u^*_t(\pi_{1,t},..,\pi'_{j,t},..,\pi_{k,t},..,\pi_{N,t}) \geq \pi'_{j,t} + \pi_{k,t} + \sum_{i=1,\neq j,\neq k}^{N}\pi_{i,t} + \beta\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[\pi'_{j,t}\pi_{k,t}u^*_{t+1}(\gamma^o_0,\gamma^v_0,\boldsymbol{\mu}'_{j,t+1})$$
$$+(1-\pi'_{j,t})\pi_{k,t}u^*_{t+1}(\gamma^o_1,\gamma^v_0,\boldsymbol{\mu}'_{j,t+1}) + \pi'_{j,t}(1-\pi_{k,t})u^*_{t+1}(\gamma^o_0,\gamma^v_1,\boldsymbol{\mu}'_{j,t+1})$$
$$+(1-\pi'_{j,t})(1-\pi_{k,t})u^*_{t+1}(\gamma^o_1,\gamma^v_1,\boldsymbol{\mu}'_{j,t+1})\big] \tag{15}$$

where left hand side represents the objective function value for the system with state $(\pi_{1,t},..,\pi'_{j,t},..,\pi_{k,t},..,\pi_{N,t})$ in period t and the right hand side of the inequality is the discounted present value following the optimal action for the system with state $(\pi_{1,t},..,\pi_{j,t},..,\pi_{k,t},..,\pi_{N,t})$.

For any realization of $\boldsymbol{\mu}'_{j,t+1}$,

$u^*_{t+1}(\gamma^o_0,\gamma^v_0,\boldsymbol{\mu}'_{j,t+1}) \geq u^*_{t+1}(\gamma^o_1,\gamma^v_0,\boldsymbol{\mu}'_{j,t+1})$ by induction hypothesis as $\gamma^o_0 \geq \gamma^o_1$. Therefore $\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[\pi_{k,t}u^*_{t+1}(\gamma^o_0,\gamma^v_0,\boldsymbol{\mu}'_{j,t+1})\big] \geq \mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[\pi_{k,t}u^*_{t+1}(\gamma^o_1,\gamma^v_0,\boldsymbol{\mu}'_{j,t+1})\big]$, and similarly $\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[(1-\pi_{k,t})u^*_{t+1}(\gamma^o_0,\gamma^v_1,\boldsymbol{\mu}'_{j,t+1})\big] \geq \mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[(1-\pi_{k,t})u^*_{t+1}(\gamma^o_1,\gamma^v_1,\boldsymbol{\mu}'_{j,t+1})\big]$.

Since $\pi_{j,t} \leq \pi'_{j,t}$,

$$\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[\pi_{j,t}\pi_{k,t}u^*_{t+1}(\gamma^o_0,\gamma^v_0,\boldsymbol{\mu}'_{j,t+1}) + (1-\pi_{j,t})\pi_{k,t}u^*_{t+1}(\gamma^o_1,\gamma^v_0,\boldsymbol{\mu}'_{j,t+1})\big]$$
$$\leq \mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[\pi'_{j,t}\pi_{k,t}u^*_{t+1}(\gamma^o_0,\gamma^v_0,\boldsymbol{\mu}'_{j,t+1}) + (1-\pi'_{j,t})\pi_{k,t}u^*_{t+1}(\gamma^o_1,\gamma^v_0,\boldsymbol{\mu}'_{j,t+1})\big] \tag{16}$$

and

$$\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[\pi_{j,t}(1-\pi_{k,t})u^*_{t+1}(\gamma^o_0,\gamma^v_1,\boldsymbol{\mu}'_{j,t+1}) + (1-\pi_{j,t})(1-\pi_{k,t})u^*_{t+1}(\gamma^o_1,\gamma^v_1,\boldsymbol{\mu}'_{j,t+1})\big]$$
$$\leq \mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[\pi'_{j,t}(1-\pi_{k,t})u^*_{t+1}(\gamma^o_0,\gamma^v_1,\boldsymbol{\mu}'_{j,t+1}) + (1-\pi'_{j,t})(1-\pi_{k,t})u^*_{t+1}(\gamma^o_1,\gamma^v_1,\boldsymbol{\mu}'_{j,t+1})\big] \tag{17}$$

Also by knowing that;

$$\sum_{i=1}^{N}\pi_{i,t} = \pi_{j,t} + \sum_{i=1,\neq j}^{N}\pi_{i,t} \leq \pi'_{j,t} + \sum_{i=1,\neq j}^{N}\pi_{i,t} \tag{18}$$

It then implies that

$$u_t^*(\pi_{1,t},..,\pi_{j,t},..,\pi_{k,t},..,\pi_{N,t}) \leq u_t^*(\pi_{1,t},..,\pi'_{j,t},..,\pi_{k,t},..,\pi_{N,t})$$

Similarly, for the system with state $(\pi_{1,t},..,\pi_{j,t},..,\pi'_{k,t},..,\pi_{N,t})$ the optimal future discounted value function is no less than the future discounted value function following any feasible policy. Thus;

$$\begin{aligned} &u_t^*(\pi_{1,t},..,\pi_{j,t},..,\pi'_{k,t},..,\pi_{N,t}) \geq \pi_{j,t} + \pi'_{k,t} + \sum_{i=1,\neq j,\neq k}^{N} \pi_{i,t} + \beta \mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}} \big[\pi_{j,t}\pi'_{k,t} u_{t+1}^*(\gamma_0^o,\gamma_0^v,\boldsymbol{\mu}'_{j,t+1}) \\ &+ (1-\pi_{j,t})\pi'_{k,t} u_{t+1}^*(\gamma_1^o,\gamma_0^v,\boldsymbol{\mu}'_{j,t+1}) + \pi_{j,t}(1-\pi'_{k,t}) u_{t+1}^*(\gamma_0^o,\gamma_1^v,\boldsymbol{\mu}'_{j,t+1}) \\ &+ (1-\pi_{j,t})(1-\pi'_{k,t}) u_{t+1}^*(\gamma_1^o,\gamma_1^v,\boldsymbol{\mu}'_{j,t+1})\big] \end{aligned} \tag{19}$$

where left hand side is the value for the system with state $(\pi_{1,t},..,\pi_{j,t},..,\pi'_{k,t},..,\pi_{I,t})$ in period t and the right hand side of the inequality is the discounted present value following the optimal action for the system with state $(\pi_{1,t},..,\pi_{j,t},..,\pi_{k,t},..,\pi_{N,t})$.

For any realization of $\boldsymbol{\mu}'_{j,t+1}$, $u_{t+1}^*(\gamma_0^o,\gamma_0^v,\boldsymbol{\mu}'_{j,t+1}) \geq u_{t+1}^*(\gamma_1^o,\gamma_0^v,\boldsymbol{\mu}'_{j,t+1})$ by induction hypothesis as $\gamma_0^o \geq \gamma_1^o$. Therefore $\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[\pi_{j,t} u_{t+1}^*(\gamma_0^o,\gamma_0^v,\boldsymbol{\mu}'_{j,t+1})\big] \geq \mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[\pi_{j,t} u_{t+1}^*(\gamma_1^o,\gamma_0^v,\boldsymbol{\mu}'_{j,t+1})\big]$, and similarly $\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[(1-\pi_{j,t}) u_{t+1}^*(\gamma_0^o,\gamma_1^v,\boldsymbol{\mu}'_{j,t+1})\big] \geq \mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[(1-\pi_{j,t}) u_{t+1}^*(\gamma_1^o,\gamma_1^v,\boldsymbol{\mu}'_{j,t+1})\big]$.

Since $\pi_{k,t} \leq \pi'_{k,t}$,

$$\begin{aligned} &\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[\pi_{j,t}\pi_{k,t} u_{t+1}^*(\gamma_0^o,\gamma_0^v,\boldsymbol{\mu}'_{j,t+1}) + \pi_{j,t}(1-\pi_{k,t}) u_{t+1}^*(\gamma_0^o,\gamma_1^v,\boldsymbol{\mu}'_{j,t+1})\big] \\ &\leq \mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[\pi_{j,t}\pi'_{k,t} u_{t+1}^*(\gamma_0^o,\gamma_0^v,\boldsymbol{\mu}'_{j,t+1}) + \pi_{j,t}(1-\pi'_{k,t}) u_{t+1}^*(\gamma_0^o,\gamma_1^v,\boldsymbol{\mu}'_{j,t+1})\big] \end{aligned} \tag{20}$$

and

$$\begin{aligned} &\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[(1-\pi_{j,t})\pi_{k,t} u_{t+1}^*(\gamma_1^o,\gamma_0^v,\boldsymbol{\mu}'_{j,t+1}) + (1-\pi_{j,t})(1-\pi_{k,t}) u_{t+1}^*(\gamma_1^o,\gamma_1^v,\boldsymbol{\mu}'_{j,t+1})\big] \\ &\leq \mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[(1-\pi_{j,t})\pi'_{k,t} u_{t+1}^*(\gamma_1^o,\gamma_0^v,\boldsymbol{\mu}'_{j,t+1}) + (1-\pi_{j,t})(1-\pi'_{k,t}) u_{t+1}^*(\gamma_1^o,\gamma_1^v,\boldsymbol{\mu}'_{j,t+1})\big] \end{aligned} \tag{21}$$

Also by knowing that;

$$\sum_{i=1}^{N} \pi_{i,t} = \pi_{k,t} + \sum_{i=1,\neq k}^{N} \pi_{i,t} \leq \pi'_{k,t} + \sum_{i=1,\neq k}^{N} \pi_{i,t} \tag{22}$$

It then implies that

$$u_t^*(\pi_{1,t},..,\pi_{j,t},..,\pi_{k,t},..,\pi_{N,t}) \leq u_t^*(\pi_{1,t},..,\pi_{j,t},..,\pi'_{k,t},..,\pi_{N,t})$$

Case(2) Patient j with information $\pi_{j,t}$ is *not* scheduled for any appointment while patient k is scheduled for a VA with information $\pi_{k,t}$

We define $\boldsymbol{\mu}'_{j,t+1}$ to represent information vector of all patients except Patients j and k as follows:

$$\begin{aligned} &\boldsymbol{\mu}'_{j,t+1} = \big(\gamma^o_{1,t},...,\gamma^o_{\boldsymbol{C^o},t},\gamma^v_{\boldsymbol{C^o}+1,t},...,\gamma^v_{k-1,t},\gamma^v_{k+1,t},...,\gamma^v_{\boldsymbol{C^o}+\boldsymbol{C^v},t},z(\pi_{\boldsymbol{C^o}+\boldsymbol{C^v}+1,t}),...,z(\pi_{j-1,t}) \\ &,z(\pi_{j+1,t}),...,z(\pi_{N,t})\big) \end{aligned} \tag{23}$$

Then the optimally equation can be stated as follows:

$$u_t^*(\pi_{1,t},..,\pi_{j,t},..,\pi_{k,t},..,\pi_{N,t}) = \sum_{i=1}^{N} \pi_{i,t} + \beta \mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[\pi_{k,t} u_{t+1}^*(z(\pi_{j,t}),\gamma_0^v,\boldsymbol{\mu}'_{j,t+1})$$

$$+ (1-\pi_{k,t})u^*_{t+1}(z(\pi_{j,t}), \gamma^v_1, \boldsymbol{\mu}'_{j,t+1})\big] \tag{24}$$

Similarly the optimality inequality for the state $(\pi_{1,t}, .., \pi'_{j,t}, .., \pi_{k,t}, .., \pi_{N,t})$ can be stated as follows:

$$u^*_t(\pi_{1,t}, .., \pi'_{j,t}, .., \pi_{k,t}, .., \pi_{N,t}) \geq \pi'_{j,t} + \sum_{i=1, i\neq j}^{N} \pi_{i,t} + \beta \mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[\pi_{k,t}u^*_{t+1}(z(\pi'_{j,t}), \gamma^v_0, \boldsymbol{\mu}'_{j,t+1})$$
$$+ (1-\pi_{k,t})u^*_{t+1}(z(\pi'_{j,t}), \gamma^v_1, \boldsymbol{\mu}'_{j,t+1})\big] \tag{25}$$

Since $z(.)$ is nondecreasing and $\pi_{j,t} \leq \pi'_{j,t}$, $z(\pi_{j,t}) \leq z(\pi'_{j,t})$. Thus, by induction hypothesis, $\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[\pi_{k,t}u^*_{t+1}(z(\pi_{j,t}), \gamma^v_0, \boldsymbol{\mu}'_{j,t+1}) + (1-\pi_{k,t})u^*_{t+1}(z(\pi_{j,t}), \gamma^v_1, \boldsymbol{\mu}'_{j,t+1})\big] \leq \mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[\pi_{k,t}u^*_{t+1}(z(\pi'_{j,t}), \gamma^v_0, \boldsymbol{\mu}'_{j,t+1}) + (1-\pi_{k,t})u^*_{t+1}(z(\pi'_{j,t}), \gamma^v_1, \boldsymbol{\mu}'_{j,t+1})\big]$.

It then implies that

$$u^*_t(\pi_{1,t}, .., \pi_{j,t}, .., \pi_{k,t}, .., \pi_{N,t}) \leq u^*_t(\pi_{1,t}, .., \pi'_{j,t}, .., \pi_{k,t}, .., \pi_{N,t})$$

Similarly, the optimality inequality for the state $(\pi_{1,t}, .., \pi_{j,t}, .., \pi'_{k,t}, .., \pi_{N,t})$ can be stated as follows:

$$u^*_t(\pi_{1,t}, .., \pi_{j,t}, .., \pi'_{k,t}, .., \pi_{I,t}) \geq \pi'_{k,t} + \sum_{i=1, i\neq k}^{N} \pi_{i,t} + \beta \mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[\pi'_{k,t}u^*_{t+1}(z(\pi_{j,t}), \gamma^v_0, \boldsymbol{\mu}'_{j,t+1})$$
$$+ (1-\pi'_{k,t})u^*_{t+1}(z(\pi_{j,t}), \gamma^v_1, \boldsymbol{\mu}'_{j,t+1})\big] \tag{26}$$

Since $\pi_{k,t} \leq \pi'_{k,t}$, $u^*_{t+1}(z(\pi_{j,t}), \gamma^v_1, \boldsymbol{\mu}'_{j,t+1}) \leq u^*_{t+1}(z(\pi_{j,t}), \gamma^v_0, \boldsymbol{\mu}'_{j,t+1})$. Thus by induction hypothesis, $\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[\pi_{k,t}u^*_{t+1}(z(\pi_{j,t}), \gamma^v_0, \boldsymbol{\mu}'_{j,t+1}) + (1-\pi_{k,t})u^*_{t+1}(z(\pi_{j,t}), \gamma^v_1, \boldsymbol{\mu}'_{j,t+1})\big] \leq \mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[\pi'_{k,t}u^*_{t+1}(z(\pi_{j,t}), \gamma^v_0, \boldsymbol{\mu}'_{j,t+1}) + (1-\pi'_{k,t})u^*_{t+1}(z(\pi_{j,t}), \gamma^v_1, \boldsymbol{\mu}'_{j,t+1})\big]$.

It then implies that

$$u^*_t(\pi_{1,t}, .., \pi_{j,t}, .., \pi_{k,t}, .., \pi_{N,t}) \leq u^*_t(\pi_{1,t}, .., \pi_{j,t}, .., \pi'_{k,t}, .., \pi_{N,t})$$

Case(3) Patient j with information $\pi_{j,t}$ is scheduled for an OA while patient k is *not* scheduled for any appointment with information $\pi_{k,t}$

We define a random realization vector $\mu'_{k,t+1}$ except patients k and j as follows:

$$\mu'_{k,t+1} = \big(\gamma^o_{1,t}, ..., \gamma^o_{j-1,t}, \gamma^o_{j+1,t}, ..., \gamma^o_{\boldsymbol{C^o},t}, \gamma^v_{\boldsymbol{C^o}+1,t}, ..., \gamma^v_{\boldsymbol{C^o}+\boldsymbol{C^v},t}, z(\pi_{\boldsymbol{C^o}+\boldsymbol{C^v}+1,t}), ..., z(\pi_{k-1,t})$$
$$, z(\pi_{k+1,t}), ..., z(\pi_{N,t})\big) \tag{27}$$

Similar to the previous cases, then the optimality equation can be stated as follows:

$$u^*_t(\pi_{1,t}, .., \pi_{j,t}, .., \pi_{k,t}, .., \pi_{N,t}) = \sum_{i=1}^{N} \pi_{i,t} + \beta \mathbb{E}_{\mu'_{k,t+1}}\big[\pi_{j,t}u^*_{t+1}(\gamma^o_0, z(\pi_{k,t}), \mu'_{k,t+1})$$
$$+ (1-\pi_{j,t})u^*_{t+1}(\gamma^o_1, z(\pi_{k,t}), \mu'_{k,t+1})\big] \tag{28}$$

Similarly, the optimality inequality for the state $(\pi_{1,t}, .., \pi_{j,t}, .., \pi'_{k,t}, .., \pi_{N,t})$ can be stated as follows:

$$u^*_t(\pi_{1,t}, .., \pi_{j,t}, .., \pi'_{k,t}, .., \pi_{N,t}) \geq \pi'_{k,t} + \sum_{i=1, i\neq k}^{N} \pi_{i,t} + \beta \mathbb{E}_{\mu'_{k,t+1}}\big[\pi_{j,t}u^*_{t+1}(z(\pi'_{k,t}), \gamma^v_0, \boldsymbol{\mu}'_{j,t+1})$$
$$+ (1-\pi_{j,t})u^*_{t+1}(z(\pi'_{k,t}), \gamma^v_1, \boldsymbol{\mu}'_{j,t+1})\big] \tag{29}$$

Since $z(.)$ is nondecreasing and $\pi_{k,t} \leq \pi'_{k,t}$, $z(\pi_{k,t}) \leq z(\pi'_{k,t})$. So $\mathbb{E}_{\mu'_{k,t+1}}\big[\pi_{j,t}u^*_{t+1}(z(\pi_{k,t}), \gamma^v_0, \mu'_{k,t+1}) + (1 - \pi_{j,t})u^*_{t+1}(z(\pi_{k,t}), \gamma^v_1, \mu'_{k,t+1})\big] \leq \mathbb{E}_{\mu'_{k,t+1}}\big[\pi_{j,t}u^*_{t+1}(z(\pi'_{k,t}), \gamma^v_0, \mu'_{k,t+1}) + (1 - \pi_{j,t})u^*_{t+1}(z(\pi'_{k,t}), \gamma^v_1, \mu'_{k,t+1})\big]$.

It then implies that:

$$u^*_t(\pi_{1,t}, .., \pi_{j,t}, .., \pi_{k,t}, .., \pi_{N,t}) \leq u^*_t(\pi_{1,t}, .., \pi_{j,t}, .., \pi'_{k,t}, .., \pi_{N,t}) \tag{30}$$

We then define the optimality inequality for the state $(\pi_{1,t}, .., \pi'_{j,t}, .., \pi_{k,t}, .., \pi_{N,t})$ as follows:

$$\begin{aligned} u^*_t(\pi_{1,t}, .., \pi'_{j,t}, .., \pi_{k,t}, .., \pi_{N,t}) \geq \pi_{j,t} + \sum_{i=1,\neq j}^{N} \pi_{i,t} + \beta\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[\pi'_{j,t}u^*_{t+1}(z(\pi_{k,t}), \gamma^v_0, \boldsymbol{\mu}'_{j,t+1}) \\ + (1 - \pi'_{j,t})u^*_{t+1}(z(\pi_{k,t}), \gamma^v_1, \boldsymbol{\mu}'_{j,t+1})\big] \end{aligned} \tag{31}$$

Since $\pi_{j,t} \leq \pi'_{j,t}$ and $u^*_{t+1}(z(\pi_{k,t}), \gamma^v_1, \boldsymbol{\mu}'_{j,t+1}) \leq u^*_{t+1}(z(\pi_{k,t}), \gamma^v_0, \boldsymbol{\mu}'_{j,t+1})$, the following inequality holds: $\mathbb{E}_{\mu'_{k,t+1}}\big[\pi_{j,t}u^*_{t+1}(z(\pi_{k,t}), \gamma^v_0, \mu'_{k,t+1}) + (1 - \pi_{j,t})u^*_{t+1}(z(\pi_{k,t}), \gamma^v_1, \mu'_{k,t+1})\big] \leq \mathbb{E}_{\mu'_{k,t+1}}\big[\pi'_{j,t}u^*_{t+1}(z(\pi_{k,t}), \gamma^v_0, \mu'_{k,t+1}) + (1 - \pi'_{j,t})u^*_{t+1}(z(\pi_{k,t}), \gamma^v_1, \mu'_{k,t+1})\big]$.

