Supplement to "Bias-corrected Common Correlated Effects Pooled estimation in dynamic panels"

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Section A of this supplement (i) provides the Jacobian matrix for the CCEPbc estimator, (ii) provides an additional discussion on the asymptotic bias of the CCEP estimator for the autoregressive parameter ρ, (iii) develops two restricted CCEPbc estimators for the single factor setting and (vi) reports Monte Carlo evidence comparing the performance of the restricted and unrestricted CCEPbc estimators. Section B introduces important notation and preliminary results for the proofs presented in Sections C and D. Section C presents proofs for $N \to \infty$ and fixed T, and Section D presents proofs for $(N, T) \to \infty$. Section E contains additional Monte Carlo simulation results for the unrestricted CCEPbc estimator.

A Additional results and discussions

A.1 Jacobian

Consider that CCEPbc estimator in eq.(21) is equivalent to

$$\widehat{\boldsymbol{\delta}}_{bc} = \underset{\boldsymbol{\delta}_0 \in \chi}{\arg\min} \frac{1}{2} \left\| \boldsymbol{\varphi}(\boldsymbol{\delta}_0) \right\|^2, \tag{A-1}$$

with $\boldsymbol{\varphi}(\boldsymbol{\delta}_0)$ given by

$$\boldsymbol{\varphi}(\boldsymbol{\delta}_0) = \frac{1}{NT} \sum_{i=1}^{N} \mathbf{w}_i' \mathbf{M} \mathbf{y}_i - \widehat{\boldsymbol{\Sigma}} \boldsymbol{\delta}_0 + \frac{1}{T} \widehat{\sigma}_{\varepsilon}^2(\boldsymbol{\delta}_0) \boldsymbol{\upsilon}(\rho_0),$$

and $\boldsymbol{\upsilon}(\rho_0) = \upsilon(\rho_0, \mathbf{H})\mathbf{q}_1$. As such, the CCEPbc estimator employs the orthogonality condition $\boldsymbol{\nabla}(\boldsymbol{\delta}_0) = \mathbf{0}$, with $\boldsymbol{\nabla}(\boldsymbol{\delta}_0)$ the gradient evaluated at $\boldsymbol{\delta}_0$,

$$\boldsymbol{\nabla}(\boldsymbol{\delta}_0) = \mathbf{J}_{\mathrm{a}}(\boldsymbol{\delta}_0)' \boldsymbol{\varphi}(\boldsymbol{\delta}_0),$$

and $\mathbf{J}_{\mathrm{a}}(\boldsymbol{\delta}_0)$ is the $k_w \times k_w$ Jacobian matrix in the sample evaluated at $\boldsymbol{\delta}_0$,

$$\mathbf{J}_{\mathrm{a}}(\boldsymbol{\delta}_0) = \frac{1}{T} \left[\left(\boldsymbol{\upsilon}(\rho_0) \otimes \dot{\boldsymbol{\sigma}}' \right) + \left(\widehat{\sigma}_{\varepsilon}^2(\boldsymbol{\delta}_0) \mathbf{q}_1 \otimes \dot{\boldsymbol{\upsilon}}' \right) \right] - \widehat{\boldsymbol{\Sigma}}, \tag{A-2}$$

with

$$\dot{\boldsymbol{\sigma}} = \frac{\partial \widehat{\sigma}_{\varepsilon}^2(\boldsymbol{\delta}_0)}{\partial \boldsymbol{\delta}_0} = 2 \frac{T}{T-c} \widehat{\boldsymbol{\Sigma}} \left(\boldsymbol{\delta}_0 - \widehat{\boldsymbol{\delta}} \right), \tag{A-3}$$

$$\dot{\boldsymbol{\upsilon}} = \frac{\partial \boldsymbol{\upsilon}(\rho_0)}{\partial \boldsymbol{\delta}_0} = \left(\sum_{t=1}^{T-1} (t-1) \rho_0^{t-2} \sum_{s=t+1}^{T} h_{s,s-t} \right) \mathbf{q}_1. \tag{A-4}$$

A.2 Discussion on the asymptotic bias of the CCEP estimator $\widehat{\rho}$

In order to gain a better understanding of the driving forces behind the asymptotic bias of the CCEP estimator for the autoregressive parameter ρ in eq.(1) of the main text, we first derive the following corollary result to Theorem 1 (with notation introduced in (C-37)).

Corollary 1. *Under the conditions of Theorem 1 and conditional on $\mathcal{C}$, the asymptotic bias of $\widehat{\rho}$ is*

$$\operatorname*{plim}_{N\to\infty}(\widehat{\rho}-\rho) = -\frac{\left[A(\rho)+D(\rho,\widetilde{\mathbf{H}})\right]}{\left[B(\rho)-E(\rho,\widetilde{\mathbf{H}})+TC\right]} = -\psi(\rho,\widetilde{\mathbf{H}},C), \tag{A-5}$$

with

- $A(\rho) = \frac{1}{1-\rho}\left(1-\frac{1}{T}\frac{1-\rho^T}{1-\rho}\right), \quad D(\rho,\widetilde{\mathbf{H}}) = \sum_{t=1}^{T-1}\rho^{t-1}\sum_{s=t+1}^{T}\widetilde{h}_{s,s-t},$
- $B(\rho) = \frac{T}{1-\rho^2}\left(1-\frac{1}{T}\frac{1+\rho}{1-\rho}-\frac{2\rho}{T^2}\frac{1-\rho^T}{(1-\rho)^2}\right), \quad E(\rho,\widetilde{\mathbf{H}}) = \frac{1}{1-\rho^2}\left[c-1+2\rho D(\rho,\widetilde{\mathbf{H}})\right],$

where $\widetilde{h}_{s,s-t}$ denotes the element on row s and column $s-t$ of $\widetilde{\mathbf{H}} = \widetilde{\mathbf{Q}}(\widetilde{\mathbf{Q}}'\widetilde{\mathbf{Q}})^{\dagger}\widetilde{\mathbf{Q}}'$, with $\mathbf{B} = \mathbf{I}_T - \boldsymbol{\iota}_T\boldsymbol{\iota}_T'/T$ and $\widetilde{\mathbf{Q}} = \mathbf{B}\mathbf{Q}$ the matrix of CSA in deviation of its column means, and

$$C = \operatorname*{plim}_{N\to\infty}\frac{\boldsymbol{\beta}'\boldsymbol{\Omega}_{\check{\mathrm{x}}}\boldsymbol{\beta}+\boldsymbol{\Lambda}'\boldsymbol{\Omega}_{\check{\mathrm{f}}}\boldsymbol{\Lambda}+2\boldsymbol{\beta}'\boldsymbol{\Omega}_{\check{\mathrm{x}},\check{\mathrm{f}}}\boldsymbol{\Lambda}}{\sigma_{\varepsilon}^2}, \tag{A-6}$$

with $\boldsymbol{\Omega}_{\check{\mathrm{x}}} = (\mathbf{X}_{-1}^{+})'\mathbb{M}_{\mathrm{X}}\mathbf{X}_{-1}^{+}/NT$, $\boldsymbol{\Omega}_{\check{\mathrm{f}}} = (\mathbb{F}_{-1}^{+})'\mathbb{M}_{\mathrm{X}}\mathbb{F}_{-1}^{+}/NT$ and $\boldsymbol{\Omega}_{\check{\mathrm{x}},\check{\mathrm{f}}} = (\mathbf{X}_{-1}^{+})'\mathbb{M}_{\mathrm{X}}\mathbb{F}_{-1}^{+}/NT$. Variables with a $+$ superscript are defined as $\mathbf{X}^{+} = (1-\rho L)^{-1}\mathbf{X}$.

The expression in eq.(A-5) shows that the inconsistency of the CCEP estimator $\widehat{\rho}$ is determined by the interplay of (i) the numerator, which is the covariance between the defactored lagged dependent variable $\check{\mathbf{y}}_{-1} = \mathbb{M}_{\mathrm{X}}\mathbf{y}_{-1}$ and the error term $\boldsymbol{\varepsilon}$, and (ii) the denominator, which is the signal that remains in the lagged dependent variable after orthogonalizing the data on the CSA $\mathbf{Q}$ through the $\mathbf{M}$ matrix. We elaborate below.

First consider the covariance terms in the numerator. The correlation between $\check{\mathbf{y}}_{-1}$ and $\boldsymbol{\varepsilon}$ originates from projecting out the nuisance parameters using the orthogonalization matrix $\mathbf{M}$. The term $A(\rho)$ is induced by the within transformation (time-demeaning implied by including $\boldsymbol{\iota}_T$ in $\mathbf{Q}$) and also appears in bias expressions for the FE estimator in dynamic models without common factors (see Nickell, 1981), whereas the additional orthogonalization on the CSA induces the CCEP-specific term $D(\rho,\widetilde{\mathbf{H}})$. The latter is stochastic as in fixed T settings the matrix $\widetilde{\mathbf{H}}$ depends, through the CSA, on the particular realization of the factors. This is the reason for why we need to condition on the σ-algebra $\mathcal{C}$ to derive Theorem 1. We expect D to be negative and smaller in magnitude[1] than A, which is positive. Hence, the asymptotic bias is expected to be negative, with the orthogonalization on

[1] This is because D is a reweighing of the sum $\sum_{t=1}^{T-1}\sum_{s=t+1}^{T}\widetilde{h}_{s,s-t} = -(c-1)/2 < 0$ in function of ρ. With positive weights ($\rho > 0$) it is therefore likely that $D < 0$. Similarly, $A + D$ is a reweighing of $\sum_{t=1}^{T-1}\sum_{s=t+1}^{T}h_{s,s-t} = (T-c)/2 > 0$ such that we can expect this sum to be positive when $\rho > 0$.

the CSA counteracting the A term in the numerator of (A-5) and therefore reducing the bias in absolute terms.

The second determinant of the bias is the denominator, which denotes the variation that remains in the lagged dependent variable after multiplying the model through with $\mathbf{M}$. The C-term represents the remaining variation due to the presence of exogenous regressors and factors, expressed relative to σ_ε^2, whereas B and E relate to the variation due to $\boldsymbol{\varepsilon}$. The positive B term is again a shared term with the FE estimator due to the within transformation, whereas the $-E$ term (which is negative) indicates that additional variation is lost compared to the FE estimator by orthogonalizing on the CSA. Including CSA will similarly reduce C. Hence, when the set of CSA cut out a relatively large amount of variation, the denominator of eq.(A-5) may decrease faster than the induced reduction in the numerator and hence result in a larger bias. For a given number of factors and regressors, increasing the number of CSA used by the CCEP estimator is therefore likely to increase its asymptotic bias. This is confirmed by the Monte Carlo simulations in Section 5 of the main paper. Finally, since we can show that $\mathbb{M}\mathbb{F}_{-1}^{+} \to^p \mathbf{0}_{T\times 1}$ for $m = 1$ (see Lemma 4) the second and last term in the numerator of C drop out in single factor settings. As such, an increase in the importance of the factors will, ceteris paribus, increase the signal in the model and reduce the asymptotic bias of the CCEP estimator, but only when more than one factor is present.

A.3 Restricted bias corrections for models with a single factor

The procedure outlined in Section 4 of the main paper is a generally applicable method in the sense that it does not require the number of factors to be known. In the single factor setting, eq.(A-5) of Corollary 1 can be used to develop more efficient restricted bias corrections, denoted CCEPbcr. Below we outline two alternative CCEPbcr estimators, depending on whether the dynamic model includes additional covariates or not.

Firstly, in a model with a single common factor ($m = 1$) and no covariates ($\boldsymbol{\beta} = \mathbf{0}$), the bias expression (A-5) simplifies considerably as $C = 0$ for $N \to \infty$. This is convenient as it is the presence of the C-term that makes bias correction from eq.(A-5) infeasible due to its dependence on the unobservable sums $\mathbf{X}_{-1}^{+}$ and $\mathbf{F}_{-1}^{+}$. Furthermore, the bias expression for $\widehat{\rho}$ no longer depends on σ_ε^2, such that ρ is the only unknown parameter in eq.(A-5). In this setting, the CCEPbcr estimator $\widehat{\boldsymbol{\delta}}_{bcr1}$ can be obtained as

$$\widehat{\boldsymbol{\delta}}_{bcr1} = \underset{|\rho_0|<1}{\arg\min} \frac{1}{2} \left\| \widehat{\rho} - \rho_0 + \psi(\rho_0, \widetilde{\mathbf{H}}, 0) \right\|^2 . \tag{A-7}$$

Secondly, adding exogenous regressors implies that $C \neq 0$ but if the single factor assumption is maintained we get the relatively simple form

$$C = \underset{N\to\infty}{\text{plim}} \frac{\boldsymbol{\beta}'\boldsymbol{\Omega}_{\check{\mathrm{x}}}\boldsymbol{\beta}}{\sigma_\varepsilon^2}, \tag{A-8}$$

which through $\boldsymbol{\Omega}_{\check{\mathrm{x}}}$ also depends on the unknown parameter ρ and on the infinite sum of explanatory variables $\mathbf{X}_{-1}^{+} = \sum_{l=0}^{\infty} \rho^l \mathbf{X}_{-1-l}$. In a finite sample, the latter can be approximated by the truncated sum $\widehat{\mathbf{X}_{-1}^{+}} = \left[\widehat{\mathbf{X}^{+}}'_{1,-1}, \ldots, \widehat{\mathbf{X}^{+}}'_{N,-1}\right]'$ where $\widehat{\mathbf{X}^{+}}_{i,-1} = \mathbf{J}^{-1}\mathbf{X}_{i,-1}$, and

$\mathbf{J}$ is a $T \times T$ matrix with ones on the main diagonal and $-\rho$ on the first sub-diagonal. The variance-covariance matrix is then estimated as $\widehat{\boldsymbol{\Omega}}_{\breve{\mathrm{x}}}(\rho) = \widehat{\mathbf{X}_{-1}^{+}}' \mathbb{M}_{\mathrm{X}} \widehat{\mathbf{X}_{-1}^{+}} / NT$. Further substituting $\widehat{\sigma}_{\varepsilon}^2(\cdot)$ as defined in (20) for σ_{ε}^2, the estimator for C is

$$\widehat{C}\left(\boldsymbol{\delta}\right) = \frac{\boldsymbol{\beta}' \widehat{\boldsymbol{\Omega}}_{\breve{\mathrm{x}}}(\rho) \boldsymbol{\beta}}{\widehat{\sigma}_{\varepsilon}^2(\boldsymbol{\delta})}, \tag{A-9}$$

which is, conditional on the unknown parameters ρ and $\boldsymbol{\beta}$, a function of the observed data only. Hence, in this setting the CCEPbcr estimator $\widehat{\boldsymbol{\delta}}_{bcr2}$ is

$$\widehat{\boldsymbol{\delta}}_{bcr2} = \underset{\boldsymbol{\delta}_0 \in \chi, |\rho_0| < 1}{\arg\min} \frac{1}{2} \left\| \widehat{\boldsymbol{\delta}} - \boldsymbol{\delta}_0 + \widehat{\boldsymbol{\nu}} \psi(\rho_0, \widetilde{\mathbf{H}}, \widehat{C}\left(\boldsymbol{\delta}_0\right)) \right\|^2, \tag{A-10}$$

where $\widehat{\boldsymbol{\nu}} = \left[1, -\widehat{\boldsymbol{\zeta}}'\right]'$ and $\widehat{\boldsymbol{\zeta}} = (\mathbf{S}_{\mathrm{x}}' \widehat{\boldsymbol{\Sigma}} \mathbf{S}_{\mathrm{x}})^{-1} \mathbf{S}_{\mathrm{x}}' \widehat{\boldsymbol{\Sigma}} \mathbf{q}_1$. This bias correction should perform well when the single factor assumption is true and the approximation of $\mathbf{X}_{-1}^{+}$ is not too inaccurate. Note that the truncation implies that $\widehat{\boldsymbol{\delta}}_{bcr2}$ is inconsistent for finite T, but in practice the bias may be negligible (depending on the size of ρ). In case more than one factor is present, eq.$(A-9)$ can be a poor approximation of C and lead to additional bias, especially when the factors have a large overall influence on the model (relative to σ_{ε}^2).

A.4 Finite sample properties of CCEPbc versus CCEPbcr

In this section we compare the performance of the unrestricted bias correction CCEPbc to that of the restricted version CCEPbcr $\widehat{\boldsymbol{\delta}}_{bcr2}$ derived in Section A.3 for a model with covariates and a single factor. As in the Monte Carlo simulation experiment presented in the main text, we also report results for variants that add the additional CSA $\bar{\mathrm{g}}_t$ to the orthogonalization matrix.

Table A-1 compares the performance of the CCEPbc estimator to that of CCEPbcr in settings with one and two common factors. The distinction between these scenarios is of interest since CCEPbcr is derived under the assumption that only one factor is present whereas CCEPbc is applicable irrespectively of the number of factors (provided that the rank condition is satisfied). In general, we find that CCEPbcr is a fairly accurate bias-correction method, even in the case of two factors. Comparing the unrestricted and restricted version shows some trade-off between bias and variance, though. CCEPbc dominates in terms of bias correction but has a downside that the estimator $\widehat{\boldsymbol{\Sigma}}$ used in eq.(18) introduces uncertainty in small samples. CCEPbcr has a smaller variance as it imposes a specific form for the denominator in (A-10) but is less effective as a bias correction method because of the truncation error made in the estimation of C and the resulting finite T inconsistency. Because this bias is offset by the lower variance (in rmse terms) in small samples (also see Table A-2 for $N = 25$), CCEPbcr may still be an interesting alternative to CCEPbc. As N grows large, however, this relative efficiency only compensates for bias when the single factor assumption is true (see upper panel of Table A-1) or when the factors are not too strong in case $m > 1$ (see lower left panel of Table A-1). Moreover, as a result of the inconsistency for finite T, CCEPbcr displays a size distortion especially when N is large. For the unrestricted version, inference is reliable in all settings (although this may require adding $\bar{\mathrm{g}}_t$), but at the cost of a higher variance.

Table A-1: Monte Carlo results for ρ : number and strength of factors ($N = 500$)

	bias	*rmse*	*size*	*bias*	*rmse*	*size*	*bias*	*rmse*	*size*	*bias*	*rmse*	*size*	*bias*	*rmse*	*size*	*bias*	*rmse*	*size*
	one factor																	
	$RI = 1$									$RI = 3$								
	$T = 10$			$T = 20$			$T = 30$			$T = 10$			$T = 20$			$T = 30$		
CCEPbc	0.000	0.057	0.06	0.001	0.014	0.04	0.000	0.008	0.05	0.001	0.055	0.06	0.000	0.014	0.04	0.000	0.008	0.04
CCEPbcr	−0.022	0.034	0.20	−0.009	0.014	0.16	−0.005	0.008	0.10	−0.023	0.034	0.20	−0.009	0.014	0.17	−0.005	0.008	0.10
CCEPbc(+g)	0.000	0.066	0.06	0.000	0.014	0.02	0.000	0.008	0.04	0.000	0.064	0.06	0.000	0.014	0.03	0.000	0.008	0.04
CCEPbcr(+g)	−0.023	0.036	0.16	−0.009	0.014	0.16	−0.005	0.009	0.10	−0.024	0.036	0.15	−0.009	0.014	0.15	−0.005	0.009	0.10
	two factors																	
	$RI = 1$									$RI = 3$								
	$T = 10$			$T = 20$			$T = 30$			$T = 10$			$T = 20$			$T = 30$		
CCEPbc	0.000	0.058	0.06	0.000	0.014	0.04	0.000	0.008	0.04	0.006	0.070	0.10	0.006	0.020	0.08	0.005	0.012	0.12
CCEPbcr	−0.022	0.033	0.19	−0.009	0.014	0.17	−0.005	0.008	0.10	0.002	0.045	0.21	0.007	0.022	0.28	0.008	0.016	0.32
CCEPbc(+g)	0.000	0.070	0.08	0.000	0.016	0.03	0.000	0.008	0.05	0.000	0.067	0.06	0.001	0.014	0.04	0.000	0.008	0.05
CCEPbcr(+g)	−0.023	0.036	0.16	−0.009	0.015	0.17	−0.005	0.009	0.10	0.003	0.047	0.18	0.007	0.020	0.25	0.007	0.014	0.28

Notes: (*i*) see notes to Table 2 in the main paper. (*ii*) CCEPbcr is the restricted CCEPbc estimator derived under the assumption of a single factor ($m = 1$).

Table A-2: Monte Carlo results for ρ : number and strength of factors ($N = 25$)

	bias	*rmse*	*size*	*bias*	*rmse*	*size*	*bias*	*rmse*	*size*	*bias*	*rmse*	*size*	*bias*	*rmse*	*size*	*bias*	*rmse*	*size*
	one factor																	
	$RI = 1$									$RI = 3$								
	$T = 10$			$T = 20$			$T = 30$			$T = 10$			$T = 20$			$T = 30$		
CCEPbc	−0.004	0.151	0.06	0.000	0.064	0.08	0.000	0.038	0.06	0.002	0.153	0.08	0.000	0.063	0.08	0.000	0.038	0.06
CCEPbcr	−0.029	0.108	0.06	−0.010	0.046	0.07	−0.005	0.032	0.07	−0.026	0.110	0.06	−0.010	0.046	0.06	−0.006	0.033	0.08
CCEPbc(+g)	−0.003	0.169	0.04	0.001	0.065	0.07	0.000	0.039	0.06	0.002	0.166	0.05	0.001	0.066	0.07	0.000	0.039	0.06
CCEPbcr(+g)	−0.026	0.116	0.04	−0.010	0.047	0.06	−0.005	0.033	0.07	−0.024	0.119	0.05	−0.010	0.048	0.06	−0.006	0.033	0.08
	two factors																	
	$RI = 1$									$RI = 3$								
	$T = 10$			$T = 20$			$T = 30$			$T = 10$			$T = 20$			$T = 30$		
CCEPbc	0.002	0.155	0.06	−0.001	0.063	0.08	−0.001	0.038	0.05	0.014	0.161	0.08	0.019	0.073	0.12	0.017	0.046	0.09
CCEPbcr	−0.027	0.106	0.06	−0.010	0.045	0.05	−0.005	0.032	0.07	−0.004	0.109	0.07	0.010	0.050	0.10	0.013	0.037	0.13
CCEPbc(+g)	−0.006	0.173	0.05	−0.001	0.067	0.07	−0.001	0.041	0.06	0.006	0.169	0.05	0.008	0.068	0.08	0.006	0.039	0.05
CCEPbcr(+g)	−0.028	0.120	0.05	−0.010	0.047	0.06	−0.006	0.033	0.07	0.001	0.118	0.05	0.007	0.047	0.08	0.008	0.034	0.09

Notes: see Table A-1.

B Notation, definitions and preliminary results

B.1 Notation

We first introduce some notation that will be used later on. In what follows we define $K = 1 + k$, $k_w = 1 + k_x$ and we set $p = 1$ for convenience but note that generalizations follow straightforwardly. With $p = 1$ model (1)-(3) can be written in VAR(1) form

$$\begin{bmatrix} 1 & -(\boldsymbol{\beta}^*)' \\ \mathbf{0}_{k\times 1} & \mathbf{I}_k \end{bmatrix} \begin{bmatrix} \mathrm{y}_{it} \\ \mathbf{z}_{it} \end{bmatrix} = \begin{bmatrix} \alpha_i \\ \mathbf{c}_{\mathrm{z},i} \end{bmatrix} + \begin{bmatrix} \rho & \mathbf{0}_{1\times k} \\ \mathbf{0}_{k\times 1} & \boldsymbol{\lambda} \end{bmatrix} \begin{bmatrix} \mathrm{y}_{it-1} \\ \mathbf{z}_{it-1} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\gamma}_i' \\ \boldsymbol{\Gamma}_i' \end{bmatrix} \mathbf{f}_t + \begin{bmatrix} \varepsilon_{it} \\ \mathbf{v}_{it} \end{bmatrix}, \tag{B-1}$$

with $\boldsymbol{\beta}^* = [\boldsymbol{\beta}', \mathbf{0}_{1\times k_g}]'$ and the associated more compact form

$$\mathbf{A}_0\mathbf{d}_{it} = \mathbf{c}_{\mathrm{d},i} + \boldsymbol{\Theta} L\mathbf{d}_{it} + \mathbf{C}_i\mathbf{f}_t + \mathbf{u}_{it},$$

where $\mathbf{c}_{\mathrm{d},i} = [\alpha_i, \mathbf{c}_{\mathrm{z},i}']'$, $\mathbf{d}_{it} = [\mathrm{y}_{it}, \mathbf{z}_{it}']'$, $\mathbf{u}_{it} = [\varepsilon_{it}, \mathbf{v}_{it}']'$ are $K \times 1$ vectors and

$$\underset{(K\times K)}{\mathbf{A}_0} = \begin{bmatrix} 1 & -(\boldsymbol{\beta}^*)' \\ \mathbf{0}_{k\times 1} & \mathbf{I}_k \end{bmatrix}, \qquad \underset{(K\times K)}{\boldsymbol{\Theta}} = \begin{bmatrix} \rho & \mathbf{0}_{1\times k} \\ \mathbf{0}_{k\times 1} & \boldsymbol{\lambda} \end{bmatrix}, \qquad \underset{(K\times m)}{\mathbf{C}_i} = \begin{bmatrix} \boldsymbol{\gamma}_i' \\ \boldsymbol{\Gamma}_i' \end{bmatrix}.$$

Since $\mathbf{A}_0$ is invertible,

$$\mathbf{d}_{it} = \mathbf{A}_0^{-1}\mathbf{c}_{\mathrm{d},i} + \mathbf{A}_0^{-1}\boldsymbol{\Theta} L\mathbf{d}_{it} + \mathbf{A}_0^{-1}\mathbf{C}_i\mathbf{f}_t + \mathbf{A}_0^{-1}\mathbf{u}_{it},$$

which can be rewritten further as

$$\begin{aligned} (\mathbf{I}_K - \boldsymbol{\Theta}^* L)\mathbf{d}_{it} &= \mathbf{c}_{\mathrm{d},i}^* + \mathbf{C}_i^*\mathbf{f}_t + \mathbf{u}_{it}^*, \\ \boldsymbol{\Theta}(L)\mathbf{d}_{it} &= \mathbf{c}_{\mathrm{d},i}^* + \mathbf{C}_i^*\mathbf{f}_t + \mathbf{u}_{it}^*, \end{aligned}$$

where the terms with an asterisk are defined as $\boldsymbol{\Theta}^* = \mathbf{A}_0^{-1}\boldsymbol{\Theta}$ and with $\boldsymbol{\Theta}(L) = \mathbf{I}_K - \boldsymbol{\Theta}^* L$. Then, as $\boldsymbol{\Theta}(L)$ is invertible by Assumption 5 we obtain the reduced form

$$\begin{aligned} \mathbf{d}_{it} &= \boldsymbol{\Theta}^{-1}(L)\mathbf{c}_{\mathrm{d},i}^* + \boldsymbol{\Theta}^{-1}(L)\mathbf{C}_i^*\mathbf{f}_t + \boldsymbol{\Theta}^{-1}(L)\mathbf{u}_{it}^*, \\ &= \ddot{\mathbf{c}}_{\mathrm{d},i} + (\ddot{\mathbf{C}}_i \otimes \mathbf{I}_K)'\,\check{\mathbf{f}}_t + \ddot{\mathbf{u}}_{it}, \end{aligned} \tag{B-2}$$

with $\ddot{\mathbf{u}}_{it} = \boldsymbol{\Theta}^{-1}(L)\mathbf{u}_{it}^*$, $\ddot{\mathbf{c}}_{\mathrm{d},i} = \boldsymbol{\Theta}^{-1}(L)\mathbf{c}_{\mathrm{d},i}^*$, $\check{\mathbf{f}}_t = vec(\mathbf{f}_t' \otimes \boldsymbol{\Theta}^{-1}(L))$ is $K^2m \times 1$ and $\ddot{\mathbf{C}}_i = vec(\mathbf{C}_i^*)$ is $Km \times 1$. Its cross-section average is

$$\bar{\mathbf{d}}_t = \ddot{\mathbf{c}}_{\mathrm{d}} + (\ddot{\mathbf{C}} \otimes \mathbf{I}_K)'\,\check{\mathbf{f}}_t + \ddot{\mathbf{u}}_t, \tag{B-3}$$

where $\ddot{\mathbf{u}}_t = \boldsymbol{\Theta}^{-1}(L)\bar{\mathbf{u}}_t^*$, $\bar{\mathbf{u}}_t^* = \frac{1}{N}\sum_{i=1}^N \mathbf{u}_{it}^*$, $\ddot{\mathbf{c}}_{\mathrm{d}} = \boldsymbol{\Theta}^{-1}(L)\bar{\mathbf{c}}_{\mathrm{d}}^*$, $\ddot{\mathbf{C}} = vec(\bar{\mathbf{C}}^*)$ and $\bar{\mathbf{C}}^* = \frac{1}{N}\sum_{i=1}^N \mathbf{C}_i^*$. Stack the observations over time into the $T \times K$ matrix $\mathbf{D}_i = [\mathbf{d}_{i1}, \ldots, \mathbf{d}_{iT}]'$ and let $\mathbf{D} = [\bar{\mathbf{d}}_1, \ldots, \bar{\mathbf{d}}_T]'$ be its cross-section average. Next, define

$$\underset{(T\times c)}{\mathbf{Q}_i} = [\boldsymbol{\iota}_T, \mathbf{D}_i, \ldots, \mathbf{D}_{i,-p^*}], \qquad \underset{(T\times c)}{\mathbf{Q}} = \frac{1}{N}\sum_{i=1}^N \mathbf{Q}_i = [\boldsymbol{\iota}_T, \mathbf{D}, \ldots, \mathbf{D}_{-p^*}], \tag{B-4}$$

with $c = 1 + K(1 + p^*)$ the number of columns of $\mathbf{Q}_i$ and $\mathbf{Q}$. Also, defining

$$\underset{(T\times 1+K^2m(1+p^*))}{\check{\mathbf{F}}} = [\boldsymbol{\iota}_T, \check{\mathbf{F}}_0, \check{\mathbf{F}}_{-1}, \ldots, \check{\mathbf{F}}_{-p^*}] \tag{B-5}$$

with $\check{\mathbf{F}}_0 = [\check{\mathbf{f}}_1, \ldots, \check{\mathbf{f}}_T]'$, $\check{\mathbf{F}}_{-1} = [\check{\mathbf{f}}_0, \ldots, \check{\mathbf{f}}_{T-1}]'$, $\check{\mathbf{F}}_{-p^*} = [\check{\mathbf{f}}_{1-p^*}, \ldots, \check{\mathbf{f}}_{T-p^*}]'$ and so on, and

$$\underset{(1+K^2m(1+p^*)\times 1+K(1+p^*))}{\ddot{\mathbf{P}}_i} = \begin{bmatrix} 1 & (\boldsymbol{\iota}'_{1+p^*} \otimes \ddot{\mathbf{c}}'_{\mathrm{d},i}) \\ \mathbf{0}_{K^2m(1+p^*)\times 1} & \mathbf{I}_{1+p^*} \otimes (\ddot{\mathbf{C}}_i \otimes \mathbf{I}_K) \end{bmatrix}, \quad \ddot{\mathbf{P}} = \frac{1}{N}\sum_{i=1}^{N} \ddot{\mathbf{P}}_i. \tag{B-6}$$

Given that $\mathbf{C}_i^* = \mathbf{A}_0^{-1}(\mathbf{C} + [\boldsymbol{\eta}_i, \boldsymbol{\nu}_i]')$ by Ass.3 we have also

$$\ddot{\mathbf{P}}_i = \mathbf{P} + \check{\mathbf{P}}_i, \tag{B-7}$$

with

$$\mathbf{P} = \begin{bmatrix} 1 & (\boldsymbol{\iota}'_{1+p^*} \otimes \mathbf{0}'_{K\times 1}) \\ \mathbf{0}_{K^2m(1+p^*)\times 1} & \mathbf{I}_{1+p^*} \otimes (\dot{\mathbf{C}} \otimes \mathbf{I}_K) \end{bmatrix}, \quad \check{\mathbf{P}}_i = \begin{bmatrix} 1 & (\boldsymbol{\iota}'_{1+p^*} \otimes \mathbf{0}'_{K\times 1}) \\ \mathbf{0}_{K^2m(1+p^*)\times 1} & \mathbf{I}_{1+p^*} \otimes (\widetilde{\mathbf{C}}_i \otimes \mathbf{I}_K) \end{bmatrix},$$

and where $\dot{\mathbf{C}} = vec(\mathbf{A}_0^{-1}\mathbf{C})$ and $\widetilde{\mathbf{C}}_i = vec(\mathbf{A}_0^{-1}[\boldsymbol{\eta}_i, \boldsymbol{\nu}_i]')$.