It then implies that

$$u^*_t(\pi_{1,t}, .., \pi_{j,t}, .., \pi_{k,t}, .., \pi_{N,t}) \leq u^*_t(\pi_{1,t}, .., \pi'_{j,t}, .., \pi_{k,t}, .., \pi_{N,t}) \tag{32}$$

Case(4) Patient j with information $\pi_{j,t}$ is *not* scheduled for any appointment while patient k is *not* scheduled for any appointment with information $\pi_{k,t}$

let us define a random realization vector $\mu'_{k,t+1}$ except patients k and j as follows:

$$\begin{aligned} \mu'_{k,t+1} = \big(\gamma^o_{1,t}, ..., \gamma^o_{\boldsymbol{C^o},t}, \gamma^v_{\boldsymbol{C^o}+1,t}, ..., \gamma^v_{\boldsymbol{C^o}+\boldsymbol{C^v},t}, z(\pi_{\boldsymbol{C^o}+\boldsymbol{C^v}+1,t}), ..., z(\pi_{j-1,t}), z(\pi_{j+1,t}), ..., z(\pi_{k-1,t}) \\ , z(\pi_{k+1,t}), ..., z(\pi_{N,t})\big) \end{aligned} \tag{33}$$

Similar to the previous cases, then the optimality equation can be stated as follows:

$$u^*_t(\pi_{1,t}, .., \pi_{j,t}, .., \pi_{k,t}, .., \pi_{N,t}) = \sum_{i=1}^{N} \pi_{i,t} + \beta\mathbb{E}_{\mu'_{k,t+1}}\big[u^*_{t+1}(z(\pi_{j,t}), z(\pi_{k,t}), \mu'_{k,t+1})\big] \tag{34}$$

Similarly, the optimality inequality for the state $(\pi_{1,t}, .., \pi_{j,t}, .., \pi'_{k,t}, .., \pi_{N,t})$ can be stated as follows:

$$u^*_t(\pi_{1,t}, .., \pi_{j,t}, .., \pi'_{k,t}, .., \pi_{N,t}) \geq \pi'_{k,t} + \sum_{i=1,i\neq k}^{N} \pi_{i,t} + \beta\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[u^*_{t+1}(z(\pi_{j,t}), z(\pi'_{k,t}), \boldsymbol{\mu}'_{j,t+1})\big] \tag{35}$$

Since $z(.)$ is nondecreasing and $\pi_{k,t} \leq \pi'_{k,t}$, $z(\pi_{k,t}) \leq z(\pi'_{k,t})$. So $\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[u^*_{t+1}(z(\pi_{j,t}), z(\pi_{k,t}), \mu'_{k,t+1})\big] \leq \mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[u^*_{t+1}(z(\pi_{j,t}), z(\pi'_{k,t}), \boldsymbol{\mu}'_{j,t+1})\big]$.

It then implies that

$$u^*_t(\pi_{1,t}, .., \pi_{j,t}, .., \pi_{k,t}, .., \pi_{N,t}) \leq u^*_t(\pi_{1,t}, .., \pi_{j,t}, .., \pi'_{k,t}, .., \pi_{N,t})$$

Similarly, we then define the optimality inequality for the state $(\pi_{1,t}, .., \pi'_{j,t}, .., \pi_{k,t}, .., \pi_{N,t})$ as follows:

$$u^*_t(\pi_{1,t}, .., \pi'_{j,t}, .., \pi_{k,t}, .., \pi_{N,t}) \geq \pi'_{j,t} + \sum_{i=1,\neq j}^{N} \pi_{i,t} + \beta\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[u^*_{t+1}(z(\pi'_{j,t}), z(\pi_{k,t}), \boldsymbol{\mu}'_{j,t+1})\big] \tag{36}$$

Since $z(.)$ is nondecreasing and $\pi_{j,t} \leq \pi'_{j,t}$, $z(\pi_{j,t}) \leq z(\pi'_{j,t})$. So $\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[u^*_{t+1}(z(\pi_{j,t}), z(\pi_{k,t}), \mu'_{k,t+1})\big] \leq \mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[u^*_{t+1}(z(\pi'_{j,t}), z(\pi_{k,t}), \mu'_{k,t+1})\big]$.

It then implies that

$$u_t^*(\pi_{1,t},..,\pi_{j,t},..,\pi_{k,t},..,\pi_{N,t}) \leq u_t^*(\pi_{1,t},..,\pi'_{j,t},..,\pi_{k,t},..,\pi_{N,t})$$

Then this completes the proof that for Model TD, $u_t^*(\pi_{1,t},..,\pi_{N,t})$ is nondecreasing in $\pi_{i,t}$ for all $i \in \{1,2,...,N\}$ and for all $t \in \{1,2,...,T\}$.

■

In the next Lemma, Lemma 2, we show that $u_t^*(\pi_{1,t},...,\pi_{N,t})$ is componentwise convex in $\pi_{i,t}$ for Model TD.

LEMMA 2. *Suppose there are N patients. For all $i \in \{1,2,...,N\}$ and for all $t \in \{1,2,...,T\}$, $u_t^*(\pi_{1,t},...,\pi_{N,t})$ is componentwise convex in $\pi_{i,t}$ for Model TD.*

Proof:

We show that the following inequality holds to prove that the $u_t^*(\pi_{1,t},...,\pi_{N,t})$ function is convex in each $\pi_{j,t}$, $\forall j \in \{1,2,...,N\}$ and all $t \in \{1,2,...,T\}$.

$$\begin{aligned}&u_t^*(\pi_{1,t},...,\nu\pi_{j,t}+(1-\nu)\pi'_{j,t},...,\pi_{N,t})\\&\leq \nu u_t^*(\pi_{1,t},...,\pi_{j,t},...,\pi_{N,t})+(1-\nu)u_t^*(\pi_{1,t},...,\pi'_{j,t},...,\pi_{N,t})\end{aligned} \tag{37}$$

We use induction to prove equation 37. For period T, $u_T^*(\pi_{1,T},...,\pi_{N,T})=\sum_{i=1}^N \pi_{i,T}$, it is linear and so convex in each $\pi_{i,T}$. With induction, we assume that $u_{t+1}^*(\pi_{1,t+1},...,\pi_{N,t+1})$ is convex in all $\pi_{i,t+1}$. We will be considering the following three cases to prove that $u_t^*(\pi_{1,t},...,\pi_{N,t})$ is convex:

Case(1) Patient j with information $\nu\pi_{j,t}+(1-\nu)\pi'_{j,t}$ is scheduled for an OA

We define $\boldsymbol{\mu}'_{j,t+1}$ to represent the information of all patients except for patient j in period $t+1$.

$$\boldsymbol{\mu}'_{j,t+1}=\left(\gamma^o_{1,t},...,\gamma^o_{j-1,t},\gamma^o_{j+1,t},...,\gamma^o_{\boldsymbol{C^o},t},...,\gamma^v_{\boldsymbol{C^o}+\boldsymbol{C^v},t},z(\pi_{\boldsymbol{C^o}+\boldsymbol{C^v}+1,t}),...,z(\pi_{N,t})\right) \tag{38}$$

Then we represent the value function as follows:

$$\begin{aligned}&u_t^*(\pi_{1,t},...,\nu\pi_{j,t}+(1-\nu)\pi'_{j,t},...,\pi_{N,t})=[\nu\pi_{j,t}+(1-\nu)\pi'_{j,t}]+\sum_{i=1,i\neq j}^N \pi_{i,t}\\&+\beta\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\left[\left(\nu\pi_{j,t}+(1-\nu)\pi'_{j,t}\right)\left[u_{t+1}^*(\gamma_0^o,\boldsymbol{\mu}'_{j,t+1})\right]+\left(1-(\nu\pi_{j,t}+(1-\nu)\pi'_{j,t})\right)\left[u_{t+1}^*(\gamma_1^o,\boldsymbol{\mu}'_{j,t+1})\right]\right] \qquad (39)\\&=\nu\pi_{j,t}+\nu\sum_{i=1,i\neq j}^N \pi_{i,t}+\beta\nu\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\left[\pi_{j,t}\left[u_{t+1}^*(\gamma_0^o,\boldsymbol{\mu}'_{j,t+1})\right]+(1-\pi_{j,t})\left[u_{t+1}^*(\gamma_1^o,\boldsymbol{\mu}'_{j,t+1})\right]\right]\\&+(1-\nu)\pi'_{j,t}+(1-\nu)\sum_{i=1,i\neq j}^N \pi_{i,t}+\beta(1-\nu)\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\left[\pi'_{j,t}\left[u_{t+1}^*(\gamma_0^o,\boldsymbol{\mu}'_{j,t+1})\right]+(1-\pi'_{j,t})\left[u_{t+1}^*(\gamma_1^o,\boldsymbol{\mu}'_{j,t+1})\right]\right] \qquad (40)\end{aligned}$$

By the definition of optimality equation, the value for the system with state $(\pi_{1,t},...,\pi_{j,t},...,\pi_{N,t})$ is greater or equal than the present discounted value if the policy is to schedule Patient $1,2,...,\boldsymbol{C^o}$ for office appointments and Patient $C_{o+1},...,C_{o+v}$ for virtual appointments in period t. Therefore;

$$u_t^*(\pi_{1,t},...,\pi_{j,t},...,\pi_{N,t})\geq \pi_{j,t}+\sum_{i=1,i\neq j}^N \pi_{i,t}+\beta\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\left[\pi_{j,t}\left[u_{t+1}^*(\gamma_0^o,\boldsymbol{\mu}'_{j,t+1})\right]+(1-\pi_{j,t})\left[u_{t+1}^*(\gamma_1^o,\boldsymbol{\mu}'_{j,t+1})\right]\right] \tag{41}$$

where the left hand side is the value of the system when system state is $(\pi_{1,t},..,\pi_{j,t},..,\pi_{N,t})$ at period t; and the right hand side is the discounted present value if the action in Period t is to schedule Patient $1,2,...,\boldsymbol{C^o}$ for office appointments and Patient $C_{o+1},...,C_{o+v}$ for virtual appointments and from Period $t+1$ on is governed by optimal policy.

Similarly;

$$u_t^*(\pi_{1,t},...,\pi'_{j,t},...,\pi_{N,t}) \geq \pi'_{j,t} + \sum_{i=1,i\neq j}^{N} \pi_{i,t} + \beta \mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\Big[\pi'_{j,t}\big[u_{t+1}^*(\gamma_0^o,\boldsymbol{\mu}'_{j,t+1})\big] + (1-\pi'_{j,t})\big[u_{t+1}^*(\gamma_1^o,\boldsymbol{\mu}'_{j,t+1})\big]\Big] \quad (42)$$

Multiplying equation 41 by ν and equation 42 by $(1-\nu)$ and summing, the resulting right hand side is equal to the right hand side of equation 40.

This implies that:

$$u_t^*(\pi_{1,t},...,\nu\pi_{j,t}+(1-\nu)\pi'_{j,t},...,\pi_{N,t}) \leq \nu u_t^*(\pi_{1,t},...,\pi_{j,t},...,\pi_{N,t}) + (1-\nu)u_t^*(\pi_{1,t},...,\pi'_{j,t},...,\pi_{N,t})$$

Case(2) Patient j with information $\nu\pi_{j,t}+(1-\nu)\pi'_{j,t}$ is scheduled for a VA

We define $\boldsymbol{\mu}'_{j,t+1}$ to represent the information of all patients except for patient j in period $t+1$.

$$\boldsymbol{\mu}'_{j,t+1} = \big(\gamma_{1,t}^o,...,\gamma_{\boldsymbol{C^o},t}^o,\gamma_{\boldsymbol{C^o}+1,t}^v,...,\gamma_{j-1,t}^v,\gamma_{j+1,t}^v,...,\gamma_{\boldsymbol{C^o}+\boldsymbol{C^v},t}^v,z(\pi_{\boldsymbol{C^o}+\boldsymbol{C^v}+1,t}),...,z(\pi_{N,t})\big) \quad (43)$$

Then we represent the value function as follows:

$$\begin{aligned}
&u_t^*(\pi_{1,t},...,\nu\pi_{j,t}+(1-\nu)\pi'_{j,t},...,\pi_{N,t}) = [\nu\pi_{j,t}+(1-\nu)\pi'_{j,t}] + \sum_{i=1,i\neq j}^{N} \pi_{i,t} \\
&+ \beta \mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\Big[\big(\nu\pi_{j,t}+(1-\nu)\pi'_{j,t}\big)\big[u_{t+1}^*(\gamma_0^v,\boldsymbol{\mu}'_{j,t+1})\big] + \big(1-(\nu\pi_{j,t}+(1-\nu)\pi'_{j,t})\big)\big[u_{t+1}^*(\gamma_1^v,\boldsymbol{\mu}'_{j,t+1})\big]\Big] \quad (44)\\
&= \nu\pi_{j,t} + \nu\sum_{i=1,i\neq j}^{N} \pi_{i,t} + \beta\nu\mathbb{E}_{\mu'_{j,t+1}}\Big[\pi_{j,t}\big[u_{t+1}^*(\gamma_0^v,\boldsymbol{\mu}'_{j,t+1})\big] + (1-\pi_{j,t})\big[u_{t+1}^*(\gamma_1^v,\boldsymbol{\mu}'_{j,t+1})\big]\Big] \\
&+ (1-\nu)\pi'_{j,t} + (1-\nu)\sum_{i=1,i\neq j}^{N} \pi_{i,t} + \beta(1-\nu)\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\Big[\pi'_{j,t}\big[u_{t+1}^*(\gamma_0^v,\boldsymbol{\mu}'_{j,t+1})\big] + (1-\pi'_{j,t})\big[u_{t+1}^*(\gamma_1^v,\boldsymbol{\mu}'_{j,t+1})\big]\Big] \quad (45)
\end{aligned}$$

By the definition of optimality equation, the value for the system with state $(\pi_{1,t},...,\pi_{j,t},...,\pi_{N,t})$ is greater or equal than the present discounted value if the policy is to schedule Patient $1,2,...,\boldsymbol{C^o}$ for office appointments and Patient $C_{o+1},...,C_{o+v}$ for virtual appointments in period t. Therefore;

$$u_t^*(\pi_{1,t},...,\pi_{j,t},...,\pi_{N,t}) \geq \pi_{j,t} + \sum_{i=1,i\neq j}^{N} \pi_{i,t} + \beta \mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\Big[\pi_{j,t}\big[u_{t+1}^*(\gamma_0^v,\boldsymbol{\mu}'_{j,t+1}) + (1-\pi_{j,t})\big[u_{t+1}^*(\gamma_1^v,\boldsymbol{\mu}'_{j,t+1})\big]\Big] \quad (46)$$

where the left hand side is the value of the system when system state is $(\pi_{1,t},...,\pi_{j,t},...,\pi_{N,t})$ at period t; and the right hand side is the discounted present value if the action in Period t is to schedule Patient $1,2,...,\boldsymbol{C^o}$ for office appointments and Patient $C_{o+1},...,C_{o+v}$ for virtual appointments and from Period $t+1$ on is governed by optimal policy.

Similarly;

$$u_t^*(\pi_{1,t},...,\pi'_{j,t},...,\pi_{N,t}) \geq \pi'_{j,t} + \sum_{i=1,i\neq j}^{N} \pi_{i,t} + \beta \mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\Big[\pi'_{j,t}\big[u_{t+1}^*(\gamma_0^v,\boldsymbol{\mu}'_{j,t+1}) + (1-\pi'_{j,t})\big[u_{t+1}^*(\gamma_1^v,\boldsymbol{\mu}'_{j,t+1})\big]\Big] \quad (47)$$

Multiplying equation 46 by ν and equation 47 by $(1-\nu)$ and summing, the resulting right hand side is equal to the right hand side of equation 45.

This implies that:

$$u^*_t(\pi_{1,t},...,\nu\pi_{j,t}+(1-\nu)\pi'_{j,t},...,\pi_{N,t}) \le \nu u^*_t(\pi_{1,t},...,\pi_{j,t},...,\pi_{N,t})+(1-\nu)u^*_t(\pi_{1,t},...,\pi'_{j,t},...,\pi_{N,t})$$

Case(3) Patient j with information $\nu\pi_{j,t}+(1-\nu)\pi'_{j,t}$ is *not* scheduled for any appointment

We define $\boldsymbol{\mu}'_{j,t+1}$ to represent information vector of all patients except patient j as follows:

$$\boldsymbol{\mu}'_{j,t+1}=\left(\gamma^o_{1,t},...,\gamma^o_{\boldsymbol{C^o},t},\gamma^v_{\boldsymbol{C^o}+1,t},...,\gamma^v_{\boldsymbol{C^o}+\boldsymbol{C^v},t},z(\pi_{\boldsymbol{C^o}+\boldsymbol{C^v}+1,t}),...,z(\pi_{j-1,t}),z(\pi_{j+1,t}),...,z(\pi_{N,t})\right) \tag{48}$$

The optimality equation can be stated as follows:

$$\begin{aligned}
&u^*_t(\pi_{1,t},...,\nu\pi_{j,t}+(1-\nu)\pi'_{j,t},...,\pi_{N,t})=[\nu\pi_{j,t}+(1-\nu)\pi'_{j,t}]+\sum_{i=1,i\neq j}^{N}\pi_{i,t}\\
&+\beta\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\left[u^*_{t+1}(z(\nu\pi_{j,t}+(1-\nu)\pi'_{j,t}),\boldsymbol{\mu}'_{j,t+1})\right] &(49)\\
&=[\nu\pi_{j,t}+(1-\nu)\pi'_{j,t}]+\sum_{i=1,i\neq j}^{N}\pi_{i,t}+\beta\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\left[u^*_{t+1}(\nu z(\pi_{j,t})+(1-\nu)z(\pi'_{j,t}),\boldsymbol{\mu}'_{j,t+1})\right] &(50)\\
&\le\nu\pi_{j,t}+\nu\sum_{i=1,i\neq j}^{N}\pi_{i,t}+\beta\nu\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\left[u^*_{t+1}(z(\pi_{j,t}),\boldsymbol{\mu}'_{j,t+1})\right]+[(1-\nu)\pi'_{j,t}]+(1-\nu)\sum_{i=1,i\neq j}^{N}\pi_{i,t}\\
&+\beta(1-\nu)\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\left[u^*_{t+1}(z(\pi'_{j,t}),\boldsymbol{\mu}'_{j,t+1})\right] &(51)
\end{aligned}$$

Equality 49 holds by the definition of the optimality equation. Equality 50 holds as $z(\pi_{i,t})$ is linear in $\pi_{i,t}$. Inequality 51 holds since $u^*_{t+1}(\cdot,...,\cdot)$ is componentwise convex by induction assumption and because expectation is a linear operator. By definition of the optimality equation and an argument similar to equations 41 and 42,

$$u^*_t(\pi_{1,t},...,\pi_{j,t},...,\pi_{N,t})\ge\pi_{j,t}+\sum_{i=1,i\neq j}^{N}\pi_{i,t}+\beta\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\left[u^*_{t+1}(z(\pi_{j,t}),\boldsymbol{\mu}'_{j,t+1})\right] \tag{52}$$

Similarly;

$$u^*_t(\pi_{1,t},...,\pi'_{j,t},...,\pi_{N,t})\ge\pi'_{j,t}+\sum_{i=1,i\neq j}^{N}\pi_{i,t}+\beta\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\left[u^*_{t+1}(z(\pi'_{j,t}),\boldsymbol{\mu}'_{j,t+1})\right] \tag{53}$$

Therefore, multiplying equation 52 by ν and 53 by $(1-\nu)$ and summing

$$\begin{aligned}
&\nu u^*_t(\pi_{1,t},...,\pi_{j,t},...,\pi_{N,t})+(1-\nu)u^*_t(\pi_{1,t},..,\pi'_{j,t},..,\pi_{k,t},..,\pi_{N,t})\\
&\ge\nu\pi_{j,t}+\nu\sum_{i=1,i\neq j}^{N}\pi_{i,t}+\beta\nu\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\left[u^*_{t+1}(z(\pi_{j,t}),\boldsymbol{\mu}'_{j,t+1})\right]+(1-\nu)\pi'_{j,t}+(1-\nu)\sum_{i=1,i\neq j}^{N}\pi_{i,t}\\
&+\beta(1-\nu)\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\left[u^*_{t+1}(z(\pi_{j,t}),\boldsymbol{\mu}'_{j,t+1})\right] &(54)\\
&\ge u^*_t(\pi_{1,t},...,\nu\pi_{j,t}+(1-\nu)\pi'_{j,t},...,\pi_{N,t}) &(55)
\end{aligned}$$

This completes the proof of convexity for Model TD.

■

In Lemma 3 and Lemma 4, we show that similar properties hold for Model T as well. Lemma 3 shows that $u^*_t(\pi_{1,t},...,\pi_{N,t})$ is nondecreasing in $\pi_{i,t}$ for Model T.

LEMMA 3. *Suppose there are N patients. For all* $i \in \{1,2,...,N\}$ *and for all* $t \in \{1,2,...,T\}$, $u_t^*(\pi_{1,t},...,\pi_{N,t})$ *is nondecreasing in* $\pi_{i,t}$ *for Model T.*

Proof:

Consider two systems, where the information for all patients at the beginning of Period t is denoted by $u_t^*(\pi_{1,t},...,\pi_{j,t},...,\pi_{N,t})$ and $u_t^*(\pi_{1,t},...,\pi'_{j,t},...\pi_{N,t})$, respectively such that $\pi_{j,t} \le \pi'_{j,t}$. We prove $u_t^*(\pi_{1,t},...,\pi_{j,t},...,\pi_{N,t}) \le u_t^*(\pi_{1,t},...,\pi_{j,t},...,\pi_{N,t})$ using induction. From the following equality $u_T(\pi_{1,T},...,\pi_{N,T}) = \sum_{i=1}^N \pi_{i,T}$, it can be shown that the value function is nondecreasing in each $\pi_{i,T}$. Now assume $u_{t+1}(\pi_{1,t+1},\pi_{2,t+1},...,\pi_{N,t+1})$ is nondecreasing in $\pi_{i,t+1}$, and let consider the following cases to show that $u_t(\pi_{1,t},\pi_{2,t},...,\pi_{N,t})$ is also nondecreasing in all $\pi_{i,t}$.

Case(1) Patient j with information $\pi_{j,t}$ is scheduled for an OA while patient k is scheduled for a VA with information $\pi_{k,t}$

We define $\boldsymbol{\mu}'_{j,t+1}$ to represent the information of all patients except for patient j and patient k in period $t+1$.

$$\begin{aligned}\boldsymbol{\mu}'_{j,t+1} = \big(&\gamma^o_{1,t},...,\gamma^o_{j-1,t},\gamma^o_{j+1,t},...,\gamma^o_{\boldsymbol{C^o},t},\gamma^v_{\boldsymbol{C^o}+1,t},...,\gamma^v_{k-1,t},\gamma^v_{k+1,t},...,\gamma^v_{\boldsymbol{C^o}+\boldsymbol{C^v},t}\\ &,z(\pi_{\boldsymbol{C^o}+\boldsymbol{C^v}+1,t}),...,z(\pi_{N,t})\big)\end{aligned} \tag{56}$$

Then, the value function in period t can be represented as follows:

$$\begin{aligned}u_t^*(\pi_{1,t},..,\pi_{j,t},..,\pi_{k,t},..,\pi_{N,t}) = &\sum_{i=1}^N \pi_{i,t} + \beta\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[\pi_{j,t}u^*_{t+1}(\gamma^o_0,\pi_{k,t}\gamma^v_0 + (1-\pi_{k,t})\gamma^v_1,\boldsymbol{\mu}'_{j,t+1})\\ &+(1-\pi_{j,t})u^*_{t+1}(\gamma^o_1,\pi_{k,t}\gamma^v_0+(1-\pi_{k,t})\gamma^v_1,\boldsymbol{\mu}'_{j,t+1})\big]\end{aligned} \tag{57}$$

For the system with state $(\pi_{1,t},..,\pi'_{j,t},..,\pi_{k,t},..,\pi_{N,t})$ the optimal future discounted value function is no less than the future discounted value function following any feasible policy. Thus;

$$\begin{aligned}u_t^*(\pi_{1,t},..,\pi'_{j,t},..,\pi_{k,t},..,\pi_{N,t}) \ge &\pi'_{j,t} + \pi_{k,t} + \sum_{i=1,\neq j,\neq k}^N \pi_{i,t}\\ &+\beta\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[\pi'_{j,t}u^*_{t+1}(\gamma^o_0,\pi_{k,t}\gamma^v_0+(1-\pi_{k,t})\gamma^v_1,\boldsymbol{\mu}'_{j,t+1})\\ &+(1-\pi'_{j,t})u^*_{t+1}(\gamma^o_1,\pi_{k,t}\gamma^v_0+(1-\pi_{k,t})\gamma^v_1,\boldsymbol{\mu}'_{j,t+1})\big]\end{aligned} \tag{58}$$

where left hand side represents the objective function value for the system with state $(\pi_{1,t},..,\pi'_{j,t},..,\pi_{k,t},..,\pi_{N,t})$ in period t and the right hand side of the inequality is the discounted present value following the optimal action for the system with state $(\pi_{1,t},..,\pi_{j,t},..,\pi_{k,t},..,\pi_{N,t})$.

For any realization of $\boldsymbol{\mu}'_{j,t+1}$,

$u^*_{t+1}(\gamma^o_0,\pi_{k,t}\gamma^v_0+(1-\pi_{k,t})\gamma^v_1,\boldsymbol{\mu}'_{j,t+1}) \ge u^*_{t+1}(\gamma^o_1,\pi_{k,t}\gamma^v_0+(1-\pi_{k,t})\gamma^v_1,\boldsymbol{\mu}'_{j,t+1})$ by induction hypothesis as $\gamma^o_0 \ge \gamma^o_1$.

Since $\pi_{j,t} \le \pi'_{j,t}$,

$$\begin{aligned}&\pi_{j,t}u^*_{t+1}(\gamma^o_0,\pi_{k,t}\gamma^v_0+(1-\pi_{k,t})\gamma^v_1,\boldsymbol{\mu}'_{j,t+1}) + (1-\pi_{j,t})u^*_{t+1}(\gamma^o_1,\pi_{k,t}\gamma^v_0+(1-\pi_{k,t})\gamma^v_1,\boldsymbol{\mu}'_{j,t+1})\\ &\le \pi'_{j,t}u^*_{t+1}(\gamma^o_0,\pi_{k,t}\gamma^v_0+(1-\pi_{k,t})\gamma^v_1,\boldsymbol{\mu}'_{j,t+1}) + (1-\pi'_{j,t})u^*_{t+1}(\gamma^o_1,\pi_{k,t}\gamma^v_0+(1-\pi_{k,t})\gamma^v_1,\boldsymbol{\mu}'_{j,t+1})\end{aligned} \tag{59}$$

Thus;

$$\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[\pi_{j,t}u^*_{t+1}(\gamma^o_0,\pi_{k,t}\gamma^v_0+(1-\pi_{k,t})\gamma^v_1,\boldsymbol{\mu}'_{j,t+1})$$

$$+(1-\pi_{j,t})u^*_{t+1}(\gamma^o_1,\pi_{k,t}\gamma^v_0+(1-\pi_{k,t})\gamma^v_1,\boldsymbol{\mu}'_{j,t+1})\big]$$
$$\leq \mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[\pi'_{j,t}u^*_{t+1}(\gamma^o_0,\pi_{k,t}\gamma^v_0+(1-\pi_{k,t})\gamma^v_1,\boldsymbol{\mu}'_{j,t+1})$$
$$+(1-\pi'_{j,t})u^*_{t+1}(\gamma^o_1,\pi_{k,t}\gamma^v_0+(1-\pi_{k,t})\gamma^v_1,\boldsymbol{\mu}'_{j,t+1})\big] \quad (60)$$

Also by knowing that;

$$\sum_{i=1}^{N}\pi_{i,t}=\pi_{j,t}+\sum_{i=1,\neq j}^{N}\pi_{i,t}\leq \pi'_{j,t}+\sum_{i=1,\neq j}^{N}\pi_{i,t} \quad (61)$$

It then implies that

$$u^*_t(\pi_{1,t},..,\pi_{j,t},..,\pi_{k,t},..,\pi_{N,t})\leq u^*_t(\pi_{1,t},..,\pi'_{j,t},..,\pi_{k,t},..,\pi_{N,t})$$

Similarly, for the system with state $(\pi_{1,t},..,\pi_{j,t},..,\pi'_{k,t},..,\pi_{N,t})$ the optimal future discounted value function is no less than the future discounted value function following any feasible policy. Thus;

$$u^*_t(\pi_{1,t},..,\pi_{j,t},..,\pi'_{k,t},..,\pi_{N,t})\geq \pi_{j,t}+\pi'_{k,t}+\sum_{i=1,\neq j,\neq k}^{N}\pi_{i,t}$$
$$+\beta\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[\pi_{j,t}u^*_{t+1}(\gamma^o_0,\pi'_{k,t}\gamma^v_0+(1-\pi'_{k,t})\gamma^v_1,\boldsymbol{\mu}'_{j,t+1})$$
$$+(1-\pi_{j,t})u^*_{t+1}(\gamma^o_1,\pi'_{k,t}\gamma^v_0+(1-\pi'_{k,t})\gamma^v_1,\boldsymbol{\mu}'_{j,t+1})\big] \quad (62)$$

where left hand side is the value for the system with state $(\pi_{1,t},..,\pi_{j,t},..,\pi'_{k,t},..,\pi_{N,t})$ in period t and the right hand side of the inequality is the discounted present value following the optimal action for the system with state $(\pi_{1,t},..,\pi_{j,t},..,\pi_{k,t},..,\pi_{N,t})$.

For any realization of $\boldsymbol{\mu}'_{j,t+1}$, $u^*_{t+1}(\gamma^o_0,\pi'_{k,t}\gamma^v_0+(1-\pi'_{k,t})\gamma^v_1,\boldsymbol{\mu}'_{j,t+1})\geq u^*_{t+1}(\gamma^o_0,\pi_{k,t}\gamma^v_0+(1-\pi_{k,t})\gamma^v_1,\boldsymbol{\mu}'_{j,t+1})$ and $u^*_{t+1}(\gamma^o_1,\pi'_{k,t}\gamma^v_0+(1-\pi'_{k,t})\gamma^v_1,\boldsymbol{\mu}'_{j,t+1})\geq u^*_{t+1}(\gamma^o_1,\pi_{k,t}\gamma^v_0+(1-\pi_{k,t})\gamma^v_1,\boldsymbol{\mu}'_{j,t+1})$ by induction hypothesis as $\gamma^v_0\geq\gamma^v_1$ and $\pi'_{k,t}\geq\pi_{k,t}$. Therefore;

$$\pi_{j,t}u^*_{t+1}(\gamma^o_0,\pi_{k,t}\gamma^v_0+(1-\pi_{k,t})\gamma^v_1,\boldsymbol{\mu}'_{j,t+1})+(1-\pi_{j,t})u^*_{t+1}(\gamma^o_1,\pi_{k,t}\gamma^v_0+(1-\pi_{k,t})\gamma^v_1,\boldsymbol{\mu}'_{j,t+1})$$
$$\leq \pi_{j,t}u^*_{t+1}(\gamma^o_0,\pi'_{k,t}\gamma^v_0+(1-\pi'_{k,t})\gamma^v_1,\boldsymbol{\mu}'_{j,t+1})+(1-\pi_{j,t})u^*_{t+1}(\gamma^o_1,\pi'_{k,t}\gamma^v_0+(1-\pi'_{k,t})\gamma^v_1,\boldsymbol{\mu}'_{j,t+1}) \quad (63)$$

and

$$\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[\pi_{j,t}u^*_{t+1}(\gamma^o_0,\pi_{k,t}\gamma^v_0+(1-\pi_{k,t})\gamma^v_1,\boldsymbol{\mu}'_{j,t+1})$$
$$+(1-\pi_{j,t})u^*_{t+1}(\gamma^o_1,\pi_{k,t}\gamma^v_0+(1-\pi_{k,t})\gamma^v_1,\boldsymbol{\mu}'_{j,t+1})\big]$$
$$\leq \mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[\pi_{j,t}u^*_{t+1}(\gamma^o_0,\pi'_{k,t}\gamma^v_0+(1-\pi'_{k,t})\gamma^v_1,\boldsymbol{\mu}'_{j,t+1})$$
$$+(1-\pi_{j,t})u^*_{t+1}(\gamma^o_1,\pi'_{k,t}\gamma^v_0+(1-\pi'_{k,t})\gamma^v_1,\boldsymbol{\mu}'_{j,t+1})\big] \quad (64)$$

Also by knowing that;

$$\sum_{i=1}^{N}\pi_{i,t}=\pi_{k,t}+\sum_{i=1,\neq k}^{N}\pi_{i,t}\leq \pi'_{k,t}+\sum_{i=1,\neq k}^{N}\pi_{i,t} \quad (65)$$

It then implies that

$$u^*_t(\pi_{1,t},..,\pi_{j,t},..,\pi_{k,t},..,\pi_{N,t})\leq u^*_t(\pi_{1,t},..,\pi_{j,t},..,\pi'_{k,t},..,\pi_{N,t})$$

Case(2) Patient j with information $\pi_{j,t}$ is *not* scheduled for any appointment while patient k is scheduled for a VA with information $\pi_{k,t}$

We define $\boldsymbol{\mu}'_{j,t+1}$ to represent information vector of all patients except Patients j and k as follows:

$$\boldsymbol{\mu}'_{j,t+1} = \big(\gamma^o_{1,t}, ..., \gamma^o_{\boldsymbol{C^o},t}, \gamma^v_{\boldsymbol{C^o}+1,t}, ..., \gamma^v_{k-1,t}, \gamma^v_{k+1,t}, ..., \gamma^v_{\boldsymbol{C^o}+\boldsymbol{C^v},t}, z(\pi_{\boldsymbol{C^o}+\boldsymbol{C^v}+1,t}), ..., z(\pi_{j-1,t}) \\ , z(\pi_{j+1,t}), ..., z(\pi_{N,t})\big) \tag{66}$$

Then the optimally equation can be stated as follows:

$$u^*_t(\pi_{1,t}, .., \pi_{j,t}, .., \pi_{k,t}, .., \pi_{N,t}) = \sum_{i=1}^{N} \pi_{i,t} + \beta \mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}} \big[u^*_{t+1}(z(\pi_{j,t}), \pi_{k,t}\gamma^v_0 + (1-\pi_{k,t})\gamma^v_1, \boldsymbol{\mu}'_{j,t+1})\big] \tag{67}$$

Similarly the optimality inequality for the state $(\pi_{1,t}, .., \pi'_{j,t}, .., \pi_{k,t}, .., \pi_{N,t})$ can be stated as follows:

$$u^*_t(\pi_{1,t}, .., \pi'_{j,t}, .., \pi_{k,t}, .., \pi_{N,t}) \geq \pi'_{j,t} + \sum_{i=1, i\neq j}^{N} \pi_{i,t} + \beta \mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}} \big[u^*_{t+1}(z(\pi'_{j,t}), \pi_{k,t}\gamma^v_0 + (1-\pi_{k,t})\gamma^v_1, \boldsymbol{\mu}'_{j,t+1})\big] \tag{68}$$

Since $z(.)$ is nondecreasing and $\pi_{j,t} \leq \pi'_{j,t}$, $z(\pi_{j,t}) \leq z(\pi'_{j,t})$. Thus, by induction hypothesis, $\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}} \big[u^*_{t+1}(z(\pi_{j,t}), \pi_{k,t}\gamma^v_0 + (1-\pi_{k,t})\gamma^v_1, \boldsymbol{\mu}'_{j,t+1})\big] \leq \mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}} \big[u^*_{t+1}(z(\pi'_{j,t}), \pi_{k,t}\gamma^v_0 + (1-\pi_{k,t})\gamma^v_1, \boldsymbol{\mu}'_{j,t+1})\big]$.

It then implies that

$$u^*_t(\pi_{1,t}, .., \pi_{j,t}, .., \pi_{k,t}, .., \pi_{N,t}) \leq u^*_t(\pi_{1,t}, .., \pi'_{j,t}, .., \pi_{k,t}, .., \pi_{N,t})$$

Similarly, the optimality inequality for the state $(\pi_{1,t}, .., \pi_{j,t}, .., \pi'_{k,t}, .., \pi_{N,t})$ can be stated as follows:

$$u^*_t(\pi_{1,t}, .., \pi_{j,t}, .., \pi'_{k,t}, .., \pi_{I,t}) \geq \pi'_{k,t} + \sum_{i=1, i\neq k}^{N} \pi_{i,t} + \beta \mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}} \big[u^*_{t+1}(z(\pi_{j,t}), \pi'_{k,t}\gamma^v_0 + (1-\pi'_{k,t})\gamma^v_1, \boldsymbol{\mu}'_{j,t+1})\big] \tag{69}$$

Since $\pi_{k,t} \leq \pi'_{k,t}$, and $\gamma^v_0 \geq \gamma^v_1$ the following inequality holds by induction hypothesis: $u^*_{t+1}(z(\pi_{j,t}), \pi_{k,t}\gamma^v_0 + (1-\pi_{k,t})\gamma^v_1, \boldsymbol{\mu}'_{j,t+1}) \leq u^*_{t+1}(z(\pi_{j,t}), \pi'_{k,t}\gamma^v_0 + (1-\pi'_{k,t})\gamma^v_1, \boldsymbol{\mu}'_{j,t+1}))$.

Thus, $\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}} \big[u^*_{t+1}(z(\pi_{j,t}), \pi_{k,t}\gamma^v_0 + (1-\pi_{k,t})\gamma^v_1, \boldsymbol{\mu}'_{j,t+1})\big] \leq \mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}} \big[u^*_{t+1}(z(\pi_{j,t}), \pi'_{k,t}\gamma^v_0 + (1-\pi'_{k,t})\gamma^v_1, \boldsymbol{\mu}'_{j,t+1})\big]$.

It then implies that

$$u^*_t(\pi_{1,t}, .., \pi_{j,t}, .., \pi_{k,t}, .., \pi_{N,t}) \leq u^*_t(\pi_{1,t}, .., \pi_{j,t}, .., \pi'_{k,t}, .., \pi_{N,t})$$

Case(3) Patient j with information $\pi_{j,t}$ is scheduled for an OA while patient k is *not* scheduled for any appointment with information $\pi_{k,t}$

We define a random realization vector $\mu'_{k,t+1}$ except patients k and j as follows:

$$\mu'_{k,t+1} = \big(\gamma^o_{1,t}, ..., \gamma^o_{j-1,t}, \gamma^o_{j+1,t}, ..., \gamma^o_{\boldsymbol{C^o},t}, \gamma^v_{\boldsymbol{C^o}+1,t}, ..., \gamma^v_{\boldsymbol{C^o}+\boldsymbol{C^v},t}, z(\pi_{\boldsymbol{C^o}+\boldsymbol{C^v}+1,t}), ..., z(\pi_{k-1,t}) \\ , z(\pi_{k+1,t}), ..., z(\pi_{N,t})\big) \tag{70}$$

Similar to the previous cases, then the optimality equation can be stated as follows:

$$u^*_t(\pi_{1,t}, .., \pi_{j,t}, .., \pi_{k,t}, .., \pi_{N,t}) = \sum_{i=1}^{N} \pi_{i,t} + \beta \mathbb{E}_{\mu'_{k,t+1}} \big[\pi_{j,t} u^*_{t+1}(\gamma^o_0, z(\pi_{k,t}), \mu'_{k,t+1}) \\ + (1-\pi_{j,t}) u^*_{t+1}(\gamma^o_1, z(\pi_{k,t}), \mu'_{k,t+1})\big] \tag{71}$$

Similarly, the optimality inequality for the state $(\pi_{1,t},..,\pi_{j,t},..,\pi'_{k,t},..,\pi_{N,t})$ can be stated as follows:

$$u_t^*(\pi_{1,t},..,\pi_{j,t},..,\pi'_{k,t},..,\pi_{N,t}) \geq \pi'_{k,t} + \sum_{i=1,i\neq k}^{N} \pi_{i,t} + \beta\mathbb{E}_{\mu'_{k,t+1}}\big[\pi_{j,t}u_{t+1}^*(\gamma_0^o, z(\pi'_{k,t}), \boldsymbol{\mu}'_{j,t+1}) + (1-\pi_{j,t})u_{t+1}^*(\gamma_1^o, z(\pi'_{k,t}), \boldsymbol{\mu}'_{j,t+1})\big] \tag{72}$$

Since $z(.)$ is nondecreasing and $\pi_{k,t} \leq \pi'_{k,t}$, $z(\pi_{k,t}) \leq z(\pi'_{k,t})$. So $\mathbb{E}_{\mu'_{k,t+1}}\big[\pi_{j,t}u_{t+1}^*(\gamma_0^o, z(\pi_{k,t}), \mu'_{k,t+1}) + (1-\pi_{j,t})u_{t+1}^*(\gamma_1^o, z(\pi_{k,t}), \mu'_{k,t+1})\big] \leq \mathbb{E}_{\mu'_{k,t+1}}\big[\pi_{j,t}u_{t+1}^*(\gamma_0^o, z(\pi'_{k,t}), \mu'_{k,t+1}) + (1-\pi_{j,t})u_{t+1}^*(\gamma_1^o, z(\pi'_{k,t}), \mu'_{k,t+1})\big]$.

It then implies that:

$$u_t^*(\pi_{1,t},..,\pi_{j,t},..,\pi_{k,t},..,\pi_{N,t}) \leq u_t^*(\pi_{1,t},..,\pi_{j,t},..,\pi'_{k,t},..,\pi_{N,t}) \tag{73}$$

We then define the optimality inequality for the state $(\pi_{1,t},..,\pi'_{j,t},..,\pi_{k,t},..,\pi_{N,t})$ as follows:

$$u_t^*(\pi_{1,t},..,\pi'_{j,t},..,\pi_{k,t},..,\pi_{N,t}) \geq \pi_{j,t} + \sum_{i=1,\neq j}^{N} \pi_{i,t} + \beta\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[\pi'_{j,t}u_{t+1}^*(\gamma_0^o, z(\pi_{k,t}), \boldsymbol{\mu}'_{j,t+1}) + (1-\pi'_{j,t})u_{t+1}^*(\gamma_1^o, z(\pi_{k,t}), \boldsymbol{\mu}'_{j,t+1})\big] \tag{74}$$

Since $\pi_{j,t} \leq \pi'_{j,t}$ and $\gamma_1^o \leq \gamma_0^o$ the following inequality holds by induction: $u_{t+1}^*(\gamma_1^v, z(\pi_{k,t}), \boldsymbol{\mu}'_{j,t+1}) \leq u_{t+1}^*(\gamma_0^o, z(\pi_{k,t}), \boldsymbol{\mu}'_{j,t+1})$.