With the definitions above, the $T \times c$ matrix of observations is

$$\mathbf{Q}_i = \check{\mathbf{F}}\ddot{\mathbf{P}}_i + \ddot{\mathbf{U}}_i, \tag{B-8}$$

such that the observed matrix of cross-section averages can similarly be decomposed into

$$\mathbf{Q} = \check{\mathbf{F}}\ddot{\mathbf{P}} + \ddot{\mathbf{U}}, \tag{B-9}$$

where

$$\underset{(T\times c)}{\ddot{\mathbf{U}}_i} = \begin{bmatrix} 0 & \ddot{\mathbf{u}}'_{i1} & \ldots & \ddot{\mathbf{u}}'_{i,1-p^*} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \ddot{\mathbf{u}}'_{iT} & \ldots & \ddot{\mathbf{u}}'_{i,T-p^*} \end{bmatrix}, \quad \underset{(T\times c)}{\ddot{\mathbf{U}}} = \frac{1}{N}\sum_{i=1}^{N} \ddot{\mathbf{U}}_i = \begin{bmatrix} 0 & \ddot{\mathbf{u}}'_{1} & \ldots & \ddot{\mathbf{u}}'_{1-p^*} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \ddot{\mathbf{u}}'_{T} & \ldots & \ddot{\mathbf{u}}'_{T-p^*} \end{bmatrix}. \tag{B-10}$$

Next, we express all the regression variables in the model in terms of $\mathbf{Q}_i$ by defining the $k \times k_x$, $c \times 1$ and $c \times k_w$ selector matrices

$$\underset{(k\times k_x)}{\mathbf{S}_x} = \begin{bmatrix} \mathbf{I}_{k_x} \\ \mathbf{0}_{k_g\times k_x} \end{bmatrix}, \qquad \underset{(c\times 1)}{\mathbf{S}_y} = \begin{bmatrix} 0 \\ 1 \\ \mathbf{0}_{(c-2)\times 1} \end{bmatrix}, \qquad \underset{(c\times k_w)}{\mathbf{S}_w} = \begin{bmatrix} \mathbf{0}_{2\times 1} & \mathbf{0}_{2\times k_x} \\ \mathbf{0}_{k\times 1} & \mathbf{S}_x \\ 1 & \mathbf{0}_{1\times k_x} \\ \mathbf{0}_{c-(3+k)\times 1} & \mathbf{0}_{c-(3+k)\times k_x} \end{bmatrix}, \tag{B-11}$$

such that

$$\mathbf{y}_i = \mathbf{Q}_i\mathbf{S}_y = \check{\mathbf{F}}\ddot{\mathbf{P}}_i\mathbf{S}_y + \ddot{\mathbf{U}}_i\mathbf{S}_y, \tag{B-12}$$

$$\mathbf{w}_i = \mathbf{Q}_i\mathbf{S}_w = \check{\mathbf{F}}\ddot{\mathbf{P}}_i\mathbf{S}_w + \boldsymbol{\epsilon}_i, \tag{B-13}$$

where notably $\boldsymbol{\epsilon}_i$ is a $T \times k_w$ matrix given by

$$\boldsymbol{\epsilon}_i = \ddot{\mathbf{U}}_i\mathbf{S}_w = [\boldsymbol{\varrho}^+_{i,-1}, \dot{\mathbf{V}}_i\mathbf{S}_{\mathrm{x}}], \tag{B-14}$$

with $\dot{\mathbf{V}}_i = [\dot{\mathbf{v}}_{i,1}, \ldots, \dot{\mathbf{v}}_{i,T}]'$ and $\dot{\mathbf{v}}_{i,t} = \boldsymbol{\lambda}(L)^{-1}\mathbf{v}_{i,t}$. Also, with $\boldsymbol{\varrho}_i = \boldsymbol{\varepsilon}_i + \dot{\mathbf{V}}_i\boldsymbol{\beta}^*$ we have $\boldsymbol{\varrho}^+_{i,-1} = \boldsymbol{\varepsilon}^+_{i,-1} + \dot{\mathbf{V}}^+_{i,-1}\boldsymbol{\beta}^*$, a $T \times 1$ vector.

B.2 Rotating the projection matrix

To proceed with the terms involving the projector $\mathbf{H}$, we extend the approach of Karabiyik et al. (2017) to dynamic settings. To that end, let $\mathbf{T}$ be a $K \times K$ orthogonal matrix such that $(\bar{\mathbf{C}}^*)'\mathbf{T} = [\bar{\mathbf{C}}_m, \bar{\mathbf{C}}_{-m}]$, with $\bar{\mathbf{C}}_m$ the full rank $m \times m$ partitioning of $(\bar{\mathbf{C}}^*)'$, and $\bar{\mathbf{C}}_{-m}$ is the $m \times (K-m)$ matrix containing the remaining $K-m$ columns. Let $\bar{\mathbf{U}}^* = [\bar{\mathbf{u}}_1^*, \ldots, \bar{\mathbf{u}}_T^*]'$ such that $\bar{\mathbf{U}}_m^*$ and $\bar{\mathbf{U}}_{-m}^*$ are the corresponding partitioning that follows from $\bar{\mathbf{U}}^*\mathbf{T} = [\bar{\mathbf{U}}_m^*, \bar{\mathbf{U}}_{-m}^*]$. Next, we introduce the $k(1+p^*) \times k(1+p^*)$ rotation matrix $\mathbf{R}$. First, let

$$\mathbf{B} = \begin{bmatrix} \bar{\mathbf{C}}_m^{-1} & -\bar{\mathbf{C}}_m^{-1}\bar{\mathbf{C}}_{-m} \\ \mathbf{0}_{(K-m)\times m} & \mathbf{I}_{K-m} \end{bmatrix} = [\mathbf{B}_m, \mathbf{B}_{-m}]. \tag{B-15}$$

In what follows it is convenient to set $p^* = 1$ in order to save on notation. However, we note that the results generalize directly. The matrices $\widetilde{\mathbf{R}}$, $\widetilde{\mathbf{T}}$ and $\mathbf{R}$, defined next, are in general[2] of dimension $K(1+p^*) \times K(1+p^*)$, $K(1+p^*) \times K(1+p^*)$ and $1+K(1+p^*) \times 1+K(1+p^*)$ respectively. In the $p^* = 1$ case we then have

$$\underset{(2K\times K)}{\mathbf{R}^*} = \begin{bmatrix} \mathbf{I}_K \\ -(\mathbf{\Theta}^*)' \end{bmatrix}, \qquad \underset{(2K\times 2K)}{\widetilde{\mathbf{R}}} = \begin{bmatrix} \mathbf{R}^*, & \mathbf{0}_{K\times K} \\ & \mathbf{I}_K \end{bmatrix}, \qquad \underset{(2K\times 2K)}{\widetilde{\mathbf{T}}} = \begin{bmatrix} \mathbf{TB} & \mathbf{0}_{K\times K} \\ \mathbf{0}_{K\times K} & \mathbf{I}_K \end{bmatrix},$$

and, accounting for the row of constants in $\mathbf{Q}$,

$$\mathbf{R} = \begin{bmatrix} 1 & \mathbf{0}_{1\times 2K} \\ \mathbf{0}_{2K\times 1} & \widetilde{\mathbf{R}}\widetilde{\mathbf{T}} \end{bmatrix}.$$

Next, since by Lemma 1 the distribution of the CCEP estimator, or all its components, is invariant to the presence of the fixed effects, we can, without loss of generality, simplify notation by setting $\mathbf{c}_{\mathrm{d},i} = \mathbf{0}_{K\times 1}$ for all i such that $\ddot{\mathbf{c}}_{\mathrm{d}} = \mathbf{0}_{K\times 1}$. Making use of (B-9) we then get the following restructuring of $\mathbf{Q}$

$$\mathbf{QR} = \check{\mathbf{F}}\ddot{\mathbf{P}}\mathbf{R} + \ddot{\mathbf{U}}\mathbf{R} = [\boldsymbol{\iota}_T, \mathbf{F}, \mathbf{0}_{T\times(K-m)}, \check{\mathbf{F}}_{-1}(\ddot{\mathbf{C}} \otimes \mathbf{I}_K)] + [\mathbf{0}_{T\times 1}, \bar{\mathbf{U}}_m, \bar{\mathbf{U}}_{-m}, \ddot{\mathbf{U}}_{-1}],$$

where $\bar{\mathbf{U}}_m = \bar{\mathbf{U}}_m^*\bar{\mathbf{C}}_m^{-1}$, $\bar{\mathbf{U}}_{-m} = \bar{\mathbf{U}}_{-m}^* - \bar{\mathbf{U}}_m^*\bar{\mathbf{C}}_m^{-1}\bar{\mathbf{C}}_{-m}$ and $\ddot{\mathbf{U}}_{-1} = [\ddot{\mathbf{u}}_0, \ldots, \ddot{\mathbf{u}}_{T-1}]'$. The matrix $\mathbf{N}$ rearranges the columns conveniently as follows

$$\mathbf{QRN} = [\boldsymbol{\iota}_T, \mathbf{F}, \check{\mathbf{F}}_{-1}(\ddot{\mathbf{C}} \otimes \mathbf{I}_K), \mathbf{0}_{T\times(K-m)}] + [\mathbf{0}_{T\times 1}, \bar{\mathbf{U}}_m, \ddot{\mathbf{U}}_{-1}, \bar{\mathbf{U}}_{-m}].$$

[2] To illustrate: for any p^* we have, with $\mathbf{L}_{(1+p^*)}$ denoting a $(1+p^*) \times (1+p^*)$ matrix of zeros with ones on the first lower sub-diagonal

$$\underset{(K(1+p^*)\times K(1+p^*))}{\widetilde{\mathbf{R}}} = \mathbf{I}_{K(1+p^*)} - (\mathbf{L}_{(1+p^*)} \otimes (\mathbf{\Theta}^*)'), \qquad \underset{(K(1+p^*)\times K(1+p^*))}{\widetilde{\mathbf{T}}} = \begin{bmatrix} \mathbf{I}_{p^*} \otimes \mathbf{TB} & \mathbf{0}_{Kp^*\times K} \\ \mathbf{0}_{K\times Kp^*} & \mathbf{I}_K \end{bmatrix},$$

and

$$\mathbf{R} = \begin{bmatrix} 1 & \mathbf{0}_{1\times K(1+p^*)} \\ \mathbf{0}_{K(1+p^*)\times 1} & \widetilde{\mathbf{R}}\widetilde{\mathbf{T}} \end{bmatrix}.$$

Note that $\check{\mathbf{F}}_{-1}(\ddot{\mathbf{C}} \otimes \mathbf{I}_K)$ is a full column rank matrix ($rk(\check{\mathbf{F}}_{-1}(\ddot{\mathbf{C}} \otimes \mathbf{I}_K)) = K$) such that $rk([\mathbf{F}, \check{\mathbf{F}}_{-1}(\ddot{\mathbf{C}} \otimes \mathbf{I}_K), \mathbf{0}_{T\times(K-m)}]) = K + m \leq c - 1 = 2K$.[3] When $m < K$, the final $K - m$ columns of $\mathbf{QRN}$ are degenerate as $\left\|[\bar{\mathbf{U}}_m, \ddot{\mathbf{U}}_{-1}, \bar{\mathbf{U}}_{-m}]\right\| = O_p(N^{-1/2})$ by Lemma 2. Hence, post-multiplying by $\mathbf{D}_N = diag(\boldsymbol{\iota}'_{(1+K+m)}, \sqrt{N}\boldsymbol{\iota}'_{(K-m)})$

$$\mathbf{Q}_0 = \mathbf{QRND}_N = [\boldsymbol{\iota}_T, \mathbf{F}, \check{\mathbf{F}}_{-1}(\ddot{\mathbf{C}} \otimes \mathbf{I}_K), \mathbf{0}_{T\times(K-m)}] + [\mathbf{0}_{T\times 1}, \bar{\mathbf{U}}_m, \ddot{\mathbf{U}}_{-1}, \sqrt{N}\bar{\mathbf{U}}_{-m}] = \mathbf{F}^0 + \bar{\mathbf{U}}^0,$$

with $\mathbf{F}^0 = [\mathbf{F}^*, \mathbf{0}_{T\times(K-m)}]$ and $\mathbf{F}^* = [\boldsymbol{\iota}_T, \mathbf{F}, \check{\mathbf{F}}_{-1}(\ddot{\mathbf{C}} \otimes \mathbf{I}_K)]$ is a $T \times (1 + K + m)$ full rank matrix. Additionally, $\bar{\mathbf{U}}^0 = [\bar{\mathbf{U}}^0_m, \bar{\mathbf{U}}^0_{-m}]$, $\bar{\mathbf{U}}^0_m = [\mathbf{0}_{T\times 1}, \bar{\mathbf{U}}_m, \ddot{\mathbf{U}}_{-1}]$ and $\bar{\mathbf{U}}^0_{-m} = \sqrt{N}\bar{\mathbf{U}}_{-m}$. Therefore, we obtain for the rotated $\mathbf{Q}$ matrix with $\mathbf{F}^+_u = [\mathbf{F}^*, \bar{\mathbf{U}}^0_{-m}]$

$$\mathbf{Q}_0 = \mathbf{F}^0 + \bar{\mathbf{U}}^0 = [\mathbf{F}^*, \bar{\mathbf{U}}^0_{-m}] + [\bar{\mathbf{U}}^0_m, \mathbf{0}_{T\times(K-m)}] = \mathbf{F}^+_u + O_p(N^{-1/2}), \tag{B-16}$$

since $\left\|\bar{\mathbf{U}}^0_m\right\| = O_p(N^{-1/2})$ and $\left\|\bar{\mathbf{U}}^0_{-m}\right\| = O_p(1)$ by Lemma 2. Hence, in contrast to $\mathbf{Q}$, the columns of $\mathbf{Q}_0$ are non-degenerate even in case $m < K$, which, given that $\mathbf{H} = \mathbf{Q}(\mathbf{Q}'\mathbf{Q})^{\dagger}\mathbf{Q} = \mathbf{Q}_0(\mathbf{Q}_0'\mathbf{Q}_0)^{-1}\mathbf{Q}_0$ will now allow us to evaluate the limit of $\mathbf{H}$.

Finally, it is convenient to define the selector matrices

$$\mathbf{S}_m = \begin{bmatrix} \mathbf{I}_{1+K+m} \\ \mathbf{0}_{(K-m)\times(1+K+m)} \end{bmatrix}, \qquad \mathbf{S}_{-m} = \begin{bmatrix} \mathbf{0}_{(1+K+m)\times(K-m)} \\ \mathbf{I}_{K-m} \end{bmatrix}, \tag{B-17}$$

such that we obtain the following key identities that will be used throughout the appendix

$$\mathbf{F}^* = \check{\mathbf{F}}\ddot{\mathbf{P}}\mathbf{RNS}_m, \tag{B-18}$$

$$\bar{\mathbf{U}}^0_m = \ddot{\mathbf{U}}\mathbf{RNS}_m, \tag{B-19}$$

$$\bar{\mathbf{U}}^0_{-m} = \sqrt{N}\ddot{\mathbf{U}}\mathbf{RNS}_{-m}. \tag{B-20}$$

B.3 Preliminary results

Assume that Ass.4 holds and $p^* \geq p$. Define next $\mathbf{R}_0$ as follows

$$\mathbf{R}_0 = \begin{bmatrix} \mathbf{0}_{1\times K} \\ \mathbf{R}^* \\ \mathbf{0}_{K(p^*-p)\times K} \end{bmatrix},$$

[3] In general, for any p^* we have

$$\begin{aligned} \mathbf{QR} = \check{\mathbf{F}}\ddot{\mathbf{P}}\mathbf{R} + \ddot{\mathbf{U}}\mathbf{R} &= [\boldsymbol{\iota}_T, \mathbf{F}, \mathbf{0}_{T\times(K-m)}, \mathbf{F}_{-1}, \mathbf{0}_{T\times(K-m)}, \ldots, \mathbf{F}_{-(p^*-1)}, \mathbf{0}_{T\times(K-m)}, \check{\mathbf{F}}_{-p^*}(\ddot{\mathbf{C}} \otimes \mathbf{I}_K)] \\ &+ [\mathbf{0}_{T\times 1}, \bar{\mathbf{U}}_m, \bar{\mathbf{U}}_{-m}, \bar{\mathbf{U}}_{m,-1}, \bar{\mathbf{U}}_{-m,-1}, \ldots, \bar{\mathbf{U}}_{m,-(p^*-1)}, \bar{\mathbf{U}}_{-m,-(p^*-1)}, \ddot{\mathbf{U}}_{-p^*}], \end{aligned}$$

and

$$\begin{aligned} \mathbf{QRN} &= [\boldsymbol{\iota}_T, \mathbf{F}, \ldots, \mathbf{F}_{-(p^*-1)}, \check{\mathbf{F}}_{-1}(\ddot{\mathbf{C}} \otimes \mathbf{I}_K), \mathbf{0}_{T\times((K-m)(p^*-1))}] \\ &+ [\mathbf{0}_{T\times 1}, \bar{\mathbf{U}}_m, \ldots, \bar{\mathbf{U}}_{m,-(p^*-1)}, \ddot{\mathbf{U}}_{-p^*}, \bar{\mathbf{U}}_{-m}, \ldots, \bar{\mathbf{U}}_{-m,-(p^*-1)}], \end{aligned}$$

with $rk([\mathbf{F}, \ldots, \mathbf{F}_{-(p^*-1)}, \check{\mathbf{F}}_{-p^*}(\ddot{\mathbf{C}} \otimes \mathbf{I}_K), \mathbf{0}_{T\times(K-m)(p^*-1)}]) = K + mp^* \leq c - 1 = K(1 + p^*)$.

such that we can write

$$\mathbf{Q}\mathbf{R}_0\mathbf{T} = \mathbf{F}[\bar{\mathbf{C}}_m, \bar{\mathbf{C}}_{-m}] + [\bar{\mathbf{U}}^*_m, \bar{\mathbf{U}}^*_{-m}].$$

This gives, multiplied by $\mathbf{B}_m$ defined in (B-15),

$$\mathbf{Q}\mathbf{R}_0\mathbf{T}\mathbf{B}_m = \mathbf{F} + \bar{\mathbf{U}}_m, \tag{B-21}$$

such that we also have the following important relation

$$\bar{\mathbf{U}}_m = \ddot{\mathbf{U}}\mathbf{R}_0\mathbf{T}\mathbf{B}_m. \tag{B-22}$$

Solving (B-21) for $\mathbf{F}$ and multiplying by $\mathbf{M}$ gives

$$\mathbf{M}\mathbf{F} = \mathbf{M}(\mathbf{Q}\mathbf{R}_0\mathbf{T}\mathbf{B}_m - \bar{\mathbf{U}}_m),$$

which in turn, given that by definition $\mathbf{M}\mathbf{Q} = \mathbf{0}_{T\times c}$, leads to the following key result

$$\mathbf{M}\mathbf{F} = -\mathbf{M}\bar{\mathbf{U}}_m. \tag{B-23}$$

C Analysis for $N \to \infty$ and T fixed

C.1 Statement of lemmas

Lemma 1. *Suppose that Ass.5 holds and a vector of constants $\boldsymbol{\iota}_T$ is included in $\mathbf{Q}$. Then, the CCEP estimator in eq.(14), or its components $\mathbf{w}_i'\mathbf{M}\mathbf{w}_i$ and $\mathbf{w}_i'\mathbf{M}\mathbf{y}_i$ are invariant to α_i and $\mathbf{c}_{z,i}$ for all sample sizes. If additionally Ass.2 holds then it is equivalent to evaluate (14) with $E(\check{\mathbf{F}}) = \mathbf{0}$ for all N and T.*

Lemma 2. *Let Ass.1 and 5 hold. Then, as $N \to \infty$ and T fixed,*

$$\left\|\ddot{\mathbf{U}}\right\| = O_p(N^{-1/2}), \qquad \left\|\bar{\mathbf{U}}^*\right\| = O_p(N^{-1/2}), \tag{C-1}$$

$$\left\|\bar{\mathbf{U}}_m\right\| = O_p(N^{-1/2}), \qquad \left\|\bar{\mathbf{U}}_m^0\right\| = O_p(N^{-1/2}), \qquad \left\|\bar{\mathbf{U}}_{-m}^0\right\| = O_p(1). \tag{C-2}$$

Lemma 3. *Let c be the number of columns in $\mathbf{Q}$. For any $N \to \infty$ and $c < \infty$,*

$$\|\mathbf{H}\| \leq M, \tag{C-3}$$

irrespective of m, with M a finite constant.

Lemma 4. *Let Ass.1-5 hold and suppose that $m = 1$ and $p = 0$, then,*

$$\mathbf{M}\mathbf{F}_{-1}^{+} \longrightarrow^{p} \mathbf{0}_{T\times 1} \qquad \text{as} \qquad N \to \infty. \tag{C-4}$$

Lemma 5. *Let Ass.1-5 hold and suppose that $p^* \geq p$, then, as $N \to \infty$*

$$\mathbf{A}^{\mathbf{F}} = \frac{1}{N}\sum_{i=1}^{N} \frac{\mathbf{w}_i'\mathbf{M}\mathbf{F}}{T}\boldsymbol{\gamma}_i = O_p\left(N^{\omega}\right), \tag{C-5}$$

with $\omega = -1$ in case $m = 1$, $p = 0$ and $\omega = -1/2$ otherwise.

Lemma 6. *Let Ass.1-3 and 5 hold, then,*

$$\widehat{\boldsymbol{\Sigma}} = \frac{1}{N}\sum_{i=1}^{N} \frac{\mathbf{w}_i'\mathbf{M}\mathbf{w}_i}{T} = O_p(1), \tag{C-6}$$

for all N and T.

Lemma 7. *Let Ass.1-3 and 5 hold. Then, as $N \to \infty$,*

$$\widehat{\boldsymbol{\Sigma}} = \boldsymbol{\Sigma} + O_p(N^{-1/2}),$$

with

$$\boldsymbol{\Sigma} = \left(vec(\mathbf{I}_{k_w})' \otimes \mathbf{I}_{k_w}\right)\Big(\mathbf{I}_{k_w} \otimes \Big[\boldsymbol{\Sigma}_{\epsilon} + \left(\mathbf{S}_w' \otimes \mathbf{S}_w'\right)\boldsymbol{\Sigma}_{\ddot{\mathbf{P}}}(\check{\mathbf{F}}' \otimes \check{\mathbf{F}}')\Big] T^{-1} vec\left(\mathbf{M}\right)\Big), \tag{C-7}$$

and where $\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}} = E(\boldsymbol{\epsilon}_i' \otimes \boldsymbol{\epsilon}_i')$ and $\boldsymbol{\Sigma}_{\ddot{\mathbf{P}}} = E(\ddot{\mathbf{P}}_i' \otimes \ddot{\mathbf{P}}_i')$.

Lemma 8. *Let Ass.1-5 hold and suppose that $p^* \geq p$. Then, for $\widehat{\sigma}_\varepsilon^2$ evaluated at $\boldsymbol{\delta}_0 \neq \boldsymbol{\delta}$ with $\|\boldsymbol{\delta} - \boldsymbol{\delta}_0\| < \infty$, as $N \to \infty$,*

$$\widehat{\sigma}_\varepsilon^2\left(\boldsymbol{\delta}_0\right) = \sigma_\varepsilon^2 - \sigma_\varepsilon^2 c_1 \boldsymbol{\upsilon}(\rho, \mathbf{H})'(\boldsymbol{\delta} - \boldsymbol{\delta}_0) + c_2(\boldsymbol{\delta} - \boldsymbol{\delta}_0)'\boldsymbol{\Sigma}(\boldsymbol{\delta} - \boldsymbol{\delta}_0) + O_p(N^{-1/2}), \tag{C-8}$$

with $c_1 = 2/(T-c)$ and $c_2 = T/(T-c)$. When evaluated at $\boldsymbol{\delta}_0 = \boldsymbol{\delta}$,

$$\widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta}) = \frac{1}{N(T-c)} \sum_{i=1}^{N} \boldsymbol{\varepsilon}_i' \mathbf{M} \boldsymbol{\varepsilon}_i + O_p(N^{-1}), \tag{C-9}$$

and also

$$\widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta}) \longrightarrow^p \sigma_\varepsilon^2, \tag{C-10}$$

for $\widehat{\sigma}_\varepsilon^2(\cdot)$ defined in eq.(20).

C.2 Proof of lemmas

C.2.1 Proof of Lemma 1

Let $\mathbf{D}_0 = \mathbf{I}_T - \boldsymbol{\iota}_T\boldsymbol{\iota}_T'/T$ and consider that $\mathbf{D}_0 = \mathbf{D}_0'$ and $\mathbf{D}_0\mathbf{D}_0 = \mathbf{D}_0$. Then, with (B-12)-(B-13) we can write the components of the CCEP estimator in (14) as

$$\mathbf{w}_i'\mathbf{M}\mathbf{y}_i = \mathbf{S}_w'\mathbf{Q}_i'\mathbf{M}\mathbf{Q}_i\mathbf{S}_y = \mathbf{S}_w'\mathbf{Q}_i'\mathbf{D}_0\mathbf{Q}_i\mathbf{S}_y - \mathbf{S}_w'\mathbf{Q}_i'\mathbf{D}_0\mathbf{Q}(\mathbf{Q}'\mathbf{D}_0\mathbf{Q})^\dagger\mathbf{Q}'\mathbf{D}_0\mathbf{Q}_i\mathbf{S}_y, \tag{C-11}$$

$$\mathbf{w}_i'\mathbf{M}\mathbf{w}_i = \mathbf{S}_w'\mathbf{Q}_i'\mathbf{M}\mathbf{Q}_i\mathbf{S}_w = \mathbf{S}_w'\mathbf{Q}_i\mathbf{D}_0\mathbf{Q}_i\mathbf{S}_w - \mathbf{S}_w'\mathbf{Q}_i'\mathbf{D}_0\mathbf{Q}(\mathbf{Q}'\mathbf{D}_0\mathbf{Q})^\dagger\mathbf{Q}'\mathbf{D}_0\mathbf{Q}_i\mathbf{S}_w. \tag{C-12}$$

Next, making use of (B-8) and (B-5)

$$\begin{aligned}\mathbf{D}_0\mathbf{Q}_i &= \mathbf{D}_0(\check{\mathbf{F}}\ddot{\mathbf{P}}_i + \ddot{\mathbf{U}}_i) = [\mathbf{D}_0\boldsymbol{\iota}_T, \mathbf{D}_0\check{\mathbf{F}}_0, \ldots, \mathbf{D}_0\check{\mathbf{F}}_{-p^*}]\ddot{\mathbf{P}}_i + \mathbf{D}_0\ddot{\mathbf{U}}_i, \\ &= \left[\mathbf{0}_{T\times 1}, \mathbf{D}_0[\check{\mathbf{F}}_0, \ldots, \check{\mathbf{F}}_{-p^*}][\mathbf{I}_{1+p^*} \otimes (\ddot{\mathbf{C}}_i \otimes \mathbf{I}_K)]\right] + \mathbf{D}_0\ddot{\mathbf{U}}_i\end{aligned} \tag{C-13}$$

because $\mathbf{D}_0\boldsymbol{\iota}_T = \mathbf{0}_{T\times 1}$, and therefore also for the CSA

$$\mathbf{D}_0\mathbf{Q} = \mathbf{D}_0(\check{\mathbf{F}}\ddot{\mathbf{P}} + \ddot{\mathbf{U}}) = \left[\mathbf{0}_{T\times 1}, \mathbf{D}_0[\check{\mathbf{F}}_0, \ldots, \check{\mathbf{F}}_{-p^*}][\mathbf{I}_{1+p^*} \otimes (\ddot{\mathbf{C}} \otimes \mathbf{I}_K)]\right] + \mathbf{D}_0\ddot{\mathbf{U}}. \tag{C-14}$$

By consequence of (C-13) and (C-14), the right hand side of (C-11)-(C-12) is devoid of the fixed effects such that both $\mathbf{w}_i'\mathbf{M}\mathbf{y}_i$ and $\mathbf{w}_i'\mathbf{M}\mathbf{w}_i$ are invariant to their presence for all sample sizes. Additionally, since from Ass.2 and 5 follows $E(\mathbf{D}_0\check{\mathbf{F}}) = \mathbf{0}$, by (C-13) and (C-14) we can without loss of generality evaluate (C-11)-(C-12) assuming $E(\check{\mathbf{F}}) = \mathbf{0}$.