Thus,

$$\begin{aligned}&\mathbb{E}_{\mu'_{k,t+1}}\big[\pi_{j,t}u_{t+1}^*(\gamma_0^o, z(\pi_{k,t}), \mu'_{k,t+1}) + (1-\pi_{j,t})u_{t+1}^*(\gamma_1^o, z(\pi_{k,t}), \mu'_{k,t+1})\big]\\ &\leq \mathbb{E}_{\mu'_{k,t+1}}\big[\pi'_{j,t}u_{t+1}^*(\gamma_0^o, z(\pi_{k,t}), \mu'_{k,t+1}) + (1-\pi'_{j,t})u_{t+1}^*(\gamma_1^o, z(\pi_{k,t}), \mu'_{k,t+1})\big]\end{aligned} \tag{75}$$

It then implies that

$$u_t^*(\pi_{1,t},..,\pi_{j,t},..,\pi_{k,t},..,\pi_{N,t}) \leq u_t^*(\pi_{1,t},..,\pi'_{j,t},..,\pi_{k,t},..,\pi_{N,t}) \tag{76}$$

Case(4) Patient j with information $\pi_{j,t}$ is *not* scheduled for any appointment while patient k is *not* scheduled for any appointment with information $\pi_{k,t}$

We define a random realization vector $\mu'_{k,t+1}$ except patients k and j as follows:

$$\mu'_{k,t+1} = \big(\gamma_{1,t}^o, ..., \gamma_{\boldsymbol{C^o},t}^o, \gamma_{\boldsymbol{C^o}+1,t}^v, ..., \gamma_{\boldsymbol{C^o}+\boldsymbol{C^v},t}^v, z(\pi_{\boldsymbol{C^o}+\boldsymbol{C^v}+1,t}), ..., z(\pi_{j-1,t}), z(\pi_{j+1,t}), ..., z(\pi_{k-1,t}), z(\pi_{k+1,t}), ..., z(\pi_{N,t})\big) \tag{77}$$

Similar to the previous cases, then the optimality equation can be stated as follows:

$$u_t^*(\pi_{1,t},..,\pi_{j,t},..,\pi_{k,t},..,\pi_{N,t}) = \sum_{i=1}^{N} \pi_{i,t} + \beta\mathbb{E}_{\mu'_{k,t+1}}\big[u_{t+1}^*(z(\pi_{j,t}), z(\pi_{k,t}), \mu'_{k,t+1})\big] \tag{78}$$

Similarly, the optimality inequality for the state $(\pi_{1,t},..,\pi_{j,t},..,\pi'_{k,t},..,\pi_{N,t})$ can be stated as follows:

$$u_t^*(\pi_{1,t},..,\pi_{j,t},..,\pi'_{k,t},..,\pi_{N,t}) \geq \pi'_{k,t} + \sum_{i=1,i\neq k}^{N} \pi_{i,t} + \beta\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[u_{t+1}^*(z(\pi_{j,t}), z(\pi'_{k,t}), \boldsymbol{\mu}'_{j,t+1})\big] \tag{79}$$

Since $z(.)$ is nondecreasing and $\pi_{k,t} \leq \pi'_{k,t}$, $z(\pi_{k,t}) \leq z(\pi'_{k,t})$. So $\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[u_{t+1}^*(z(\pi_{j,t}), z(\pi_{k,t}), \mu'_{k,t+1})\big] \leq \mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\big[u_{t+1}^*(z(\pi_{j,t}), z(\pi'_{k,t}), \boldsymbol{\mu}'_{j,t+1})\big]$.

It then implies that

$$u_t^*(\pi_{1,t},..,\pi_{j,t},..,\pi_{k,t},..,\pi_{N,t}) \leq u_t^*(\pi_{1,t},..,\pi_{j,t},..,\pi'_{k,t},..,\pi_{N,t})$$

Similarly, we then define the optimality inequality for the state $(\pi_{1,t},..,\pi'_{j,t},..,\pi_{k,t},..,\pi_{N,t})$ as follows:

$$u_t^*(\pi_{1,t},..,\pi'_{j,t},..,\pi_{k,t},..,\pi_{N,t}) \geq \pi'_{j,t} + \sum_{i=1,\neq j}^{N} \pi_{i,t} + \beta \mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}} \left[u_{t+1}^*(z(\pi'_{j,t}), z(\pi_{k,t}), \boldsymbol{\mu}'_{j,t+1}) \right] \tag{80}$$

Since $z(.)$ is nondecreasing and $\pi_{j,t} \leq \pi'_{j,t}$, $z(\pi_{j,t}) \leq z(\pi'_{j,t})$. So $\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}} \left[u_{t+1}^*(z(\pi_{j,t}), z(\pi_{k,t}), \mu'_{k,t+1}) \right] \leq \mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}} \left[u_{t+1}^*(z(\pi'_{j,t}), z(\pi_{k,t}), \mu'_{k,t+1}) \right]$.

It then implies that

$$u_t^*(\pi_{1,t},..,\pi_{j,t},..,\pi_{k,t},..,\pi_{N,t}) \leq u_t^*(\pi_{1,t},..,\pi'_{j,t},..,\pi_{k,t},..,\pi_{N,t})$$

Then this completes the proof that for Model T, $u_t^*(\pi_{1,t},..,\pi_{N,t})$ is nondecreasing in $\pi_{i,t}$ for all $i \in \{1,2,...,N\}$ and for all $t \in \{1,2,...,T\}$.

■

Similarly, in Lemma 4, we show that $u_t^*(\pi_{1,t},...,\pi_{N,t})$ is componentwise convex in $\pi_{i,t}$ for Model T.

Lemma 4. *Suppose there are N patients. For all $i \in \{1,2,...,N\}$ and for all $t \in \{1,2,...,T\}$, $u_t^*(\pi_{1,t},...,\pi_{N,t})$ is componentwise convex in $\pi_{i,t}$ for Model T.*

Proof:

We show that the following inequality holds to prove that the $u_t^*(\pi_{1,t},...,\pi_{N,t})$ function is convex in each $\pi_{j,t}$, $\forall j \in \{1,2,...,N\}$ and all $t \in \{1,2,...,T\}$.

$$\begin{aligned} &u_t^*(\pi_{1,t},..,\nu\pi_{j,t} + (1-\nu)\pi'_{j,t},..,\pi_{k,t},..,\pi_{N,t}) \\ &\leq \nu u_t^*(\pi_{1,t},..,\pi_{j,t},..,\pi_{k,t},..,\pi_{N,t}) + (1-\nu) u_t^*(\pi_{1,t},..,\pi'_{j,t},..,\pi_{k,t},..,\pi_{N,t}) \end{aligned} \tag{81}$$

We use induction to prove equation 81. For period T, $u_T^*(\pi_{1,T},...,\pi_{N,T}) = \sum_{i=1}^{N} \pi_{i,T}$, it is linear and so convex in each $\pi_{i,T}$. With induction, we assume that $u_{t+1}^*(\pi_{1,t+1},...,\pi_{N,t+1})$ is convex in all $\pi_{i,t+1}$. We will be considering the following three cases to prove that $u_t^*(\pi_{1,t},...,\pi_{N,t})$ is convex:

Case(1) Patient j with information $\nu\pi_{j,t} + (1-\nu)\pi'_{j,t}$ is scheduled for an OA

We define $\boldsymbol{\mu}'_{j,t+1}$ to represent the information of all patients except for patient j in period $t+1$.

$$\boldsymbol{\mu}'_{j,t+1} = \left(\gamma^o_{1,t}, ..., \gamma^o_{j-1,t}, \gamma^o_{j+1,t}, ..., \gamma^o_{\boldsymbol{C^o},t}, ..., \gamma^v_{\boldsymbol{C^o}+\boldsymbol{C^v},t}, z(\pi_{\boldsymbol{C^o}+\boldsymbol{C^v}+1,t}), ..., z(\pi_{N,t}) \right) \tag{82}$$

Then we represent the value function as follows:

$$\begin{aligned} &u_t^*(\pi_{1,t},...,\nu\pi_{j,t} + (1-\nu)\pi'_{j,t},...,\pi_{N,t}) = [\nu\pi_{j,t} + (1-\nu)\pi'_{j,t}] + \sum_{i=1,i\neq j}^{N} \pi_{i,t} \\ &+ \beta \mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}} \left[\left(\nu\pi_{j,t} + (1-\nu)\pi'_{j,t}\right) \left[u_{t+1}^*(\gamma^o_0, \boldsymbol{\mu}'_{j,t+1}) \right] + \left(1 - (\nu\pi_{j,t} + (1-\nu)\pi'_{j,t})\right) \left[u_{t+1}^*(\gamma^o_1, \boldsymbol{\mu}'_{j,t+1}) \right] \right] \\ &= \nu\pi_{j,t} + \nu \sum_{i=1,i\neq j}^{N} \pi_{i,t} + \beta\nu \mathbb{E}_{\mu'_{j,t+1}} \left[\pi_{j,t} \left[u_{t+1}^*(\gamma^o_0, \boldsymbol{\mu}'_{j,t+1}) \right] + (1-\pi_{j,t}) \left[u_{t+1}^*(\gamma^o_1, \boldsymbol{\mu}'_{j,t+1}) \right] \right] \end{aligned} \tag{83}$$

$$+(1-\nu)\pi'_{j,t}+(1-\nu)\sum_{i=1,i\neq j}^{N}\pi_{i,t}+\beta(1-\nu)\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\Big[\pi'_{j,t}\big[u^*_{t+1}(\gamma^o_0,\boldsymbol{\mu}'_{j,t+1})\big]+(1-\pi'_{j,t})\big[u^*_{t+1}(\gamma^o_1,\boldsymbol{\mu}'_{j,t+1})\big]\Big] \tag{84}$$

By the definition of optimality equation, the value for the system with state $(\pi_{1,t},...,\pi_{j,t},...,\pi_{N,t})$ is greater or equal than the present discounted value if the policy is to schedule Patient $1,2,...,\boldsymbol{C^o}$ for office appointments and Patient $C_{o+1},...,C_{o+v}$ for virtual appointments in period t. Therefore;

$$u^*_t(\pi_{1,t},...,\pi_{j,t},...,\pi_{N,t})\geq\pi_{j,t}+\sum_{i=1,i\neq j}^{N}\pi_{i,t}+\beta\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\Big[\pi_{j,t}\big[u^*_{t+1}(\gamma^o_0,\boldsymbol{\mu}'_{j,t+1})\big]+(1-\pi_{j,t})\big[u^*_{t+1}(\gamma^o_1,\boldsymbol{\mu}'_{j,t+1})\big]\Big] \tag{85}$$

where the left hand side is the value of the system when system state is $(\pi_{1,t},..,\pi_{j,t},..,\pi_{N,t})$ at period t; and the right hand side is the discounted present value if the action in Period t is to schedule Patient $1,2,...,\boldsymbol{C^o}$ for office appointments and Patient $C_{o+1},...,C_{o+v}$ for virtual appointments and from Period $t+1$ on is governed by optimal policy.

Similarly;

$$u^*_t(\pi_{1,t},...,\pi'_{j,t},...,\pi_{N,t})\geq\pi'_{j,t}+\sum_{i=1,i\neq j}^{N}\pi_{i,t}+\beta\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\Big[\pi'_{j,t}\big[u^*_{t+1}(\gamma^o_0,\boldsymbol{\mu}'_{j,t+1})\big]+(1-\pi'_{j,t})\big[u^*_{t+1}(\gamma^o_1,\boldsymbol{\mu}'_{j,t+1})\big]\Big] \tag{86}$$

Multiplying equation 85 by ν and equation 86 by $(1-\nu)$ and summing, the resulting right hand side is equal to the right hand side of equation 84.

This implies that:

$$u^*_t(\pi_{1,t},...,\nu\pi_{j,t}+(1-\nu)\pi'_{j,t},...,\pi_{N,t})\leq\nu u^*_t(\pi_{1,t},...,\pi_{j,t},...,\pi_{N,t})+(1-\nu)u^*_t(\pi_{1,t},...,\pi'_{j,t},...,\pi_{N,t})$$

Case(2) Patient j with information $\nu\pi_{j,t}+(1-\nu)\pi'_{j,t}$ is scheduled for a VA

We define $\boldsymbol{\mu}'_{j,t+1}$ to represent the information of all patients except for patient j in period $t+1$.

$$\boldsymbol{\mu}'_{j,t+1}=\big(\gamma^o_{1,t},...,\gamma^o_{\boldsymbol{C^o},t},\gamma^v_{\boldsymbol{C^o}+1,t},...,\gamma^v_{j-1,t},\gamma^v_{j+1,t},...,\gamma^v_{\boldsymbol{C^o}+\boldsymbol{C^v},t},z(\pi_{\boldsymbol{C^o}+\boldsymbol{C^v}+1,t}),...,z(\pi_{N,t})\big) \tag{87}$$

Then we represent the value function as follows:

$$\begin{aligned}u^*_t(\pi_{1,t},...,\nu\pi_{j,t}+(1-\nu)\pi'_{j,t},...,\pi_{N,t})&=[\nu\pi_{j,t}+(1-\nu)\pi'_{j,t}]+\sum_{i=1,i\neq j}^{N}\pi_{i,t}\\&+\beta\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\Big[\big[u^*_{t+1}(\big(\nu\pi_{j,t}+(1-\nu)\pi'_{j,t}\big)\gamma^v_0+\big(1-(\nu\pi_{j,t}+(1-\nu)\pi'_{j,t}\big)\gamma^v_1,\boldsymbol{\mu}'_{j,t+1}))\big]\Big]\end{aligned} \tag{88}$$

By the definition of optimality equation, the value for the system with state $(\pi_{1,t},...,\pi_{j,t},...,\pi_{N,t})$ is greater or equal than the present discounted value if the policy is to schedule Patient $1,2,...,\boldsymbol{C^o}$ for office appointments and Patient $C_{o+1},...,C_{o+v}$ for virtual appointments in period t. Therefore;

$$u^*_t(\pi_{1,t},...,\pi_{j,t},...,\pi_{N,t})\geq\pi_{j,t}+\sum_{i=1,i\neq j}^{N}\pi_{i,t}+\beta\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\Big[\big[u^*_{t+1}(\pi_{j,t}\gamma^v_0+(1-\pi_{j,t})\gamma^v_1,\boldsymbol{\mu}'_{j,t+1})\big]\Big] \tag{89}$$

where the left hand side is the value of the system when system state is $(\pi_{1,t},...,\pi_{j,t},...,\pi_{N,t})$ at period t; and the right hand side is the discounted present value if the action in Period t is to schedule Patient $1,2,...,\boldsymbol{C^o}$ for office appointments and Patient $C_{o+1},...,C_{o+v}$ for virtual appointments and from Period $t+1$ on is governed by optimal policy.

Similarly;

$$u_t^*(\pi_{1,t},...,\pi'_{j,t},...,\pi_{N,t}) \geq \pi'_{j,t} + \sum_{i=1,i\neq j}^{N} \pi_{i,t} + \beta\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\left[\left[u_{t+1}^*(\pi'_{j,t}\gamma_0^v + (1-\pi'_{j,t})\gamma_1^v, \boldsymbol{\mu}'_{j,t+1})\right]\right] \tag{90}$$

Multiplying equation 89 by ν and equation 90 by $(1-\nu)$ and summing, the resulting right hand side is equal to the following equation:

$$\begin{aligned}&(1-\nu)\pi'_{j,t} + (1-\nu)\sum_{i=1,i\neq j}^{N} \pi_{i,t} + (1-\nu)\beta\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\left[\left[u_{t+1}^*(\pi'_{j,t}\gamma_0^v + (1-\pi'_{j,t})\gamma_1^v, \boldsymbol{\mu}'_{j,t+1})\right]\right]\\ &\nu\pi_{j,t} + \nu\sum_{i=1,i\neq j}^{N} \pi_{i,t} + \nu\beta\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\left[\left[u_{t+1}^*(\pi_{j,t}\gamma_0^v + (1-\pi_{j,t})\gamma_1^v, \boldsymbol{\mu}'_{j,t+1})\right]\right] \qquad (91)\end{aligned}$$

Since $u_{t+1}^*(\cdot,...,\cdot)$ is componentwise convex by induction the following inequality holds.

$$\begin{aligned}&(1-\nu)\pi'_{j,t} + (1-\nu)\sum_{i=1,i\neq j}^{N} \pi_{i,t} + (1-\nu)\beta\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\left[\left[u_{t+1}^*(\pi'_{j,t}\gamma_0^v + (1-\pi'_{j,t})\gamma_1^v, \boldsymbol{\mu}'_{j,t+1})\right]\right]\\ &\nu\pi_{j,t} + \nu\sum_{i=1,i\neq j}^{N} \pi_{i,t} + \nu\beta\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\left[\left[u_{t+1}^*(\pi_{j,t}\gamma_0^v + (1-\pi_{j,t})\gamma_1^v, \boldsymbol{\mu}'_{j,t+1})\right]\right] \qquad (92)\\ &\geq [\nu\pi_{j,t} + (1-\nu)\pi'_{j,t}] + \sum_{i=1,i\neq j}^{N} \pi_{i,t}\\ &+\beta\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\left[\left[u_{t+1}^*(\big(\nu\pi_{j,t} + (1-\nu)\pi'_{j,t}\big)\gamma_0^v + \big(1-(\nu\pi_{j,t} + (1-\nu)\pi'_{j,t}\big)\gamma_1^v, \boldsymbol{\mu}'_{j,t+1}))\right]\right] \qquad (93)\end{aligned}$$

This implies that:

$$u_t^*(\pi_{1,t},...,\nu\pi_{j,t} + (1-\nu)\pi'_{j,t},...,\pi_{N,t}) \leq \nu u_t^*(\pi_{1,t},...,\pi_{j,t},...,\pi_{N,t}) + (1-\nu)u_t^*(\pi_{1,t},...,\pi'_{j,t},...,\pi_{N,t})$$

Case(3) Patient j with information $\nu\pi_{j,t} + (1-\nu)\pi'_{j,t}$ is *not* scheduled for any appointment

We define $\boldsymbol{\mu}'_{j,t+1}$ to represent information vector of all patients except patient j as follows:

$$\boldsymbol{\mu}'_{j,t+1} = \big(\gamma_{1,t}^o,...,\gamma_{\boldsymbol{C^o},t}^o, \gamma_{\boldsymbol{C^o}+1,t}^v,...,\gamma_{\boldsymbol{C^o}+\boldsymbol{C^v},t}^v, z(\pi_{\boldsymbol{C^o}+\boldsymbol{C^v}+1,t}),...,z(\pi_{j-1,t}), z(\pi_{j+1,t}),...,z(\pi_{N,t})\big) \tag{94}$$

The optimality equation can be stated as follows:

$$\begin{aligned}&u_t^*(\pi_{1,t},...,\nu\pi_{j,t} + (1-\nu)\pi'_{j,t},...,\pi_{N,t}) = [\nu\pi_{j,t} + (1-\nu)\pi'_{j,t}] + \sum_{i=1,i\neq j}^{N} \pi_{i,t}\\ &+\beta\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\left[u_{t+1}^*(z(\nu\pi_{j,t} + (1-\nu)\pi'_{j,t}), \boldsymbol{\mu}'_{j,t+1})\right] \qquad (95)\\ &= [\nu\pi_{j,t} + (1-\nu)\pi'_{j,t}] + \sum_{i=1,i\neq j}^{N} \pi_{i,t} + \beta\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\left[u_{t+1}^*(\nu z(\pi_{j,t}) + (1-\nu)z(\pi'_{j,t}), \boldsymbol{\mu}'_{j,t+1})\right] \qquad (96)\\ &\leq \nu\pi_{j,t} + \nu\sum_{i=1,i\neq j}^{N} \pi_{i,t} + \beta\nu\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\left[u_{t+1}^*(z(\pi_{j,t}), \boldsymbol{\mu}'_{j,t+1})\right] + [(1-\nu)\pi'_{j,t}] + (1-\nu)\sum_{i=1,i\neq j}^{N} \pi_{i,t}\\ &+\beta(1-\nu)\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\left[u_{t+1}^*(z(\pi'_{j,t}), \boldsymbol{\mu}'_{j,t+1})\right] \qquad (97)\end{aligned}$$

Equality 95 holds by the definition of the optimality equation. Equality 96 holds as $z(\pi_{i,t})$ is linear in $\pi_{i,t}$. Inequality 97 holds since $u_{t+1}^*(\cdot,...,\cdot)$ is componentwise convex by induction assumption and because expectation is a linear operator. By definition of the optimality equation and an argument similar to equations 41 and 42,

$$u_t^*(\pi_{1,t},...,\pi_{j,t},...,\pi_{N,t}) \geq \pi_{j,t} + \sum_{i=1,i\neq j}^{N} \pi_{i,t} + \beta\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\left[u_{t+1}^*(z(\pi_{j,t}), \boldsymbol{\mu}'_{j,t+1})\right] \tag{98}$$

Similarly;

$$u_t^*(\pi_{1,t},...,\pi'_{j,t},...,\pi_{N,t}) \geq \pi'_{j,t} + \sum_{i=1,i\neq j}^{N} \pi_{i,t} + \beta \mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\left[u_{t+1}^*(z(\pi'_{j,t}),\boldsymbol{\mu}'_{j,t+1})\right] \tag{99}$$

Therefore, multiplying equation 98 by ν and 99 by $(1-\nu)$ and summing

$$\begin{aligned}
&\nu u_t^*(\pi_{1,t},...,\pi_{j,t},...,\pi_{N,t}) + (1-\nu)u_t^*(\pi_{1,t},..,\pi'_{j,t},..,\pi_{k,t},..,\pi_{N,t}) \\
&\geq \nu\pi_{j,t} + \nu\sum_{i=1,i\neq j}^{N}\pi_{i,t} + \beta\nu\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\left[u_{t+1}^*(z(\pi_{j,t}),\boldsymbol{\mu}'_{j,t+1})\right] + (1-\nu)\pi'_{j,t} + (1-\nu)\sum_{i=1,i\neq j}^{N}\pi_{i,t} \\
&+ \beta(1-\nu)\mathbb{E}_{\boldsymbol{\mu}'_{j,t+1}}\left[u_{t+1}^*(z(\pi_{j,t}),\boldsymbol{\mu}'_{j,t+1})\right] \qquad (100) \\
&\geq u_t^*(\pi_{1,t},...,\nu\pi_{j,t}+(1-\nu)\pi'_{j,t},...,\pi_{N,t}) \qquad (101)
\end{aligned}$$

This completes the proof of convexity for Model T.