C.2.2 Proof of Lemma 2

From the definition we have $\bar{\mathbf{U}}^* = [\bar{\mathbf{u}}_1^*, \ldots, \bar{\mathbf{u}}_T^*]'$ such that its t-th row can be written as $\bar{\mathbf{u}}_t^* = N^{-1}\sum_{i=1}^N \mathbf{u}_{it}^* = N^{-1}\sum_{i=1}^N \mathbf{A}_0^{-1}\mathbf{u}_{it}$, where $\mathbf{A}_0^{-1}$ always exists and has fixed and finite entries. From Ass.1 follows $E(\mathbf{u}_{it}) = \mathbf{0}$ and therefore $E(\bar{\mathbf{u}}_t^*) = \mathbf{0}$. Consider now the variance

$$Var\left(\bar{\mathbf{u}}_t^*\right) = E\left(\frac{1}{N}\sum_{i=1}^N \mathbf{u}_{it}^*\right)\left(\frac{1}{N}\sum_{j=1}^N \mathbf{u}_{jt}^*\right)' = E\left(\frac{1}{N^2}\sum_{i=1}^N \mathbf{u}_{it}^*\mathbf{u}_{it}^{*\prime}\right),$$

$$= \mathbf{A}_0^{-1}\left(\frac{1}{N^2}\sum_{i=1}^{N}E(\mathbf{u}_{it}\mathbf{u}_{it}')\right)(\mathbf{A}_0^{-1})' = \mathbf{A}_0^{-1}\left(\frac{1}{N^2}\sum_{i=1}^{N}\boldsymbol{\Omega}_{\mathbf{u}}\right)(\mathbf{A}_0^{-1})' = O\left(\frac{1}{N}\right),$$

because by Ass.1 the $\mathbf{u}_{it}$ are independent over i and the entries of $\boldsymbol{\Omega}_{\mathbf{u}} = \begin{bmatrix}\sigma_\varepsilon^2 & \mathbf{0}_{k\times1}' \\ \mathbf{0}_{k\times1} & \boldsymbol{\Omega}_{\mathbf{v}}\end{bmatrix}$ are bounded for all i. Consequently, $\|\bar{\mathbf{u}}_t^*\| = O_p(N^{-1/2})$ and $\left\|\bar{\mathbf{U}}^*\right\| = O_p(N^{-1/2})$. Consider next $\ddot{\mathbf{U}}$ defined in (B-10) and let $\boldsymbol{\xi}_q = [0, \ddot{\mathbf{u}}_q', \ddot{\mathbf{u}}_{q-1}', \dots, \ddot{\mathbf{u}}_{q-p^*}']'$ be its q-th row. Since its entries are defined as $\ddot{\mathbf{u}}_t = \boldsymbol{\Theta}^{-1}(L)\bar{\mathbf{u}}_t^*$, with $\boldsymbol{\Theta}^{-1}(L)$ a fixed and stable lag polynomial by Ass.5 such that $\ddot{\mathbf{u}}_t$ is stationary, it follows from the above that $\|\ddot{\mathbf{u}}_t\| = O_p(N^{-1/2})$ and $E(\boldsymbol{\xi}_q) = \mathbf{0}$. This in turn implies that $E\left\|\boldsymbol{\xi}_q\right\|^2 = \sum_{l=0}^{p^*}E(\ddot{\mathbf{u}}_{q-l}'\ddot{\mathbf{u}}_{q-l}) \leq O(N^{-1})$, which establishes that $\left\|\boldsymbol{\xi}_q\right\| = O_p(N^{-1/2})$ and $\left\|\ddot{\mathbf{U}}\right\| = O_p(N^{-1/2})$. Combining this result with eqs.(B-19), (B-20) and (B-22) gives

$$\begin{aligned}
\left\|\bar{\mathbf{U}}_m\right\| &\leq \left\|\ddot{\mathbf{U}}\right\| \|\mathbf{R}_0\| \|\mathbf{T}\| \|\mathbf{B}_m\| = O_p(N^{-1/2}), \\
\left\|\bar{\mathbf{U}}_m^0\right\| &\leq \left\|\ddot{\mathbf{U}}\right\| \|\mathbf{R}\| \|\mathbf{N}\| \|\mathbf{S}_m\| = O_p(N^{-1/2}), \\
\left\|\bar{\mathbf{U}}_{-m}^0\right\| &\leq \sqrt{N}\left\|\ddot{\mathbf{U}}\right\| \|\mathbf{R}\| \|\mathbf{N}\| \|\mathbf{S}_{-m}\| = O_p(1),
\end{aligned}$$

which ends the proof.

C.2.3 Proof of Lemma 3

Recall that $\mathbf{Q} = \check{\mathbf{F}}\ddot{\mathbf{P}} + \ddot{\mathbf{U}}$ is a $T \times c$ real stochastic matrix with $T \geq c$ and $\ddot{\mathbf{U}} = O_p(N^{-1/2})$ by Lemma 2. Let r be the rank of $\mathbf{Q}$ and note that $r_0 = rk(\check{\mathbf{F}}\ddot{\mathbf{P}}) \leq r$ depending on m and k. Despite that $r_0 \leq r$, Feng and Zhang (2007) show that $r \overset{a.s.}{\to} c$ as $N \to \infty$ irrespective of r_0 (also see Karabiyik et al., 2017). Accordingly, $rk(\mathbf{H}) \overset{a.s.}{\to} c$ with $N \to \infty$ such that, by the property $rk(\mathbf{H}) = tr(\mathbf{H})$ of idempotent matrices, also $tr(\mathbf{H}) \overset{a.s.}{\to} c$. Consider next the matrix norm of $\mathbf{H}$. Given the above

$$\|\mathbf{H}\| = \sqrt{tr\left(\mathbf{H}\mathbf{H}'\right)} = \sqrt{tr\left(\mathbf{H}\right)} = \sqrt{c}, \tag{C-15}$$

and therefore $\mathbf{H}$ is bounded for any N irrespective of r_0 since c does not depend on N.

C.2.4 Proof of Lemma 4

Suppose that $p = 0$, $m = 1$ and write the one period lag of (1) as

$$\begin{aligned}
(1-\rho L)\mathrm{y}_{i,t-1} &= \alpha_i + \mathbf{x}_{i,t-1}'\boldsymbol{\beta} + \gamma_i \mathrm{f}_{t-1} + \varepsilon_{i,t-1}, \\
&= (\alpha_i + \mathbf{c}_{\mathrm{z,i}}'\boldsymbol{\beta}^*) + (\gamma_i + \boldsymbol{\beta}^{*\prime}\boldsymbol{\Gamma}_i')\mathrm{f}_{t-1} + (\varepsilon_{i,t-1} + \mathbf{v}_{i,t-1}'\boldsymbol{\beta}^*), \\
&= \alpha_i^* + \gamma_i^*\mathrm{f}_{t-1} + \varepsilon_{i,t-1}^*,
\end{aligned}$$

where $\mathbf{x}_{i,t-1}'\boldsymbol{\beta} = \mathbf{z}_{i,t-1}'\boldsymbol{\beta}^* = \mathbf{c}_{\mathrm{z,i}}'\boldsymbol{\beta}^* + \mathrm{f}_{t-1}\boldsymbol{\Gamma}_i\boldsymbol{\beta}^* + \mathbf{v}_{i,t-1}'\boldsymbol{\beta}^*$ was substituted in. Solve for f_t

$$\mathrm{f}_{t-1} = \frac{1}{\gamma_i^*}\left((1-\rho L)\mathrm{y}_{i,t-1} - \alpha_i^* - \varepsilon_{i,t-1}^*\right),$$

with $\gamma_i^* = \gamma_i + \boldsymbol{\beta}^{*\prime}\boldsymbol{\Gamma}_i'$ and multiply both sides with $(1-\rho L)^{-1}$

$$
\begin{aligned}
(1-\rho L)^{-1}\mathrm{f}_{t-1} &= \frac{(1-\rho L)^{-1}}{\gamma_i^*}\left((1-\rho L)\mathrm{y}_{i,t-1} - \alpha_i^* - \varepsilon_{i,t-1}^*\right), \\
\mathrm{f}_{t-1}^{+} &= \frac{1}{\gamma_i^*}\left((\mathrm{y}_{i,t-1} - (1-\rho L)^{-1}\alpha_i^* - (1-\rho L)^{-1}\varepsilon_{i,t-1}^*\right),
\end{aligned}
$$

where $\mathrm{f}_{t-1}^{+} = (1-\rho L)^{-1}\mathrm{f}_{t-1}$. Next, averaging over i gives

$$
\mathrm{f}_{t-1}^{+} = \frac{1}{\bar{\gamma}^*}\left(\bar{\mathrm{y}}_{t-1} - \bar{\alpha}^*/(1-\rho) - (1-\rho L)^{-1}\bar{\varepsilon}_{t-1}^*\right),
$$

where barred variables are averages and it follows from Lemma 2 that $(1-\rho L)^{-1}\bar{\varepsilon}_{t-1}^* = O_p(N^{-1/2})$. Given the above we can write $\mathbf{F}_{-1}^{+} = (1-\rho L)^{-1}\mathbf{F}_{-1} = [\mathrm{f}_0^{+},\ldots,\mathrm{f}_{T-1}^{+}]'$ using $\bar{\boldsymbol{\varepsilon}}_{-1}^{*+} = (1-\rho L)^{-1}[\bar{\varepsilon}_0^*,\ldots,\bar{\varepsilon}_{T-1}^*]'$ as

$$
\mathbf{F}_{-1}^{+} = \mathbf{Q}^*\begin{bmatrix}-\bar{\alpha}^* \\ 1-\rho\end{bmatrix}\frac{1}{\bar{\gamma}^*(1-\rho)} - \frac{\bar{\boldsymbol{\varepsilon}}_{-1}^{*+}}{\bar{\gamma}^*} = \mathbf{Q}^*\mathbf{P}^* + O_p\left(\frac{1}{\sqrt{N}}\right), \tag{C-16}
$$

with $\mathbf{Q}^* = [\boldsymbol{\iota}_T, \bar{\mathbf{y}}_{-1}]$ and obvious definition for $\mathbf{P}^*$. Provided a constant and $\bar{\mathbf{y}}_{-1}$ are included in $\mathbf{Q}$, we have

$$
\mathbf{M}\mathbf{F}_{-1}^{+} = O_p(N^{-1/2}), \tag{C-17}
$$

because in this case $\mathbf{M}\mathbf{Q}^* = \mathbf{0}$ by definition and $\mathbf{M}$ is bounded in norm by Lemma 3. Note that (C-17) does not go through in the multiple factor case or with $p > 0$ since, lagging (9) and multiplying both sides with $\rho(L)^{-1} = (1-\rho L)^{-1}$ yields

$$
\mathbf{f}_{t-1}^{+} = \left(\mathbf{C}'\mathbf{C}\right)^{-1}\mathbf{C}'\left(\begin{bmatrix}1 & -\rho(L)^{-1}(\boldsymbol{\beta}^*)' \\ 0 & \rho(L)^{-1}\boldsymbol{\lambda}(L)\end{bmatrix}\begin{bmatrix}\bar{\mathrm{y}}_{t-1} \\ \bar{\mathbf{z}}_{t-1}\end{bmatrix} - \rho(L)^{-1}\begin{bmatrix}\bar{\alpha} \\ \bar{\mathbf{c}}_{\mathrm{z}}\end{bmatrix}\right) + O_p(N^{-1/2}),
$$

which shows that an infinite number of lags of $\bar{\mathbf{z}}_{t-1}$ are required to approximate $\mathbf{f}_t^{+}$.

C.2.5 Proof of Lemma 5

Let $\mathbf{A}^{\mathbf{F}} = \frac{1}{NT}\sum_{i=1}^N \mathbf{w}_i'\mathbf{M}\mathbf{F}\boldsymbol{\gamma}_i$. Since the rank condition holds by Ass.4 we have substituting in (B-23) and using $\boldsymbol{\gamma}_i = \boldsymbol{\gamma} + \boldsymbol{\eta}_i$ from Ass.3

$$
\mathbf{A}^{\mathbf{F}} = -\frac{1}{NT}\sum_{i=1}^N \mathbf{w}_i'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\gamma}_i = -\frac{1}{T}\bar{\mathbf{w}}'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\gamma} - \frac{1}{NT}\sum_{i=1}^N \mathbf{w}_i'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i = -\frac{1}{NT}\sum_{i=1}^N \mathbf{w}_i'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i,
$$

since $\bar{\mathbf{w}} \subset \mathbf{Q}$ such that $\mathbf{M}\bar{\mathbf{w}} = \mathbf{0}_{T\times k_w}$. We next make use of $\mathbf{M} = \mathbf{I}_T - \mathbf{H}$ to write the matrix norm of $\mathbf{A}^{\mathbf{F}}$ as

$$
\left\|\mathbf{A}^{\mathbf{F}}\right\| \le \left\|\frac{1}{NT}\sum_{i=1}^N \mathbf{w}_i'\bar{\mathbf{U}}_m\boldsymbol{\eta}_i\right\| + \left\|\frac{1}{NT}\sum_{i=1}^N \mathbf{w}_i'\mathbf{H}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i\right\|. \tag{C-18}
$$

Turning to the first term gives

$$\left\| \frac{1}{NT} \sum_{i=1}^{N} \mathbf{w}_i' \bar{\mathbf{U}}_m \boldsymbol{\eta}_i \right\| \leq \left\| \frac{1}{NT} \sum_{i=1}^{N} (\boldsymbol{\eta}_i' \otimes \mathbf{w}_i') \right\| \left\| \bar{\mathbf{U}}_m \right\| = O_p\left(\frac{1}{\sqrt{N}}\right),$$

since $\left\| \bar{\mathbf{U}}_m \right\| = O_p(N^{-1/2})$ by Lemma 2 and since substituting in $\mathbf{w}_i = \check{\mathbf{F}}\ddot{\mathbf{P}}_i\mathbf{S}_w + \boldsymbol{\epsilon}_i$ by (B-13) leads to

$$\begin{aligned} \left\| \frac{1}{NT} \sum_{i=1}^{N} (\boldsymbol{\eta}_i' \otimes \mathbf{w}_i') \right\| &\leq \left\| \frac{1}{NT} \sum_{i=1}^{N} \left(\boldsymbol{\eta}_i' \otimes \mathbf{S}_w' \ddot{\mathbf{P}}_i' \check{\mathbf{F}}'\right) \right\| + \left\| \frac{1}{NT} \sum_{i=1}^{N} (\boldsymbol{\eta}_i' \otimes \boldsymbol{\epsilon}_i') \right\|, \\ &= \left\| \frac{1}{NT} \sum_{i=1}^{N} \left(\boldsymbol{\eta}_i' \otimes \mathbf{S}_w' \ddot{\mathbf{P}}_i' \check{\mathbf{F}}'\right) \right\| + O_p\left(\frac{1}{\sqrt{N}}\right) = O_p(1), \end{aligned} \tag{C-19}$$

because $\boldsymbol{\epsilon}_i$ and $\boldsymbol{\eta}_i$ are independent and loadings are i.i.d. with bounded fourth moments by Ass.3. For the second term, we find with (C-19),

$$\left\| \frac{1}{NT} \sum_{i=1}^{N} \mathbf{w}_i' \mathbf{H} \bar{\mathbf{U}}_m \boldsymbol{\eta}_i \right\| \leq \left\| \frac{1}{NT} \sum_{i=1}^{N} (\boldsymbol{\eta}_i' \otimes \mathbf{w}_i') \right\| \|\mathbf{H}\| \left\| \bar{\mathbf{U}}_m \right\| = O_p\left(\frac{1}{\sqrt{N}}\right), \tag{C-20}$$

since $\|\mathbf{H}\|$ is bounded by Lemma 3. Combining results in (C-18) gives

$$\left\| \mathbf{A}^{\mathbf{F}} \right\| = O_p(N^{-1/2}),$$

which proves that in general $\left\| \mathbf{A}^{\mathbf{F}} \right\| = O_p(N^{\omega})$ with $\omega = -1/2$.

It remains to show that $\omega = -1$ when $m = 1$ and $p = 0$. Write $\mathbf{A}^{\mathbf{F}}$ explicitly as

$$\mathbf{A}^{\mathbf{F}} = -\frac{1}{N} \sum_{i=1}^{N} \frac{\mathbf{w}_i' \mathbf{M} \bar{\mathbf{U}}_m}{T} \boldsymbol{\eta}_i = -\frac{1}{NT} \sum_{i=1}^{N} \begin{bmatrix} \mathbf{y}_{i,-1}' \mathbf{M} \bar{\mathbf{U}}_m \boldsymbol{\eta}_i \\ \mathbf{X}_i' \mathbf{M} \bar{\mathbf{U}}_m \boldsymbol{\eta}_i \end{bmatrix}. \tag{C-21}$$

Suppose that $m = 1$, $p = 0$. We can then write $\mathbf{M}\mathbf{y}_{i,-1}$ more explicitly by inverting eq.(6) and employing (C-16) of Lemma 4

$$\mathbf{M}\mathbf{y}_{i,-1} = \mathbf{M}\left(\mathbf{F}_{-1}^{+}\gamma_i + \mathbf{X}_{i,-1}^{+}\boldsymbol{\beta} + \boldsymbol{\varepsilon}_{i,-1}^{+}\right) = \mathbf{M}\left(\mathbf{X}_{i,-1}^{+}\boldsymbol{\beta} + \boldsymbol{\varepsilon}_{i,-1}^{+} - \frac{\gamma_i}{\bar{\gamma}^*}\bar{\boldsymbol{\varepsilon}}_{-1}^{*+}\right), \tag{C-22}$$

and since $p = 0$ (no dynamics in $\mathbf{z}_{it}$) we can also write (3) in matrix notation as

$$\mathbf{Z}_i = [\mathbf{X}_i, \mathbf{G}_i] = \boldsymbol{\iota}_T \mathbf{c}_{z,i}' + \mathbf{F}\boldsymbol{\Gamma}_i + \mathbf{V}_i,$$

where $\mathbf{V}_i = [\mathbf{v}_{i1}, \ldots, \mathbf{v}_{iT}]'$. Defining $\mathbf{S}_x = [\mathbf{I}_{k_x}, \mathbf{0}_{k_x \times k_g}]'$ as the matrix selecting $\mathbf{X}_i$ from $\mathbf{Z}_i$ and substituting in (B-23) gives

$$\mathbf{M}\mathbf{X}_i = \mathbf{M}\mathbf{Z}_i\mathbf{S}_x = \mathbf{M}(\mathbf{F}\boldsymbol{\Gamma}_i + \mathbf{V}_i)\mathbf{S}_x = \mathbf{M}(\mathbf{V}_i - \bar{\mathbf{U}}_m\boldsymbol{\Gamma}_i)\mathbf{S}_x. \tag{C-23}$$

Similarly, from (C-16) in Lemma 4

$$\mathbf{M}\mathbf{X}_{i,-1}^{+} = \mathbf{M}\rho(L)^{-1}\mathbf{Z}_{i,-1}\mathbf{S}_x = \mathbf{M}(\mathbf{F}_{-1}^{+}\boldsymbol{\Gamma}_i + \mathbf{V}_{i,-1}^{+})\mathbf{S}_x = \mathbf{M}(\mathbf{V}_{i,-1}^{+} - \bar{\gamma}^{*-1}\bar{\boldsymbol{\varepsilon}}_{-1}^{*+}\boldsymbol{\Gamma}_i)\mathbf{S}_x. \tag{C-24}$$

Consider the first row of (C-21), substituting in (C-22) gives

$$\frac{1}{NT}\sum_{i=1}^{N}\mathbf{y}_{i,-1}'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i = \frac{1}{NT}\sum_{i=1}^{N}\left(\boldsymbol{\beta}'\mathbf{X}_{i,-1}^{+\prime} + \boldsymbol{\varepsilon}_{i,-1}^{+\prime} - \frac{\gamma_i}{\bar{\gamma}^*}\bar{\boldsymbol{\varepsilon}}_{-1}^{*+\prime}\right)\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i, \tag{C-25}$$

where since $\bar{\mathbf{U}}_m$ and $\bar{\boldsymbol{\varepsilon}}_{-1}^{+*}$ are $O_p(N^{-1/2})$ and loadings and errors are independent

$$\left\|\frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\varepsilon}_{i,-1}^{+\prime}\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i\right\| \leq \frac{1}{T}\left\|\frac{1}{N}\sum_{i=1}^{N}\left(\boldsymbol{\eta}_i'\otimes\boldsymbol{\varepsilon}_{i,-1}^{+\prime}\right)\right\|\|\mathbf{M}\|\left\|\bar{\mathbf{U}}_m\right\| = O_p(N^{-1}),$$

$$\left\|\frac{1}{NT}\sum_{i=1}^{N}\frac{\gamma_i}{\bar{\gamma}^*}\bar{\boldsymbol{\varepsilon}}_{-1}^{+*\prime}\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i\right\| \leq \frac{1}{T}\left\|\frac{1}{N}\sum_{i=1}^{N}\left(\boldsymbol{\eta}_i'\otimes\frac{\gamma_i}{\bar{\gamma}^*}\right)\right\|\left\|\bar{\boldsymbol{\varepsilon}}_{-1}^{+*}\right\|\|\mathbf{M}\|\left\|\bar{\mathbf{U}}_m\right\| = O_p(N^{-1}),$$

and we find for first term of (C-25), after substituting in (C-24),

$$\frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\beta}'\mathbf{S}_x'\left(\mathbf{V}_{i,-1}^{+\prime} - \bar{\gamma}^{*-1}\boldsymbol{\Gamma}_i'\bar{\boldsymbol{\varepsilon}}_{-1}^{*+\prime}\right)\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i = O_p(N^{-1}),$$

because

$$\left\|\frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\beta}'\mathbf{S}_x'\mathbf{V}_{i,-1}^{+\prime}\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i\right\| \leq \frac{1}{T}\left\|\frac{1}{N}\sum_{i=1}^{N}\left(\boldsymbol{\eta}_i'\otimes\boldsymbol{\beta}'\mathbf{S}_x'\mathbf{V}_{i,-1}^{+\prime}\right)\right\|\|\mathbf{M}\|\left\|\bar{\mathbf{U}}_m\right\| = O_p\left(\frac{1}{N}\right),$$

$$\left\|\frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\beta}'\mathbf{S}_x'\frac{\boldsymbol{\Gamma}_i'}{\bar{\gamma}^*}\bar{\boldsymbol{\varepsilon}}_{-1}^{+*\prime}\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i\right\| \leq \frac{1}{T}\left\|\frac{1}{N}\sum_{i=1}^{N}\left(\boldsymbol{\eta}_i'\otimes\boldsymbol{\beta}'\mathbf{S}_x'\frac{\boldsymbol{\Gamma}_i'}{\bar{\gamma}^*}\right)\right\|\left\|\bar{\boldsymbol{\varepsilon}}_{-1}^{+*}\right\|\|\mathbf{M}\|\left\|\bar{\mathbf{U}}_m\right\| = O_p\left(\frac{1}{N}\right),$$

where we note that the last bound can be sharpened to $O_p(N^{-3/2})$ when $\boldsymbol{\gamma}_i$ and $\boldsymbol{\Gamma}_i$ are independent. Regardless, combining results in (C-25) gives

$$\frac{1}{NT}\sum_{i=1}^{N}\mathbf{y}_{i,-1}'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i = O_p(N^{-1}). \tag{C-26}$$

For rows 2 to k_w of (C-21) we find, after substituting in (C-23) and using similar arguments as before

$$\frac{1}{NT}\sum_{i=1}^{N}\mathbf{X}_i'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i = \frac{1}{NT}\sum_{i=1}^{N}\mathbf{S}_x'(\mathbf{V}_i' - \boldsymbol{\Gamma}_i'\bar{\mathbf{U}}_m')\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i = O_p(N^{-1}). \tag{C-27}$$

Combining (C-26)-(C-27) in (C-21) leads to $\mathbf{A}^{\mathbf{F}} = O_p(N^{\omega})$ with $\omega = -1$, as required.

C.2.6 Proof of Lemma 6

Recall from eq.(B-13) that $\mathbf{w}_i = \check{\mathbf{F}}\ddot{\mathbf{P}}_i\mathbf{S}_w + \boldsymbol{\epsilon}_i$ with $\mathbf{S}_w$ the selector matrix defined in (B-11) and $\check{\mathbf{F}}$, $\ddot{\mathbf{P}}_i$ and $\boldsymbol{\epsilon}_i$ are defined in eq.(B-5), (B-6) and (B-14) respectively. Let $\boldsymbol{\vartheta}_{i,s}$ be the s-th column of $\mathbf{w}_i$ and note that by Ass.1-3 and 5 the $\ddot{\mathbf{P}}_i$, $\boldsymbol{\epsilon}_i$ and $\check{\mathbf{F}}$ are independent and stationary with finite variance such that $\boldsymbol{\vartheta}_{i,s} = O_p(1)$ for every i and s and $\|\boldsymbol{\vartheta}_{i,s}\| = O_p(\sqrt{T})$. Consider the matrix $\widehat{\boldsymbol{\Sigma}} = \sum_{i=1}^{N}\mathbf{w}_i'\mathbf{M}\mathbf{w}_i/NT$ and note that element s on its diagonal is $\frac{1}{NT}\sum_{i=1}^{N}\|\mathbf{M}\boldsymbol{\vartheta}_{i,s}\|^2 = O_p(1)$, since $\|\mathbf{M}\boldsymbol{\vartheta}_{i,s}\| \le \|\boldsymbol{\vartheta}_{i,s}\| = O_p(\sqrt{T})$ for all i and s. Using the same argument we have for the off-diagonal element on row s and column $s' \neq s$

$$\left\|\frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\vartheta}_{i,s}'\mathbf{M}\boldsymbol{\vartheta}_{i,s'}\right\| \le \frac{1}{NT}\sum_{i=1}^{N}\left\|\boldsymbol{\vartheta}_{i,s}'\mathbf{M}\boldsymbol{\vartheta}_{i,s'}\right\| \le \frac{1}{NT}\sum_{i=1}^{N}\|\mathbf{M}\boldsymbol{\vartheta}_{i,s}\|\,\|\mathbf{M}\boldsymbol{\vartheta}_{i,s'}\| = O_p(1),$$

such that $\widehat{\boldsymbol{\Sigma}} = O_p(1)$ and the lemma is proved.

C.2.7 Proof of Lemma 7

Consider the following decomposition of $\widehat{\boldsymbol{\Sigma}}$ obtained by substituting in eq.(B-13)

$$\begin{aligned}\widehat{\boldsymbol{\Sigma}} = \frac{1}{NT}\sum_{i=1}^{N}\mathbf{w}_i'\mathbf{M}\mathbf{w}_i =& \frac{1}{NT}\sum_{i=1}^{N}\mathbf{S}_w'\ddot{\mathbf{P}}_i'\check{\mathbf{F}}'\mathbf{M}\check{\mathbf{F}}\ddot{\mathbf{P}}_i\mathbf{S}_w + \frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\epsilon}_i'\mathbf{M}\boldsymbol{\epsilon}_i \\ &+ \frac{1}{NT}\sum_{i=1}^{N}\mathbf{S}_w'\ddot{\mathbf{P}}_i'\check{\mathbf{F}}'\mathbf{M}\boldsymbol{\epsilon}_i + \frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\epsilon}_i'\mathbf{M}\check{\mathbf{F}}\ddot{\mathbf{P}}_i\mathbf{S}_w.\end{aligned}$$

By Ass.1 and 3, the $\boldsymbol{\epsilon}_i$ and $\ddot{\mathbf{P}}_i$ are independent of each other and over i such that

$$\left\|\frac{1}{N}\sum_{i=1}^{N}\mathbf{S}_w'\ddot{\mathbf{P}}_i'\check{\mathbf{F}}'\mathbf{M}\boldsymbol{\epsilon}_i\right\| \le \left\|\frac{1}{N}\sum_{i=1}^{N}\left(\boldsymbol{\epsilon}_i'\otimes\mathbf{S}_w'\ddot{\mathbf{P}}_i'\right)\right\|\left\|\check{\mathbf{F}}\right\|\|\mathbf{M}\| = O_p(N^{-1/2}),$$

which also uses Lemma 3. Substituting in this result and noting that the summation operates only on $\boldsymbol{\epsilon}_i$ and $\ddot{\mathbf{P}}_i$ we can write

$$vec(\widehat{\boldsymbol{\Sigma}}) = T^{-1}\left[\widehat{\boldsymbol{\Sigma}}_\epsilon + (\mathbf{S}_w'\otimes\mathbf{S}_w')\,\widehat{\boldsymbol{\Sigma}}_{\ddot{\mathbf{P}}}(\check{\mathbf{F}}'\otimes\check{\mathbf{F}}')\right]vec\,(\mathbf{M}) + O_p(N^{-1/2}),$$

where $\widehat{\boldsymbol{\Sigma}}_\epsilon = \frac{1}{N}\sum_{i=1}^{N}(\boldsymbol{\epsilon}_i'\otimes\boldsymbol{\epsilon}_i')$ and $\widehat{\boldsymbol{\Sigma}}_{\ddot{\mathbf{P}}} = \frac{1}{N}\sum_{i=1}^{N}(\ddot{\mathbf{P}}_i'\otimes\ddot{\mathbf{P}}_i')$. For the latter, since by Ass.1 and 3 the $\boldsymbol{\epsilon}_i$ and $\ddot{\mathbf{P}}_i$ are independent over i with bounded moments up to the fourth order

$$\begin{aligned}\widehat{\boldsymbol{\Sigma}}_\epsilon &= E\left(\boldsymbol{\epsilon}_i'\otimes\boldsymbol{\epsilon}_i'\right) + O_p(N^{-1/2}) = \boldsymbol{\Sigma}_\epsilon + O_p(N^{-1/2}),\\ \widehat{\boldsymbol{\Sigma}}_{\ddot{\mathbf{P}}} &= E(\ddot{\mathbf{P}}_i'\otimes\ddot{\mathbf{P}}_i') + O_p(N^{-1/2}) = \boldsymbol{\Sigma}_{\ddot{\mathbf{P}}} + O_p(N^{-1/2}),\end{aligned}$$

with $\boldsymbol{\Sigma}_{\ddot{\mathbf{P}}} = E(\ddot{\mathbf{P}}_i'\otimes\ddot{\mathbf{P}}_i')$ and $\boldsymbol{\Sigma}_\epsilon = E(\boldsymbol{\epsilon}_i'\otimes\boldsymbol{\epsilon}_i')$. Therefore, matricising $vec(\widehat{\boldsymbol{\Sigma}})$ yields

$$\widehat{\boldsymbol{\Sigma}} = (vec(\mathbf{I}_{k_w})'\otimes\mathbf{I}_{k_w})\left(\mathbf{I}_{k_w}\otimes\left[\boldsymbol{\Sigma}_\epsilon + (\mathbf{S}_w'\otimes\mathbf{S}_w')\,\boldsymbol{\Sigma}_{\ddot{\mathbf{P}}}(\check{\mathbf{F}}'\otimes\check{\mathbf{F}}')\right]T^{-1}vec\,(\mathbf{M})\right) + O_p(N^{-1/2}),$$

which is the result stated in the lemma.