■

We next characterize the optimal scheduling policy for VAs when the OA scheduling decisions are fixed in Model TD and T. We use the following Proposition 1 Item 2 to show the proof of Theorem 1.

Proof of Proposition 1 Item 2 Suppose there are N patients and Condition 1 and Condition 2 in Section 4 are met. Consider any arbitrary allocation of $\boldsymbol{C^o}$ patients to OAs capacity in a period $t \in \{1,2,...,T-1\}$. Reindex the remaining patients by their information vectors such that $\pi_{1,t} \leq \pi_{2,t} \leq ... \leq \pi_{N-\boldsymbol{C^o},t}$. Then the optimal policy is to schedule patients $1,2,...,\boldsymbol{C^v}$ for VAs in a period $t \in \{1,2,...,T-1\}$. Without loss of generality, let us define two policies;

Policy 1: Patients $\{1,3,...,C_{v+1}\}$ are scheduled for VAs and patients $\{2,C_{v+2},...,C_{N-\boldsymbol{C^o}}\}$ are not scheduled (NS) in period t.

Policy 2: Patients $\{2,3,...,C_{v+1}\}$ are scheduled for VAs and patients $\{1,C_{v+2},...,C_{N-\boldsymbol{C^o}}\}$ are not scheduled (NS) in period t.

We define $\boldsymbol{\mu}'_{t+1}$ which represents the information vector of all patients except patient 1 and patient 2 in period $t+1$.

$$\boldsymbol{\mu}'_{t+1} = \left(\gamma^v_{3,t},...,\gamma^v_{\boldsymbol{C^v}+1,t}, z(\pi_{\boldsymbol{C^v}+2,t}), z(\pi_{\boldsymbol{C^v}+3,t}),...,z(\pi_{N-\boldsymbol{C^o},t})\right) \tag{102}$$

We prove this item for TD and T models separately.

1-Both Treatment and Diagnosis (TD) Model

If optimal policy is followed from period $t+1$ to the end of the horizon, we define the value of Policy 1 as follows:

$$\begin{aligned}
u_t\left(\pi_{1,t},...,\pi_{N,t}; 1\in VA, 2\in \text{NS}\right) &= \sum_{i=1}^{N}\pi_{i,t} + \beta\mathbb{E}_{\boldsymbol{\mu}'_{t+1}}\left[\pi_{1,t}u_{t+1}^*(\gamma^v_0, z(\pi_{2,t}),\boldsymbol{\mu}'_{t+1})\right. \\
&\left.+ (1-\pi_{1,t})u_{t+1}^*(\gamma^v_1, z(\pi_{2,t}),\boldsymbol{\mu}'_{t+1})\right] \qquad (103) \\
&= \sum_{i=1}^{N}\pi_{i,t} + \beta\mathbb{E}_{\boldsymbol{\mu}'_{t+1}}\left[\pi_{1,t}u_{t+1}^*(p, z(\pi_{2,t}),\boldsymbol{\mu}'_{t+1}) + (1-\pi_{1,t})u_{t+1}^*(q_v p, z(\pi_{2,t}),\boldsymbol{\mu}'_{t+1})\right] \\
&\qquad (104)
\end{aligned}$$

Similarly, if Patient 2 is scheduled instead of Patient 1 in period t and the optimal policy is followed from period $t+1$ until the end of the horizon, the value of Policy 2 can be expressed as follows:

$$\begin{aligned}u_t\big(\pi_{1,t},...,\pi_{N,t};2\in VA,1\in \text{NS}\big) &= \sum_{i=1}^{N}\pi_{i,t}+\beta\mathbb{E}_{\boldsymbol{\mu}'_{t+1}}\big[\pi_{2,t}u^*_{t+1}(z(\pi_{1,t}),\gamma_0^v,\boldsymbol{\mu}'_{t+1})\\ &\quad (1-\pi_{2,t})u^*_{t+1}(z(\pi_{1,t}),\gamma_0^v,\boldsymbol{\mu}'_{t+1})\big] \qquad (105)\\ &= \sum_{i=1}^{N}\pi_{i,t}+\beta\mathbb{E}_{\boldsymbol{\mu}'_{t+1}}\big[\pi_{2,t}u^*_{t+1}(z(\pi_{1,t}),p,\boldsymbol{\mu}'_{t+1})+(1-\pi_{2,t})u^*_{t+1}(z(\pi_{1,t}),q_vp,\boldsymbol{\mu}'_{t+1})\big] \qquad (106)\end{aligned}$$

Now we show that $u_t\big(\pi_{1,t},...,\pi_{N,t};1\in VA,2\in \text{NS}\big)\geq u_t\big(\pi_{1,t},...,\pi_{N,t};2\in VA,1\in \text{NS}\big)$ by showing that for any given realization $\boldsymbol{\mu}'_{t+1}$ the following inequality holds:

$$\begin{aligned}&\pi_{1,t}u^*_{t+1}(p,z(\pi_{2,t}),\boldsymbol{\mu}'_{t+1})+(1-\pi_{1,t})u^*_{t+1}(q_vp,z(\pi_{2,t}),\boldsymbol{\mu}'_{t+1})\\ &\geq \pi_{2,t}u^*_{t+1}(z(\pi_{1,t}),p,\boldsymbol{\mu}'_{t+1})+(1-\pi_{2,t})u^*_{t+1}(z(\pi_{1,t}),q_vp,\boldsymbol{\mu}'_{t+1}) \qquad (107)\end{aligned}$$

let us define $f((\pi_{1,t},\pi_{2,t});\boldsymbol{\mu}'_{t+1}), f((\pi_{2,t},\pi_{1,t});\boldsymbol{\mu}'_{t+1})$ and $g((\pi_{1,t},\pi_{2,t});\boldsymbol{\mu}'_{t+1})$

$$\begin{aligned}f((\pi_{1,t},\pi_{2,t});\boldsymbol{\mu}'_{t+1}) &= \pi_{1,t}u^*_{t+1}(p,z(\pi_{2,t}),\boldsymbol{\mu}'_{t+1})+(1-\pi_{1,t})u^*_{t+1}(q_vp,z(\pi_{2,t}),\boldsymbol{\mu}'_{t+1})\\ f((\pi_{2,t},\pi_{1,t});\boldsymbol{\mu}'_{t+1}) &= \pi_{2,t}u^*_{t+1}(z(\pi_{1,t}),p,\boldsymbol{\mu}'_{t+1})+(1-\pi_{2,t})u^*_{t+1}(z(\pi_{1,t}),q_vp,\boldsymbol{\mu}'_{t+1})\\ g((\pi_{1,t},\pi_{2,t});\boldsymbol{\mu}'_{t+1}) &= f((\pi_{1,t},\pi_{2,t});\boldsymbol{\mu}'_{t+1})-f((\pi_{2,t},\pi_{1,t};\boldsymbol{\mu}'_{t+1}) \qquad (108)\end{aligned}$$

Note that $u^*_{t+1}(z(\pi_{1,t}),p,\boldsymbol{\mu}'_{t+1})=u^*_{t+1}(p,z(\pi_{1,t}),\boldsymbol{\mu}'_{t+1})$ and $u^*_{t+1}(z(\pi_{1,t}),q_vp,\boldsymbol{\mu}'_{t+1})=u^*_{t+1}(q_vp,z(\pi_{1,t}),\boldsymbol{\mu}'_{t+1})$. To prove that the inequality 107 holds we need to show that $g((\pi_{1,t},\pi_{2,t});\boldsymbol{\mu}'_{t+1})\geq 0$

Next, we show Leamma 5. Lemma 5 presents an intermediate result which is used to prove Proposition 1 Item 2.

Lemma 5. *Given any* $0\leq\pi_{1,t}\leq 1$ *there are two possible cases for* $g((\pi_{1,t},1);\boldsymbol{\mu}'_{t+1})$;

(i) When $\pi_{1,t}=1$, $g((\pi_{1,t},1);\boldsymbol{\mu}'_{t+1})=0$

(ii) When $\pi_{1,t}<1$, $g(\pi_{1,t},1);\boldsymbol{\mu}'_{t+1})\geq 0$

where $g((\pi_{1,t},1);\boldsymbol{\mu}'_{t+1})$ is defined as follows:

$$\begin{aligned}g((\pi_{1,t},1);\boldsymbol{\mu}'_{t+1}) &= \big[\pi_{1,t}u^*_{t+1}(p,z(\pi_{2,t}),\boldsymbol{\mu}'_{t+1})+(1-\pi_{1,t})u^*_{t+1}(q_vp,z(\pi_{2,t}),\boldsymbol{\mu}'_{t+1})\big]-u^*_{t+1}(z(\pi_{1,t}),p,\boldsymbol{\mu}'_{t+1})\\ &= \big[\pi_{1,t}u^*_{t+1}(p,p,\boldsymbol{\mu}'_{t+1})+(1-\pi_{1,t})u^*_{t+1}(q_vp,p,\boldsymbol{\mu}'_{t+1})\big]-u^*_{t+1}(z(\pi_{1,t}),p,\boldsymbol{\mu}'_{t+1}) \qquad (109)\end{aligned}$$

Proof:

(i) When $\pi_{1,t}=1$, $g((\pi_{1,t},1);\boldsymbol{\mu}'_{t+1})=u^*_{t+1}(p,p,\boldsymbol{\mu}'_{t+1})-\big[u^*_{t+1}(p,p,\boldsymbol{\mu}'_{t+1})\big]=0$

(ii) When $\pi_{1,t}<1$, the following inequality holds: $g((\pi_{1,t},1);\boldsymbol{\mu}'_{t+1})\geq 0$, since both $u^*_{t+1}(p,p,\boldsymbol{\mu}'_{t+1})$ and $u^*_{t+1}(q_vp,p,\boldsymbol{\mu}'_{t+1})$ are greater and equal to the third term in Equation 109, $u^*_{t+1}(z(\pi_{1,t}),p,\boldsymbol{\mu}'_{t+1})$. Note that $u^*_{t+1}(\cdot,...,\cdot)$ is increasing and according to the Condition 2, the worst possible information value of patient i in period $t+1$ ($\pi_{i,t+1}$) if he was scheduled for a VA in period t ($\pi_{i,t+1}=q_vp$), is better than the best possible information value of patient k in period $t+1$ if he was not scheduled in any appointment in period t, $\pi_{k,t+1}=(z(\pi_{k,t}))$ where $q_v\geq p$.

■

Through Lemma 6, we define another intermediate result that is used in Proposition 1 Item 2.

LEMMA 6. *Given any $0 \le \pi_{1,t} \le 1$, there are two possible cases for $g((\pi_{1,t},0);\boldsymbol{\mu}'_{t+1})$;*

(i) When $\pi_{1,t}=0$, $g((\pi_{1,t},0);\boldsymbol{\mu}'_{t+1})=0$

(ii) When $0 \le \pi_{1,t} \le 1$ $g((\pi_{1,t},0);\boldsymbol{\mu}'_{t+1}) \le 0$

where $g((\pi_{1,t},0);\boldsymbol{\mu}'_{t+1})$ is defined as follows:

$$g((\pi_{1,t},0);\boldsymbol{\mu}'_{t+1}) = \pi_{1,t}u^*_{t+1}(p,z(\pi_{2,t}),\boldsymbol{\mu}'_{t+1}) + (1-\pi_{1,t})u^*_{t+1}(q_v p, z(\pi_{2,t}),\boldsymbol{\mu}'_{t+1}) - u^*_{t+1}(z(\pi_{1,t}),q_v p,\boldsymbol{\mu}'_{t+1}) \tag{110}$$

$$g((\pi_{1,t},0);\boldsymbol{\mu}'_{t+1}) = \pi_{1,t}u^*_{t+1}(p,0,\boldsymbol{\mu}'_{t+1}) + (1-\pi_{1,t})u^*_{t+1}(q_v p,0,\boldsymbol{\mu}'_{t+1}) - u^*_{t+1}(z(\pi_{1,t}),q_v p,\boldsymbol{\mu}'_{t+1}) \tag{111}$$

Proof:

(i) When $\pi_{1,t}=0$, $g((\pi_{1,t},0);\boldsymbol{\mu}'_{t+1})$ can be expressed as follows: $g((\pi_{1,t},0);\boldsymbol{\mu}'_{t+1}) = u^*_{t+1}(q_v p,0,\mu'_{t+1}) - u^*_{t+1}(0,q_v p,\mu'_{t+1}) = 0$, since $u^*_{t+1}(q_v p,0,\mu'_{t+1}) = u^*_{t+1}(0,q_v p,\mu'_{t+1})$.

(ii) We prove this item by contradiction. Suppose $g((\pi_{1,t},0),\boldsymbol{\mu}'_{t+1}) \ge 0$

– Let consider the case where $\pi_{2,t} \in [\pi_{1,t},1]$

when $\pi_{1,t}=\pi_{2,t}$, $g((\pi_{1,t},\pi_{1,t});\boldsymbol{\mu}'_{t+1})=0$ and $g((\pi_{1,t};\pi_{2,t});\boldsymbol{\mu}'_{t+1})>0$ for $\pi_{2,t}\in[\pi_{1,t},1]$ by Lemma 5. By symmetry when $\pi_{2,t}\le\pi_{1,t}$, the following equation holds $g((\pi_{1,t},\pi_{2,t});\boldsymbol{\mu}'_{t+1})\le 0$. Then, for $\pi_{2,t}=0$ the equation becomes $g((\pi_{1,t},0);\boldsymbol{\mu}'_{t+1})\le 0$ which contradicts with the assumption that $g((\pi_{1,t},0);\boldsymbol{\mu}'_{t+1})>0$

– Let consider the case where $\pi_{2,t}\in[0,\pi_{1,t}]$ and according to the assumption, the following inequality should hold: $g((\pi_{1,t},\pi_{2,t});\boldsymbol{\mu}'_{t+1})>0$

If this assumption is true, similarly by symmetry when $\pi_{2,t}\ge\pi_{1,t}$ $g((\pi_{1,t},\pi_{2,t});\boldsymbol{\mu}'_{t+1})\le 0$. When $\pi_{2,t}=1$ it becomes as follows: $g((\pi_{1,t},1);\boldsymbol{\mu}'_{t+1})\le 0$ which contradicts Lemma 5 where $g((\pi_{1,t},1);\boldsymbol{\mu}'_{t+1})\ge 0$.

Lemma 7 presents an intermediate result which is used to prove Proposition 1 Item 2. ■

LEMMA 7. *For any given $\pi_{1,t}$ and $\boldsymbol{\mu}'_{t+1}$, we show that the following inequality holds:*

$$g((\pi_{1,t},(\alpha x+(1-\alpha)y));\boldsymbol{\mu}'_{t+1}) \le \alpha g((\pi_{1,t},x);\boldsymbol{\mu}'_{t+1}) + (1-\alpha)g((\pi_{1,t},y);\boldsymbol{\mu}'_{t+1}) \tag{112}$$

where $\forall\alpha\in[0,1]$ and $x,y\in[0,1]$.

Proof:

By definition of $g((\pi_{1,t},.);\boldsymbol{\mu}'_{t+1})$, the right hand side of Equation 112 can be written as follows:

$$\begin{aligned}
\alpha g((\pi_{1,t},x);\boldsymbol{\mu}'_{t+1}) + (1-\alpha)g((\pi_{1,t},y);\boldsymbol{\mu}'_{t+1}) &= \pi_{1,t}[\alpha u^*_{t+1}(p,z(x),\boldsymbol{\mu}'_{t+1}) + (1-\alpha)u^*_{t+1}(p,z(y),\boldsymbol{\mu}'_{t+1})] \\
&+ (1-\pi_{1,t})[\alpha u^*_{t+1}(q_v p,z(x),\boldsymbol{\mu}'_{t+1}) \\
&+ (1-\alpha)u^*_{t+1}(q_v p,z(y),\boldsymbol{\mu}'_{t+1})] \\
&- [\alpha x+(1-\alpha)y]u^*_{t+1}(z(\pi_{1,t}),p,\boldsymbol{\mu}'_{t+1}) \\
&- [\alpha(1-x)+(1-\alpha)(1-y)]u^*_{t+1}(z(\pi_{1,t}),q_v p,\boldsymbol{\mu}'_{t+1})
\end{aligned} \tag{113}$$

Also, by definition of $g((\pi_{1,t},.);\boldsymbol{\mu}'_{t+1})$, $g((\pi_{1,t},(\alpha x+(1-\alpha)y));\boldsymbol{\mu}'_{t+1})$ can be written as follows:

$$\begin{aligned}
g((\pi_{1,t},(\alpha x+(1-\alpha)y));\boldsymbol{\mu}'_{t+1}) &= \pi_{1,t}u^*_{t+1}(p,\alpha z(x)+(1-\alpha)z(y),\boldsymbol{\mu}'_{t+1}) \\
&+ (1-\pi_{1,t})u^*_{t+1}(q_v p,\alpha z(x)+(1-\alpha)z(y),\boldsymbol{\mu}'_{t+1})
\end{aligned}$$

$$-[\alpha x+(1-\alpha)y]u^*_{t+1}(z(\pi_{1,t}),p,\mu'_{t+1})$$
$$-(1-[\alpha x+(1-\alpha)y])u^*_{t+1}(z(\pi_{1,t}),q_vp,\mu'_{t+1}) \tag{114}$$

Since $u^*_{t+1}(p,z(x),\boldsymbol{\mu}'_{t+1})$ is componentwise convex in each element then;

$$\alpha u^*_{t+1}(p,z(x),\boldsymbol{\mu}'_{t+1})+(1-\alpha)u^*_{t+1}(p,z(y),\boldsymbol{\mu}'_{t+1})\geq u^*_{t+1}(p,\alpha z(x)+(1-\alpha)z(y),\boldsymbol{\mu}'_{t+1}) \tag{115}$$
$$\alpha u^*_{t+1}(q_vp,z(x),\boldsymbol{\mu}'_{t+1})+(1-\alpha)u^*_{t+1}(q_vp,z(y),\boldsymbol{\mu}'_{t+1})\geq u^*_{t+1}(q_vp,\alpha z(x)+(1-\alpha)z(y),\boldsymbol{\mu}'_{t+1}) \tag{116}$$

By substituting Equations 115 and 116 into the right hand side of 113, we prove Equation 112.

By Lemmas 5, 6, and 7, $g((\pi_{1,t},\pi_{2,t});\boldsymbol{\mu}'_{t+1})$ is a univariate continuous and convex function, which takes value $g((\pi_{1,t},\pi_{2,t});\boldsymbol{\mu}'_{t+1})\geq 0$ when $\pi_{2,t}=1$ and $g((\pi_{1,t},\pi_{2,t});\boldsymbol{\mu}'_{t+1})\leq 0$ when $\pi_{2,t}=0$. We also know that $g((\pi_{1,t},\pi_{2,t});\boldsymbol{\mu}'_{t+1})=0$ for $\pi_{1,t}=\pi_{2,t}$. Thus, $\forall\ \pi_{2,t}\geq\pi_{1,t}$, when Condition 2 is met $g((\pi_{1,t},\pi_{2,t});\boldsymbol{\mu}'_{t+1})\geq 0$.

Therefore by the argument in the beginning of the proof, every patient scheduled for VAs must have a smaller information value than patients in the unscheduled group and Condition 2 should be met.