C.2.8 Proof of Lemma 8

Consider the estimator $\widehat{\sigma}_\varepsilon^2(\cdot)$ defined in equation (20) evaluated at $\boldsymbol{\delta}_0 \neq \boldsymbol{\delta}$, with $\boldsymbol{\delta} = [\rho, \boldsymbol{\beta}']'$ the true parameter vector. Suppose that $p^* \ge p$ and Ass.1-5 hold. We can then make use of eqs.(6) and (B-23) to obtain

$$\widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta}_0) = \frac{1}{N(T-c)}\sum_{i=1}^{N}\|\mathbf{M}(\mathbf{y}_i - \mathbf{w}_i\boldsymbol{\delta}_0)\|^2 = \frac{1}{N(T-c)}\sum_{i=1}^{N}\|\mathbf{M}(\mathbf{w}_i(\boldsymbol{\delta}-\boldsymbol{\delta}_0) + \mathbf{F}\boldsymbol{\gamma}_i + \boldsymbol{\varepsilon}_i)\|^2,$$

$$
\begin{aligned}
=&\frac{1}{N(T-c)}\sum_{i=1}^{N}\left\|\mathbf{M}\left(\mathbf{w}_i(\boldsymbol{\delta}-\boldsymbol{\delta}_0)-\bar{\mathbf{U}}_m\boldsymbol{\gamma}_i+\boldsymbol{\varepsilon}_i\right)\right\|^2,\\
=&\frac{1}{N(T-c)}\sum_{i=1}^{N}\left\|\mathbf{M}\left(\mathbf{w}_i(\boldsymbol{\delta}-\boldsymbol{\delta}_0)+\boldsymbol{\varepsilon}_i\right)\right\|^2+O_p(N^{-1/2}),\\
=&\frac{T}{T-c}(\boldsymbol{\delta}-\boldsymbol{\delta}_0)'\widehat{\boldsymbol{\Sigma}}(\boldsymbol{\delta}-\boldsymbol{\delta}_0)+\frac{2}{N(T-c)}\sum_{i=1}^{N}(\boldsymbol{\delta}-\boldsymbol{\delta}_0)'\mathbf{w}_i'\mathbf{M}\boldsymbol{\varepsilon}_i\\
&+\frac{1}{N(T-c)}\sum_{i=1}^{N}\boldsymbol{\varepsilon}_i'\mathbf{M}\boldsymbol{\varepsilon}_i+O_p(N^{-1/2}),
\end{aligned}
\tag{C-28}
$$

since we have $\left\|\frac{1}{N}\sum_{i=1}^{N}\mathbf{w}_i'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\gamma}_i\right\|=\left\|\frac{1}{N}\sum_{i=1}^{N}\mathbf{w}_i'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i\right\|=O_p(N^{-1/2})$ as proved in Lemma 5 such that for any $\|\boldsymbol{\delta}-\boldsymbol{\delta}_0\|<\infty$,

$$
\left\|\frac{1}{N}\sum_{i=1}^{N}\mathbf{w}_i'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i(\boldsymbol{\delta}-\boldsymbol{\delta}_0)\right\|\le\left\|\frac{1}{N}\sum_{i=1}^{N}\mathbf{w}_i'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i\right\|\|\boldsymbol{\delta}-\boldsymbol{\delta}_0\|=O_p\left(\frac{1}{\sqrt{N}}\right),
$$

and because

$$
\left\|\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\varepsilon}_i'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\gamma}_i\right\|\le\left\|\frac{1}{N}\sum_{i=1}^{N}(\boldsymbol{\gamma}_i'\otimes\boldsymbol{\varepsilon}_i')\right\|\left\|\bar{\mathbf{U}}_m\right\|\|\mathbf{M}\|=O_p\left(\frac{1}{N}\right),
\tag{C-29}
$$

$$
\left\|\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\gamma}_i'\bar{\mathbf{U}}_m'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\gamma}_i\right\|\le\left\|\frac{1}{N}\sum_{i=1}^{N}(\boldsymbol{\gamma}_i'\otimes\boldsymbol{\gamma}_i')\right\|\left\|\bar{\mathbf{U}}_m\right\|^2\|\mathbf{M}\|=O_p\left(\frac{1}{N}\right),
\tag{C-30}
$$

due to $\left\|\frac{1}{N}\sum_{i=1}^{N}(\boldsymbol{\gamma}_i'\otimes\boldsymbol{\gamma}_i')\right\|=O_p(1)$ and $\left\|\frac{1}{N}\sum_{i=1}^{N}(\boldsymbol{\gamma}_i'\otimes\boldsymbol{\varepsilon}_i')\right\|=O_p(N^{-1/2})$ by Ass.1 and 3, $\left\|\bar{\mathbf{U}}_m\right\|=O_p(N^{-1/2})$ by Lemma 2 and $\|\mathbf{M}\|=O(1)$ by Lemma 3. Next, we take the first two remaining terms in (C-28) individually as $N\to\infty$,

$$
\frac{T}{T-c}(\boldsymbol{\delta}-\boldsymbol{\delta}_0)'\widehat{\boldsymbol{\Sigma}}(\boldsymbol{\delta}-\boldsymbol{\delta}_0)=c_2(\boldsymbol{\delta}-\boldsymbol{\delta}_0)'\boldsymbol{\Sigma}(\boldsymbol{\delta}-\boldsymbol{\delta}_0)+O_p(N^{-1/2}),
\tag{C-31}
$$

$$
\frac{2}{T-c}\frac{1}{N}\sum_{i=1}^{N}(\boldsymbol{\delta}-\boldsymbol{\delta}_0)'\mathbf{w}_i'\mathbf{M}\boldsymbol{\varepsilon}_i=-c_1\sigma_\varepsilon^2\boldsymbol{\upsilon}(\rho,\mathbf{H})'(\boldsymbol{\delta}-\boldsymbol{\delta}_0)+O_p(N^{-1/2}),
\tag{C-32}
$$

where $c_1=\frac{2}{T-c}$ and $c_2=\frac{T}{T-c}$. The first result follows from Lemma 7 and the second from Theorem 1. Also, letting $h_{t,s}$ denote the element on row t and column s of $\mathbf{H}$, $\widetilde{c}=T-c$ and with $\bar{\varepsilon}_{t,s}=\frac{1}{N}\sum_{i=1}^{N}\varepsilon_{i,t}\varepsilon_{i,s}$,

$$
\begin{aligned}
\frac{1}{T-c}\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\varepsilon}_i'\mathbf{M}\boldsymbol{\varepsilon}_i&=\widetilde{c}^{-1}\frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}\varepsilon_{i,t}^2-\widetilde{c}^{-1}\frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}h_{t,s}\varepsilon_{i,t}\varepsilon_{i,s},\\
&=\widetilde{c}^{-1}\sum_{t=1}^{T}\bar{\varepsilon}_{t,t}-\widetilde{c}^{-1}\sum_{t=1}^{T}\sum_{s=1}^{T}h_{t,s}\bar{\varepsilon}_{t,s},\\
&=\widetilde{c}^{-1}\sum_{t=1}^{T}\bar{\varepsilon}_{t,t}-\widetilde{c}^{-1}\sum_{t=1}^{T}h_{t,t}\bar{\varepsilon}_{t,t}+O_p(N^{-1/2}),
\end{aligned}
$$

$$= \widetilde{c}^{-1}\left[T\sigma_\varepsilon^2 + \sum_{t=1}^{T}\left(\bar{\varepsilon}_{t,t} - \sigma_\varepsilon^2\right)\right] - \widetilde{c}^{-1}\left[\sigma_\varepsilon^2\sum_{t=1}^{T} h_{t,t} + \sum_{t=1}^{T} h_{t,t}\left(\bar{\varepsilon}_{t,t} - \sigma_\varepsilon^2\right)\right] + O_p(N^{-1/2}),$$

$$= \widetilde{c}^{-1}\sigma_\varepsilon^2\left(T - \sum_{t=1}^{T} h_{t,t}\right) + O_p(N^{-1/2}) = \sigma_\varepsilon^2\widetilde{c}^{-1}\widetilde{c} + O_p(N^{-1/2}),$$

$$= \sigma_\varepsilon^2 + O_p(N^{-1/2}), \tag{C-33}$$

since $\sum_{t=1}^{T} h_{t,t} = tr(\mathbf{H}) = c$ and by Ass.1 $\bar{\varepsilon}_{t,s} = O_p(N^{-1/2})$ for $t \neq s$ and $\bar{\varepsilon}_{t,t} = \sigma_\varepsilon^2 + O_p(N^{-1/2})$. Combining results gives

$$\widehat{\sigma}_\varepsilon^2\left(\boldsymbol{\delta}_0\right) = \sigma_\varepsilon^2 - \sigma_\varepsilon^2 c_1 \boldsymbol{\upsilon}(\rho, \mathbf{H})'(\boldsymbol{\delta} - \boldsymbol{\delta}_0) + c_2(\boldsymbol{\delta} - \boldsymbol{\delta}_0)'\boldsymbol{\Sigma}(\boldsymbol{\delta} - \boldsymbol{\delta}_0) + O_p(N^{-1/2}). \tag{C-34}$$

This proves (C-8).

Finally, evaluating eq.(20) at $\boldsymbol{\delta}_0 = \boldsymbol{\delta}$ we have, employing again (B-23) in $\widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta})$,

$$\widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta}) = \frac{1}{N(T-c)}\sum_{i=1}^{N}\left\|\mathbf{M}(\mathbf{w}_i(\boldsymbol{\delta} - \boldsymbol{\delta}) + \mathbf{F}\boldsymbol{\gamma}_i + \boldsymbol{\varepsilon}_i)\right\|^2 = \frac{1}{N(T-c)}\sum_{i=1}^{N}\left\|\mathbf{M}(\boldsymbol{\varepsilon}_i - \bar{\mathbf{U}}_m\boldsymbol{\gamma}_i)\right\|^2,$$

$$= \frac{1}{N(T-c)}\sum_{i=1}^{N}\boldsymbol{\varepsilon}_i'\mathbf{M}\boldsymbol{\varepsilon}_i - 2\frac{1}{N(T-c)}\sum_{i=1}^{N}\boldsymbol{\varepsilon}_i'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\gamma}_i + \frac{1}{N(T-c)}\sum_{i=1}^{N}\boldsymbol{\gamma}_i'\bar{\mathbf{U}}_m'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\gamma}_i,$$

which makes, by (C-29) and (C-30),

$$\widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta}) = \frac{1}{N(T-c)}\sum_{i=1}^{N}\boldsymbol{\varepsilon}_i'\mathbf{M}\boldsymbol{\varepsilon}_i + O_p(N^{-1}),$$

and proves (C-9) of the lemma. Finally, eq.(C-10) in the lemma follows directly from (C-34) evaluated at $\boldsymbol{\delta}_0 = \boldsymbol{\delta}$ and letting $N \to \infty$.

C.3 Statement of theorems

Below we state the theorems that are not presented in the main text or in section A.2.

Theorem 3. *Let $\boldsymbol{\phi}(\cdot) = \widehat{\boldsymbol{\delta}} - \widehat{\mathbf{m}}(\cdot)$, $\widetilde{\boldsymbol{\phi}}(\cdot) = \lim_{N\to\infty}\boldsymbol{\phi}(\cdot)$ and suppose that $p^* \geq p$ and Ass.1-5 hold. Assuming that $\widetilde{\boldsymbol{\phi}}(\boldsymbol{\delta}_0) = \mathbf{0}$ implies $\boldsymbol{\delta}_0 = \boldsymbol{\delta}$, and that $\chi \subseteq \mathbb{R}^{k_w}$ is compact with $\boldsymbol{\delta} \in \chi$,*

$$\widehat{\boldsymbol{\delta}}_{bc} \longrightarrow^p \boldsymbol{\delta} \qquad as \qquad N \to \infty,$$

with $\widehat{\boldsymbol{\delta}}_{bc}$ defined in eq.(21).

C.4 Proof of theorems and corollaries

C.4.1 Proof of Theorem 1

The CCEP estimator for $\boldsymbol{\delta}$ defined in (14) is

$$\widehat{\boldsymbol{\delta}} = \left(\frac{1}{N}\sum_{i=1}^{N}\frac{\mathbf{w}_i'\mathbf{M}\mathbf{w}_i}{T}\right)^{-1}\frac{1}{N}\sum_{i=1}^{N}\frac{\mathbf{w}_i'\mathbf{M}\mathbf{y}_i}{T}.$$

Substituting in eq.(6) gives

$$\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta} = \widehat{\boldsymbol{\Sigma}}^{-1}\left(\mathbf{A}^{\varepsilon} + \mathbf{A}^{\mathbf{F}}\right) = \widehat{\boldsymbol{\Sigma}}^{-1}\mathbf{A}^{\varepsilon} + O_p(N^{-1/2}), \tag{C-35}$$

where $\mathbf{A}^{\varepsilon} = \frac{1}{NT}\sum_{i=1}^{N}\mathbf{w}_i'\mathbf{M}\boldsymbol{\varepsilon}_i$ and because $\widehat{\boldsymbol{\Sigma}} = \frac{1}{NT}\sum_{i=1}^{N}\mathbf{w}_i'\mathbf{M}\mathbf{w}_i = O_p(1)$ by Lemma 6 and $\mathbf{A}^{\mathbf{F}} = \frac{1}{NT}\sum_{i=1}^{N}\mathbf{w}_i'\mathbf{M}\mathbf{F}\boldsymbol{\gamma}_i = O_p(N^{-1/2})$ by Lemma 5. Substituting in (B-13), we can decompose $\mathbf{A}^{\varepsilon}$ as

$$\mathbf{A}^{\varepsilon} = \frac{1}{NT}\sum_{i=1}^{N}\mathbf{S}_w'\ddot{\mathbf{P}}_i'\check{\mathbf{F}}'\mathbf{M}\boldsymbol{\varepsilon}_i + \frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\epsilon}_i'\mathbf{M}\boldsymbol{\varepsilon}_i,$$

where by the independence of $\boldsymbol{\varepsilon}_i$ and $\ddot{\mathbf{P}}_i$ by Ass.1 and 3, and Lemma 3

$$\left\|\frac{1}{NT}\sum_{i=1}^{N}\mathbf{S}_w'\ddot{\mathbf{P}}_i'\check{\mathbf{F}}'\mathbf{M}\boldsymbol{\varepsilon}_i\right\| \le \frac{1}{T}\left\|\frac{1}{N}\sum_{i=1}^{N}\left(\boldsymbol{\varepsilon}_i'\otimes\mathbf{S}_w'\ddot{\mathbf{P}}_i'\right)\right\|\left\|\check{\mathbf{F}}\right\|\|\mathbf{M}\| = O_p(N^{-1/2}).$$

Next, note that we can write, with $h_{t,s}$ denoting the element on row t and column s of $\mathbf{H}$, and with $\bar{\boldsymbol{\epsilon}}_{t,s} = \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\epsilon}_{it}\varepsilon_{is}$,

$$\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\epsilon}_i'\mathbf{M}\boldsymbol{\varepsilon}_i = \sum_{t=1}^{T}\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\epsilon}_{it}\varepsilon_{it} - \sum_{t=1}^{T}\sum_{s=1}^{T}h_{t,s}\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\epsilon}_{it}\varepsilon_{is} = \sum_{t=1}^{T}\bar{\boldsymbol{\epsilon}}_{t,t} - \sum_{t=1}^{T}\sum_{s=1}^{T}h_{t,s}\bar{\boldsymbol{\epsilon}}_{t,s},$$

where making use of (B-14) and Ass.1 and 5, for all t and s

$$\left[\bar{\boldsymbol{\epsilon}}_{t,s} - \sigma_{\varepsilon}^2\mathbf{q}_1\rho^{t-1-s}\mathbb{1}_{(t-1\ge s)}\right] = O_p(N^{-1/2}),$$

with $\mathbf{q}_1 = [1, \mathbf{0}_{k_x\times 1}']'$ and $\mathbb{1}_a$ is the indicator function returning 1 if the condition a is true, and zero otherwise. This gives, since by Lemma 3 all $h_{t,s}$ are bounded and $\bar{\boldsymbol{\epsilon}}_{t,t} = O_p(N^{-1/2})$,

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\epsilon}_i'\mathbf{M}\boldsymbol{\varepsilon}_i &= \sum_{t=1}^{T}\bar{\boldsymbol{\epsilon}}_{t,t} - \sum_{t=1}^{T}\sum_{s=1}^{T}h_{t,s}\left[\bar{\boldsymbol{\epsilon}}_{t,s} - \sigma_{\varepsilon}^2\mathbf{q}_1\rho^{t-1-s}\mathbb{1}_{(t-1\ge s)}\right] - \sigma_{\varepsilon}^2\mathbf{q}_1\sum_{t=1}^{T}\sum_{s=1}^{T}h_{t,s}\rho^{t-1-s}\mathbb{1}_{(t-1\ge s)},\\ &= -\sigma_{\varepsilon}^2\mathbf{q}_1\sum_{t=1}^{T}\sum_{s=1}^{T}h_{t,s}\rho^{t-1-s}\mathbb{1}_{(t-1\ge s)} + O_p(N^{-1/2}),\end{aligned}$$

and in turn leads to the conclusion

$$\operatorname*{plim}_{N\to\infty}\mathbf{A}^{\varepsilon} = -T^{-1}\sigma_{\varepsilon}^2\mathbf{q}_1\sum_{t=1}^{T}\sum_{s=1}^{T}h_{t,s}\rho^{t-1-s}\mathbb{1}_{(t-1\ge s)} = -T^{-1}\sigma_{\varepsilon}^2\boldsymbol{\upsilon}(\rho,\mathbf{H}),$$

with $\boldsymbol{\upsilon}(\rho,\mathbf{H}) = \upsilon(\rho,\mathbf{H})\mathbf{q}_1$ and $\upsilon(\rho,\mathbf{H}) = \sum_{t=1}^{T-1}\rho^{t-1}\sum_{s=t+1}^{T}h_{s,s-t}$. Next up is the denominator. From Lemma 7,

$$\begin{aligned}\widehat{\boldsymbol{\Sigma}} &= (vec(\mathbf{I}_{k_w})'\otimes\mathbf{I}_{k_w})\left(\mathbf{I}_{k_w}\otimes\left[\boldsymbol{\Sigma}_{\epsilon} + (\mathbf{S}_w'\otimes\mathbf{S}_w')\boldsymbol{\Sigma}_{\ddot{\mathbf{P}}}(\check{\mathbf{F}}'\otimes\check{\mathbf{F}}')\right]T^{-1}vec(\mathbf{M})\right) + O_p(N^{-1/2}),\\ &= \boldsymbol{\Sigma} + O_p(N^{-1/2}),\end{aligned}$$

with $\mathbf{\Sigma}_{\ddot{\mathbf{P}}} = E(\ddot{\mathbf{P}}_i' \otimes \ddot{\mathbf{P}}_i')$ and $\mathbf{\Sigma}_{\epsilon} = E(\boldsymbol{\epsilon}_i' \otimes \boldsymbol{\epsilon}_i')$, which are all $O(1)$ terms by Ass.1, 3 and 5. Hence, combining results gives

$$\operatorname*{plim}_{N\to\infty}(\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta}) = -\frac{\sigma_\varepsilon^2}{T}\mathbf{\Sigma}^{-1}\boldsymbol{\upsilon}(\rho, \mathbf{H}), \tag{C-36}$$

which is the result stated in eq.(15). Equations (16)-(17) in Theorem 1 are a reformulation of (C-36) obtained by application of the Frisch-Waugh-Lovell theorem and defining $\boldsymbol{\zeta} = \operatorname{plim}_{N\to\infty}(\breve{\mathbf{X}}'\breve{\mathbf{X}})^{-1}\breve{\mathbf{X}}'\mathbf{y}_{-1} = (\mathbf{S}_{\mathrm{x}}'\mathbf{\Sigma}\mathbf{S}_{\mathrm{x}})^{-1}\mathbf{S}_{\mathrm{x}}'\mathbf{\Sigma}\mathbf{q}_1$ and $\sigma^2_{\breve{\mathrm{y}}_{-1}} = \operatorname{plim}_{N\to\infty}\frac{\breve{\mathbf{y}}'_{-1}\breve{\mathbf{y}}_{-1}}{NT}$, with $\breve{\mathbf{y}}_{-1} = \mathbb{M}_x[\mathbf{y}'_{1,-1}, \ldots, \mathbf{y}'_{N,-1}]'$, $\mathbb{M}_x = \mathbb{M} - \breve{\mathbf{X}}(\breve{\mathbf{X}}'\breve{\mathbf{X}})^{-1}\breve{\mathbf{X}}'$, $\breve{\mathbf{X}} = \mathbb{M}[\mathbf{X}_1', \ldots, \mathbf{X}_N']'$ *and* $\mathbb{M} = \mathbf{I}_N \otimes \mathbf{M}$.

C.4.2 Proof for Corollary 1

It will be useful for the derivation of the explicit bias expression in eq.(A-5) to stack eq.(6) over individuals as

$$\mathbf{y} \;=\; (\mathbf{I}_N \otimes \boldsymbol{\iota}_T)\,\boldsymbol{\alpha} + \rho\mathbf{y}_{-1} + \mathbf{X}\boldsymbol{\beta} + \mathbb{F}\mathbf{\Lambda} + \boldsymbol{\varepsilon}, \tag{C-37}$$

with $\mathbb{F} = (\mathbf{I}_N \otimes \mathbf{F})$, $\mathbf{y} = [\mathbf{y}_1', \ldots, \mathbf{y}_N']'$, $\mathbf{X} = [\mathbf{X}_1', \ldots, \mathbf{X}_N']'$, $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_N]'$, $\mathbf{\Lambda} = [\boldsymbol{\gamma}_1', \ldots, \boldsymbol{\gamma}_N']'$ and $\boldsymbol{\varepsilon} = [\boldsymbol{\varepsilon}_1', \ldots, \boldsymbol{\varepsilon}_N']'$. With Ass.5 expression (C-37) can be inverted to get

$$\mathbf{y} \;=\; (\mathbf{I}_N \otimes \boldsymbol{\iota}_T)\,\boldsymbol{\alpha}^+ + \mathbf{X}^+\boldsymbol{\beta} + \mathbb{F}^+\mathbf{\Lambda} + \boldsymbol{\varepsilon}^+, \tag{C-38}$$

with $\mathbb{F}^+ = (\mathbf{I}_N \otimes \mathbf{F}^+)$ and variables with a + superscript defined as $\mathbf{X}^+ = (1-\rho L)^{-1}\mathbf{X}$. Using eq.(C-37) and the Frisch-Waugh-Lovell theorem, write the CCEP estimator as

$$\widehat{\rho} \;=\; (\mathbf{y}'_{-1}\mathbb{M}_{\mathrm{X}}\mathbf{y}_{-1})^{-1}\mathbf{y}'_{-1}\mathbb{M}_{\mathrm{X}}\mathbf{y}, \tag{C-39}$$

$$\widehat{\boldsymbol{\beta}} \;=\; (\mathbf{X}'\mathbb{M}\mathbf{X})^{-1}\,\mathbf{X}'\mathbb{M}\,(\mathbf{y} - \widehat{\rho}\mathbf{y}_{-1})\,, \tag{C-40}$$

with $\mathbb{M}_{\mathrm{X}} = \mathbb{M}_x\mathbb{M}$, $\mathbb{M} = \mathbf{I}_N \otimes \mathbf{M}$ and $\mathbb{M}_x = \mathbf{I}_{NT} - \mathbb{M}\mathbf{X}\,(\mathbf{X}'\mathbb{M}\mathbf{X})^{-1}\,\mathbf{X}'\mathbb{M}$. Eq.(C-35) implies

$$\operatorname*{plim}_{N\to\infty}(\widehat{\rho} - \rho) = \operatorname*{plim}_{N\to\infty}\frac{\mathbf{y}'_{-1}\mathbb{M}_{\mathrm{X}}\boldsymbol{\varepsilon}}{\mathbf{y}'_{-1}\mathbb{M}_{\mathrm{X}}\mathbf{y}_{-1}}, \tag{C-41}$$

$$\operatorname*{plim}_{N\to\infty}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}) = \operatorname*{plim}_{N\to\infty}(\mathbf{X}'\mathbb{M}\mathbf{X})^{-1}\,\mathbf{X}'\mathbb{M}\mathbf{y}_{-1}\,(\rho - \widehat{\rho})\,, \tag{C-42}$$

such that, defining $\boldsymbol{\zeta} = \operatorname{plim}_{N\to\infty}(\mathbf{X}'\mathbb{M}\mathbf{X})^{-1}\,\mathbf{X}'\mathbb{M}\mathbf{y}_{-1}$ we obtain for (C-42)

$$\operatorname*{plim}_{N\to\infty}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}) = -\boldsymbol{\zeta}\operatorname*{plim}_{N\to\infty}(\rho - \widehat{\rho}), \tag{C-43}$$

which is the expression in eq.(17).

Next, consider that lagging eq.(C-38) one period gives the following expression for $\mathbf{y}_{-1}$

$$\mathbf{y}_{-1} \;=\; (\mathbf{I}_N \otimes \boldsymbol{\iota}_T)\,\boldsymbol{\alpha}^+ + \mathbf{X}^+_{-1}\boldsymbol{\beta} + (\mathbf{I}_N \otimes \mathbf{F}^+_{-1})\mathbf{\Lambda} + \boldsymbol{\varepsilon}^+_{-1}.$$

This leads to

$$\mathbb{M}\mathbf{y}_{-1} \;=\; \mathbb{M}\mathbf{X}^+_{-1}\boldsymbol{\beta} + (\mathbf{I}_N \otimes \mathbf{M}\mathbf{F}^+_{-1})\mathbf{\Lambda} + \mathbb{M}\boldsymbol{\varepsilon}^+_{-1}. \tag{C-44}$$

We will use this result to evaluate (C-41) conditional on $\mathcal{C} = \sigma\{\check{\mathbf{F}}, \mathbf{Q}\}$. From the strict exogeneity of $\mathbf{X}$ (Ass.1) and the independence of $\boldsymbol{\Lambda}$ and $\boldsymbol{\varepsilon}$ (Ass.3) follows

$$\begin{aligned}
\operatorname*{plim}_{N\to\infty} \frac{1}{NT}\mathbf{y}_{-1}'\mathbb{M}_{\mathrm{X}}\boldsymbol{\varepsilon} = \operatorname*{plim}_{N\to\infty} \frac{1}{NT}\mathbf{y}_{-1}'\mathbb{M}\boldsymbol{\varepsilon} &= \operatorname*{plim}_{N\to\infty} \frac{1}{NT}(\boldsymbol{\varepsilon}_{-1}^{+})'\mathbb{M}\boldsymbol{\varepsilon}, \\
&= \operatorname*{plim}_{N\to\infty} \frac{1}{NT}\sum_{i=1}^{N}(\boldsymbol{\varepsilon}_{i,-1}^{+})'\mathbf{M}\boldsymbol{\varepsilon}_i. \qquad \text{(C-45)}
\end{aligned}$$

Defining $\widetilde{\mathbf{Q}} = \mathbf{B}\mathbf{Q}$, with $\mathbf{Q}$ a fixed matrix conditional on $\mathcal{C}$, and $\mathbf{B} = \mathbf{I}_T - \boldsymbol{\iota}_T\boldsymbol{\iota}_T'/T$, the numerator of (C-41) is

$$\begin{aligned}
\operatorname*{plim}_{N\to\infty} \frac{1}{NT}\sum_{i=1}^{N}(\boldsymbol{\varepsilon}_{i,-1}^{+})'\mathbf{M}\boldsymbol{\varepsilon}_i &= \operatorname*{plim}_{N\to\infty} \frac{1}{NT}\sum_{i=1}^{N}(\boldsymbol{\varepsilon}_{i,-1}^{+})'\left[(\boldsymbol{\varepsilon}_i - \bar{\boldsymbol{\varepsilon}}_i) - \widetilde{\mathbf{Q}}(\widetilde{\mathbf{Q}}'\widetilde{\mathbf{Q}})^{\dagger}\widetilde{\mathbf{Q}}'\left(\boldsymbol{\varepsilon}_i - \bar{\boldsymbol{\varepsilon}}_i\right)\right], \\
&= -\operatorname*{plim}_{N\to\infty} \frac{1}{NT}\sum_{i=1}^{N}(\boldsymbol{\varepsilon}_{i,-1}^{+})'\bar{\boldsymbol{\varepsilon}}_i - \operatorname*{plim}_{N\to\infty} \frac{1}{NT}\sum_{i=1}^{N}(\boldsymbol{\varepsilon}_{i,-1}^{+})'\widetilde{\mathbf{Q}}(\widetilde{\mathbf{Q}}'\widetilde{\mathbf{Q}})^{\dagger}\widetilde{\mathbf{Q}}'\boldsymbol{\varepsilon}_i, \\
&= -\frac{\sigma_\varepsilon^2}{T}A\left(\rho\right) - \frac{\sigma_\varepsilon^2}{T}\sum_{t=1}^{T-1}\rho^{t-1}\sum_{s=t+1}^{T}\widetilde{h}_{s,s-t}, \\
&= -\frac{\sigma_\varepsilon^2}{T}A\left(\rho\right) - \frac{\sigma_\varepsilon^2}{T}D(\rho, \widetilde{\mathbf{H}}), \qquad \text{(C-46)}
\end{aligned}$$

with $A(\rho) = \frac{1}{1-\rho}\left(1 - \frac{1}{T}\frac{1-\rho^T}{1-\rho}\right)$, $D(\rho, \widetilde{\mathbf{H}}) = \sum_{t=1}^{T-1}\rho^{t-1}\sum_{s=t+1}^{T}\widetilde{h}_{s,s-t}$ and $\widetilde{\mathbf{H}} = \widetilde{\mathbf{Q}}(\widetilde{\mathbf{Q}}'\widetilde{\mathbf{Q}})^{\dagger}\widetilde{\mathbf{Q}}'$.