■

2-Only Treatment Case

Consider same policies, i.e., Policy 1 and Policy 2, for Model T. If optimal policy is followed from period $t+1$ to the end of the horizon, Policy 1 can be defined for Model T as follows;

$$\begin{aligned}u_t\big(\pi_{1,t},...,\pi_{N,t};1\in VA,2\in \text{NS}\big)&=\sum_{i=1}^N\pi_{i,t}+\beta\mathbb{E}_{\boldsymbol{\mu}'_{t+1}}\big[u^*_{t+1}(\pi_{1,t}\gamma^v_0+(1-\pi_{1,t})\gamma^v_1,z(\pi_{2,t}),\boldsymbol{\mu}'_{t+1})\big]\\&=\sum_{i=1}^N\pi_{i,t}+\beta\mathbb{E}_{\boldsymbol{\mu}'_{t+1}}\big[u^*_{t+1}(\pi_{1,t}p+(1-\pi_{1,t})q_vp,z(\pi_{2,t}),\boldsymbol{\mu}'_{t+1})\big]\end{aligned} \tag{117}$$

Similarly, if optimal policy is followed from period $t+1$ to the end of the horizon, Policy 2 can be defined for Model T as follows;

$$\begin{aligned}u_t\big(\pi_{1,t},...,\pi_{N,t};2\in VA,1\in \text{NS}\big)&=\sum_{i=1}^N\pi_{i,t}+\beta\mathbb{E}_{\boldsymbol{\mu}'_{t+1}}\big[u^*_{t+1}(z(\pi_{1,t}),\pi_{2,t}\gamma^v_0+(1-\pi_{2,t})\gamma^v_1,\boldsymbol{\mu}'_{t+1})\big]\\&=\sum_{i=1}^N\pi_{i,t}+\beta\mathbb{E}_{\boldsymbol{\mu}'_{t+1}}\big[u^*_{t+1}(z(\pi_{1,t}),\pi_{2,t}p+(1-\pi_{2,t})q_vp,\boldsymbol{\mu}'_{t+1})\big]\end{aligned} \tag{118}$$

Now we show that $u_t\big(\pi_{1,t},...,\pi_{N,t};1\in VA,2\in \text{NS}\big)\geq u_t\big(\pi_{1,t},...,\pi_{N,t};2\in VA,1\in \text{NS}\big)$ by showing that for any given realization $\boldsymbol{\mu}'_{t+1}$ the following inequality holds:

$$u^*_{t+1}(\pi_{1,t}p+(1-\pi_{1,t})q_vp,z(\pi_{2,t}),\boldsymbol{\mu}'_{t+1})\geq u^*_{t+1}(z(\pi_{1,t}),\pi_{2,t}p+(1-\pi_{2,t})q_vp,\boldsymbol{\mu}'_{t+1}) \tag{119}$$

let us define $f((\pi_{1,t},\pi_{2,t});\boldsymbol{\mu}'_{t+1}),f((\pi_{2,t},\pi_{1,t});\boldsymbol{\mu}'_{t+1})$ and $g((\pi_{1,t},\pi_{2,t});\boldsymbol{\mu}'_{t+1})$

$$\begin{aligned}f((\pi_{1,t},\pi_{2,t});\boldsymbol{\mu}'_{t+1})&=u^*_{t+1}(\pi_{1,t}p+(1-\pi_{1,t})q_vp,z(\pi_{2,t}),\boldsymbol{\mu}'_{t+1})\\f((\pi_{2,t},\pi_{1,t});\boldsymbol{\mu}'_{t+1})&=u^*_{t+1}(z(\pi_{1,t}),\pi_{2,t}p+(1-\pi_{2,t})q_vp,\boldsymbol{\mu}'_{t+1})\\g((\pi_{1,t},\pi_{2,t});\boldsymbol{\mu}'_{t+1})&=f((\pi_{1,t},\pi_{2,t}))-f((\pi_{2,t},\pi_{1,t})\end{aligned} \tag{120}$$

■

Lemma 8 presents an intermediate result which is used to prove Proposition 1 Item 2.

LEMMA 8. *Given any* $0 \leq \pi_{1,t} \leq 1$, *there are two possible cases for* $g((\pi_{1,t}, 1); \boldsymbol{\mu}'_{t+1})$;

(i) When $\pi_{1,t} = 1$, $g((\pi_{1,t}, 1); \boldsymbol{\mu}'_{t+1}) = 0$

(ii) When $\pi_{1,t} < 1$, $g(\pi_{1,t}, 1); \boldsymbol{\mu}'_{t+1}) \geq 0$

Proof: (i) When $\pi_{1,t} = 1$, $g((\pi_{1,t}, 1); \boldsymbol{\mu}'_{t+1}) = 0$

$$g((\pi_{1,t}, 1); \boldsymbol{\mu}'_{t+1}) = \left[u^*_{t+1}(\pi_{1,t}p + (1-\pi_{1,t})q_v p, z(\pi_{2,t}), \boldsymbol{\mu}'_{t+1})\right] - u^*_{t+1}(z(\pi_{1,t}), \pi_{2,t}p + (1-\pi_{2,t})q_v p, \boldsymbol{\mu}'_{t+1})$$

$$= \left[u^*_{t+1}(\pi_{1,t}p + (1-\pi_{1,t})q_v p, p, \boldsymbol{\mu}'_{t+1})\right] - u^*_{t+1}(\pi_{1,t}p, p, \boldsymbol{\mu}'_{t+1}) \tag{121}$$

$$= u^*_{t+1}(p, p, \boldsymbol{\mu}'_{t+1}) - \left[u^*_{t+1}(p, p, \boldsymbol{\mu}'_{t+1})\right] = 0 \tag{122}$$

(ii) When $\pi_{1,t} < 1$; the following inequality holds;

$$g((\pi_{1,t}, 1); \boldsymbol{\mu}'_{t+1}) = \left[u^*_{t+1}(\pi_{1,t}p + (1-\pi_{1,t})q_v p, z(\pi_{2,t}), \boldsymbol{\mu}'_{t+1})\right] - u^*_{t+1}(z(\pi_{1,t}), \pi_{2,t}p + (1-\pi_{2,t})q_v p, \boldsymbol{\mu}'_{t+1}) \tag{123}$$

$$\left[u^*_{t+1}(\pi_{1,t}p + (1-\pi_{1,t})q_v p, p, \boldsymbol{\mu}'_{t+1})\right] - u^*_{t+1}(\pi_{1,t}p, p, \boldsymbol{\mu}'_{t+1}) \geq 0 \tag{124}$$

The inequality 124 holds since the value function is a nondecreasing componentwise convex function and $\pi_{1,t}p + (1-\pi_{1,t})q_v p \geq \pi_{1,t}p$ by Condition 2 where it ensures that $q_v p \geq p$.

■

Lemma 9 presents an intermediate result which is used to prove Proposition 1 Item 2.

LEMMA 9. *Given any* $0 \leq \pi_{1,t} \leq 1$, *there are two possible cases for* $g((\pi_{1,t}, 0); \boldsymbol{\mu}'_{t+1})$;

(i) When $\pi_{1,t} = 0$, $g((\pi_{1,t}, 0); \boldsymbol{\mu}'_{t+1}) = 0$

(ii) When $0 \leq \pi_{1,t} \leq 1$, $g((\pi_{1,t}, 0); \boldsymbol{\mu}'_{t+1}) \leq 0$

Proof:

$$g((\pi_{1,t}, 0); \boldsymbol{\mu}'_{t+1}) = \left[u^*_{t+1}(\pi_{1,t}p + (1-\pi_{1,t})q_v p, 0, \boldsymbol{\mu}'_{t+1})\right] - u^*_{t+1}(z(\pi_{1,t}), q_v p, \boldsymbol{\mu}'_{t+1}) \tag{125}$$

(i) When $\pi_{1,t} = 0$, $g((\pi_{1,t}, 0); \boldsymbol{\mu}'_{t+1}) = u^*_{t+1}(q_v p, 0, \boldsymbol{\mu}'_{t+1}) - u^*_{t+1}(0, q_v p, \boldsymbol{\mu}'_{t+1}) = 0$, since $u^*_{t+1}(q_v p, 0, \boldsymbol{\mu}'_{t+1}) = u^*_{t+1}(0, q_v p, \boldsymbol{\mu}'_{t+1})$

(ii) When $0 \leq \pi_{1,t} \leq 1$, $g((\pi_{1,t}, 0), \boldsymbol{\mu}'_{t+1}) \leq 0$.

We prove this item by contradiction. Suppose $g((\pi_{1,t}, 0), \boldsymbol{\mu}'_{t+1}) > 0$.

– Let consider the case where $\pi_{2,t} \in [\pi_{1,t}, 1]$

when $\pi_{1,t} = \pi_{2,t}$, $g((\pi_{1,t}, \pi_{1,t}); \boldsymbol{\mu}'_{t+1}) = 0$. For the cases where $\pi_{2,t} \in [\pi_{1,t}, 1]$, $g((\pi_{1,t}; \pi_{2,t}); \boldsymbol{\mu}'_{t+1}) \geq 0$ by Lemma 8. By symmetry when $\pi_{2,t} \leq \pi_{1,t}$, the following equation holds $g((\pi_{1,t}, \pi_{2,t}); \boldsymbol{\mu}'_{t+1}) \leq 0$. Then, for $\pi_{2,t} = 0$ this inequality becomes $g((\pi_{1,t}, 0); \boldsymbol{\mu}'_{t+1}) \leq 0$ which contradicts with the assumption that $g((\pi_{1,t}, 0); \boldsymbol{\mu}'_{t+1}) > 0$

– Let consider the case where $\pi_{2,t} \in [0, \pi_{1,t}]$

With the assumption that the following inequality should hold: $g((\pi_{1,t}, \pi_{2,t}); \boldsymbol{\mu}'_{t+1}) > 0$ when $\pi_{2,t} \leq \pi_{1,t}$. If this assumption is true, similarly by symmetry the following statement should be true; when $\pi_{2,t} \geq \pi_{1,t}$, $g((\pi_{1,t}, \pi_{2,t}); \boldsymbol{\mu}'_{t+1}) \leq 0$. For $\pi_{2,t} = 1$ this inequality becomes $g((\pi_{1,t}, 1); \boldsymbol{\mu}'_{t+1}) \leq 0$ which contradicts the Lemma 8 which states that $g((\pi_{1,t}, 1); \boldsymbol{\mu}'_{t+1}) \geq 0$ when $\pi_{2,t} = 1$. With Lemma 8 $g((\pi_{1,t}, 1); \boldsymbol{\mu}'_{t+1}) \geq 0$ therefore $g((1, \pi_{1,t}); \boldsymbol{\mu}'_{t+1}) \leq 0$ (since $g((1, \pi_{1,t}) = -g((1, \pi_{1,t}))$, so $g((\pi_{1,t}, 0); \boldsymbol{\mu}'_{t+1}) \leq 0$ that contradicts with the assumption that $g((\pi_{1,t}, 0); \boldsymbol{\mu}'_{t+1}) \geq 0$.

■

Lemma 10 presents an intermediate result which is used to prove Proposition 1 Item 2.

LEMMA 10. *For any given $\pi_{2,t}$ and $\boldsymbol{\mu}'_{t+1}$, we show that $g((\pi_{1,t},\pi_{2,t});\boldsymbol{\mu}'_{t+1})$ is an increasing function in $\pi_{2,t}$ by showing that the following inequality holds for $\pi'_{2,t}\leq\pi_{2,t}$:*

$$g((\pi_{1,t},\pi_{2,t});\boldsymbol{\mu}'_{t+1})-g((\pi_{1,t},\pi'_{2,t});\boldsymbol{\mu}'_{t+1})>0 \tag{126}$$

Proof:

For $\pi_{1,t}=\pi_{2,t}>\pi'_{2,t}$ the inequality 126 can be expressed as follows:

$$g((\pi_{1,t},\pi_{2,t});\boldsymbol{\mu}'_{t+1})-g((\pi_{1,t},\pi'_{2,t});\boldsymbol{\mu}'_{t+1})>0 \tag{127}$$

The left hand side of the inequality 127 is stated as follows:

$$\begin{aligned}
&g((\pi_{1,t},\pi_{2,t});\boldsymbol{\mu}'_{t+1})-g((\pi_{1,t},\pi'_{2,t});\boldsymbol{\mu}'_{t+1})\\
&=u^*_{t+1}(\pi_{1,t}p+(1-\pi_{1,t})q_vp,z(\pi_{1,t}),\boldsymbol{\mu}'_{t+1})-u^*_{t+1}(z(\pi_{1,t}),\pi_{1,t}p+(1-\pi_{1,t})q_vp,\boldsymbol{\mu}'_{t+1})\\
&-\left[u^*_{t+1}(\pi_{1,t}p+(1-\pi_{1,t})q_vp,z(\pi'_{2,t}),\boldsymbol{\mu}'_{t+1})-u^*_{t+1}(z(\pi_{1,t}),\pi'_{2,t}p+(1-\pi'_{2,t})q_vp,\boldsymbol{\mu}'_{t+1})\right] &&(128)\\
&=0-\left[u^*_{t+1}(\pi_{1,t}p+(1-\pi_{1,t})q_vp,z(\pi'_{2,t}),\boldsymbol{\mu}'_{t+1})-u^*_{t+1}(z(\pi_{1,t}),\pi'_{2,t}p+(1-\pi'_{2,t})q_vp,\boldsymbol{\mu}'_{t+1})\right] &&(129)
\end{aligned}$$

where $u^*_{t+1}(\pi_{1,t}p+(1-\pi_{1,t})q_vp,z(\pi_{1,t}),\boldsymbol{\mu}'_{t+1})=u^*_{t+1}(z(\pi_{1,t}),\pi_{1,t}p+(1-\pi_{1,t})q_vp,\boldsymbol{\mu}'_{t+1})$. To show that $g((\pi_{1,t},\pi'_{2,t});\boldsymbol{\mu}'_{t+1})=u^*_{t+1}(\pi_{1,t}p+(1-\pi_{1,t})q_vp,z(\pi'_{2,t}),\boldsymbol{\mu}'_{t+1})-u^*_{t+1}(z(\pi_{1,t}),\pi'_{2,t}p+(1-\pi'_{2,t})q_vp,\boldsymbol{\mu}'_{t+1})\leq0$ we use contradiction. Suppose $g((\pi_{1,t},\pi'_{2,t});\boldsymbol{\mu}'_{t+1})>0$ when $\pi_{1,t}>\pi'_{2,t}$. This assumption contradicts Lemma 9 where $g((\pi_{1,t},0);\boldsymbol{\mu}'_{t+1})\leq0$.

This shows that $g((\pi_{1,t},\pi_{2,t});\boldsymbol{\mu}'_{t+1})$ increases as $\pi_{2,t}$ increases.

■

By Lemmas 8, 9, 10 $g((\pi_{1,t},\pi_{2,t});\boldsymbol{\mu}'_{t+1})$ is a univariate continuous and increasing function in $\pi_{2,t}$, which takes value $g((\pi_{1,t},\pi_{2,t});\boldsymbol{\mu}'_{t+1})\geq0$ when $\pi_{2,t}=1$ and $g((\pi_{1,t},\pi_{2,t});\boldsymbol{\mu}'_{t+1})\leq0$ when $\pi_{2,t}=0$. Therefore, $g((\pi_{1,t},\pi_{2,t});\boldsymbol{\mu}'_{t+1})\leq0$ when $\pi_{2,t}\geq\pi_{1,t}$ and the Conditions 1.2 is met.

Therefore, by the argument in the beginning of the proof, every patient scheduled for VAs must have a smaller information value than the patients who are not scheduled for any medical intervention and the effectiveness of the VAs (q_v) should be greater than or equal to disease progression (p).

■

Lemma 11 characterizes the optimal scheduling policy for OAs and VAs when the non-scheduled patients are fixed in Models TD and T. We use Lemma 1 in the proof of Theorem 1.

LEMMA 11. *Consider any arbitrary $N-\boldsymbol{C^o}-\boldsymbol{C^v}$ patients are not scheduled for any appointments in a period $t\in\{1,2,...,T-1\}$. Reindex the remaining patients by their information vectors such that $\pi_{1,t}\leq\pi_{2,t}\leq...\leq\pi_{\boldsymbol{C^o}+C_{v,t}}$. Then the optimal policy is to schedule patients $1,2,...,\boldsymbol{C^o}$ for OA and patients $\boldsymbol{C^o}+1,\boldsymbol{C^o}+2,...,\boldsymbol{C^o}+\boldsymbol{C^v}$ for VAs in any period $t\in\{1,2,...,T-1\}$.*

Proof:

Without loss of generality, we define the following two policies;

Policy 1: Patients $\{1,3,...,C_{o+1}\}$ are scheduled for OAs and patients $\{2,C_{o+2},...,C_{\boldsymbol{C^o}+\boldsymbol{C^v}}\}$ are scheduled for VAs in period t.

Policy 2: Patients $\{2,3,...,C_{o+1}\}$ are scheduled for OAs and patients $\{1,C_{o+2},...,C_{\boldsymbol{C^o}+\boldsymbol{C^v}}\}$ are scheduled for VAs in period t.

We define $\boldsymbol{\mu}'_{t+1}$ which represents the information vector of all patients except Patient 1 and Patient 2 in period $t+1$.

$$\boldsymbol{\mu}'_{t+1} = \left(\gamma^o_{3,t}, \gamma^o_{4,t}, ..., \gamma^o_{\boldsymbol{C^o}+1,t}, \gamma^v_{\boldsymbol{C^o}+2,t}, \gamma^v_{\boldsymbol{C^o}+3,t}, ..., \gamma^v_{\boldsymbol{C^o}+\boldsymbol{C^v},t}\right) \tag{130}$$

We perform our analyses for the following two model (Model TD and Model D) as follows:

1-Both Treatment and Diagnosis Model (Model TD)

If optimal policy is followed from period $t+1$ to the end of the horizon, we define the value of Policy 1 as follows:

$$\begin{aligned}
u_t\big(\pi_{1,t},...,\pi_{N,t};1\in OA,2\in \text{VA}\big) &= \sum_{i=1}^{N}\pi_{i,t} + \beta\mathbb{E}_{\boldsymbol{\mu}'_{t+1}}\big[\pi_{2,t}u^*_{t+1}(\gamma^o_0,\gamma^v_0,\boldsymbol{\mu}'_{t+1}) \\
&\quad + (1-\pi_{2,t})u^*_{t+1}(\gamma^o_0,\gamma^v_1,\boldsymbol{\mu}'_{t+1})\big] && (131)\\
&= \sum_{i=1}^{N}\pi_{i,t} + \beta\mathbb{E}_{\boldsymbol{\mu}'_{t+1}}\big[\pi_{2,t}u^*_{t+1}(p,p,\boldsymbol{\mu}'_{t+1}) + (1-\pi_{2,t})u^*_{t+1}(p,q_vp,\boldsymbol{\mu}'_{t+1})\big] && (132)
\end{aligned}$$

where $\gamma^o_0=\gamma^o_1=p$, since OAs provide perfect treatment (Condition 1).