Turning to the denominator of equation (C-41), using (C-44) we get

$$\begin{aligned}
\mathbf{y}_{-1}'\mathbb{M}_{\mathrm{X}}\mathbf{y}_{-1} &= \left\|\mathbb{M}_x\left(\mathbb{M}\mathbf{X}_{-1}^{+}\boldsymbol{\beta} + (\mathbf{I}_N \otimes \mathbf{M}\mathbf{F}_{-1}^{+})\boldsymbol{\Lambda} + \mathbb{M}\boldsymbol{\varepsilon}_{-1}^{+}\right)\right\|^2, \\
&= \left\|\mathbb{M}_{\mathrm{X}}\mathbf{X}_{-1}^{+}\boldsymbol{\beta}\right\|^2 + \left\|\mathbb{M}_{\mathrm{X}}(\mathbf{I}_N \otimes \mathbf{F}_{-1}^{+})\boldsymbol{\Lambda}\right\|^2 + \left\|\mathbb{M}_{\mathrm{X}}\boldsymbol{\varepsilon}_{-1}^{+}\right\|^2 \\
&\quad + 2\boldsymbol{\beta}'(\mathbf{X}_{-1}^{+})'\mathbb{M}_{\mathrm{X}}\boldsymbol{\varepsilon}_{-1}^{+} + 2\boldsymbol{\beta}'(\mathbf{X}_{-1}^{+})'\mathbb{M}_{\mathrm{X}}(\mathbf{I}_N \otimes \mathbf{F}_{-1}^{+})\boldsymbol{\Lambda} \\
&\quad + 2\boldsymbol{\Lambda}'(\mathbf{I}_N \otimes (\mathbf{F}_{-1}^{+})')\mathbb{M}_{\mathrm{X}}\boldsymbol{\varepsilon}_{-1}^{+}.
\end{aligned}$$

Defining first

$$\begin{aligned}
C^{+} &= \left\|\mathbb{M}_{\mathrm{X}}\mathbf{X}_{-1}^{+}\boldsymbol{\beta}\right\|^2 + \left\|\mathbb{M}_{\mathrm{X}}(\mathbf{I}_N \otimes \mathbf{F}_{-1}^{+})\boldsymbol{\Lambda}\right\|^2 + 2\boldsymbol{\beta}'(\mathbf{X}_{-1}^{+})'\mathbb{M}_{\mathrm{X}}\boldsymbol{\varepsilon}_{-1}^{+} \\
&\quad + 2\boldsymbol{\beta}'(\mathbf{X}_{-1}^{+})'\mathbb{M}_{\mathrm{X}}(\mathbf{I}_N \otimes \mathbf{F}_{-1}^{+})\boldsymbol{\Lambda} + 2\boldsymbol{\Lambda}'(\mathbf{I}_N \otimes (\mathbf{F}_{-1}^{+})')\mathbb{M}_{\mathrm{X}}\boldsymbol{\varepsilon}_{-1}^{+},
\end{aligned}$$

and taking the limit (conditional on $\mathcal{C}$) gives

$$\begin{aligned}
\operatorname*{plim}_{N\to\infty} \frac{1}{NT}C^{+} &= \operatorname*{plim}_{N\to\infty} \frac{1}{NT}\left\|\mathbb{M}_{\mathrm{X}}\mathbf{X}_{-1}^{+}\boldsymbol{\beta}\right\|^2 + \operatorname*{plim}_{N\to\infty} \frac{1}{NT}\left\|\mathbb{M}_{\mathrm{X}}(\mathbf{I}_N \otimes \mathbf{F}_{-1}^{+})\boldsymbol{\Lambda}\right\|^2 \\
&\quad + \operatorname*{plim}_{N\to\infty} \frac{1}{NT}2\boldsymbol{\beta}'(\mathbf{X}_{-1}^{+})'\mathbb{M}_{\mathrm{X}}(\mathbf{I}_N \otimes \mathbf{F}_{-1}^{+})\boldsymbol{\Lambda}, \qquad \text{(C-47)}
\end{aligned}$$

because by Ass.1 and 3

$$\operatorname*{plim}_{N\to\infty} \frac{1}{NT}2\boldsymbol{\Lambda}'(\mathbf{I}_N \otimes (\mathbf{F}_{-1}^{+})')\mathbb{M}_{\mathrm{X}}\boldsymbol{\varepsilon}_{-1}^{+} = 0,$$

$$\operatorname*{plim}_{N\to\infty}\frac{1}{NT}2\boldsymbol{\beta}'(\mathbf{X}_{-1}^{+})'\mathbb{M}_{\mathrm{X}}\boldsymbol{\varepsilon}_{-1}^{+} \quad = \quad 0.$$

Hence

$$\begin{aligned}
\operatorname*{plim}_{N\to\infty}\frac{1}{NT}\mathbf{y}'_{-1}\mathbb{M}_{\mathrm{X}}\mathbf{y}_{-1} \quad &= \quad \operatorname*{plim}_{N\to\infty}\frac{1}{NT}\left\|\mathbb{M}_{\mathrm{X}}\boldsymbol{\varepsilon}_{-1}^{+}\right\|^{2}+\operatorname*{plim}_{N\to\infty}\frac{1}{NT}C^{+},\\
&= \quad \operatorname*{plim}_{N\to\infty}\frac{1}{NT}\left\|\mathbb{M}\boldsymbol{\varepsilon}_{-1}^{+}\right\|^{2}-\operatorname*{plim}_{N\to\infty}\frac{1}{NT}(\boldsymbol{\varepsilon}_{-1}^{+})'\mathbb{M}\mathbf{X}\left(\mathbf{X}'\mathbb{M}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbb{M}\boldsymbol{\varepsilon}_{-1}^{+}\\
&\qquad +\operatorname*{plim}_{N\to\infty}\frac{1}{NT}C^{+},\\
&= \quad \operatorname*{plim}_{N\to\infty}\frac{1}{NT}\sum_{i=1}^{N}(\boldsymbol{\varepsilon}_{i,-1}^{+})'\mathbf{M}\boldsymbol{\varepsilon}_{i,-1}^{+}+\operatorname*{plim}_{N\to\infty}\frac{1}{NT}C^{+}. \qquad \text{(C-48)}
\end{aligned}$$

Focusing on the first term of (C-48) and using earlier definitions gives

$$\begin{aligned}
\operatorname*{plim}_{N\to\infty}\frac{1}{NT}\sum_{i=1}^{N}(\boldsymbol{\varepsilon}_{i,-1}^{+})'\mathbf{M}\boldsymbol{\varepsilon}_{i,-1}^{+} &= \operatorname*{plim}_{N\to\infty}\frac{1}{NT}\sum_{i=1}^{N}(\boldsymbol{\varepsilon}_{i,-1}^{+})'\Big[\left(\boldsymbol{\varepsilon}_{i,-1}^{+}-\bar{\boldsymbol{\varepsilon}}_{i,-1}^{+}\right)\\
&\qquad -\widetilde{\mathbf{Q}}(\widetilde{\mathbf{Q}}'\widetilde{\mathbf{Q}})^{\dagger}\widetilde{\mathbf{Q}}'\left(\boldsymbol{\varepsilon}_{i,-1}^{+}-\bar{\boldsymbol{\varepsilon}}_{i,-1}^{+}\right)\Big],\\
&= \operatorname*{plim}_{N\to\infty}\frac{1}{NT}\sum_{t=1}^{T}\sum_{i=1}^{N}\left(\varepsilon_{i,t-1}^{+}-\bar{\varepsilon}_{i,-1}^{+}\right)^{2}\\
&\qquad -\operatorname*{plim}_{N\to\infty}\frac{1}{NT}\sum_{i=1}^{N}(\boldsymbol{\varepsilon}_{i,-1}^{+})'\widetilde{\mathbf{H}}\boldsymbol{\varepsilon}_{i,-1}^{+},\\
&= \frac{\sigma_{\varepsilon}^{2}}{T}B\left(\rho\right)-\frac{\sigma_{\varepsilon}^{2}}{1-\rho^{2}}\frac{1}{T}\left[tr(\widetilde{\mathbf{H}})+2\rho\sum_{t=1}^{T-1}\rho^{t-1}\sum_{s=t+1}^{T}\widetilde{h}_{s,s-t}\right],\\
&= \frac{\sigma_{\varepsilon}^{2}}{T}\left(B\left(\rho\right)-\frac{1}{1-\rho^{2}}\left[c-1+2\rho D(\rho,\widetilde{\mathbf{H}})\right]\right), \qquad \text{(C-49)}
\end{aligned}$$

where $B(\rho)=\frac{T}{1-\rho^{2}}\left(1-\frac{1}{T}\frac{1+\rho}{1-\rho}-\frac{2\rho}{T^{2}}\frac{1-\rho^{T}}{(1-\rho)^{2}}\right)$.

Combining (C-46), (C-47) and (C-49)

$$\begin{aligned}
\operatorname*{plim}_{N\to\infty}\left(\widehat{\rho}-\rho\right) &= \frac{-\frac{\sigma_{\varepsilon}^{2}}{T}\left(A\left(\rho\right)+D(\rho,\widetilde{\mathbf{H}})\right)}{\frac{\sigma_{\varepsilon}^{2}}{T}\left(B\left(\rho\right)-\frac{1}{1-\rho^{2}}\left[c-1+2\rho D(\rho,\widetilde{\mathbf{H}})\right]\right)+\operatorname{plim}_{N\to\infty}\frac{1}{NT}C^{+}},\\
&= \frac{-A\left(\rho\right)-D(\rho,\widetilde{\mathbf{H}})}{B\left(\rho\right)-\frac{1}{1-\rho^{2}}\left[c-1+2\rho D(\rho,\widetilde{\mathbf{H}})\right]+\operatorname{plim}_{N\to\infty}\frac{1}{N\sigma_{\varepsilon}^{2}}C^{+}}, \qquad \text{(C-50)}
\end{aligned}$$

which we reformulate to

$$\operatorname*{plim}_{N\to\infty}\left(\widehat{\rho}-\rho\right) \quad = \quad -\frac{\left[A\left(\rho\right)+D(\rho,\widetilde{\mathbf{H}})\right]}{\left[B\left(\rho\right)-E(\rho,\widetilde{\mathbf{H}})+TC\right]}, \qquad \text{(C-51)}$$

where $E(\rho, \widetilde{\mathbf{H}}) = \frac{1}{1-\rho^2}\left[c - 1 + 2\rho D(\rho, \widetilde{\mathbf{H}})\right]$ and

$$C = \underset{N\to\infty}{\text{plim}} \frac{1}{NT\sigma_\varepsilon^2}\left(\left\|\mathbb{M}_{\mathrm{X}}\mathbf{X}_{-1}^{+}\boldsymbol{\beta}\right\|^2 + \left\|\mathbb{M}_{\mathrm{X}}(\mathbf{I}_N \otimes \mathbf{F}_{-1}^{+})\boldsymbol{\Lambda}\right\|^2 + 2\boldsymbol{\beta}'(\mathbf{X}_{-1}^{+})'\mathbb{M}_{\mathrm{X}}(\mathbf{I}_N \otimes \mathbf{F}_{-1}^{+})\boldsymbol{\Lambda}\right),$$
$$= \underset{N\to\infty}{\text{plim}} \frac{1}{NT\sigma_\varepsilon^2}\left(\boldsymbol{\beta}'(\mathbf{X}_{-1}^{+})'\mathbb{M}_{\mathrm{X}}\mathbf{X}_{-1}^{+}\boldsymbol{\beta} + \boldsymbol{\Lambda}'(\mathbb{F}_{-1}^{+})'\mathbb{M}_{\mathrm{X}}\mathbb{F}_{-1}^{+}\boldsymbol{\Lambda} + 2\boldsymbol{\beta}'(\mathbf{X}_{-1}^{+})'\mathbb{M}_{\mathrm{X}}\mathbb{F}_{-1}^{+}\boldsymbol{\Lambda}\right),$$
$$= \underset{N\to\infty}{\text{plim}} \frac{\boldsymbol{\beta}'\boldsymbol{\Omega}_{\breve{\mathrm{x}}}\boldsymbol{\beta} + \boldsymbol{\Lambda}'\boldsymbol{\Omega}_{\breve{\mathrm{f}}}\boldsymbol{\Lambda} + 2\boldsymbol{\beta}'\boldsymbol{\Omega}_{\breve{\mathrm{x}},\breve{\mathrm{f}}}\boldsymbol{\Lambda}}{\sigma_\varepsilon^2},$$

with $\boldsymbol{\Omega}_{\breve{\mathrm{x}}} = (\mathbf{X}_{-1}^{+})'\mathbb{M}_{\mathrm{X}}\mathbf{X}_{-1}^{+}/NT$, $\boldsymbol{\Omega}_{\breve{\mathrm{f}}} = (\mathbb{F}_{-1}^{+})'\mathbb{M}_{\mathrm{X}}\mathbb{F}_{-1}^{+}/NT$, $\boldsymbol{\Omega}_{\breve{\mathrm{x}},\breve{\mathrm{f}}} = (\mathbf{X}_{-1}^{+})'\mathbb{M}_{\mathrm{X}}\mathbb{F}_{-1}^{+}/NT$ and $\mathbb{F}_{-1}^{+} = (\mathbf{I}_N \otimes \mathbf{F}_{-1}^{+})$.

C.4.3 Proof of Theorem 3

Let $\boldsymbol{\phi}(\boldsymbol{\delta}_0)$ be the vector of moment conditions employed by CCEPbc in (21) evaluated at $\boldsymbol{\delta}_0 \neq \boldsymbol{\delta}$, with $\boldsymbol{\delta}$ the population parameter vector $\boldsymbol{\delta} = [\rho, \boldsymbol{\beta}']'$. Multiplying by $\widehat{\boldsymbol{\Sigma}}$ and solving in eq.(6) gives

$$\widehat{\boldsymbol{\Sigma}}\boldsymbol{\phi}(\boldsymbol{\delta}_0) = \frac{1}{NT}\sum_{i=1}^{N}\mathbf{w}_i'\mathbf{M}\mathbf{y}_i - \frac{1}{NT}\sum_{i=1}^{N}\mathbf{w}_i'\mathbf{M}\mathbf{w}_i\boldsymbol{\delta}_0 + \frac{1}{T}\widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta}_0)\boldsymbol{\upsilon}(\rho_0),$$
$$= \widehat{\boldsymbol{\Sigma}}\left(\boldsymbol{\delta} - \boldsymbol{\delta}_0\right) + \frac{1}{NT}\sum_{i=1}^{N}\mathbf{w}_i'\mathbf{M}\boldsymbol{\varepsilon}_i + \frac{1}{NT}\sum_{i=1}^{N}\mathbf{w}_i'\mathbf{M}\mathbf{F}\boldsymbol{\gamma}_i + \frac{1}{T}\widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta}_0)\boldsymbol{\upsilon}(\rho_0),$$
$$= \widehat{\boldsymbol{\Sigma}}(\boldsymbol{\delta} - \boldsymbol{\delta}_0) + \frac{1}{T}\left(\frac{1}{N}\sum_{i=1}^{N}\mathbf{w}_i'\mathbf{M}\boldsymbol{\varepsilon}_i + \widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta}_0)\boldsymbol{\upsilon}(\rho)\right) + O_p(N^{-1/2}), \tag{C-52}$$

because $\frac{1}{NT}\sum_{i=1}^{N}\mathbf{w}_i'\mathbf{M}\mathbf{F}\boldsymbol{\gamma}_i = O_p(N^{-1/2})$ by Lemma 5. Note that we have dropped the dependence of $\boldsymbol{\upsilon}(\cdot)$ on $\mathbf{H}$ for simplicity.

Consider the middle term. From Lemma 8 with $\|\boldsymbol{\delta} - \boldsymbol{\delta}_0\| < \infty$ given compactness of χ,

$$\widehat{\sigma}_\varepsilon^2\left(\boldsymbol{\delta}_0\right) = \sigma_\varepsilon^2 - \sigma_\varepsilon^2 c_1\boldsymbol{\upsilon}(\rho)'(\boldsymbol{\delta} - \boldsymbol{\delta}_0) + c_2(\boldsymbol{\delta} - \boldsymbol{\delta}_0)'\boldsymbol{\Sigma}(\boldsymbol{\delta} - \boldsymbol{\delta}_0) + O_p(N^{-1/2}),$$

with $\boldsymbol{\Sigma}$ defined in eq.(C-7) of Lemma 7, and where $c_1 = \frac{2}{T-c}$ and $c_2 = \frac{T}{T-c}$. We also have

$$\frac{1}{N}\sum_{i=1}^{N}\mathbf{w}_i'\mathbf{M}\boldsymbol{\varepsilon}_i = -\sigma_\varepsilon^2\boldsymbol{\upsilon}(\rho) + O_p(N^{-1/2}),$$

by Theorem 1. As such, by combining results we can write as $N \to \infty$ that

$$\frac{1}{N}\sum_{i=1}^{N}\mathbf{w}_i'\mathbf{M}\boldsymbol{\varepsilon}_i + \widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta}_0)\boldsymbol{\upsilon}(\rho_0)$$
$$= -\sigma_\varepsilon^2\boldsymbol{\upsilon}(\rho) + \boldsymbol{\upsilon}(\rho_0)\left[\sigma_\varepsilon^2 - \sigma_\varepsilon^2 c_1\boldsymbol{\upsilon}(\rho)'(\boldsymbol{\delta} - \boldsymbol{\delta}_0) + c_2(\boldsymbol{\delta} - \boldsymbol{\delta}_0)'\boldsymbol{\Sigma}(\boldsymbol{\delta} - \boldsymbol{\delta}_0)\right] + o_p(1),$$
$$= -\sigma_\varepsilon^2\left[\boldsymbol{\upsilon}(\rho) - \boldsymbol{\upsilon}(\rho_0)\right] - \sigma_\varepsilon^2 c_1\boldsymbol{\upsilon}(\rho_0)\boldsymbol{\upsilon}(\rho)'(\boldsymbol{\delta} - \boldsymbol{\delta}_0) + c_2\boldsymbol{\upsilon}(\rho_0)(\boldsymbol{\delta} - \boldsymbol{\delta}_0)'\boldsymbol{\Sigma}(\boldsymbol{\delta} - \boldsymbol{\delta}_0) + o_p(1),$$

Substituting this result in (C-52) gives $\left\|\boldsymbol{\phi}(\boldsymbol{\delta}_0) - \widetilde{\boldsymbol{\phi}}(\boldsymbol{\delta}_0)\right\| = o_p(1)$ for $\|\boldsymbol{\delta} - \boldsymbol{\delta}_0\| < \infty$, with

$$\widetilde{\boldsymbol{\phi}}(\boldsymbol{\delta}_0) = (\boldsymbol{\delta} - \boldsymbol{\delta}_0) - \frac{1}{T}\Bigg[\sigma_\varepsilon^2 \boldsymbol{\Sigma}^{-1}\left[\boldsymbol{v}(\rho) - \boldsymbol{v}(\rho_0)\right] + \sigma_\varepsilon^2 c_1 \boldsymbol{\Sigma}^{-1}\boldsymbol{v}(\rho_0)\boldsymbol{v}(\rho)'(\boldsymbol{\delta} - \boldsymbol{\delta}_0)$$

$$- c_2 \boldsymbol{\Sigma}^{-1}\boldsymbol{v}(\rho_0)(\boldsymbol{\delta} - \boldsymbol{\delta}_0)'\boldsymbol{\Sigma}(\boldsymbol{\delta} - \boldsymbol{\delta}_0)\Bigg]. \quad \text{(C-53)}$$

In (C-53) we note that $\|\boldsymbol{v}(\rho_0)\| < \infty$ since $\|\mathbf{H}\| = \sqrt{c}$ from Lemma 3 such that $v(\rho_0, \mathbf{H}) < \infty$ for any finite ρ_0 (and where $|\rho|< 1$ by Ass.5 ensures $\|\boldsymbol{v}(\rho)\| < \infty$ also as $T \to \infty$). Also, $c_2 = O(1)$ and since $T > c$, $c_1 = O(1)$. $\|\boldsymbol{\Sigma}\| = O(1)$ is shown in Lemma 7 and $\sigma_\varepsilon^2 < \infty$ by Ass.1. This implies that $\left\|\widetilde{\boldsymbol{\phi}}(\boldsymbol{\delta}_0)\right\| < \infty$ provided $\|\boldsymbol{\delta} - \boldsymbol{\delta}_0\| < \infty$. Also, clearly from (C-53),

$$\widetilde{\boldsymbol{\phi}}(\boldsymbol{\delta}_0) = \mathbf{0}_{k_w \times 1}, \qquad \text{for} \qquad \boldsymbol{\delta}_0 = \boldsymbol{\delta}.$$

Finally, since $[\boldsymbol{v}(\rho) - \boldsymbol{v}(\rho_0)]$ is determined only by $\rho_0 - \rho$ and is zero only for $\rho_0 = \rho$ we take that $\widetilde{\boldsymbol{\phi}}(\boldsymbol{\delta}_0) = \mathbf{0}_{k_w \times 1}$ implies $\boldsymbol{\delta}_0 = \boldsymbol{\delta}$ such that, assuming that the admissible parameter space $\chi \subseteq \mathbb{R}^{k_w}$ in (21) is compact with $\boldsymbol{\delta}$ contained in its interior, we have as in Newey and McFadden (1994) (Section 2.5) that

$$\widehat{\boldsymbol{\delta}}_{bc} \longrightarrow^p \boldsymbol{\delta},$$

as $N \to \infty$.

D Analysis for $(N,T) \to \infty$

D.1 Preliminary results

Consider the decomposition

$$\begin{aligned}\mathbf{M_F} - \mathbf{M} &= \bar{\mathbf{U}}^0[\mathbf{Q}_0'\mathbf{Q}_0]^{-1}(\bar{\mathbf{U}}^0)' + \bar{\mathbf{U}}^0[\mathbf{Q}_0'\mathbf{Q}_0]^{-1}(\mathbf{F}^0)' + \mathbf{F}^0[\mathbf{Q}_0'\mathbf{Q}_0]^{-1}(\bar{\mathbf{U}}^0)' \\ &\quad + \mathbf{F}^0\left([\mathbf{Q}_0'\mathbf{Q}_0]^{-1} - [(\mathbf{F}^0)'\mathbf{F}^0]^{-1}\right)(\mathbf{F}^0)',\end{aligned}$$

and note that, since $\mathbf{F}^0 = [\mathbf{F}^*, \mathbf{0}_{T\times(K-m)}]$ and $\bar{\mathbf{U}}^0 = [\bar{\mathbf{U}}^0_m, \bar{\mathbf{U}}^0_{-m}]$, we have using similar steps as in the proof of Lemma S.1 of Karabiyik et al. (2017)

$$\begin{aligned}\mathbf{M_F} - \mathbf{M} &= T^{-1}\bar{\mathbf{U}}^0_{-m}[T^{-1}(\bar{\mathbf{U}}^0_{-m})'\bar{\mathbf{U}}^0_{-m}]^{-1}(\bar{\mathbf{U}}^0_{-m})' + T^{-1}\bar{\mathbf{U}}^0_m[T^{-1}(\mathbf{F}^*)'\mathbf{F}^*]^{-1}(\bar{\mathbf{U}}^0_m)' \\ &\quad + T^{-1}\mathbf{F}^*[T^{-1}(\mathbf{F}^*)'\mathbf{F}^*]^{-1}(\bar{\mathbf{U}}^0_m)' + T^{-1}\bar{\mathbf{U}}^0_m[T^{-1}(\mathbf{F}^*)'\mathbf{F}^*]^{-1}(\mathbf{F}^*)' \\ &\quad + T^{-1}\mathbf{Q}_0\left(\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1} - \widehat{\boldsymbol{\Sigma}}_{\mathbf{F}_u^+}^{-1}\right)\mathbf{Q}_0',\end{aligned} \tag{D-1}$$

with $\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}} = T^{-1}\mathbf{Q}_0'\mathbf{Q}_0$ and

$$\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}_u^+} = \frac{1}{T}\begin{bmatrix}(\mathbf{F}^*)'\mathbf{F}^* & \mathbf{0}_{(1+K+m)\times(K-m)} \\ \mathbf{0}_{(K-m)\times(1+K+m)} & (\bar{\mathbf{U}}^0_{-m})'\bar{\mathbf{U}}^0_{-m}\end{bmatrix} = \begin{bmatrix}\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^*} & \mathbf{0}_{(1+K+m)\times(K-m)} \\ \mathbf{0}_{(K-m)\times(1+K+m)} & \widehat{\boldsymbol{\Sigma}}_{\mathbf{u}^0_{-m}}\end{bmatrix}, \tag{D-2}$$

where $\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^*} = T^{-1}(\mathbf{F}^*)'\mathbf{F}^*$ and $\widehat{\boldsymbol{\Sigma}}_{\mathbf{u}^0_{-m}} = T^{-1}(\bar{\mathbf{U}}^0_{-m})'\bar{\mathbf{U}}^0_{-m}$.

D.2 Statement of lemmas

Lemma 9. *Suppose Assumptions 1-3 and 5 hold, then, as $(N,T) \to \infty$,*

$$\frac{\ddot{\mathbf{U}}'\ddot{\mathbf{U}}}{T} = O_p\left(\frac{1}{N}\right), \tag{D-3}$$

$$\frac{\ddot{\mathbf{U}}'\check{\mathbf{F}}}{T} = O_p\left(\frac{1}{\sqrt{NT}}\right), \qquad \frac{\check{\mathbf{F}}'\check{\mathbf{F}}}{T} = O_p(1), \tag{D-4}$$

$$\frac{\ddot{\mathbf{U}}_i'\check{\mathbf{F}}}{T} = O_p\left(\frac{1}{\sqrt{T}}\right), \qquad \frac{\boldsymbol{\varepsilon}_i'\check{\mathbf{F}}}{T} = O_p\left(\frac{1}{\sqrt{T}}\right), \qquad \frac{\boldsymbol{\epsilon}_i'\check{\mathbf{F}}}{T} = O_p\left(\frac{1}{\sqrt{T}}\right), \tag{D-5}$$

$$\frac{\boldsymbol{\varepsilon}_i'\ddot{\mathbf{U}}}{T} = O_p\left(\frac{1}{N}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right), \qquad \frac{\boldsymbol{\epsilon}_i'\ddot{\mathbf{U}}}{T} = O_p\left(\frac{1}{N}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right). \tag{D-6}$$

Lemma 10. *Suppose Assumptions 1-5 hold, then, as $(N,T) \to \infty$,*

$$\frac{(\bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}^0_m}{T} = O_p\left(\frac{1}{N}\right), \qquad \frac{(\bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}^0_{-m}}{T} = O_p\left(\frac{1}{\sqrt{N}}\right), \qquad \frac{(\bar{\mathbf{U}}^0_{-m})'\bar{\mathbf{U}}^0_{-m}}{T} = O_p(1), \tag{D-7}$$

$$\frac{(\mathbf{F}^*)'\mathbf{F}^*}{T} = O_p(1), \qquad \frac{(\bar{\mathbf{U}}^0_m)'\mathbf{F}^*}{T} = O_p\left(\frac{1}{\sqrt{NT}}\right), \qquad \frac{(\bar{\mathbf{U}}^0_{-m})'\mathbf{F}^*}{T} = O_p\left(\frac{1}{\sqrt{T}}\right), \tag{D-8}$$

$$\frac{(\bar{\mathbf{U}}^0)'\mathbf{F}^*}{T} = O_p\left(\frac{1}{\sqrt{T}}\right), \qquad \frac{\boldsymbol{\varepsilon}_i'\mathbf{F}^*}{T} = O_p\left(\frac{1}{\sqrt{T}}\right), \qquad \frac{\boldsymbol{\epsilon}_i'\mathbf{F}^*}{T} = O_p\left(\frac{1}{\sqrt{T}}\right), \tag{D-9}$$

$$\frac{\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}^0_m}{T} = O_p\left(\frac{1}{N}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right), \qquad \frac{\boldsymbol{\varepsilon}_i'\bar{\mathbf{U}}^0_m}{T} = O_p\left(\frac{1}{N}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right), \tag{D-10}$$

$$\frac{\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}^0_{-m}}{T} = O_p\left(\frac{1}{\sqrt{N}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right), \qquad \frac{\boldsymbol{\varepsilon}_i'\bar{\mathbf{U}}^0_{-m}}{T} = O_p\left(\frac{1}{\sqrt{N}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right), \tag{D-11}$$

$$\frac{\boldsymbol{\epsilon}_i'\mathbf{Q}_0}{T} = O_p\left(\frac{1}{\sqrt{T}}\right), \qquad \frac{\boldsymbol{\varepsilon}_i'\mathbf{Q}_0}{T} = O_p\left(\frac{1}{\sqrt{T}}\right), \qquad \frac{\mathbf{Q}_0'\bar{\mathbf{U}}^0_m}{T} = O_p\left(\frac{1}{\sqrt{N}}\right). \tag{D-12}$$

Lemma 11. *Suppose Assumptions 1-5 hold and let* $\widetilde{\mathbf{P}}_i = \ddot{\mathbf{P}}_i - N^{-1}\sum_{i=1}^N \ddot{\mathbf{P}}_i$. *Then, as* $(N,T)\to\infty$,

$$N^{-1}\sum_{i=1}^N \left(T^{-1}\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}^0_m \otimes \mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right) = O_p\left(\frac{1}{N^{3/2}}\right) + O_p\left(\frac{1}{N\sqrt{T}}\right), \tag{D-13}$$

$$N^{-1}\sum_{i=1}^N \left(T^{-1}\boldsymbol{\varepsilon}_i'\bar{\mathbf{U}}^0_m \otimes \mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right) = O_p\left(\frac{1}{N^{3/2}}\right) + O_p\left(\frac{1}{N\sqrt{T}}\right), \tag{D-14}$$

$$N^{-1}\sum_{i=1}^N \left(T^{-1}\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}^0_{-m} \otimes \mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right) = O_p\left(\frac{1}{N}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right), \tag{D-15}$$

$$N^{-1}\sum_{i=1}^N \left(T^{-1}\boldsymbol{\varepsilon}_i'\bar{\mathbf{U}}^0_{-m} \otimes \mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right) = O_p\left(\frac{1}{N}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right), \tag{D-16}$$

$$N^{-1}\sum_{i=1}^N \left(T^{-1}\boldsymbol{\epsilon}_i'\check{\mathbf{F}} \otimes \mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right) = O_p\left(\frac{1}{\sqrt{NT}}\right), \tag{D-17}$$

$$N^{-1}\sum_{i=1}^N \left(T^{-1}\boldsymbol{\varepsilon}_i'\check{\mathbf{F}} \otimes \mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right) = O_p\left(\frac{1}{\sqrt{NT}}\right), \tag{D-18}$$

$$N^{-1}\sum_{i=1}^N \left(T^{-1}\boldsymbol{\varepsilon}_i'\mathbf{Q}_0 \otimes \mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right) = O_p\left(\frac{1}{N}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right), \tag{D-19}$$

where the results hold similarly if $\mathbf{S}_w'\widetilde{\mathbf{P}}_i'$ *is substituted for* $\boldsymbol{\eta}_i'$.

Lemma 12. *Let Assumptions 1-5 hold. Then, as* $(N,T)\to\infty$,

$$\begin{aligned}
\widehat{\boldsymbol{\Sigma}}_{\check{\mathbf{F}}} &= T^{-1}\check{\mathbf{F}}'\check{\mathbf{F}} = \boldsymbol{\Sigma}_{\check{\mathbf{F}}} + O_p(T^{-1/2}),\\
\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^*} &= T^{-1}(\mathbf{F}^*)'\mathbf{F}^* = \boldsymbol{\Sigma}_{\mathbf{F}^*} + O_p(N^{-1/2}) + O_p(T^{-1/2}),\\
\widehat{\boldsymbol{\Sigma}}_{\check{\mathbf{F}}\mathbf{F}^*} &= T^{-1}(\mathbf{F}^*)'\mathbf{F}^* = \boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*} + O_p(N^{-1/2}) + O_p(T^{-1/2}),\\
\widehat{\boldsymbol{\Sigma}}_{\mathbf{u}^0_{-m}} &= T^{-1}(\bar{\mathbf{U}}^0_{-m})'\bar{\mathbf{U}}^0_{-m} = \boldsymbol{\Sigma}_{\mathbf{u}^0_{-m}} + O_p(T^{-1/2}),
\end{aligned}$$

$$\widehat{\boldsymbol{\Sigma}}_{\mathbf{u}^0_{-m}\mathbf{u}^0_m} = T^{-1}(\bar{\mathbf{U}}^0_{-m})'\sqrt{N}\bar{\mathbf{U}}^0_m = \boldsymbol{\Sigma}_{\mathbf{u}^0_{-m}\mathbf{u}^0_m} + O_p(T^{-1/2}),$$

and also

$$\left\|\widehat{\boldsymbol{\Sigma}}^{-1}_{\mathbf{F}^*} - \boldsymbol{\Sigma}^{-1}_{\mathbf{F}^*}\right\| = O_p(N^{-1/2}) + O_p(T^{-1/2}),$$
$$\left\|\widehat{\boldsymbol{\Sigma}}^{-1}_{\mathbf{u}^0_{-m}} - \boldsymbol{\Sigma}^{-1}_{\mathbf{u}^0_{-m}}\right\| = O_p(T^{-1/2}),$$

where

$$\boldsymbol{\Sigma}_{\check{\mathbf{F}}} = E(\check{\mathbf{F}}'\check{\mathbf{F}}/T), \qquad \boldsymbol{\Sigma}_{\ddot{\mathbf{U}}} = E(\ddot{\mathbf{U}}_i'\ddot{\mathbf{U}}_i/T), \qquad \boldsymbol{\Sigma}_{\mathbf{F}^*} = \mathbf{S}_m'\mathbf{N}'\mathbf{R}'\mathbf{P}'\boldsymbol{\Sigma}_{\check{\mathbf{F}}}\mathbf{PRNS}_m,$$
$$\boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*} = \boldsymbol{\Sigma}_{\check{\mathbf{F}}}\mathbf{PRNS}_m, \qquad \boldsymbol{\Sigma}_{\mathbf{u}^0_{-m}} = \mathbf{B}_{-m}'\mathbf{T}'\mathbf{A}_0^{-1}\boldsymbol{\Omega}_{\mathbf{u}}(\mathbf{A}_0^{-1})'\mathbf{TB}_{-m},$$
$$\boldsymbol{\Sigma}_{\mathbf{u}^0_{-m}\mathbf{u}^0_m} = \mathbf{S}_{-m}'\mathbf{N}'\mathbf{R}'\boldsymbol{\Sigma}_{\ddot{\mathbf{U}}}\mathbf{RNS}_m, \qquad \boldsymbol{\Omega}_{\mathbf{u}} = \begin{bmatrix}\sigma^2_{\varepsilon} & \mathbf{0}'_{k\times 1}\\ \mathbf{0}_{k\times 1} & \boldsymbol{\Omega}_{\mathbf{v}}\end{bmatrix}.$$

Lemma 13. *Let $\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}} = T^{-1}\mathbf{Q}_0'\mathbf{Q}_0$ and suppose Assumptions 1-5 hold. Then, as $(N,T)\to\infty$, with $\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^+_u}$ defined in eq.(D-2),*

$$\left\|\widehat{\boldsymbol{\Sigma}}^{-1}_{\mathbf{Q}} - \widehat{\boldsymbol{\Sigma}}^{-1}_{\mathbf{F}^+_u}\right\| = O_p(N^{-1/2}) + O_p(T^{-1/2}), \tag{D-20}$$

and also

$$\sqrt{T}\left[\widehat{\boldsymbol{\Sigma}}^{-1}_{\mathbf{Q}} - \widehat{\boldsymbol{\Sigma}}^{-1}_{\mathbf{F}^+_u}\right] = -\begin{bmatrix}\mathbf{0}_{(1+K+m)\times(1+K+m)} & T^{-1/2}\widehat{\boldsymbol{\Sigma}}_{\mathbf{Fu}}\\ T^{-1/2}\widehat{\boldsymbol{\Sigma}}'_{\mathbf{Fu}} & \mathbf{0}_{(K-m)\times(K-m)}\end{bmatrix} + O_p\left(\frac{1}{\sqrt{N}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right) + O_p\left(\frac{\sqrt{T}}{N}\right), \tag{D-21}$$

where $\widehat{\boldsymbol{\Sigma}}_{\mathbf{Fu}} = \boldsymbol{\Sigma}^{-1}_{\mathbf{F}^*}(\mathbf{F}^* + \bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}^0_{-m}\boldsymbol{\Sigma}^{-1}_{\mathbf{u}^0_{-m}}$.

Lemma 14. *Suppose Assumptions 1-5 hold and let $p^* \geq p$. Then, as $(N,T)\to\infty$,*

$$\mathbf{A}^{\mathbf{F}} = \frac{1}{NT}\sum_{i=1}^{N}\mathbf{w}_i'\mathbf{MF}\boldsymbol{\gamma}_i = O_p\left(\frac{1}{N}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right), \tag{D-22}$$

and letting $\mathbf{A}^{\mathbf{F}}_{NT} = \sqrt{NT}\mathbf{A}^{\mathbf{F}}$, provided that $T/N \to M < \infty$,

$$\mathbf{A}^{\mathbf{F}}_{NT} = \boldsymbol{\Psi}_{\mathbf{F}} vec\left(\frac{\check{\mathbf{F}}'\sqrt{N}\ddot{\mathbf{U}}}{\sqrt{T}}\right) + \sqrt{\frac{T}{N}}\left(\mathbf{b}^{\mathbf{F}}_0 - \mathbf{b}^{\mathbf{F}}_1\right) + O_p\left(\frac{1}{\sqrt{N}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right), \tag{D-23}$$

with

$$\mathbf{b}^{\mathbf{F}}_0 = \boldsymbol{\Sigma}_{\boldsymbol{\eta}}\left[\mathbf{B}_m'\mathbf{T}'\mathbf{R}_0' \otimes \boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*}\boldsymbol{\Sigma}^{-1}_{\mathbf{F}^*}\mathbf{S}_m'\mathbf{N}'\mathbf{R}'\right] vec\left(\boldsymbol{\Sigma}_{\ddot{\mathbf{U}}}\right),$$

$$
\begin{aligned}
\mathbf{b}_1^{\mathbf{F}} &= \mathbf{\Sigma}_{\boldsymbol{\eta}} \left[\mathbf{\Sigma}'_{\mathbf{u}^0_{-m}\mathbf{u}_m} \mathbf{\Sigma}^{-1}_{\mathbf{u}^0_{-m}} \mathbf{S}'_{-m}\mathbf{N}'\mathbf{R}' \otimes \mathbf{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*} \mathbf{\Sigma}^{-1}_{\mathbf{F}^*} \mathbf{S}'_m \mathbf{N}'\mathbf{R}' \right] vec\left(\mathbf{\Sigma}_{\ddot{\mathbf{U}}}\right), \\
\mathbf{\Psi}_{\mathbf{F}} &= -\mathbf{V}_{\mathbf{F},1} + \mathbf{V}_{\mathbf{F},2} + \mathbf{V}_{\mathbf{F},3} - \mathbf{V}_{\mathbf{F},4}, \\
\mathbf{V}_{\mathbf{F},1} &= \mathbf{\Sigma}_{\boldsymbol{\eta}} \left[\mathbf{B}'_m \mathbf{T}'\mathbf{R}'_0 \otimes \mathbf{I}_{1+K^2 m(1+p^*)} \right], \\
\mathbf{V}_{\mathbf{F},2} &= \mathbf{\Sigma}_{\boldsymbol{\eta}} \left[\mathbf{B}'_m \mathbf{T}'\mathbf{R}'_0 \otimes \mathbf{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*} \mathbf{\Sigma}^{-1}_{\mathbf{F}^*} \mathbf{S}'_m \mathbf{N}'\mathbf{R}'\mathbf{P}' \right], \\
\mathbf{V}_{\mathbf{F},3} &= \mathbf{\Sigma}_{\boldsymbol{\eta}} \left[\mathbf{B}'_m \mathbf{T}'\mathbf{R}'_0 \mathbf{\Sigma}_{\ddot{\mathbf{U}}} \mathbf{R}\mathbf{N}\mathbf{S}_{-m} \mathbf{\Sigma}^{-1}_{\mathbf{u}^0_{-m}} \mathbf{S}'_{-m}\mathbf{N}'\mathbf{R}' \otimes \mathbf{I}_{1+k^2 m(1+p^*)} \right], \\
\mathbf{V}_{\mathbf{F},4} &= \mathbf{\Sigma}_{\boldsymbol{\eta}} \left[\mathbf{\Sigma}'_{\mathbf{u}^0_{-m}\mathbf{u}_m} \mathbf{\Sigma}^{-1}_{\mathbf{u}^0_{-m}} \mathbf{S}'_{-m}\mathbf{N}'\mathbf{R}' \otimes \mathbf{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*} \mathbf{\Sigma}^{-1}_{\mathbf{F}^*} \mathbf{S}'_m \mathbf{N}'\mathbf{R}'\mathbf{P}' \right].
\end{aligned}
$$

and where $\mathbf{\Sigma}_{\boldsymbol{\eta}} = E(\boldsymbol{\eta}'_i \otimes \mathbf{S}'_w \ddot{\mathbf{P}}'_i)$.

Lemma 15. *Suppose Assumptions 1-5 hold. Then, as* $(N,T) \to \infty$,

$$
\mathbf{A}^{\varepsilon} = \frac{1}{NT} \sum_{i=1}^{N} \mathbf{w}'_i \mathbf{M} \boldsymbol{\varepsilon}_i = O_p\left(\frac{1}{N}\right) + O_p\left(\frac{1}{T}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right), \tag{D-24}
$$

and letting $\mathbf{A}^{\varepsilon}_{NT} = \sqrt{NT}\mathbf{A}^{\varepsilon}$,

$$
\mathbf{A}^{\varepsilon}_{NT} = O_p(1) + O_p(\sqrt{T}N^{-1/2}) + O_p(\sqrt{N}T^{-1/2}). \tag{D-25}
$$

Lemma 16. *Suppose Assumptions 1-5 hold and let* $p^* \geq p$. *Then, for any* $\boldsymbol{\delta}_0 \neq \boldsymbol{\delta}$ *such that* $\|\boldsymbol{\delta} - \boldsymbol{\delta}_0\| < \infty$ *we have as* $(N,T) \to \infty$

$$
\widehat{\sigma}^2_{\varepsilon}\left(\boldsymbol{\delta}_0\right) = \sigma^2_{\varepsilon} + (\boldsymbol{\delta} - \boldsymbol{\delta}_0)'\widehat{\mathbf{\Sigma}}(\boldsymbol{\delta} - \boldsymbol{\delta}_0) + O_p(N^{-1}) + O_p(T^{-1}) + O_p((NT)^{-1/2}), \tag{D-26}
$$

whereas if $\boldsymbol{\delta}_0 = \boldsymbol{\delta}$ *then*

$$
\widehat{\sigma}^2_{\varepsilon}\left(\boldsymbol{\delta}\right) = \sigma^2_{\varepsilon} + O_p(N^{-1}) + O_p((NT)^{-1/2}), \tag{D-27}
$$

with $\widehat{\sigma}^2(\cdot)$ *defined in (20).*

Lemma 17. *Suppose Assumptions 1-5 hold and let* $p^* \geq p$. *Then, as* $(N,T) \to \infty$ *with* $\boldsymbol{\upsilon} = \upsilon(\rho, \mathbf{H})\mathbf{q}_1$ *and* $\widehat{\sigma}^2(\cdot)$ *defined in (20)*

$$
\mathbf{A}^{c} = \frac{1}{NT} \sum_{i=1}^{N} \mathbf{w}'_i \mathbf{M} \boldsymbol{\varepsilon}_i + \frac{1}{T}\widehat{\sigma}^2_{\varepsilon}(\boldsymbol{\delta})\boldsymbol{\upsilon} = O_p(N^{-1}) + O_p((NT)^{-1/2}). \tag{D-28}
$$

Letting $\mathbf{A}^{c}_{NT} = \sqrt{NT}\mathbf{A}^{c}$ *and* $T/N \to M < \infty$,

$$
\mathbf{A}^{c}_{NT} = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{\boldsymbol{\epsilon}'_i \boldsymbol{\varepsilon}_i}{\sqrt{T}} + \mathbf{\Psi}_{\boldsymbol{\epsilon}} vec\left[\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left(\frac{\boldsymbol{\varepsilon}'_i \check{\mathbf{F}}}{\sqrt{T}} \otimes \mathbf{S}'_w \widetilde{\mathbf{P}}'_i \right) \right] - \sqrt{T}N^{-1/2}\mathbf{b}^{\mathbf{U}} \tag{D-29}
$$

$$+ O_p\left(\frac{1}{\sqrt{N}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right),$$

with $\mathbf{b}^{\mathbf{U}} = \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}\mathbf{U}_{-m}}\boldsymbol{\Sigma}^{-1}_{\mathbf{u}^0_{-m}}\boldsymbol{\Sigma}'_{\boldsymbol{\varepsilon}\mathbf{U}_{-m}}$, $\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}\mathbf{U}_{-m}} = \mathbf{S}'_w\boldsymbol{\Sigma}_{\ddot{\mathbf{U}}}\mathbf{RNS}_{-m}$, $\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}\mathbf{U}_{-m}} = E(\boldsymbol{\varepsilon}'_i\ddot{\mathbf{U}}_i/T)\mathbf{RNS}_{-m}$, $\mathbf{B}^{\mathbf{F}} = \mathbf{I}_{1+K^2m(1+p^*)} - \boldsymbol{\Sigma}_{\check{\mathbf{F}}}\mathbf{PRNS}_m\boldsymbol{\Sigma}^{-1}_{\mathbf{F}^*}\mathbf{S}'_m\mathbf{N}'\mathbf{R}'\mathbf{P}'$ *and* $\boldsymbol{\Psi}_{\boldsymbol{\epsilon}} = \left[vec\left(\mathbf{B}^{\mathbf{F}}\right)' \otimes \mathbf{I}_{k_w}\right]$.
Finally, for $\widetilde{\mathbf{A}}^c(\boldsymbol{\delta}_0)$ *the vector* $\mathbf{A}^c$ *evaluated at* $\boldsymbol{\delta}_0 \neq \boldsymbol{\delta}$

$$\widetilde{\mathbf{A}}^c(\boldsymbol{\delta}_0) = \frac{1}{T}(\boldsymbol{\delta}-\boldsymbol{\delta}_0)'\widehat{\boldsymbol{\Sigma}}(\boldsymbol{\delta}-\boldsymbol{\delta}_0)\boldsymbol{\upsilon}(\rho_0) + \frac{1}{T}\sigma^2_{\varepsilon}\left[\boldsymbol{\upsilon}(\rho_0) - \boldsymbol{\upsilon}\right] + O_p\left(\frac{1}{N}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right),$$

where $\boldsymbol{\upsilon}(\rho_0) = \upsilon(\rho_0, \mathbf{H})\mathbf{q}_1$.

Lemma 18. *Suppose Assumptions 1-5 hold. Then, as* $(N,T) \to \infty$,

$$\widehat{\boldsymbol{\Sigma}} \longrightarrow^p \dot{\boldsymbol{\Sigma}} = \boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{P}} + \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}, \tag{D-30}$$

where $\boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{P}} = (vec(\mathbf{I}_{k_w})' \otimes \mathbf{I}_{k_w})\left(\mathbf{I}_{k_w} \otimes \boldsymbol{\Sigma}_{\widetilde{\mathbf{P}}}vec(\mathbf{V}^{\mathbf{F}})\right)$, $\boldsymbol{\Sigma}_{\widetilde{\mathbf{P}}} = E(\mathbf{S}'_w\widetilde{\mathbf{P}}'_i \otimes \mathbf{S}'_w\widetilde{\mathbf{P}}'_i)$, $\mathbf{V}^{\mathbf{F}} = \boldsymbol{\Sigma}_{\check{\mathbf{F}}} - \boldsymbol{\Sigma}_{\check{\mathbf{F}}}\mathbf{PRNS}_m\boldsymbol{\Sigma}^{-1}_{\mathbf{F}^*}\mathbf{S}'_m\mathbf{N}'\mathbf{R}'\mathbf{P}'\boldsymbol{\Sigma}_{\check{\mathbf{F}}}$ *and* $\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}} = E(\boldsymbol{\epsilon}'_i\boldsymbol{\epsilon}_i/T)$.

D.3 Proof of lemmas

Proof of Lemma 9

The proof for this Lemma is, under Ass.1-3 and 5, identical to that of Lemmas 1 and 2 in Pesaran (2006). The proof is therefore omitted.

Proof of Lemma 10

To prove this lemma, recall from eqs.(B-18)-(B-20) that $\mathbf{F}^* = \check{\mathbf{F}}\ddot{\mathbf{P}}\mathbf{RNS}_m$, $\bar{\mathbf{U}}^0_m = \ddot{\mathbf{U}}\mathbf{RNS}_m$, and $\bar{\mathbf{U}}^0_{-m} = \sqrt{N}\ddot{\mathbf{U}}\mathbf{RNS}_{-m}$. Hence, we have

$$\left\|T^{-1}(\bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}^0_m\right\| = \left\|\mathbf{S}'_m\mathbf{N}'\mathbf{R}'T^{-1}\ddot{\mathbf{U}}'\ddot{\mathbf{U}}\mathbf{RNS}_m\right\| \leq \|\mathbf{RNS}_m\|^2\left\|T^{-1}\ddot{\mathbf{U}}'\ddot{\mathbf{U}}\right\| = O_p(N^{-1}),$$

since $\left\|T^{-1}\ddot{\mathbf{U}}'\ddot{\mathbf{U}}\right\| = O_p(N^{-1})$ by (D-3) of Lemma 9 and we have by definition that $\|\mathbf{R}\| = O_p(1)$ and $\|\mathbf{N}\|$ and $\|\mathbf{S}_m\|$ are $O(1)$. Similarly we obtain

$$\begin{aligned}
\left\|T^{-1}(\bar{\mathbf{U}}^0_{-m})'\bar{\mathbf{U}}^0_{-m}\right\| &= N\left\|\mathbf{S}'_{-m}\mathbf{N}'\mathbf{R}'T^{-1}\ddot{\mathbf{U}}'\ddot{\mathbf{U}}\mathbf{RNS}_{-m}\right\| \leq \|\mathbf{RNS}_{-m}\|^2 N\left\|T^{-1}\ddot{\mathbf{U}}'\ddot{\mathbf{U}}\right\| = O_p(1), \\
\left\|T^{-1}(\bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}^0_{-m}\right\| &= \sqrt{N}\left\|\mathbf{S}'_m\mathbf{N}'\mathbf{R}'T^{-1}\ddot{\mathbf{U}}'\ddot{\mathbf{U}}\mathbf{RNS}_{-m}\right\|, \\
&\leq \|\mathbf{RN}\|^2\|\mathbf{S}_m\|\|\mathbf{S}_{-m}\|\sqrt{N}\left\|T^{-1}\ddot{\mathbf{U}}'\ddot{\mathbf{U}}\right\| = O_p(N^{-1/2}),
\end{aligned}$$

which proves (D-7). Moving on to (D-8), we have, noting that $\left\|\ddot{\mathbf{P}}\right\| = O_p(1)$,

$$\left\|T^{-1}(\mathbf{F}^*)'\mathbf{F}^*\right\| = \left\|\mathbf{S}'_m\mathbf{N}'\mathbf{R}'\ddot{\mathbf{P}}'T^{-1}\check{\mathbf{F}}'\check{\mathbf{F}}\ddot{\mathbf{P}}\mathbf{RNS}_m\right\| \leq \left\|\ddot{\mathbf{P}}\mathbf{RNS}_m\right\|^2\left\|T^{-1}\check{\mathbf{F}}'\check{\mathbf{F}}\right\| = O_p(1),$$

$$\left\|T^{-1}(\bar{\mathbf{U}}_m^0)'\mathbf{F}^*\right\| = \left\|\mathbf{S}_m'\mathbf{N}'\mathbf{R}'T^{-1}\ddot{\mathbf{U}}'\check{\mathbf{F}}\ddot{\mathbf{P}}\mathbf{RNS}_m\right\| \leq \|\mathbf{RNS}_m\|^2 \left\|\ddot{\mathbf{P}}\right\| \left\|T^{-1}\ddot{\mathbf{U}}'\check{\mathbf{F}}\right\| = O_p((NT)^{-1/2}),$$

$$\begin{aligned}\left\|T^{-1}(\bar{\mathbf{U}}_{-m}^0)'\mathbf{F}^*\right\| &= \sqrt{N}\left\|\mathbf{S}_{-m}'\mathbf{N}'\mathbf{R}'T^{-1}\ddot{\mathbf{U}}'\check{\mathbf{F}}\ddot{\mathbf{P}}\mathbf{RNS}_m\right\|, \\ &\leq \|\mathbf{RN}\|^2 \|\mathbf{S}_m\| \|\mathbf{S}_{-m}\| \left\|\ddot{\mathbf{P}}\right\| \sqrt{N} \left\|T^{-1}\ddot{\mathbf{U}}'\check{\mathbf{F}}\right\| = O_p(T^{-1/2}),\end{aligned}$$

where we have made use of (D-4) of Lemma 9. The second and third result in (D-9) follow analogously from (D-5) of Lemma 9 and, given (D-8), the first result in (D-9) follows from the definition $\bar{\mathbf{U}}^0 = [\bar{\mathbf{U}}_m^0, \bar{\mathbf{U}}_{-m}^0]$. Next up, making use of (D-6) gives

$$\left\|T^{-1}\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}_m^0\right\| = \left\|T^{-1}\boldsymbol{\epsilon}_i'\ddot{\mathbf{U}}\mathbf{RNS}_m\right\| \leq \|\mathbf{RNS}_m\| \left\|T^{-1}\boldsymbol{\epsilon}_i'\ddot{\mathbf{U}}\right\| = O_p(N^{-1}) + O_p((NT)^{-1/2}),$$

and similarly for $\left\|T^{-1}\boldsymbol{\varepsilon}_i'\bar{\mathbf{U}}_m^0\right\|$. Also

$$\left\|T^{-1}\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}_{-m}^0\right\| = \left\|T^{-1}\boldsymbol{\epsilon}_i'\ddot{\mathbf{U}}\mathbf{RNS}_{-m}\right\| \leq \|\mathbf{RNS}_{-m}\| \sqrt{N} \left\|T^{-1}\boldsymbol{\epsilon}_i'\ddot{\mathbf{U}}\right\| = O_p(N^{-1/2}) + O_p(T^{-1/2}),$$

with the argument being identical for $\left\|T^{-1}\boldsymbol{\varepsilon}_i'\bar{\mathbf{U}}_{-m}^0\right\|$. This establishes (D-10) and (D-11). Turning next to $T^{-1}\boldsymbol{\epsilon}_i'\mathbf{Q}_0$ of (D-12) we find making use of the definition in (B-16)

$$\left\|T^{-1}\boldsymbol{\epsilon}_i'\mathbf{Q}_0\right\| = \left\|T^{-1}\boldsymbol{\epsilon}_i'(\mathbf{F}^0 + \bar{\mathbf{U}}^0)\right\| \leq \left\|T^{-1}\boldsymbol{\epsilon}_i'\mathbf{F}^0\right\| + \left\|T^{-1}\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}^0\right\| = O_p(T^{-1/2}),$$

because $\|T^{-1}\boldsymbol{\epsilon}_i'\mathbf{F}^0\| = \left\|T^{-1}\boldsymbol{\epsilon}_i'[\mathbf{F}^*, \mathbf{0}_{T\times(K-m)}]\right\| = \left\|T^{-1}\boldsymbol{\epsilon}_i'\check{\mathbf{F}}\ddot{\mathbf{P}}\mathbf{RNS}_m\right\| \leq \left\|T^{-1}\boldsymbol{\epsilon}_i'\check{\mathbf{F}}\right\| \left\|\ddot{\mathbf{P}}\mathbf{RNS}_m\right\| = O_p(T^{-1/2})$ by (D-5) of Lemma 9 and because $\left\|T^{-1}\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}^0\right\| = \left\|T^{-1}\boldsymbol{\epsilon}_i'[\bar{\mathbf{U}}_m^0, \bar{\mathbf{U}}_{-m}^0]\right\| = O_p(N^{-1/2}) + O_p(T^{-1/2})$ by (D-10) and (D-11). $\|T^{-1}\boldsymbol{\varepsilon}_i'\mathbf{Q}_0\| = O_p(T^{-1/2})$ of (D-12) can be established in the same way. Finally, for $\left\|T^{-1}\mathbf{Q}_0'\bar{\mathbf{U}}_m^0\right\|$, making use of (D-7) and (D-8)

$$\begin{aligned}\left\|T^{-1}\mathbf{Q}_0'\bar{\mathbf{U}}_m^0\right\| &= \left\|T^{-1}(\mathbf{F}^0 + \bar{\mathbf{U}}^0)'\bar{\mathbf{U}}_m^0\right\| \leq \left\|T^{-1}[\mathbf{F}^*, \mathbf{0}_{T\times(K-m)}]'\bar{\mathbf{U}}_m^0\right\| + \left\|T^{-1}[\bar{\mathbf{U}}_m^0, \bar{\mathbf{U}}_{-m}^0]'\bar{\mathbf{U}}_m^0\right\|, \\ &\leq \left\|T^{-1}(\mathbf{F}^*)'\bar{\mathbf{U}}_m^0\right\| + \left\|T^{-1}[\bar{\mathbf{U}}_m^0, \bar{\mathbf{U}}_{-m}^0]'\bar{\mathbf{U}}_m^0\right\| = O_p(N^{-1/2}).\end{aligned}$$

which then proves the final statement in (D-12), and therefore the lemma. □

Proof of Lemma 11

Note that substituting in $\ddot{\mathbf{P}}_i = \mathbf{P} + \check{\mathbf{P}}_i$ from eq.(B-7) gives by Ass.3 that $\widetilde{\mathbf{P}}_i = \check{\mathbf{P}}_i + O_p(N^{-1/2})$. Then, since the following matrix norms are identical

$$\left\|\frac{1}{N}\sum_{i=1}^N \left(T^{-1}\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}_m^0 \otimes \mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right)\right\| = \left\|\frac{1}{N}\sum_{i=1}^N \left(\mathbf{S}_w'\widetilde{\mathbf{P}}_i' \otimes T^{-1}\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}_m^0\right)\right\|,$$

we will evaluate the second. Let $\widetilde{p}_{i,r,d}$ denote the element on row $r = 1, \ldots, k_w$ and column $d = 1, \ldots, 1 + K^2m(1+p^*)$ of $\mathbf{S}_w'\widetilde{\mathbf{P}}_i'$. Then the elements on rows $k_w(r-1)+1$ to k_wr and columns $k_w(d-1)+1$ to k_wd of the second Kronecker product are given by $\frac{1}{N}\sum_{i=1}^N \widetilde{p}_{i,r,d}\frac{\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}_m^0}{T}$. To evaluate these terms, consider that we can write, making use of (B-19) and (B-10),

$$\bar{\mathbf{U}}_m^0 = \bar{\mathbf{U}}_{m,-i}^0 + \frac{1}{N}\mathbf{U}_{m,i}^0,$$

where $\bar{\mathbf{U}}^0_{m,-i} = N^{-1}\sum_{j=1,j\neq i}^{N} \ddot{\mathbf{U}}_j\mathbf{RNS}_m$ and $\mathbf{U}^0_{m,i} = \ddot{\mathbf{U}}_i\mathbf{RNS}_m$. Hence

$$\left\|\frac{1}{N}\sum_{i=1}^{N}\widetilde{p}_{i,r,d}\frac{\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}^0_m}{T}\right\| \leq \left\|\frac{1}{N}\sum_{i=1}^{N}\widetilde{p}_{i,r,d}\left(\frac{\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}^0_{m,-i}}{T}\right)\right\| + \left\|\frac{1}{N^2}\sum_{i=1}^{N}\widetilde{p}_{i,r,d}\left(\frac{\boldsymbol{\epsilon}_i'\mathbf{U}^0_{m,i}}{T}\right)\right\|,$$
$$= O_p\left(\frac{1}{N\sqrt{T}}\right) + O_p\left(\frac{1}{N^{3/2}}\right),$$

because $T^{-1}\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}^0_{m,-i} = O_p((NT)^{-1/2})$, $T^{-1}\boldsymbol{\epsilon}_i'\mathbf{U}^0_{m,i} = O_p(1)$ and by Ass.3 $\frac{1}{N}\sum_{i=1}^{N}\widetilde{p}_{i,r,d} = O_p(N^{-1/2})$ with $\widetilde{p}_{i,r,d}$ independent of the other variables. Since this applies for all $r = 1,\ldots,k_w$ and $d = 1,\ldots,1+K^2m(1+p^*)$ we have

$$\left\|N^{-1}\sum_{i=1}^{N}\left(T^{-1}\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}^0_m \otimes \mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right)\right\| = O_p\left(\frac{1}{N^{3/2}}\right) + O_p\left(\frac{1}{N\sqrt{T}}\right),$$

which is the result in (D-13), and (D-14) follows in similar fashion. In turn, to prove (D-15) we note that

$$\bar{\mathbf{U}}^0_{-m} = \sqrt{N}\bar{\mathbf{U}}^0_{-m,-i} + \frac{1}{\sqrt{N}}\mathbf{U}^0_{-m,i},$$

with $\bar{\mathbf{U}}^0_{-m,-i} = N^{-1}\sum_{j=1,j\neq i}^{N}\ddot{\mathbf{U}}_j\mathbf{RNS}_{-m}$ and $\mathbf{U}^0_{-m,i} = \ddot{\mathbf{U}}_i\mathbf{RNS}_{-m}$ such that for $r = 1,\ldots,k_w$ and column $d = 1,\ldots,1+K^2m(1+p^*)$ we have for the corresponding elements in the Kronecker product

$$\left\|\frac{1}{N}\sum_{i=1}^{N}\widetilde{p}_{i,r,d}\frac{\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}^0_{-m}}{T}\right\| \leq \sqrt{N}\left\|\frac{1}{N}\sum_{i=1}^{N}\widetilde{p}_{i,r,d}\left(\frac{\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}^0_{-m,-i}}{T}\right)\right\| + \sqrt{N}\left\|\frac{1}{N^2}\sum_{i=1}^{N}\widetilde{p}_{i,r,d}\left(\frac{\boldsymbol{\epsilon}_i'\mathbf{U}^0_{-m,i}}{T}\right)\right\|,$$
$$= O_p\left(\frac{1}{N}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right),$$

since also $T^{-1}\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}^0_{-m,-i} = O_p((NT)^{-1/2})$ and $T^{-1}\boldsymbol{\epsilon}_i'\mathbf{U}^0_{-m,i} = O_p(1)$. This implies (D-15) and the result in (D-16) can be established in the same way. Next up is (D-17). The elements on rows $k_w(r-1)+1$ to k_wr and columns $(1+K^2m(1+p^*))(d-1)+1$ to $(1+K^2m(1+p^*))d$ of $\frac{1}{N}\sum_{i=1}^{N}\left(\mathbf{S}_w'\widetilde{\mathbf{P}}_i' \otimes T^{-1}\boldsymbol{\epsilon}_i'\check{\mathbf{F}}\right)$ are given by

$$\frac{1}{N}\sum_{i=1}^{N}\widetilde{p}_{i,r,d}\frac{\boldsymbol{\epsilon}_i'\check{\mathbf{F}}}{T} = \frac{\bar{\mathbf{a}}_{r,d}'\check{\mathbf{F}}}{T} = O_p\left(\frac{1}{\sqrt{NT}}\right), \tag{D-31}$$

with $\bar{\mathbf{a}}_{r,d} = \frac{1}{N}\sum_{i=1}^{N}\widetilde{p}_{i,r,d}\boldsymbol{\epsilon}_i$ and $\|\bar{\mathbf{a}}_{r,d}\| = O_p(N^{-1/2})$ by the independence of $\widetilde{p}_{i,r,d}$ and $\boldsymbol{\epsilon}_i$ from Ass.1 and 3. The result then follows because also $\bar{\mathbf{a}}_{r,d}$ and $\check{\mathbf{F}}$ are independent stationary variables. Since (D-31) holds for every sub-matrix

$$\left\|\frac{1}{N}\sum_{i=1}^{N}\left(T^{-1}\boldsymbol{\epsilon}_i'\check{\mathbf{F}} \otimes \mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right)\right\| = O_p\left(\frac{1}{\sqrt{NT}}\right),$$

with again an analogous argument for (D-18). The final result is found by noting that

$$\frac{1}{N}\sum_{i=1}^{N}\left(T^{-1}\boldsymbol{\varepsilon}_i'\mathbf{Q}_0 \otimes \mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right) = \frac{1}{N}\sum_{i=1}^{N}\left(T^{-1}\boldsymbol{\varepsilon}_i'\left([\mathbf{F}^*, \mathbf{0}_{T\times(K-m)}] + [\bar{\mathbf{U}}_m^0, \bar{\mathbf{U}}_{-m}^0]\right) \otimes \mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right),$$

such that since $\mathbf{F}^* = \check{\mathbf{F}}\ddot{\mathbf{P}}\mathbf{R}\mathbf{N}\mathbf{S}_m$ from (B-18), inserting the preceding results gives

$$\frac{1}{N}\sum_{i=1}^{N}\left(T^{-1}\boldsymbol{\varepsilon}_i'\mathbf{Q}_0 \otimes \mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right) = O_p\left(\frac{1}{N}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right).$$