Similarly, if optimal policy is followed from period $t+1$ to the end of the horizon, Policy 2 can be defined as follows:

$$\begin{aligned}
u_t\big(\pi_{1,t},...,\pi_{N,t};2\in OA,1\in \text{VA}\big) &= \sum_{i=1}^{N}\pi_{i,t} + \beta\mathbb{E}_{\boldsymbol{\mu}'_{t+1}}\big[\pi_{1,t}u^*_{t+1}(\gamma^v_0,\gamma^o_0,\boldsymbol{\mu}'_{t+1}) \\
&\quad (1-\pi_{1,t})u^*_{t+1}(\gamma^v_1,\gamma^o_0,\boldsymbol{\mu}'_{t+1})\big] && (133)\\
&= \sum_{i=1}^{I}\pi_{i,t} + \beta\mathbb{E}_{\boldsymbol{\mu}'_{t+1}}\big[\pi_{1,t}u^*_{t+1}(p,p,\boldsymbol{\mu}'_{t+1}) + (1-\pi_{1,t})u^*_{t+1}(q_vp,p,\boldsymbol{\mu}'_{t+1})\big] && (134)
\end{aligned}$$

We show that $u_t\big(\pi_{1,t},...,\pi_{N,t};1\in OA,2\in \text{VA}\big) \geq u_t\big(\pi_{1,t},...,\pi_{N,t};2\in OA,1\in \text{VA}\big)$ by showing that for any given realization of $\boldsymbol{\mu}'_{t+1}$, the following inequality holds:

$$\begin{aligned}
&\pi_{2,t}u^*_{t+1}(p,p,\boldsymbol{\mu}'_{t+1}) + (1-\pi_{2,t})u^*_{t+1}(p,q_vp,\boldsymbol{\mu}'_{t+1}) \\
&\geq \pi_{1,t}u^*_{t+1}(p,p,\boldsymbol{\mu}'_{t+1}) + (1-\pi_{1,t})u^*_{t+1}(q_vp,p,\boldsymbol{\mu}'_{t+1}) && (135)
\end{aligned}$$

Since $u^*_{t+1}(q_vp,p,\boldsymbol{\mu}'_{t+1}) = u^*_{t+1}(p,q_vp,\boldsymbol{\mu}'_{t+1})$ and $u^*_{t+1}(p,p,\boldsymbol{\mu}'_{t+1}) \geq u^*_{t+1}(q_vp,p,\boldsymbol{\mu}'_{t+1})$, inequality 135 holds if and only if $\pi_{2,t} \geq \pi_{1,t}$. Therefore, by the argument in the beginning of the proof, every patient scheduled for OAs must have a smaller information value $(\pi_{i,t})$ than the patients scheduled for VAs when OAs provide perfect treatment. ■

2-Only Treatment Model (Model T)

We use the same policies (i.e., Policy 1 and Policy 2) described in Model TD above. If optimal policy is followed from period $t+1$ to the end of the horizon, Policy 1 can be defined as follows:

$$u_t\big(\pi_{1,t},...,\pi_{N,t};1\in \text{OA},2\in \text{VA}\big)=\sum_{i=1}^{N}\pi_{i,t}+\beta\mathbb{E}_{\boldsymbol{\mu}'_{t+1}}\left[u^*_{t+1}(\gamma^o_0,\pi_{2,t}\gamma^v_0+(1-\pi_{2,t})\gamma^v_1,\boldsymbol{\mu}'_{t+1})\right] \tag{136}$$

$$=\sum_{i=1}^{N}\pi_{i,t}+\beta\mathbb{E}_{\boldsymbol{\mu}'_{t+1}}\left[u^*_{t+1}(p,\pi_{2,t}p+(1-\pi_{2,t})q_vp,\boldsymbol{\mu}'_{t+1})\right] \tag{137}$$

Similarly, if optimal policy is followed from period $t+1$ to the end of the horizon, Policy 2 can be defined as follows:

$$u_t\big(\pi_{1,t},...,\pi_{N,t};2\in \text{OA},1\in \text{VA}\big)=\sum_{i=1}^{N}\pi_{i,t}+\beta\mathbb{E}_{\boldsymbol{\mu}'_{t+1}}\left[u^*_{t+1}(\pi_{1,t}\gamma^v_0+(1-\pi_{1,t})\gamma^v_1,\gamma^o_0,\boldsymbol{\mu}'_{t+1})\right] \tag{138}$$

$$=\sum_{i=1}^{N}\pi_{i,t}+\beta\mathbb{E}_{\boldsymbol{\mu}'_{t+1}}\left[u^*_{t+1}(\pi_{1,t}p+(1-\pi_{1,t})q_vp,p,\boldsymbol{\mu}'_{t+1})\right] \tag{139}$$

We show that $u_t\big(\pi_{1,t},...,\pi_{N,t};1\in \text{OA},2\in \text{VA}\big)\geq u_t\big(\pi_{1,t},...,\pi_{N,t};2\in \text{OA},1\in \text{VA}\big)$ by showing that for any given realization of $\boldsymbol{\mu}'_{t+1}$, the following inequality holds:

$$u^*_{t+1}(p,\pi_{2,t}p+(1-\pi_{2,t})q_vp,\boldsymbol{\mu}'_{t+1})\geq u^*_{t+1}(\pi_{1,t}p+(1-\pi_{1,t})q_vp,p,\boldsymbol{\mu}'_{t+1}) \tag{140}$$

where $\pi_{2,t}p+(1-\pi_{2,t})q_vp\geq \pi_{1,t}p+(1-\pi_{1,t})q_vp$ when $\pi_{2,t}\geq\pi_{1,t}$. Then $u^*_{t+1}(p,\pi_{2,t}p+(1-\pi_{2,t})q_vp,\boldsymbol{\mu}'_{t+1})\geq u^*_{t+1}(\pi_{1,t}p+(1-\pi_{1,t})q_vp,p,\boldsymbol{\mu}'_{t+1})$ since $u^*_{t+1}(.,.,\boldsymbol{\mu}'_{t+1})$ is a univariate continuous convex and increasing function. Therefore, by the argument in the beginning of the proof, every patient scheduled for OAs must have a smaller information value than the patients who are scheduled for VAs.

■

Through Theorem 2, we characterize the optimal scheduling policy for OAs in Model D.

Proof of Theorem 2 Suppose there are N patients and OAs provide perfect treatment (i.e., $q_o=1$). Reindex the patients by their information vectors such that $\pi_{1,t}\leq\pi_{2,t}\leq...\leq\pi_{N,t}$. The optimal scheduling policy for OAs in any period $t\in\{1,2,...,T-1\}$ is to schedule patients $1,2,...,\boldsymbol{C^o}$.

The proof of this item is same with the proof of Proposition 1 Item 1 and Lemma 11 since Model D has a similar structure with Model TD when $q_v=0$.

■

Next, Corollary 1 is used to characterize the optimal scheduling policy for VAs in Model D.

Proof of Corollary 1 Suppose there are N patients and OAs provide perfect treatment (i.e., $q_o=1$). If $\boldsymbol{C^o}+\boldsymbol{C^v}=N-1$ and $\boldsymbol{C^v}=1$, it is optimal to schedule either patient for VA among the patients who are not scheduled for OAs.

Since OAs provide perfect treatment, any patient scheduled for an OA will be the healthiest patient group in the next period. When the sickest patients are scheduled for OAs according to the optimal policy stated in Theorem 2 Item 1, there remains two patients and one of the remaining two patients can be scheduled for the VA slot. This item states that, with this setting it is optimal to schedule any of these remaining patients to the VA. Suppose that the office appointment scheduling decisions are fixed and we define two policies without loss of generality. If these two policies are found to be equal to each other, this means that it is indifferent to schedule any of the remaining patient for the VA with this setting.

We define the following two policies:

Policy 1: Patients $\{1,4,...,C_{o+1}\}$ are scheduled for OAs, Patient 2 is scheduled for VA and Patient 3 is not scheduled (NS) for any type of appointment in period t.

Policy 2: Patients $\{1,4,...,C_{o+1}\}$ are scheduled for OAs, Patient 3 is scheduled for VA and Patient 2 is not scheduled (NS) for any type of appointment in period t.

We define $\boldsymbol{\mu}'_{t+1}$ which represents the information vector of all patients except Patient 1, Patient 2 and Patient 3 in period $t+1$.

$$\boldsymbol{\mu}'_{t+1} = \left(\gamma^o_{4,t}, \gamma^o_{5,t} ..., \gamma^o_{N,t}\right) \tag{141}$$

If optimal policy is followed from period $t+1$ to the end of the horizon, Policy 1 can be defined as follows;

$$\begin{aligned} u_t\big(\pi_{1,t},...,\pi_{N,t}; 1\in OA, 2\in VA, 3\in NS\big) &= \sum_{i=1}^{N} \pi_{i,t} + \beta \mathbb{E}_{\boldsymbol{\mu}'_{t+1}}\big[\pi_{2,t} u^*_{t+1}(\gamma^o_0, \gamma^v_0, z(\pi_{3,t}), \boldsymbol{\mu}'_{t+1}) \\ &\quad + (1-\pi_{2,t}) u^*_{t+1}(\gamma^o_0, \gamma^v_1, z(\pi_{3,t}), \boldsymbol{\mu}'_{t+1})\big] && (142) \\ &= \sum_{i=1}^{N} \pi_{i,t} + \beta \mathbb{E}_{\boldsymbol{\mu}'_{t+1}}\big[\pi_{2,t} u^*_{t+1}(p, p, \pi_{3,t} p, \boldsymbol{\mu}'_{t+1}) \\ &\quad + (1-\pi_{2,t}) u^*_{t+1}(p, 0, \pi_{3,t} p, \boldsymbol{\mu}'_{t+1})\big] && (143) \end{aligned}$$

Note that VAs in Model D only provide diagnosis, so $\gamma^v_0 = p$ and $\gamma^v_1 = 0$.

Similarly, if optimal policy is followed from period $t+1$ to the end of the horizon, Policy 2 can be expressed as follows:

$$\begin{aligned} u_t\big(\pi_{1,t},...,\pi_{N,t}; 1\in OA, 2\in NS, 3\in VA\big) &= \sum_{i=1}^{N} \pi_{i,t} + \beta \mathbb{E}_{\boldsymbol{\mu}'_{t+1}}\big[\pi_{3,t} u^*_{t+1}(\gamma^o_0, z(\pi_{2,t}), \gamma^v_0, \boldsymbol{\mu}'_{t+1})\big] && (144) \\ &\quad + (1-\pi_{3,t}) u^*_{t+1}(\gamma^o_0, z(\pi_{2,t}), \gamma^v_1, \boldsymbol{\mu}'_{t+1})\big] && (145) \\ &= \sum_{i=1}^{N} \pi_{i,t} + \beta \mathbb{E}_{\boldsymbol{\mu}'_{t+1}}\big[\pi_{3,t} u^*_{t+1}(p, \pi_{2,t} p, p, \boldsymbol{\mu}'_{t+1}) \\ &\quad + (1-\pi_{3,t}) u^*_{t+1}(p, \pi_{2,t} p, 0, \boldsymbol{\mu}'_{t+1})\big] && (146) \end{aligned}$$

Note that $\boldsymbol{\mu}'_{t+1} = \{p, p, ..., p\}$, as OAs are providing perfect treatment. We show that $u_t\big(\pi_{1,t},...,\pi_{N,t}; 1\in OA, 2\in VA, 3\in \text{NS}\big) = u_t\big(\pi_{1,t},...,\pi_{N,t}; 1\in OA, 2\in \text{NS}, 3\in VA\big)$ by showing that the following equality holds:

$$\begin{aligned} &\pi_{2,t} u^*_{t+1}(p, p, \pi_{3,t} p, \boldsymbol{\mu}'_{t+1}) + (1-\pi_{2,t}) u^*_{t+1}(p, 0, \pi_{3,t} p, \boldsymbol{\mu}'_{t+1}) \\ &= \pi_{3,t} u^*_{t+1}(p, \pi_{2,t} p, p, \boldsymbol{\mu}'_{t+1}) + (1-\pi_{3,t}) u^*_{t+1}(p, \pi_{2,t} p, 0, \boldsymbol{\mu}'_{t+1}) && (147) \end{aligned}$$

We define $g((\pi_{2,t}, \pi_{3,t}); p, \boldsymbol{\mu}'_{t+1})$

$$\begin{aligned} &g((\pi_{2,t}, \pi_{3,t}); p, \boldsymbol{\mu}'_{t+1}) = \big[\pi_{2,t} u^*_{t+1}(p, p, \pi_{3,t} p, \boldsymbol{\mu}'_{t+1}) + (1-\pi_{2,t}) u^*_{t+1}(p, 0, \pi_{3,t} p, \boldsymbol{\mu}'_{t+1})\big] \\ &- \big[\pi_{3,t} u^*_{t+1}(p, \pi_{2,t} p, p, \boldsymbol{\mu}'_{t+1}) + (1-\pi_{3,t}) u^*_{t+1}(p, \pi_{2,t} p, 0, \boldsymbol{\mu}'_{t+1})\big] && (148) \end{aligned}$$

To prove equality 147, we need to show $g((\pi_{2,t}; \pi_{3,t}); p, \boldsymbol{\mu}'_{t+1}) = 0$

Lemma 12 presents an intermediate result which is used to prove Corollary 1.

Lemma 12. *Given any* $0 \le \pi_{2,t} \le 1$ *there are three possible cases for* $g((\pi_{2,t},1);p,\boldsymbol{\mu}'_{t+1})$ *:*

(i) When $\pi_{2,t}=1,\ g((\pi_{2,t},1);p,\boldsymbol{\mu}'_{t+1})=0$

(ii) When $\pi_{2,t}=0,\ g(\pi_{2,t},1);p,\boldsymbol{\mu}'_{t+1})=0$

(iii) When $0<\pi_{2,t}<1,\ g(\pi_{2,t},1);p,\boldsymbol{\mu}'_{t+1})=0$

Proof:

$$g((\pi_{2,t},1);p,\boldsymbol{\mu}'_{t+1})=\left[\pi_{2,t}u^*_{t+1}(p,p,p,\boldsymbol{\mu}'_{t+1})+(1-\pi_{2,t})u^*_{t+1}(p,0,p,\boldsymbol{\mu}'_{t+1})\right]-u^*_{t+1}(p,\pi_{2,t}p,p,\boldsymbol{\mu}'_{t+1}) \tag{149}$$

(i) When $\pi_{2,t}=1,\ g((\pi_{2,t},1);p,\boldsymbol{\mu}'_{t+1})=0$

$$g((\pi_{2,t},1);p,\boldsymbol{\mu}'_{t+1})=u^*_{t+1}(p,p,p,\boldsymbol{\mu}'_{t+1})-\left[u^*_{t+1}(p,p,p,\boldsymbol{\mu}'_{t+1})\right]=0 \tag{150}$$

(ii) When $\pi_{2,t}=0,\ g((\pi_{2,t},1);p,\boldsymbol{\mu}'_{t+1})=0$

$$g((\pi_{2,t},1);p,\boldsymbol{\mu}'_{t+1})=u^*_{t+1}(p,0,p,\boldsymbol{\mu}'_{t+1})-\left[u^*_{t+1}(p,0,p,\boldsymbol{\mu}'_{t+1})\right]=0 \tag{151}$$

(iii) We show one more decision period to clearly observe the impact of VAs when they provide only diagnosis. For the defined policies, when $\pi_{3,t}=1$, in period $t+1$, except the sickest patient (Patient 2) all the other patients will have the health state of "p", which is the healthiest state as it is stated through Policy 1 ($\pi_{2,t}u^*_{t+1}(p,p,p,\boldsymbol{\mu}'_{t+1})+(1-\pi_{2,t})u^*_{t+1}(p,0,p,\boldsymbol{\mu}'_{t+1})$) and Policy 2 ($u^*_{t+1}(p,\pi_{2,t}p,p,\boldsymbol{\mu}'_{t+1})$). In both policies through Theorem 2 item 2 the sickest patient (Patient 2) will be certainly scheduled for an OA in period $t+2$. Since all other patients have the health state of p, any patient can be scheduled for VA slot in period $t+1$. For that reason we define $\boldsymbol{\mu}'_{t+2}$ to represent the information vector of all patients except Patient 2 in period $t+2$.

$$\boldsymbol{\mu}'_{t+2}=\left(\gamma^v_{4,t},\gamma^o_{3,t},\gamma^o_{5,t}...,\gamma^o_{N,t}\right) \tag{152}$$

Thus, we rewrite the value function considering that the optimal policy is applied starting from period $t+2$ to the end of the horizon. With Theorem 2 item 2 the sickest patient will be scheduled for OA in period $t+1$.

$$\begin{aligned} u_t\big(\pi_{1,t},...,\pi_{N,t};1\in OA,2\in VA,3\in NS\big) &= \sum_{i=1}^N \pi_{i,t}+\sum_{i=1}^N \pi_{i,t+1}+\beta\mathbb{E}_{\boldsymbol{\mu}'_{t+2}}\big[\pi_{2,t}u^*_{t+2}(\boldsymbol{\mu}'_{t+2},z(p)) \\ &\quad +(1-\pi_{2,t})u^*_{t+2}(\boldsymbol{\mu}'_{t+2},z(p))\big] \qquad (153) \\ &= \sum_{i=1}^N \pi_{i,t}+\sum_{i=1}^N \pi_{i,t+1}+\beta\mathbb{E}_{\boldsymbol{\mu}'_{t+2}}\big[u^*_{t+2}(\boldsymbol{\mu}'_{t+2},z(p))\big] \qquad (154) \end{aligned}$$

Similarly Policy 2 also can be expressed as follows:

$$u_t\big(\pi_{1,t},...,\pi_{N,t};1\in OA,2\in NS,3\in VA\big)=\sum_{i=1}^N \pi_{i,t}+\sum_{i=1}^N \pi_{i,t+1}+\beta\mathbb{E}_{\boldsymbol{\mu}'_{t+2}}\big[u^*_{t+2}(\boldsymbol{\mu}'_{t+2},z(p))\big] \tag{155}$$

Then it is observed that Equation 154 is same with Equation 155. Thus, when $0<\pi_{2,t}<1,\ g(\pi_{2,t},1);\boldsymbol{\mu}'_{t+1})=0$

Lemma 13 presents an intermediate result which is used to prove Corollary 1.

Lemma 13. *Given any* $0 \le \pi_{2,t} \le 1$ *there are three possible cases for* $g((\pi_{2,t},0);p,\boldsymbol{\mu}'_{t+1})$ *;*

(i) When $\pi_{2,t}=1\ g((\pi_{2,t},0);p,\boldsymbol{\mu}'_{t+1})=0$

(ii) When $\pi_{2,t}=0,\ g(\pi_{2,t},0);p,\boldsymbol{\mu}'_{t+1})=0$

(iii) When $0<\pi_{2,t}<1,\ g(\pi_{2,t},0);p,\boldsymbol{\mu}'_{t+1})=0$

Proof:

$$g((\pi_{1,t},0);p,\boldsymbol{\mu}'_{t+1}) = \left[\pi_{2,t}u^*_{t+1}(p,p,0,\boldsymbol{\mu}'_{t+1}) + (1-\pi_{2,t})u^*_{t+1}(p,0,0,\boldsymbol{\mu}'_{t+1})\right] - u^*_{t+1}(p,\pi_{2,t}p,0,\boldsymbol{\mu}'_{t+1}) \tag{156}$$

(i) When $\pi_{2,t}=1$, $g((\pi_{2,t},0);p,\boldsymbol{\mu}'_{t+1})=0$

$$g((\pi_{2,t},0);p,\boldsymbol{\mu}'_{t+1}) = u^*_{t+1}(p,p,0,\boldsymbol{\mu}'_{t+1}) - \left[u^*_{t+1}(p,p,0,\boldsymbol{\mu}'_{t+1})\right] = 0 \tag{157}$$

(ii) When $\pi_{2,t}=0$, $g((\pi_{2,t},0);p,\boldsymbol{\mu}'_{t+1})=0$

$$g((\pi_{2,t},0);p,\boldsymbol{\mu}'_{t+1}) = u^*_{t+1}(p,0,0,\boldsymbol{\mu}'_{t+1}) - \left[u^*_{t+1}(p,0,0,\boldsymbol{\mu}'_{t+1})\right] = 0 \tag{158}$$

(iii) When $0<\pi_{2,t}<1$, $g((\pi_{2,t},0);p,\boldsymbol{\mu}'_{t+1})=0$.

We prove this item by contradiction. Suppose $g((\pi_{2,t},0);p,\boldsymbol{\mu}'_{t+1})\neq 0$.

– For the cases where $\pi_{3,t}\in[\pi_{2,t},1]$, $g((\pi_{2,t},\pi_{3,t});p,\boldsymbol{\mu}'_{t+1})=0$ by Lemma 12. By symmetry when $\pi_{3,t}\leq\pi_{2,t}$, the following equality should hold as well: $g((\pi_{2,t},\pi_{3,t});p,\boldsymbol{\mu}'_{t+1})=0$. Then, when $\pi_{3,t}=0$; $g((\pi_{2,t},0);p,\boldsymbol{\mu}'_{t+1})=0$ which contradicts the assumption that $g((\pi_{2,t},0);p,\boldsymbol{\mu}'_{t+1})\neq 0$.

– For the cases where $\pi_{3,t}\in[0,\pi_{2,t}]$, according to the assumption the following equality should hold: $g((\pi_{2,t},\pi_{3,t});p,\boldsymbol{\mu}'_{t+1})\neq 0$. If this assumption is true, by symmetry $g((\pi_{2,t},\pi_{3,t});p,\boldsymbol{\mu}'_{t+1})\neq 0$ should also hold, when $\pi_{3,t}\geq\pi_{2,t}$. Then, when $\pi_{3,t}=1$; $g((\pi_{2,t},1);p,\boldsymbol{\mu}'_{t+1})\neq 0$ which contradicts the Lemma 12 that $g((\pi_{2,t},1);p,\boldsymbol{\mu}'_{t+1})=0$.

Therefore, by the argument in the beginning of the proof, if $\boldsymbol{C^o}+\boldsymbol{C^v}=N1$, $\boldsymbol{C^v}=1$, and $q_o=1$ it is optimal to schedule either patient for VA among the patients who are not scheduled for OAs.

■

Theorem 3 Item 1 characterizes the cases where VAs having only diagnosis do not improve the overall health status of the patients.

Proof of Theorem 3 Item 1 Suppose there are N patients $1,2,...,N\in\mathcal{I}$ and OAs provide perfect treatment (i.e., $q_o=1$). If $\boldsymbol{C^o}+\boldsymbol{C^v}=N$ and $\boldsymbol{C^o}\geq\boldsymbol{C^v}$, then the VAs do not improve overall health status of patient population. Through these conditions, it is guaranteed that every patient is scheduled for either an office or a virtual appointment in a period t and all patients scheduled for a VA in period t will be scheduled for an OA in period $t+1$ according to optimal policy since there are enough number of OA slots.

In order to show the impact of VAs that provide only diagnosis, without loss of generality we define two policies as follows:

Policy 1: Patients $\{2,3,...,\boldsymbol{C^o}+1\}$ are scheduled for OAs and patients $\{1,\boldsymbol{C^o}+2,\boldsymbol{C^o}+3,...,N\}$ are scheduled for VAs with diagnosis

Policy 2: Patients $\{2,3,...,\boldsymbol{C^o}+1\}$ are scheduled for OAs, patients $\{\boldsymbol{C^o}+2,\boldsymbol{C^o}+3,...,N\}$ are scheduled for VAs with diagnosis and patient 1 is not scheduled for any appointment.