Finally, given the independence of $\boldsymbol{\eta}_i$ from $\boldsymbol{\epsilon}_j, \boldsymbol{\varepsilon}_j$ and $\check{\mathbf{F}}$ for all i, j, t by Ass.3 all the stated results also hold true when $\mathbf{S}_w'\widetilde{\mathbf{P}}_i'$ is substituted for $\boldsymbol{\eta}_i'$. This establishes the lemma.

Proof of Lemma 12

Consider that by Assumptions 2 and 5, $\check{\mathbf{F}}$ is a matrix of covariance stationary variables with finite fourth moments. As such, the first result $\widehat{\boldsymbol{\Sigma}}_{\check{\mathbf{F}}} = T^{-1}\check{\mathbf{F}}'\check{\mathbf{F}} = \boldsymbol{\Sigma}_{\check{\mathbf{F}}} + O_p(T^{-1/2})$, with $\boldsymbol{\Sigma}_{\check{\mathbf{F}}} = E(T^{-1}\check{\mathbf{F}}'\check{\mathbf{F}})$ follows directly. Similarly, from Ass.1 and 5 follows that $\widehat{\boldsymbol{\Sigma}}_{\ddot{\mathbf{U}}} = T^{-1}N\ddot{\mathbf{U}}'\ddot{\mathbf{U}} = \boldsymbol{\Sigma}_{\ddot{\mathbf{U}}} + O_p(T^{-1/2})$, with $\boldsymbol{\Sigma}_{\ddot{\mathbf{U}}} = E(\ddot{\mathbf{U}}_i'\ddot{\mathbf{U}}_i/T)$ since error terms are independent over i. The second and third statements of the lemma are obtained by substituting in (B-18) and by making use of the first statement and $\ddot{\mathbf{P}} = \mathbf{P} + O_p(N^{-1/2})$ by Ass.3

$$\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^*} = \frac{(\mathbf{F}^*)'\mathbf{F}^*}{T} = \mathbf{S}_m'\mathbf{N}'\mathbf{R}'\ddot{\mathbf{P}}'\widehat{\boldsymbol{\Sigma}}_{\check{\mathbf{F}}}\ddot{\mathbf{P}}\mathbf{R}\mathbf{N}\mathbf{S}_m = \mathbf{S}_m'\mathbf{N}'\mathbf{R}'\mathbf{P}'\boldsymbol{\Sigma}_{\check{\mathbf{F}}}\mathbf{P}\mathbf{R}\mathbf{N}\mathbf{S}_m + O_p(N^{-1/2}) + O_p(T^{-1/2}),$$
$$\widehat{\boldsymbol{\Sigma}}_{\check{\mathbf{F}}\mathbf{F}^*} = \frac{\check{\mathbf{F}}'\mathbf{F}^*}{T} = \widehat{\boldsymbol{\Sigma}}_{\check{\mathbf{F}}}\ddot{\mathbf{P}}\mathbf{R}\mathbf{N}\mathbf{S}_m = \boldsymbol{\Sigma}_{\check{\mathbf{F}}}\mathbf{P}\mathbf{R}\mathbf{N}\mathbf{S}_m + O_p(N^{-1/2}) + O_p(T^{-1/2}).$$

Since $\mathbf{F}^{*\prime}\mathbf{F}^*/T$ is by construction a $1+K+m \times 1+K+m$ full rank matrix we also have $\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^*}^{-1} = \boldsymbol{\Sigma}_{\mathbf{F}^*}^{-1} + O_p(N^{-1/2}) + O_p(T^{-1/2})$. For the next result, consider that $\bar{\mathbf{U}}_{-m}^0 = \sqrt{N}\bar{\mathbf{U}}^*\mathbf{T}\mathbf{B}_{-m}$, with $\bar{\mathbf{U}}^* = [\bar{\mathbf{u}}_1^*, \ldots, \bar{\mathbf{u}}_T^*]'$ and $\bar{\mathbf{u}}_t^* = \mathbf{A}_0^{-1}\bar{\mathbf{u}}_t$. Therefore, by Ass.1

$$\begin{aligned}\widehat{\boldsymbol{\Sigma}}_{\mathbf{u}_{-m}^0} = T^{-1}(\bar{\mathbf{U}}_{-m}^0)'\bar{\mathbf{U}}_{-m}^0 &= \mathbf{B}_{-m}'\mathbf{T}'\mathbf{A}_0^{-1}\left(NT^{-1}\sum_{t=1}^{T}\bar{\mathbf{u}}_t\bar{\mathbf{u}}_t'\right)(\mathbf{A}_0^{-1})'\mathbf{T}\mathbf{B}_{-m},\\ &= \mathbf{B}_{-m}'\mathbf{T}'\mathbf{A}_0^{-1}\boldsymbol{\Omega}_{\mathbf{u}}(\mathbf{A}_0^{-1})'\mathbf{T}\mathbf{B}_{-m} + O_p(T^{-1/2}),\\ &= \boldsymbol{\Sigma}_{\mathbf{u}_{-m}^0} + O_p(T^{-1/2}),\end{aligned}$$

where $\boldsymbol{\Sigma}_{\mathbf{u}_{-m}^0} = \mathbf{B}_{-m}'\mathbf{T}'\mathbf{A}_0^{-1}\boldsymbol{\Omega}_{\mathbf{u}}(\mathbf{A}_0^{-1})'\mathbf{T}\mathbf{B}_{-m}$ is a $(K-m)\times(K-m)$ positive definite matrix because Ass.1 implies $\boldsymbol{\Omega}_{\mathbf{u}} = E(\mathbf{u}_{i,t}\mathbf{u}_{i,t}') = \begin{bmatrix}\sigma_\varepsilon^2 & \mathbf{0}_{k\times 1}' \\ \mathbf{0}_{k\times 1} & \boldsymbol{\Omega}_{\mathbf{v}}\end{bmatrix}$. Consequently also $\widehat{\boldsymbol{\Sigma}}_{\mathbf{u}_{-m}^0}^{-1} = \boldsymbol{\Sigma}_{\mathbf{u}_{-m}^0}^{-1} + O_p(T^{-1/2})$. Finally, the last result can be obtained by substituting in (B-19)-(B-20) and $\widehat{\boldsymbol{\Sigma}}_{\ddot{\mathbf{U}}} = \boldsymbol{\Sigma}_{\ddot{\mathbf{U}}} + O_p(T^{-1/2})$ as follows

$$\widehat{\boldsymbol{\Sigma}}_{\mathbf{u}_{-m}^0\mathbf{u}_m^0} = T^{-1}(\bar{\mathbf{U}}_{-m}^0)'\sqrt{N}\bar{\mathbf{U}}_m^0 = \mathbf{S}_{-m}'\mathbf{N}'\mathbf{R}'\widehat{\boldsymbol{\Sigma}}_{\ddot{\mathbf{U}}}\mathbf{R}\mathbf{N}\mathbf{S}_m = \mathbf{S}_{-m}'\mathbf{N}'\mathbf{R}'\boldsymbol{\Sigma}_{\ddot{\mathbf{U}}}\mathbf{R}\mathbf{N}\mathbf{S}_m + O_p(T^{-1/2}).$$

Proof of Lemma 13

Consider that by definition

$$\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}} = T^{-1}\mathbf{Q}_0'\mathbf{Q}_0 = T^{-1}(\mathbf{F}^0)'\mathbf{F}^0 + T^{-1}(\bar{\mathbf{U}}^0)'\mathbf{F}^0 + T^{-1}(\mathbf{F}^0)'\bar{\mathbf{U}}^0 + T^{-1}(\bar{\mathbf{U}}^0)'\bar{\mathbf{U}}^0,$$

with, since $\mathbf{F}^0 = [\mathbf{F}^*, \mathbf{0}_{T\times(K-m)}]$,

$$T^{-1}(\mathbf{F}^0)'\mathbf{F}^0 = \begin{bmatrix} \widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^*} & \mathbf{0}_{(1+K+m)\times(K-m)} \\ \mathbf{0}_{(K-m)\times(1+K+m)} & \mathbf{0}_{(K-m)\times(K-m)} \end{bmatrix},$$

and also because by Lemma 10 we have $\left\|T^{-1}(\bar{\mathbf{U}}^0_{-m})'\mathbf{F}^*\right\| = O_p(T^{-1/2})$ and $\left\|T^{-1}(\bar{\mathbf{U}}^0_{m})'\mathbf{F}^*\right\| = O_p((NT)^{-1/2})$, it follows that

$$T^{-1}(\bar{\mathbf{U}}^0)'\mathbf{F}^0 + T^{-1}(\mathbf{F}^0)'\bar{\mathbf{U}}^0 = \begin{bmatrix} T^{-1}(\mathbf{F}^*)'\bar{\mathbf{U}}^0_m + T^{-1}(\bar{\mathbf{U}}^0_m)'\mathbf{F}^* & T^{-1}(\mathbf{F}^*)'\bar{\mathbf{U}}^0_{-m} \\ T^{-1}(\bar{\mathbf{U}}^0_{-m})'\mathbf{F}^* & \mathbf{0}_{(K-m)\times(K-m)} \end{bmatrix} = O_p\left(\frac{1}{\sqrt{T}}\right).$$

Next, making use of Lemma 10

$$\begin{aligned} T^{-1}(\bar{\mathbf{U}}^0)'\bar{\mathbf{U}}^0 &= \frac{1}{T}\begin{bmatrix} (\bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}^0_m & (\bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}^0_{-m} \\ (\bar{\mathbf{U}}^0_{-m})'\bar{\mathbf{U}}^0_m & (\bar{\mathbf{U}}^0_{-m})'\bar{\mathbf{U}}^0_{-m} \end{bmatrix}, \\ &= \begin{bmatrix} \mathbf{0}_{(1+K+m)\times(1+K+m)} & \mathbf{0}_{(1+K+m)\times(K-m)} \\ \mathbf{0}_{(K-m)\times(1+K+m)} & \widehat{\boldsymbol{\Sigma}}_{\mathbf{u}^0_{-m}} \end{bmatrix} + O_p(N^{-1/2}), \end{aligned}$$

and recalling from (D-2) that

$$\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^+_u} = \begin{bmatrix} \widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^*} & \mathbf{0}_{(1+K+m)\times(K-m)} \\ \mathbf{0}_{(K-m)\times(1+K+m)} & \widehat{\boldsymbol{\Sigma}}_{\mathbf{u}^0_{-m}} \end{bmatrix}, \tag{D-32}$$

we have, given the results above

$$\begin{aligned} \widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}} - \widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^+_u} &= \begin{bmatrix} T^{-1}(\mathbf{F}^*)'\bar{\mathbf{U}}^0_m + T^{-1}(\bar{\mathbf{U}}^0_m)'\mathbf{F}^* & T^{-1}(\mathbf{F}^*)'\bar{\mathbf{U}}^0_{-m} \\ T^{-1}(\bar{\mathbf{U}}^0_{-m})'\mathbf{F}^* & \mathbf{0}_{(K-m)\times(K-m)} \end{bmatrix} \\ &\quad + T^{-1}\begin{bmatrix} (\bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}^0_m & (\bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}^0_{-m} \\ (\bar{\mathbf{U}}^0_{-m})'\bar{\mathbf{U}}^0_m & \mathbf{0}_{(K-m)\times(K-m)} \end{bmatrix}, \\ &= O_p(N^{-1/2}) + O_p(T^{-1/2}). \end{aligned} \tag{D-33}$$

Then, since $p^* = 1$ we have $rk(\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}) = 1 + K(1+p^*) = 1 + 2K$, $rk(\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^*}) = 1 + K + m$ and $rk(\widehat{\boldsymbol{\Sigma}}_{\mathbf{u}^0_{-m}}) = K - m$, such that for the block diagonal matrix $rk(\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^+_u}) = rk(\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^*}) + rk(\widehat{\boldsymbol{\Sigma}}_{\mathbf{u}^0_{-m}}) = 1 + 2K$. Therefore, by Theorem 1 of Karabiyik et al. (2017)

$$\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1} = \widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^+_u}^{-1} + O_p(N^{-1/2}) + O_p(T^{-1/2}).$$

This proves (D-20) of the lemma.

Moving on to the second statement, consider that from Lemma 12

$$\left\|\widehat{\mathbf{\Sigma}}_{\mathbf{u}^0_{-m}} - \mathbf{\Sigma}_{\mathbf{u}^0_{-m}}\right\| = O_p(T^{-1/2}),$$

where $\mathbf{\Sigma}_{\mathbf{u}^0_{-m}} = \mathbf{B}'_{-m}\mathbf{T}'\mathbf{A}_0^{-1}\mathbf{\Omega}_{\mathbf{u}}(\mathbf{A}_0^{-1})'\mathbf{T}\mathbf{B}_{-m}$ is a $(K-m)\times(K-m)$ positive definite matrix. Consider also from Lemma 12 that $\widehat{\mathbf{\Sigma}}_{\mathbf{F}^*} = \mathbf{\Sigma}_{\mathbf{F}^*} + O_p(N^{-1/2}) + O_p(T^{-1/2})$, with $\mathbf{\Sigma}_{\mathbf{F}^*}$ a $(1+K+m)\times(1+K+m)$ full rank matrix. Accordingly, we have denoting

$$\mathbf{\Sigma}_{\mathbf{F}_u^+} = \begin{bmatrix} \mathbf{\Sigma}_{\mathbf{F}^*} & \mathbf{0}_{(1+K+m)\times(K-m)} \\ \mathbf{0}_{(K-m)\times(1+K+m)} & \mathbf{\Sigma}_{\mathbf{u}^0_{-m}} \end{bmatrix},$$

that

$$\widehat{\mathbf{\Sigma}}_{\mathbf{F}_u^+} = \mathbf{\Sigma}_{\mathbf{F}_u^+} + O_p(N^{-1/2}) + O_p(T^{-1/2}),$$

and since $rk(\widehat{\mathbf{\Sigma}}_{\mathbf{F}_u^+}) = rk(\mathbf{\Sigma}_{\mathbf{F}_u^+})$ also

$$\widehat{\mathbf{\Sigma}}^{-1}_{\mathbf{F}_u^+} = \mathbf{\Sigma}^{-1}_{\mathbf{F}_u^+} + O_p(N^{-1/2}) + O_p(T^{-1/2}),$$

with

$$\mathbf{\Sigma}^{-1}_{\mathbf{F}_u^+} = \begin{bmatrix} \mathbf{\Sigma}^{-1}_{\mathbf{F}^*} & \mathbf{0}_{(1+K+m)\times(K-m)} \\ \mathbf{0}_{(K-m)\times(1+K+m)} & \mathbf{\Sigma}^{-1}_{\mathbf{u}^0_{-m}} \end{bmatrix}, \tag{D-34}$$

which implies in turn, making use of (D-20) that

$$\widehat{\mathbf{\Sigma}}^{-1}_{\mathbf{Q}} = \mathbf{\Sigma}^{-1}_{\mathbf{F}_u^+} + O_p(N^{-1/2}) + O_p(T^{-1/2}).$$

Consider then the following identity

$$\sqrt{T}\left[\widehat{\mathbf{\Sigma}}^{-1}_{\mathbf{Q}} - \widehat{\mathbf{\Sigma}}^{-1}_{\mathbf{F}_u^+}\right] = -\widehat{\mathbf{\Sigma}}^{-1}_{\mathbf{Q}}\sqrt{T}\left[\widehat{\mathbf{\Sigma}}_{\mathbf{Q}} - \widehat{\mathbf{\Sigma}}_{\mathbf{F}_u^+}\right]\widehat{\mathbf{\Sigma}}^{-1}_{\mathbf{F}_u^+}, \tag{D-35}$$

such that by the results above

$$\sqrt{T}\left[\widehat{\mathbf{\Sigma}}^{-1}_{\mathbf{Q}} - \widehat{\mathbf{\Sigma}}^{-1}_{\mathbf{F}_u^+}\right] = -\mathbf{\Sigma}^{-1}_{\mathbf{F}_u^+}\sqrt{T}\left[\widehat{\mathbf{\Sigma}}_{\mathbf{Q}} - \widehat{\mathbf{\Sigma}}_{\mathbf{F}_u^+}\right]\mathbf{\Sigma}^{-1}_{\mathbf{F}_u^+} + O_p(N^{-1/2}) + O_p(T^{-1/2}).$$

Using (D-33) we find for the middle term, also making use of Lemma 10,

$$\begin{aligned}\sqrt{T}\left[\widehat{\mathbf{\Sigma}}_{\mathbf{Q}} - \widehat{\mathbf{\Sigma}}_{\mathbf{F}_u^+}\right] &= \frac{1}{\sqrt{T}}\begin{bmatrix} (\mathbf{F}^*)'\bar{\mathbf{U}}^0_m + (\bar{\mathbf{U}}^0_m)'\mathbf{F}^* & (\mathbf{F}^*)'\bar{\mathbf{U}}^0_{-m} \\ (\bar{\mathbf{U}}^0_{-m})'\mathbf{F}^* & \mathbf{0}_{(K-m)\times(K-m)} \end{bmatrix} \\ &\quad + \frac{1}{\sqrt{T}}\begin{bmatrix} (\bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}^0_m & (\bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}^0_{-m} \\ (\bar{\mathbf{U}}^0_{-m})'\bar{\mathbf{U}}^0_m & \mathbf{0}_{(K-m)\times(K-m)} \end{bmatrix}, \\ &= \frac{1}{\sqrt{T}}\begin{bmatrix} \mathbf{0}_{(1+K+m)\times(1+K+m)} & (\mathbf{F}^*)'\bar{\mathbf{U}}^0_{-m} \\ (\bar{\mathbf{U}}^0_{-m})'\mathbf{F}^* & \mathbf{0}_{(K-m)\times(K-m)} \end{bmatrix} + O_p\left(\frac{1}{\sqrt{N}}\right) \\ &\quad + \frac{1}{\sqrt{T}}\begin{bmatrix} \mathbf{0}_{(1+K+m)\times(1+K+m)} & (\bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}^0_{-m} \\ (\bar{\mathbf{U}}^0_{-m})'\bar{\mathbf{U}}^0_m & \mathbf{0}_{(K-m)\times(K-m)} \end{bmatrix} + O_p\left(\frac{\sqrt{T}}{N}\right),\end{aligned}$$

$$= \frac{1}{\sqrt{T}} \begin{bmatrix} \mathbf{0}_{(1+K+m)\times(1+K+m)} & (\mathbf{F}^* + \bar{\mathbf{U}}_m^0)'\bar{\mathbf{U}}_{-m}^0 \\ (\bar{\mathbf{U}}_{-m}^0)'(\mathbf{F}^* + \bar{\mathbf{U}}_m^0) & \mathbf{0}_{(K-m)\times(K-m)} \end{bmatrix} + O_p\left(\frac{1}{\sqrt{N}}\right) + O_p\left(\frac{\sqrt{T}}{N}\right).$$

Hence

$$\sqrt{T}\left[\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1} - \widehat{\boldsymbol{\Sigma}}_{\mathbf{F}_u^+}^{-1}\right] = -\boldsymbol{\Sigma}_{\mathbf{F}_u^+}^{-1} \frac{1}{\sqrt{T}} \begin{bmatrix} \mathbf{0}_{(1+K+m)\times(1+K+m)} & (\mathbf{F}^* + \bar{\mathbf{U}}_m^0)'\bar{\mathbf{U}}_{-m}^0 \\ (\bar{\mathbf{U}}_{-m}^0)'(\mathbf{F}^* + \bar{\mathbf{U}}_m^0) & \mathbf{0}_{(K-m)\times(K-m)} \end{bmatrix} \boldsymbol{\Sigma}_{\mathbf{F}_u^+}^{-1}$$
$$+ O_p\left(\frac{1}{\sqrt{N}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right) + O_p\left(\frac{\sqrt{T}}{N}\right),$$

and making use of (D-34)

$$\sqrt{T}\left[\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1} - \widehat{\boldsymbol{\Sigma}}_{\mathbf{F}_u^+}^{-1}\right] = -\frac{1}{\sqrt{T}} \begin{bmatrix} \mathbf{0}_{(1+K+m)\times(1+K+m)} & \boldsymbol{\Sigma}_{\mathbf{F}^*}^{-1}(\mathbf{F}^* + \bar{\mathbf{U}}_m^0)'\bar{\mathbf{U}}_{-m}^0 \\ \boldsymbol{\Sigma}_{\mathbf{u}_{-m}^0}^{-1}(\bar{\mathbf{U}}_{-m}^0)'(\mathbf{F}^* + \bar{\mathbf{U}}_m^0) & \mathbf{0}_{(K-m)\times(K-m)} \end{bmatrix} \boldsymbol{\Sigma}_{\mathbf{F}_u^+}^{-1}$$
$$+ O_p\left(\frac{1}{\sqrt{N}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right) + O_p\left(\frac{\sqrt{T}}{N}\right),$$
$$= -\frac{1}{\sqrt{T}} \begin{bmatrix} \mathbf{0}_{(1+K+m)\times(1+K+m)} & \boldsymbol{\Sigma}_{\mathbf{F}^*}^{-1}(\mathbf{F}^* + \bar{\mathbf{U}}_m^0)'\bar{\mathbf{U}}_{-m}^0\boldsymbol{\Sigma}_{\mathbf{u}_{-m}^0}^{-1} \\ \boldsymbol{\Sigma}_{\mathbf{u}_{-m}^0}^{-1}(\bar{\mathbf{U}}_{-m}^0)'(\mathbf{F}^* + \bar{\mathbf{U}}_m^0)\boldsymbol{\Sigma}_{\mathbf{F}^*}^{-1} & \mathbf{0}_{(K-m)\times(K-m)} \end{bmatrix}$$
$$+ O_p\left(\frac{1}{\sqrt{N}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right) + O_p\left(\frac{\sqrt{T}}{N}\right),$$

which, by defining $\widehat{\boldsymbol{\Sigma}}_{\mathbf{Fu}} = \boldsymbol{\Sigma}_{\mathbf{F}^*}^{-1}(\mathbf{F}^* + \bar{\mathbf{U}}_m^0)'\bar{\mathbf{U}}_{-m}^0\boldsymbol{\Sigma}_{\mathbf{u}_{-m}^0}^{-1}$, can be written more compactly as

$$\sqrt{T}\left[\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1} - \widehat{\boldsymbol{\Sigma}}_{\mathbf{F}_u^+}^{-1}\right] = -\begin{bmatrix} \mathbf{0}_{(1+K+m)\times(1+K+m)} & T^{-1/2}\widehat{\boldsymbol{\Sigma}}_{\mathbf{Fu}} \\ T^{-1/2}\widehat{\boldsymbol{\Sigma}}_{\mathbf{Fu}}' & \mathbf{0}_{(K-m)\times(K-m)} \end{bmatrix}$$
$$+ O_p\left(\frac{1}{\sqrt{N}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right) + O_p\left(\frac{\sqrt{T}}{N}\right).$$

This is the result in (D-21).