We define $\boldsymbol{\mu}'_{t+1}$ which represents the information vector of all patients except patient 1 in period $t+1$.

$$\boldsymbol{\mu}'_{t+1} = \left(\gamma^o_{2,t},...,\gamma^o_{\boldsymbol{C^o}+1,t},\gamma^v_{\boldsymbol{C^o}+2,t},...,\gamma^v_{C_N,t}\right) \tag{159}$$

If optimal policy is followed from period $t+1$ to end of the horizon, the value of Policy 1 can be defined as follows:

$$u_t\big(\pi_{1,t},...,\pi_{N,t};1\in VA\big)=\sum_{i=1}^{N}\pi_{i,t}+\beta\mathbb{E}_{\boldsymbol{\mu}'_{t+1}}\big[\pi_{1,t}u^*_{t+1}(\gamma^v_0,\boldsymbol{\mu}'_{t+1})+(1-\pi_{1,t})u^*_{t+1}(\gamma^v_1,\boldsymbol{\mu}'_{t+1})\big] \tag{160}$$

$$=\sum_{i=1}^{N}\pi_{i,t}+\beta\mathbb{E}_{\boldsymbol{\mu}'_{t+1}}\big[\pi_{1,t}u^*_{t+1}(p,\boldsymbol{\mu}'_{t+1})+(1-\pi_{1,t})u^*_{t+1}(0,\boldsymbol{\mu}'_{t+1})\big] \tag{161}$$

Similarly, if optimal policy is followed from period $t+1$ to the end of the horizon, we define the value of Policy 2 as follows;

$$u_t\big(\pi_{1,t},...,\pi_{N,t};1\in NS\big)=\sum_{i=1}^{N}\pi_{i,t}+\beta\mathbb{E}_{\boldsymbol{\mu}'_{t+1}}\big[u^*_{t+1}(z(\pi_{1,t}),\boldsymbol{\mu}'_{t+1})\big] \tag{162}$$

$$=\sum_{i=1}^{N}\pi_{i,t}+\beta\mathbb{E}_{\boldsymbol{\mu}'_{t+1}}\big[u^*_{t+1}(\pi_{1,t}p,\boldsymbol{\mu}'_{t+1})\big] \tag{163}$$

Now we show that $u_t\big(\pi_{1,t},...,\pi_{N,t};1\in VA\big)=u_t\big(\pi_{1,t},...,\pi_{N,t};1\in NS\big)$ by showing that for any given realization of $\boldsymbol{\mu}'_{t+1}$ the following inequality holds:

$$\pi_{1,t}u^*_{t+1}(p,\boldsymbol{\mu}'_{t+1})+(1-\pi_{1,t})u^*_{t+1}(0,\boldsymbol{\mu}'_{t+1})=u^*_{t+1}(\pi_{1,t}p,\boldsymbol{\mu}'_{t+1}) \tag{164}$$

We define $g((\pi_{1,t});p,\boldsymbol{\mu}'_{t+1})$ as follows:

$$\begin{aligned}g((\pi_{1,t});p,\boldsymbol{\mu}'_{t+1})=&\pi_{1,t}u^*_{t+1}(p,\boldsymbol{\mu}'_{t+1})+(1-\pi_{1,t})u^*_{t+1}(0,\boldsymbol{\mu}'_{t+1})\\&-u^*_{t+1}(\pi_{1,t}p,\boldsymbol{\mu}'_{t+1})\end{aligned} \tag{165}$$

To prove equality 164, we need to show that $g((\pi_{1,t});p,\boldsymbol{\mu}'_{t+1})=0$

Lemma 14 presents an intermediate result which is used to prove Theorem 3 Item 1.

Lemma 14. *There are three possible cases for* $g((\pi_{1,t});p,\boldsymbol{\mu}'_{t+1})$*;*

(i) When $\pi_{1,t}=0$*;* $g((\pi_{1,t});p,\boldsymbol{\mu}'_{t+1})=0$*;*

(ii) When $\pi_{1,t}=1$*;* $g((\pi_{1,t});p,\boldsymbol{\mu}'_{t+1})=0$*;*

(iii) When $0<\pi_{1,t}<1$*;* $g((\pi_{1,t});p,\boldsymbol{\mu}'_{t+1})=0$*;*

Proof:

(i) When $\pi_{1,t}=0$; $g((\pi_{1,t});p,\boldsymbol{\mu}'_{t+1})=0$;

$$g((0);p,\boldsymbol{\mu}'_{t+1})=u^*_{t+1}(0,\boldsymbol{\mu}'_{t+1})-u^*_{t+1}(0,\boldsymbol{\mu}'_{t+1})=0 \tag{166}$$

(ii) When $\pi_{1,t}=1$; $g((\pi_{1,t});p,\boldsymbol{\mu}'_{t+1})=0$;

$$g((1);p,\boldsymbol{\mu}'_{t+1})=u^*_{t+1}(p,\boldsymbol{\mu}'_{t+1})-u^*_{t+1}(p,\boldsymbol{\mu}'_{t+1})=0 \tag{167}$$

(iii) We show one more decision period to clearly observe the impact of VAs when they provide only diagnosis. For Policy 1 and Policy 2 in period $t+1$, except the patients scheduled for VAs (i.e., Patients 1, $\boldsymbol{C^o}+2,\boldsymbol{C^o}+3,...N$) all the other patients will have the health status of "p", which is the healthiest status. Since the sickest patients will be scheduled for OAs according to Theorem 2 Item 2, patients that are scheduled for the VA in period t, will be scheduled for the OA in period $t+1$ and the remaning patients will be scheduled for VAs.

We define $\boldsymbol{\mu}'_{t+2}$ which represents the information vector of all patients except Patient 1 in period $t+2$, as they will have the same information vector for both policies.

$$\boldsymbol{\mu}'_{t+2} = \left(\gamma^v_{2,t}, ..., \gamma^v_{\boldsymbol{C^v}+1,t}, \gamma^o_{\boldsymbol{C^v}+2,t}, ..., \gamma^o_{C_N,t}\right) \tag{168}$$

If optimal policy is followed starting at period $t+2$ to the end of horizon, the value of Policy 1 can be defined as follows:

$$\begin{aligned} u_t\left(\pi_{1,t}, ..., \pi_{N,t}; 1 \in VA\right) &= \sum_{i=1}^{N} \pi_{i,t} + \sum_{i=1}^{N} \pi_{i,t+1} + \beta \mathbb{E}_{\boldsymbol{\mu}'_{t+2}}\left[\pi_{1,t} u^*_{t+2}(\gamma^o_0, \boldsymbol{\mu}'_{t+2}) + (1-\pi_{1,t}) u^*_{t+2}(\gamma^o_0, \boldsymbol{\mu}'_{t+2})\right] && (169) \\ &= \sum_{i=1}^{N} \pi_{i,t} + \sum_{i=1}^{N} \pi_{i,t+1} + \beta \mathbb{E}_{\boldsymbol{\mu}'_{t+2}}\left[\pi_{1,t} u^*_{t+2}(p, \boldsymbol{\mu}'_{t+2}) + (1-\pi_{1,t}) u^*_{t+2}(p, \boldsymbol{\mu}'_{t+2})\right] && (170) \end{aligned}$$

Similarly, if optimal policy is followed starting at period $t+2$ to the end of horizon, the value of Policy 2 can be defined as follows:

$$\begin{aligned} u_t\left(\pi_{1,t}, ..., \pi_{N,t}; 1 \in NS\right) &= \sum_{i=1}^{N} \pi_{i,t} + \sum_{i=1}^{N} \pi_{i,t+1} + \beta \mathbb{E}_{\mu'_{t+2}}\left[u^*_{t+2}(\gamma^o_0, \mu'_{t+2})\right] && (171) \\ &= \sum_{i=1}^{N} \pi_{i,t} + \sum_{i=1}^{N} \pi_{i,t+1} + \beta \mathbb{E}_{\boldsymbol{\mu}'_{t+2}}\left[u^*_{t+2}(p, \boldsymbol{\mu}'_{t+2})\right] && (172) \end{aligned}$$

Thus, it can be shown that:

$$\begin{aligned} g((\pi_{1,t}); p, \boldsymbol{\mu}'_{t+1}) &= \sum_{i=1}^{N} \pi_{i,t} + \sum_{i=1}^{N} \pi_{i,t+1} + \beta \mathbb{E}_{\boldsymbol{\mu}'_{t+2}}\left[\pi_{1,t} u^*_{t+2}(p, \boldsymbol{\mu}'_{t+2}) + (1-\pi_{1,t}) u^*_{t+2}(p, \boldsymbol{\mu}'_{t+2})\right] \\ &\quad - \sum_{i=1}^{N} \pi_{i,t} + \sum_{i=1}^{N} \pi_{i,t+1} + \beta \mathbb{E}_{\boldsymbol{\mu}'_{t+2}}\left[u^*_{t+2}(p, \boldsymbol{\mu}'_{t+2})\right] \\ &= 0 && (173) \end{aligned}$$

■

By Lemma 14, $g((\pi_{1,t}); p, \boldsymbol{\mu}'_{t+1})$ is a univariate continuous function which takes values of $g((\pi_{1,t}); p, \boldsymbol{\mu}'_{t+1}) = 0$ for any $0 \le \pi_{i,t} \le 1$. Therefore, VAs do not improve overall health status of patient population, when OAs provide perfect treatment (i.e., $q_o = 1$) and $\boldsymbol{C^o} + \boldsymbol{C^v} = N$ and $\boldsymbol{C^o} \ge \boldsymbol{C^v}$.

■

Through Theorem 3 Item 2, we define the cases where VAs providing only diagnosis can improve the overall health status of the patient population by providing information.

Proof of Theorem 3 Item 2 Suppose there are N patients and OAs provide perfect treatment (i.e., $q_o = 1$). If $\boldsymbol{C^o} + \boldsymbol{C^v} < N$ and $\boldsymbol{C^o} < \boldsymbol{C^v}$, then VAs can improve overall health status of patient population. To show the improvement impact of VAs, we define two policies where VAs provide only diagnosis. Without loss of generality, the policies are defined as follows:

Policy 1: Patients $\{1, 3, 4, ..., \boldsymbol{C^v} + 1\}$ are scheduled for a VA with diagnosis and Patients $\{\boldsymbol{C^v} + 2, ..., \boldsymbol{C^o} + \boldsymbol{C^v} + 1\}$ are scheduled for OA and Patients $\{2, \boldsymbol{C^o} + \boldsymbol{C^v} + 2, ..., C_N\}$ are not scheduled for any appointment.

Policy 2: Patients $\{3, 4, ..., \boldsymbol{C^v} + 1\}$ are scheduled for a VA with diagnosis and Patients $\{\boldsymbol{C^v} + 2, ..., \boldsymbol{C^o} + \boldsymbol{C^v} + 1\}$ are scheduled for OA and Patients $\{1, 2, \boldsymbol{C^o} + \boldsymbol{C^v} + 2, ..., C_N\}$ are not scheduled for any appointment.

We define $\boldsymbol{\mu}'_{t+1}$ which represents the information vector of all patients except Patient 1 and Patient 2 in period $t+1$.

$$\boldsymbol{\mu}'_{t+1} = \big(\gamma^v_{3,t}, ..., \gamma^v_{\boldsymbol{C^v}+1,t}, \gamma^o_{\boldsymbol{C^v}+2,t}, ..., \gamma^o_{\boldsymbol{C^o}+\boldsymbol{C^v}+1,t}, z(\pi_{\boldsymbol{C^o}+\boldsymbol{C^v}+2,t}), ..., z(\pi_{N,t})\big) \tag{174}$$

If optimal policy is followed from period $t+1$ to the end of the horizon, the value of Policy 1 is defined as follows:

$$\begin{aligned} u_t\big(\pi_{1,t}, ..., \pi_{N,t}; 1 \in VA, 2 \in NS\big) &= \sum_{i=1}^{N} \pi_{i,t} + \beta \mathbb{E}_{\boldsymbol{\mu}'_{t+1}} \big[\pi_{1,t} u^*_{t+1}(\gamma^v_0, z(\pi_{2,t}), \boldsymbol{\mu}'_{t+1}) \\ &\quad + (1-\pi_{1,t}) u^*_{t+1}(\gamma^v_1, z(\pi_{2,t}), \boldsymbol{\mu}'_{t+1})\big] \qquad (175) \\ &= \sum_{i=1}^{N} \pi_{i,t} + \beta \mathbb{E}_{\boldsymbol{\mu}'_{t+1}} \big[\pi_{1,t} u^*_{t+1}(p, \pi_{2,t}p, \boldsymbol{\mu}'_{t+1}) + (1-\pi_{1,t}) u^*_{t+1}(0, \pi_{2,t}p, \boldsymbol{\mu}'_{t+1})\big] \qquad (176) \end{aligned}$$

Similarly if optimal policy is followed from period $t+1$ to the end of the horizon, the value of Policy 2 is described as follows:

$$\begin{aligned} u_t\big(\pi_{1,t}, ..., \pi_{N,t}; 1 \in NS, 2 \in NS\big) &= \sum_{i=1}^{N} \pi_{i,t} + \beta \mathbb{E}_{\boldsymbol{\mu}'_{t+1}} \big[u^*_{t+1}(z(\pi_{1,t}), z(\pi_{2,t}), \boldsymbol{\mu}'_{t+1})\big] \qquad (177) \\ &= \sum_{i=1}^{N} \pi_{i,t} + \beta \mathbb{E}_{\boldsymbol{\mu}'_{t+1}} \big[u^*_{t+1}(\pi_{1,t}p, \pi_{2,t}p, \boldsymbol{\mu}'_{t+1})\big] \qquad (178) \end{aligned}$$

If VAs can improve overall health status of patient population; the following inequality should hold: $u_t\big(\pi_{1,t}, ..., \pi_{N,t}; 1 \in VA, 2 \in NS\big) \geq u_t\big(\pi_{1,t}, ..., \pi_{N,t}; 1 \in NS, 2 \in NS\big)$

We define $g((\pi_{1,t}, \pi_{2,t}); \boldsymbol{\mu}'_{t+1}) = \pi_{1,t} u^*_{t+1}(p, \pi_{2,t}p, \boldsymbol{\mu}'_{t+1}) + (1-\pi_{1,t}) u^*_{t+1}(0, \pi_{2,t}p, \boldsymbol{\mu}'_{t+1}) - u^*_{t+1}(\pi_{1,t}p, \pi_{2,t}p, \boldsymbol{\mu}'_{t+1})$

By definition, $u^*_{t+1}(.,., \boldsymbol{\mu}'_{t+1})$ is componentwise convex in each element then;

$$\pi_{1,t} u^*_{t+1}(p, \pi_{2,t}p, \boldsymbol{\mu}'_{t+1}) + (1-\pi_{1,t}) u^*_{t+1}(0, \pi_{2,t}p, \boldsymbol{\mu}'_{t+1}) \geq u^*_{t+1}(\pi_{1,t}p, \pi_{2,t}p, \boldsymbol{\mu}'_{t+1}) \tag{179}$$

Thus, $g((\pi_{1,t}, \pi_{2,t}); \boldsymbol{\mu}'_{t+1}) \geq 0$

By the argument in the beginning of the proof, VAs can improve the overall health status of patients when OAs provide perfect treatment (i.e., $q_o = 1$) if $\boldsymbol{C^o} + \boldsymbol{C^v} < N$ and $\boldsymbol{C^o} < \boldsymbol{C^v}$.

■

Through Theorem 3 Item 3, we next define another condition under which the VAs that provide only diagnosis can improve the health status of the population.

Proof of Theorem 3 Item 3 If $\boldsymbol{C^o} + \boldsymbol{C^v} = N$ and $\boldsymbol{C^o} \geq \boldsymbol{C^v}$, and the treatment is not perfect, $q_o = 1 - \epsilon$ where $0 < \epsilon < 1$, then VAs can improve overall health status of patient population.

Since there are N patients and total of $\boldsymbol{C^o} + \boldsymbol{C^v} = N$ appointments, all patients are scheduled to either an office or a virtual appointment in a period t. In Theorem 3 item 1, it is shown that VAs do not improve the overall health status of patients when OAs provide perfect treatment if $\boldsymbol{C^o} + \boldsymbol{C^v} = N$ and $\boldsymbol{C^o} \geq \boldsymbol{C^v}$. In this item, on the other hand, OAs do not provide perfect treatment. Without loss of generality, we define two policies where VAs provide only diagnosis as follows:

Policy 1: Patient 1 is scheduled for an OA and Patient 2 is scheduled for a VA.

Policy 2: Patient 1 is scheduled for an OA and Patient 2 is not scheduled for any appointment.

We define $\boldsymbol{\mu}'_{t+1}$ which represents the information vector of all patients except Patients 1 and 2 in period $t+1$.

$$\boldsymbol{\mu}'_{t+1} = \left(\gamma^o_{3,t}, ..., \gamma^o_{\boldsymbol{C^o}+2,t}, \gamma^v_{\boldsymbol{C^o}+3,t}, ..., \gamma^v_{C_N,t}\right) \tag{180}$$

If optimal policy is followed from period $t+1$ to the end of horizon, Policy 1 can be defined as follows:

$$\begin{aligned} u_t\big(\pi_{1,t}, ..., \pi_{N,t}; 1 \in OA, 2 \in VA\big) = &\sum_{i=1}^{N} \pi_{i,t} + \beta \mathbb{E}_{\boldsymbol{\mu}'_{t+1}} \big[\pi_{1,t}\pi_{2,t} u^*_{t+1}(\gamma^o_0, \gamma^v_0, \boldsymbol{\mu}'_{t+1}) \\ &+ \pi_{1,t}(1-\pi_{2,t}) u^*_{t+1}(\gamma^o_0, \gamma^v_1, \boldsymbol{\mu}'_{t+1}) + (1-\pi_{1,t})\pi_{2,t} u^*_{t+1}(\gamma^o_1, \gamma^v_0, \boldsymbol{\mu}'_{t+1}) \\ &+ (1-\pi_{1,t})(1-\pi_{2,t}) u^*_{t+1}(\gamma^o_1, \gamma^v_1, \boldsymbol{\mu}'_{t+1})\big] \qquad (181) \\ = &\sum_{i=1}^{N} \pi_{i,t} + \beta \mathbb{E}_{\boldsymbol{\mu}'_{t+1}} \big[\pi_{1,t}\pi_{2,t} u^*_{t+1}(p, p, \boldsymbol{\mu}'_{t+1}) + \pi_{1,t}(1-\pi_{2,t}) u^*_{t+1}(p, 0, \boldsymbol{\mu}'_{t+1}) \\ &+ (1-\pi_{1,t})\pi_{2,t} u^*_{t+1}(q_o p, p, \boldsymbol{\mu}'_{t+1}) + (1-\pi_{1,t})(1-\pi_{2,t}) u^*_{t+1}(q_o p, 0, \boldsymbol{\mu}'_{t+1})\big] \qquad (182) \end{aligned}$$

Similarly, if optimal policy is followed from $t+1$ to the end of the horizon, Policy 2 can be defined as follows:

$$\begin{aligned} u_t\big(\pi_{1,t}, ..., \pi_{N,t}; 1 \in OA, 2 \in NS\big) = &\sum_{i=1}^{N} \pi_{i,t} + \beta \mathbb{E}_{\boldsymbol{\mu}'_{t+1}} \big[\pi_{1,t} u^*_{t+1}(\gamma^o_0, z(\pi_{2,t}), \boldsymbol{\mu}'_{t+1}) \\ &+ (1-\pi_{1,t}) u^*_{t+1}(\gamma^o_1, z(\pi_{2,t}), \boldsymbol{\mu}'_{t+1})\big] \qquad (183) \\ = &\sum_{i=1}^{N} \pi_{i,t} + \beta \mathbb{E}_{\boldsymbol{\mu}'_{t+1}} \big[\pi_{1,t} u^*_{t+1}(p, \pi_{2,t} p, \boldsymbol{\mu}'_{t+1}) \\ &+ (1-\pi_{1,t}) u^*_{t+1}(q_o p, \pi_{2,t} p, \boldsymbol{\mu}'_{t+1})\big] \qquad (184) \end{aligned}$$

Now we show that $u_t\big(\pi_{1,t}, ..., \pi_{N,t}; 1 \in OA, 2 \in VA\big) \geq u_t\big(\pi_{1,t}, ..., \pi_{N,t}; 1 \in OA, 2 \in NS\big)$ if the following inequality holds:

$$\begin{aligned} &\pi_{1,t}\pi_{2,t} u^*_{t+1}(p, p, \boldsymbol{\mu}'_{t+1}) + \pi_{1,t}(1-\pi_{2,t}) u^*_{t+1}(p, 0, \boldsymbol{\mu}'_{t+1}) + (1-\pi_{1,t})\pi_{2,t} u^*_{t+1}(q_o p, p, \boldsymbol{\mu}'_{t+1}) \\ &+ (1-\pi_{1,t})(1-\pi_{2,t}) u^*_{t+1}(q_o p, 0, \boldsymbol{\mu}'_{t+1}) \geq \pi_{1,t} u^*_{t+1}(p, \pi_{2,t} p, \boldsymbol{\mu}'_{t+1}) + (1-\pi_{1,t}) u^*_{t+1}(q_o p, \pi_{2,t} p, \boldsymbol{\mu}'_{t+1}) \end{aligned} \tag{185}$$

By definition, $u^*_{t+1}(., ., \boldsymbol{\mu}'_{t+1})$ is componentwise convex in each element. Then, Equation 185 holds.