Proof of Lemma 14

Consider $\mathbf{A}^{\mathbf{F}} = \frac{1}{NT}\sum_{i=1}^{N} \mathbf{w}_i'\mathbf{M}\mathbf{F}\boldsymbol{\gamma}_i$. Under Ass.4 and assuming that $p^* \geq p$ we can substitute in (B-23) to obtain

$$\mathbf{A}^{\mathbf{F}} = \frac{1}{NT}\sum_{i=1}^{N} \mathbf{w}_i'\mathbf{M}\mathbf{F}\boldsymbol{\gamma}_i = -\frac{1}{NT}\sum_{i=1}^{N} \mathbf{w}_i'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\gamma}_i,$$

and also, by eq.(5) of Ass.3,

$$\mathbf{A}^{\mathbf{F}} = -\frac{1}{NT}\sum_{i=1}^{N} \mathbf{w}_i'\mathbf{M}\bar{\mathbf{U}}_m(\boldsymbol{\gamma} + \boldsymbol{\eta}_i) = -\frac{1}{T}\bar{\mathbf{w}}'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\gamma} - \frac{1}{NT}\sum_{i=1}^{N} \mathbf{w}_i'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i,$$
$$= -\frac{1}{NT}\sum_{i=1}^{N} \mathbf{w}_i'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i,$$

because $\mathbf{M}\bar{\mathbf{w}} = \mathbf{0}_{T\times k_w}$ since $\bar{\mathbf{w}} \subseteq \mathbf{Q}$. Substituting in (B-13) gives

$$\mathbf{A}^{\mathbf{F}} = -(\mathbf{A}_1^{\mathbf{F}} + \mathbf{A}_2^{\mathbf{F}}), \tag{D-36}$$

with $\mathbf{A}_1^{\mathbf{F}} = \frac{1}{NT}\sum_{i=1}^N \mathbf{S}_w'\ddot{\mathbf{P}}_i'\check{\mathbf{F}}'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i$ and $\mathbf{A}_2^{\mathbf{F}} = \frac{1}{NT}\sum_{i=1}^N \boldsymbol{\epsilon}_i'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i$.
We start by evaluating $\mathbf{A}_2^{\mathbf{F}}$, and make use of $\mathbf{M} = \mathbf{I}_T - \mathbf{Q}(\mathbf{Q}'\mathbf{Q})^{\dagger}\mathbf{Q}' = \mathbf{I}_T - \mathbf{Q}_0(\mathbf{Q}_0'\mathbf{Q}_0)^{-1}\mathbf{Q}_0'$

$$\left\|\mathbf{A}_2^{\mathbf{F}}\right\| \le \left\|\frac{1}{N}\sum_{i=1}^N \frac{\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}_m}{T}\boldsymbol{\eta}_i\right\| + \left\|\frac{1}{N}\sum_{i=1}^N \frac{\boldsymbol{\epsilon}_i'\mathbf{Q}_0}{T}\left(\frac{\mathbf{Q}_0'\mathbf{Q}_0}{T}\right)^{-1}\frac{\mathbf{Q}_0'\bar{\mathbf{U}}_m}{T}\boldsymbol{\eta}_i\right\| = \left\|\mathbf{A}_{21}^{\mathbf{F}}\right\| + \left\|\mathbf{A}_{22}^{\mathbf{F}}\right\|$$

with obvious definitions for $\mathbf{A}_{21}^{\mathbf{F}}$ and $\mathbf{A}_{22}^{\mathbf{F}}$. Taking on first $\mathbf{A}_{21}^{\mathbf{F}}$, note that

$$\mathbf{A}_{21}^{\mathbf{F}} = \left[\frac{1}{N}\sum_{i=1}^N\left(\boldsymbol{\eta}_i' \otimes \frac{\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}_m}{T}\right)\right] vec(\mathbf{I}_m),$$

and therefore, by eq.(D-13) of Lemma 11,

$$\left\|\mathbf{A}_{21}^{\mathbf{F}}\right\| \le \left\|\frac{1}{N}\sum_{i=1}^N\left(\boldsymbol{\eta}_i' \otimes T^{-1}\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}_m\right)\right\| \|\mathbf{I}_m\| = O_p(N^{3/2}) + O_p(N^{-1}T^{-1/2}).$$

Next up is, $\mathbf{A}_{22}^{\mathbf{F}}$. We find

$$\begin{aligned}
\left\|\mathbf{A}_{22}^{\mathbf{F}}\right\| &\le \left\|\frac{1}{N}\sum_{i=1}^N\left(\boldsymbol{\eta}_i' \otimes \frac{\boldsymbol{\epsilon}_i'\mathbf{Q}_0}{T}\right)\right\| \left\|\left(\frac{\mathbf{Q}_0'\mathbf{Q}_0}{T}\right)^{-1}\right\| \left\|\frac{\mathbf{Q}_0'\bar{\mathbf{U}}_m}{T}\right\|, \\
&\le \left\|\frac{1}{N}\sum_{i=1}^N\left(\boldsymbol{\eta}_i' \otimes \frac{\boldsymbol{\epsilon}_i'(\mathbf{F}^0 + \bar{\mathbf{U}}^0)}{T}\right)\right\| \left\|\left(\frac{\mathbf{Q}_0'\mathbf{Q}_0}{T}\right)^{-1}\right\| \left\|\frac{\mathbf{Q}_0'\bar{\mathbf{U}}_m}{T}\right\|, \\
&= O_p\left(\frac{1}{N^{3/2}}\right) + O_p\left(\frac{1}{N\sqrt{T}}\right),
\end{aligned}$$

because $\|(T^{-1}\mathbf{Q}_0'\mathbf{Q}_0)^{-1}\| = O_p(1)$, $\left\|T^{-1}\mathbf{Q}_0'\bar{\mathbf{U}}_m^0\right\| = O_p(N^{-1/2})$ by (D-12) of Lemma 10 and

$$\begin{aligned}
&\left\|\frac{1}{N}\sum_{i=1}^N\left(\boldsymbol{\eta}_i' \otimes T^{-1}\boldsymbol{\epsilon}_i'(\mathbf{F}^0 + \bar{\mathbf{U}}^0)\right)\right\| \\
&\quad\le \left\|\frac{1}{N}\sum_{i=1}^N\left(\boldsymbol{\eta}_i' \otimes T^{-1}\boldsymbol{\epsilon}_i'\mathbf{F}^0\right)\right\| + \left\|\frac{1}{N}\sum_{i=1}^N\left(\boldsymbol{\eta}_i' \otimes T^{-1}\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}^0\right)\right\|, \\
&\quad\le \left\|\frac{1}{N}\sum_{i=1}^N\left(\boldsymbol{\eta}_i' \otimes T^{-1}\boldsymbol{\epsilon}_i'[\mathbf{F}^*, \mathbf{0}_{T\times(K-m)}]\right)\right\| + \left\|\frac{1}{N}\sum_{i=1}^N\left(\boldsymbol{\eta}_i' \otimes T^{-1}\boldsymbol{\epsilon}_i'[\bar{\mathbf{U}}_m^0, \bar{\mathbf{U}}_{-m}^0]\right)\right\|, \\
&\quad= O_p\left(\frac{1}{N}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right),
\end{aligned}$$

by (D-13), (D-15) and (D-17) of Lemma 11. It follows that

$$\left\|\mathbf{A}_2^{\mathbf{F}}\right\| = O_p\left(\frac{1}{N^{3/2}}\right) + O_p\left(\frac{1}{N\sqrt{T}}\right). \tag{D-37}$$

Next up is $\mathbf{A}_1^{\mathbf{F}}$. Recalling that $\mathbf{M}_{\mathbf{F}} = \mathbf{I}_T - \mathbf{H}_{\mathbf{F}}$ and $\mathbf{H}_{\mathbf{F}} = \mathbf{F}^*((\mathbf{F}^*)'\mathbf{F}^*)^{-1}(\mathbf{F}^*)'$ we can decompose it as

$$\begin{aligned}\mathbf{A}_1^{\mathbf{F}} &= \frac{1}{NT}\sum_{i=1}^N \mathbf{S}_w'\ddot{\mathbf{P}}_i'\check{\mathbf{F}}'\bar{\mathbf{U}}_m\boldsymbol{\eta}_i - \frac{1}{NT}\sum_{i=1}^N \mathbf{S}_w'\ddot{\mathbf{P}}_i'\check{\mathbf{F}}'\mathbf{H}_{\mathbf{F}}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i - \frac{1}{NT}\sum_{i=1}^N \mathbf{S}_w'\ddot{\mathbf{P}}_i'\check{\mathbf{F}}'(\mathbf{M}_{\mathbf{F}} - \mathbf{M})\bar{\mathbf{U}}_m\boldsymbol{\eta}_i,\\ &= \mathbf{A}_{11}^{\mathbf{F}} - \mathbf{A}_{12}^{\mathbf{F}} - \mathbf{A}_{13}^{\mathbf{F}},\end{aligned}$$

with obvious definitions for $\mathbf{A}_{11}^{\mathbf{F}}$, $\mathbf{A}_{12}^{\mathbf{F}}$ and $\mathbf{A}_{13}^{\mathbf{F}}$. For the first two terms we find

$$\begin{aligned}\left\|\mathbf{A}_{11}^{\mathbf{F}}\right\| &\le \left\|\frac{1}{N}\sum_{i=1}^N \left(\boldsymbol{\eta}_i' \otimes \mathbf{S}_w'\ddot{\mathbf{P}}_i'\right)\right\| \left\|\frac{\check{\mathbf{F}}'\bar{\mathbf{U}}_m}{T}\right\| = O_p\left(\frac{1}{\sqrt{NT}}\right),\\ \left\|\mathbf{A}_{12}^{\mathbf{F}}\right\| &\le \left\|\frac{1}{N}\sum_{i=1}^N \left(\boldsymbol{\eta}_i' \otimes \mathbf{S}_w'\ddot{\mathbf{P}}_i'\right)\right\| \left\|\frac{\check{\mathbf{F}}'\mathbf{F}^*}{T}\right\| \left\|\left(\frac{(\mathbf{F}^*)'\mathbf{F}^*}{T}\right)^{-1}\right\| \left\|\frac{(\mathbf{F}^*)'\bar{\mathbf{U}}_m}{T}\right\| = O_p\left(\frac{1}{\sqrt{NT}}\right),\end{aligned}$$

since $\left\|\frac{1}{N}\sum_{i=1}^N \left(\boldsymbol{\eta}_i' \otimes \mathbf{S}_w'\ddot{\mathbf{P}}_i'\right)\right\| = O_p(1)$, and Lemmas 9 and 10 show that $\left\|T^{-1}(\mathbf{F}^*)'\bar{\mathbf{U}}_m\right\| = O_p((NT)^{-1/2})$, $\left\|T^{-1}\check{\mathbf{F}}'\mathbf{F}^*\right\| = O_p(1)$ and $\|(T^{-1}(\mathbf{F}^*)'\mathbf{F}^*)^{-1}\| = O_p(1)$.

Next is $\mathbf{A}_{13}^{\mathbf{F}}$. Making use of (D-1) gives the following decomposition

$$\mathbf{A}_{13}^{\mathbf{F}} = \frac{1}{N}\sum_{i=1}^N \mathbf{S}_w'\ddot{\mathbf{P}}_i'\left[\mathbf{A}_{131}^{\mathbf{F}} + \mathbf{A}_{132}^{\mathbf{F}} + \mathbf{A}_{133}^{\mathbf{F}} + \mathbf{A}_{134}^{\mathbf{F}} + \mathbf{A}_{135}^{\mathbf{F}}\right]\boldsymbol{\eta}_i,$$

with, defining $\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}} = [T^{-1}\mathbf{Q}_0'\mathbf{Q}_0]$,

$$\begin{aligned}\mathbf{A}_{131}^{\mathbf{F}} &= T^{-1}\check{\mathbf{F}}'\bar{\mathbf{U}}_{-m}^0[T^{-1}(\bar{\mathbf{U}}_{-m}^0)'\bar{\mathbf{U}}_{-m}^0]^{-1}T^{-1}(\bar{\mathbf{U}}_{-m}^0)'\bar{\mathbf{U}}_m,\\ \mathbf{A}_{132}^{\mathbf{F}} &= T^{-1}\check{\mathbf{F}}'\bar{\mathbf{U}}_m^0[T^{-1}(\mathbf{F}^*)'\mathbf{F}^*]^{-1}T^{-1}(\bar{\mathbf{U}}_m^0)'\bar{\mathbf{U}}_m,\\ \mathbf{A}_{133}^{\mathbf{F}} &= T^{-1}\check{\mathbf{F}}'\mathbf{F}^*[T^{-1}(\mathbf{F}^*)'\mathbf{F}^*]^{-1}T^{-1}(\bar{\mathbf{U}}_m^0)'\bar{\mathbf{U}}_m,\\ \mathbf{A}_{134}^{\mathbf{F}} &= T^{-1}\check{\mathbf{F}}'\bar{\mathbf{U}}_m^0[T^{-1}(\mathbf{F}^*)'\mathbf{F}^*]^{-1}T^{-1}(\mathbf{F}^*)'\bar{\mathbf{U}}_m,\\ \mathbf{A}_{135}^{\mathbf{F}} &= T^{-1}\check{\mathbf{F}}'\mathbf{Q}_0\left(\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1} - \widehat{\boldsymbol{\Sigma}}_{\mathbf{F}_u^+}^{-1}\right)T^{-1}\mathbf{Q}_0'\bar{\mathbf{U}}_m,\end{aligned}$$

which yields, by Lemma 10,

$$\begin{aligned}\left\|\mathbf{A}_{131}^{\mathbf{F}}\right\| &\le \left\|T^{-1}\check{\mathbf{F}}'\bar{\mathbf{U}}_{-m}^0\right\| \left\|[T^{-1}(\bar{\mathbf{U}}_{-m}^0)'\bar{\mathbf{U}}_{-m}^0]^{-1}\right\| \left\|T^{-1}(\bar{\mathbf{U}}_{-m}^0)'\bar{\mathbf{U}}_m\right\| = O_p((NT)^{-1/2}),\\ \left\|\mathbf{A}_{132}^{\mathbf{F}}\right\| &\le \left\|T^{-1}\check{\mathbf{F}}'\bar{\mathbf{U}}_m^0\right\| \left\|[T^{-1}(\mathbf{F}^*)'\mathbf{F}^*]^{-1}\right\| \left\|T^{-1}(\bar{\mathbf{U}}_m^0)'\bar{\mathbf{U}}_m\right\| = O_p(N^{-3/2}T^{-1/2}),\\ \left\|\mathbf{A}_{133}^{\mathbf{F}}\right\| &\le \left\|T^{-1}\check{\mathbf{F}}'\mathbf{F}^*\right\| \left\|[T^{-1}(\mathbf{F}^*)'\mathbf{F}^*]^{-1}\right\| \left\|T^{-1}(\bar{\mathbf{U}}_m^0)'\bar{\mathbf{U}}_m\right\| = O_p(N^{-1}),\\ \left\|\mathbf{A}_{134}^{\mathbf{F}}\right\| &\le \left\|T^{-1}\check{\mathbf{F}}'\bar{\mathbf{U}}_m^0\right\| \left\|[T^{-1}(\mathbf{F}^*)'\mathbf{F}^*]^{-1}\right\| \left\|T^{-1}(\mathbf{F}^*)'\bar{\mathbf{U}}_m\right\| = O_p((NT)^{-1}),\\ \left\|\mathbf{A}_{135}^{\mathbf{F}}\right\| &\le \left\|T^{-1}\check{\mathbf{F}}'\mathbf{Q}_0\right\| \left\|\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1} - \widehat{\boldsymbol{\Sigma}}_{\mathbf{F}_u^+}^{-1}\right\| \left\|T^{-1}\mathbf{Q}_0'\bar{\mathbf{U}}_m\right\| = O_p(N^{-1}) + O_p((NT)^{-1/2}),\end{aligned}$$

because also $\left\|T^{-1}\check{\mathbf{F}}'\mathbf{Q}_0\right\| = \left\|T^{-1}\check{\mathbf{F}}'([\mathbf{F}^*, \mathbf{0}_{T\times(K-m)}] + \bar{\mathbf{U}}^0)\right\| = O_p(1)$ and $\left\|\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1} - \widehat{\boldsymbol{\Sigma}}_{\mathbf{F}_u^+}^{-1}\right\| = O_p(N^{-1/2}) + O_p(T^{-1/2})$ from (D-20) of Lemma 13. Hence,

$$\left\|\mathbf{A}_{13}^{\mathbf{F}}\right\| = O_p(N^{-1}) + O_p((NT)^{-1/2}),$$

which implies, in turn $\left\|\mathbf{A}_1^{\mathbf{F}}\right\| = O_p(N^{-1}) + O_p((NT)^{-1/2})$, and therefore, combining results for $\left\|\mathbf{A}_1^{\mathbf{F}}\right\|$ and $\left\|\mathbf{A}_2^{\mathbf{F}}\right\|$ in (D-36)

$$\left\|\mathbf{A}^{\mathbf{F}}\right\| \leq \left\|\mathbf{A}_1^{\mathbf{F}}\right\| + \left\|\mathbf{A}_2^{\mathbf{F}}\right\| = O_p(N^{-1}) + O_p((NT)^{-1/2}),$$

which is the result stated in eq.(D-22).

Next, let $\mathbf{A}_{NT}^{\mathbf{F}} = \sqrt{NT}\mathbf{A}^{\mathbf{F}}$ such that by the results above

$$\mathbf{A}_{NT}^{\mathbf{F}} = \mathbf{A}_{NT,1}^{\mathbf{F}} + \mathbf{A}_{NT,2}^{\mathbf{F}} + \mathbf{A}_{NT,3}^{\mathbf{F}} + \mathbf{A}_{NT,4}^{\mathbf{F}} + \mathbf{A}_{NT,5}^{\mathbf{F}} + O_p(N^{-1/2}) + O_p(\sqrt{T}N^{-1}),$$

with

$$\begin{aligned}
\mathbf{A}_{NT,1}^{\mathbf{F}} &= -\frac{1}{N}\sum_{i=1}^{N}\mathbf{S}_w'\ddot{\mathbf{P}}_i'\frac{\check{\mathbf{F}}'\sqrt{N}\bar{\mathbf{U}}_m}{\sqrt{T}}\boldsymbol{\eta}_i,\\
\mathbf{A}_{NT,2}^{\mathbf{F}} &= \frac{1}{N}\sum_{i=1}^{N}\mathbf{S}_w'\ddot{\mathbf{P}}_i'\frac{\check{\mathbf{F}}'\mathbf{F}^*}{T}\left(\frac{(\mathbf{F}^*)'\mathbf{F}^*}{T}\right)^{-1}\frac{(\mathbf{F}^*)'\sqrt{N}\bar{\mathbf{U}}_m}{\sqrt{T}}\boldsymbol{\eta}_i,\\
\mathbf{A}_{NT,3}^{\mathbf{F}} &= \frac{1}{N}\sum_{i=1}^{N}\mathbf{S}_w'\ddot{\mathbf{P}}_i'\frac{\check{\mathbf{F}}'\bar{\mathbf{U}}_{-m}^0}{\sqrt{T}}\left(\frac{(\bar{\mathbf{U}}_{-m}^0)'\bar{\mathbf{U}}_{-m}^0}{T}\right)^{-1}\frac{(\bar{\mathbf{U}}_{-m}^0)'\sqrt{N}\bar{\mathbf{U}}_m}{T}\boldsymbol{\eta}_i,\\
\mathbf{A}_{NT,4}^{\mathbf{F}} &= \frac{1}{N}\sum_{i=1}^{N}\mathbf{S}_w'\ddot{\mathbf{P}}_i'\frac{\check{\mathbf{F}}'\mathbf{F}^*}{T}\left(\frac{(\mathbf{F}^*)'\mathbf{F}^*}{T}\right)^{-1}\frac{(\bar{\mathbf{U}}_m^0)'\sqrt{N}\bar{\mathbf{U}}_m}{\sqrt{T}}\boldsymbol{\eta}_i,\\
\mathbf{A}_{NT,5}^{\mathbf{F}} &= \frac{1}{N}\sum_{i=1}^{N}\mathbf{S}_w'\ddot{\mathbf{P}}_i'\frac{\check{\mathbf{F}}'\mathbf{Q}_0}{T}\sqrt{T}\left[\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1} - \widehat{\boldsymbol{\Sigma}}_{\mathbf{F}_u^+}^{-1}\right]\frac{\mathbf{Q}_0'\sqrt{N}\bar{\mathbf{U}}_m}{T}\boldsymbol{\eta}_i.
\end{aligned}$$

Taking on first $\mathbf{A}_{NT,1}^{\mathbf{F}}$ we substitute in $\bar{\mathbf{U}}_m = \ddot{\mathbf{U}}\mathbf{R}_0\mathbf{T}\mathbf{B}_m$ by eq.(B-22) and write

$$\begin{aligned}
\mathbf{A}_{NT,1}^{\mathbf{F}} &= -\left[\frac{1}{N}\sum_{i=1}^{N}\left(\boldsymbol{\eta}_i'\otimes\mathbf{S}_w'\ddot{\mathbf{P}}_i'\right)\right]vec\left(\frac{\check{\mathbf{F}}'\sqrt{N}\bar{\mathbf{U}}_m}{\sqrt{T}}\right)\\
&= -\left[\frac{1}{N}\sum_{i=1}^{N}\left(\boldsymbol{\eta}_i'\otimes\mathbf{S}_w'\ddot{\mathbf{P}}_i'\right)\right]\left[\mathbf{B}_m'\mathbf{T}'\mathbf{R}_0'\otimes\mathbf{I}_{1+K^2m(1+p^*)}\right]vec\left(\frac{\check{\mathbf{F}}'\sqrt{N}\ddot{\mathbf{U}}}{\sqrt{T}}\right),
\end{aligned}$$

such that denoting

$$\boldsymbol{\Sigma}_{\boldsymbol{\eta}} = E\left(\boldsymbol{\eta}_i'\otimes\mathbf{S}_w'\ddot{\mathbf{P}}_i'\right),$$

which we note exists and is bounded by Ass.3, we have $\left[\frac{1}{N}\sum_{i=1}^{N}\left(\boldsymbol{\eta}_i'\otimes\mathbf{S}_w'\ddot{\mathbf{P}}_i'\right)\right] = \boldsymbol{\Sigma}_{\boldsymbol{\eta}} + O_p(N^{-1/2})$, and therefore

$$\mathbf{A}_{NT,1}^{\mathbf{F}} = -\boldsymbol{\Sigma}_{\boldsymbol{\eta}}\left[\mathbf{B}_m'\mathbf{T}'\mathbf{R}_0'\otimes\mathbf{I}_{1+K^2m(1+p^*)}\right]vec\left(T^{-1/2}\check{\mathbf{F}}'\sqrt{N}\ddot{\mathbf{U}}\right) + O_p(N^{-1/2}).$$

Next, for $\mathbf{A}_{NT,2}^{\mathbf{F}}$ we can write using (B-22) and (B-18) that $(\mathbf{F}^*)'\bar{\mathbf{U}}_m = \mathbf{S}_m'\mathbf{N}'\mathbf{R}'\ddot{\mathbf{P}}'\check{\mathbf{F}}'\ddot{\mathbf{U}}\mathbf{R}_0\mathbf{T}\mathbf{B}_m$ and substitute it into the expression to give

$$\mathbf{A}_{NT,2}^{\mathbf{F}} = \left[\frac{1}{N}\sum_{i=1}^{N}\left(\boldsymbol{\eta}_i'\otimes\mathbf{S}_w'\ddot{\mathbf{P}}_i'\right)\right]\left[\mathbf{I}_m\otimes\frac{\check{\mathbf{F}}'\mathbf{F}^*}{T}\left(\frac{(\mathbf{F}^*)'\mathbf{F}^*}{T}\right)^{-1}\right]\left[\mathbf{B}_m'\mathbf{T}'\mathbf{R}_0'\otimes\mathbf{S}_m'\mathbf{N}'\mathbf{R}'\ddot{\mathbf{P}}'\right]$$

$$\times\, vec\left(\frac{\check{\mathbf{F}}'\sqrt{N}\ddot{\mathbf{U}}}{\sqrt{T}}\right).$$

Since by Lemma 12

$$T^{-1}\check{\mathbf{F}}'\mathbf{F}^* = \boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*} + O_p(T^{-1/2}) + O_p(N^{-1/2}),$$
$$T^{-1}\mathbf{F}^{*\prime}\mathbf{F}^* = \boldsymbol{\Sigma}_{\mathbf{F}^*} + O_p(T^{-1/2}) + O_p(N^{-1/2}),$$

with $\boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*} = \boldsymbol{\Sigma}_{\check{\mathbf{F}}}\mathbf{PRNS}_m$ and $\boldsymbol{\Sigma}_{\mathbf{F}^*} = \mathbf{S}_m'\mathbf{N}'\mathbf{R}'\mathbf{P}'\boldsymbol{\Sigma}_{\check{\mathbf{F}}}\mathbf{PRNS}_m$, we get

$$\mathbf{A}^{\mathbf{F}}_{NT,2} = \boldsymbol{\Sigma}_{\boldsymbol{\eta}}\left[\mathbf{B}_m'\mathbf{T}'\mathbf{R}_0' \otimes \boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*}\boldsymbol{\Sigma}_{\mathbf{F}^*}^{-1}\mathbf{S}_m'\mathbf{N}'\mathbf{R}'\mathbf{P}'\right] vec\left(\frac{\check{\mathbf{F}}'\sqrt{N}\ddot{\mathbf{U}}}{\sqrt{T}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right) + O_p\left(\frac{1}{\sqrt{N}}\right).$$

Continuing on to the next term, substituting in (B-20) and (B-22) gives

$$\mathbf{A}^{\mathbf{F}}_{NT,3} = \frac{1}{N}\sum_{i=1}^{N}\mathbf{S}_w'\ddot{\mathbf{P}}_i'\left(\frac{\check{\mathbf{F}}'\sqrt{N}\ddot{\mathbf{U}}}{\sqrt{T}}\right)\mathbf{RNS}_{-m}\widehat{\boldsymbol{\Sigma}}^{-1}_{\mathbf{u}^0_{-m}}\mathbf{S}_{-m}'\mathbf{N}'\mathbf{R}'\left(\frac{N\ddot{\mathbf{U}}'\ddot{\mathbf{U}}}{T}\right)\mathbf{R}_0\mathbf{TB}_m\boldsymbol{\eta}_i,$$

where by Lemma 12

$$\left\|\widehat{\boldsymbol{\Sigma}}^{-1}_{\mathbf{u}^0_{-m}} - \boldsymbol{\Sigma}^{-1}_{\mathbf{u}^0_{-m}}\right\| = O_p(T^{-1/2}),$$

with $\boldsymbol{\Sigma}_{\mathbf{u}^0_{-m}} = \mathbf{B}_{-m}'\mathbf{T}'\mathbf{A}_0^{-1}\boldsymbol{\Omega}_{\mathbf{u}}(\mathbf{A}_0^{-1})'\mathbf{TB}_{-m}$ and $\boldsymbol{\Omega}_{\mathbf{u}} = \begin{bmatrix}\sigma_\varepsilon^2 & \mathbf{0}_{k\times 1}' \\ \mathbf{0}_{k\times 1} & \boldsymbol{\Omega}_{\mathbf{v}}\end{bmatrix}$. Also, from the proof of Lemma 12 we have $N\ddot{\mathbf{U}}'\ddot{\mathbf{U}}/T = \boldsymbol{\Sigma}_{\ddot{\mathbf{U}}} + O_p(T^{-1/2})$, with $\boldsymbol{\Sigma}_{\ddot{\mathbf{U}}} = E(\ddot{\mathbf{U}}_i'\ddot{\mathbf{U}}_i/T)$. As such,

$$\mathbf{A}^{\mathbf{F}}_{NT,3} = \frac{1}{N}\sum_{i=1}^{N}\mathbf{S}_w'\ddot{\mathbf{P}}_i'\left(\frac{\check{\mathbf{F}}'\sqrt{N}\ddot{\mathbf{U}}}{\sqrt{T}}\right)\mathbf{RNS}_{-m}\boldsymbol{\Sigma}^{-1}_{\mathbf{u}^0_{-m}}\mathbf{S}_{-m}'\mathbf{N}'\mathbf{R}'\boldsymbol{\Sigma}_{\ddot{\mathbf{U}}}\mathbf{R}_0\mathbf{TB}_m\boldsymbol{\eta}_i + O_p\left(\frac{1}{\sqrt{T}}\right),$$

and, as before

$$\mathbf{A}^{\mathbf{F}}_{NT,3} = \boldsymbol{\Sigma}_{\boldsymbol{\eta}}\left[\mathbf{B}_m'\mathbf{T}'\mathbf{R}_0'\boldsymbol{\Sigma}_{\ddot{\mathbf{U}}}\mathbf{RNS}_{-m}\boldsymbol{\Sigma}^{-1}_{\mathbf{u}^0_{-m}}\mathbf{S}_{-m}'\mathbf{N}'\mathbf{R}' \otimes \mathbf{I}_{1+k^2m(1+p^*)}\right] vec\left(\frac{\check{\mathbf{F}}'\sqrt{N}\ddot{\mathbf{U}}}{\sqrt{T}}\right)$$
$$+ O_p(N^{-1/2}) + O_p(T^{-1/2}).$$

For the next term, since using earlier results $\mathbf{A}^{\mathbf{F}}_{NT,4} = O_p(\sqrt{T}N^{-1/2})$, we define first

$$\mathbf{B}^{\mathbf{F}}_{NT,4} = \sum_{i=1}^{N}\mathbf{S}_w'\ddot{\mathbf{P}}_i'\frac{\check{\mathbf{F}}'\mathbf{F}^*}{T}\left(\frac{(\mathbf{F}^*)'\mathbf{F}^*}{T}\right)^{-1}\frac{(\bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}_m}{T}\boldsymbol{\eta}_i,$$

and note that $\mathbf{A}^{\mathbf{F}}_{NT,4} = \sqrt{\frac{T}{N}}\mathbf{B}^{\mathbf{F}}_{NT,4}$. Substituting in (B-19) and (B-22) gives $(\bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}_m = \mathbf{S}_m'\mathbf{N}'\mathbf{R}'\ddot{\mathbf{U}}'\ddot{\mathbf{U}}\mathbf{R}_0\mathbf{TB}_m$ and therefore

$$\mathbf{B}^{\mathbf{F}}_{NT,4} = \frac{1}{N}\sum_{i=1}^{N}\mathbf{S}_w'\ddot{\mathbf{P}}_i'\frac{\check{\mathbf{F}}'\mathbf{F}^*}{T}\left(\frac{(\mathbf{F}^*)'\mathbf{F}^*}{T}\right)^{-1}\frac{N(\bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}_m}{T}\boldsymbol{\eta}_i,$$

$$= \mathbf{\Sigma}_{\boldsymbol{\eta}} \left[\mathbf{I}_m \otimes \mathbf{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*} \mathbf{\Sigma}_{\mathbf{F}^*}^{-1}\right] \left[\mathbf{B}_m' \mathbf{T}' \mathbf{R}_0' \otimes \mathbf{S}_m' \mathbf{N}' \mathbf{R}'\right] vec\left(\mathbf{\Sigma}_{\ddot{\mathbf{U}}}\right) + O_p(T^{-1/2}) + O_p(N^{-1/2}),$$
$$= \mathbf{\Sigma}_{\boldsymbol{\eta}} \left[\mathbf{B}_m' \mathbf{T}' \mathbf{R}_0' \otimes \mathbf{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*} \mathbf{\Sigma}_{\mathbf{F}^*}^{-1} \mathbf{S}_m' \mathbf{N}' \mathbf{R}'\right] vec\left(\mathbf{\Sigma}_{\ddot{\mathbf{U}}}\right) + O_p(T^{-1/2}) + O_p(N^{-1/2}).$$

Hence, we have for $T/N \to M < \infty$ (implying $\sqrt{T}N^{-1} \to 0$)

$$\mathbf{A}_{NT,4}^{\mathbf{F}} = \sqrt{T}N^{-1/2}\mathbf{B}_{NT,4}^{\mathbf{F}} = \sqrt{T}N^{-1/2}\mathbf{b}_0^{\mathbf{F}} + O_p(N^{-1/2}), \tag{D-38}$$

with $\mathbf{b}_0^{\mathbf{F}} = \mathbf{\Sigma}_{\boldsymbol{\eta}} \left[\mathbf{B}_m' \mathbf{T}' \mathbf{R}_0' \otimes \mathbf{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*} \mathbf{\Sigma}_{\mathbf{F}^*}^{-1} \mathbf{S}_m' \mathbf{N}' \mathbf{R}'\right] vec\left(\mathbf{\Sigma}_{\ddot{\mathbf{U}}}\right)$.

Last up is $\mathbf{A}_{NT,5}^{\mathbf{F}}$, given by

$$\mathbf{A}_{NT,5}^{\mathbf{F}} = \frac{1}{N}\sum_{i=1}^{N} \mathbf{S}_w' \ddot{\mathbf{P}}_i' \frac{\check{\mathbf{F}}'\mathbf{Q}_0}{T} \sqrt{T}\left[\widehat{\mathbf{\Sigma}}_{\mathbf{Q}}^{-1} - \widehat{\mathbf{\Sigma}}_{\mathbf{F}_u^+}^{-1}\right] \frac{\mathbf{Q}_0'\sqrt{N}\bar{\mathbf{U}}_m}{T} \boldsymbol{\eta}_i.$$

First we decompose it into 4 parts using $\mathbf{Q}_0 = \mathbf{F}^0 + \bar{\mathbf{U}}^0$,

$$\mathbf{A}_{NT,5}^{\mathbf{F}} = \frac{1}{N}\sum_{i=1}^{N} \mathbf{S}_w' \ddot{\mathbf{P}}_i' \left(\sum_{l=1}^{4} \mathbf{A}_{NT,5,l}^{\mathbf{F}}\right) \boldsymbol{\eta}_i,$$

with

$$\left\|\mathbf{A}_{NT,5,1}^{\mathbf{F}}\right\| \leq \sqrt{NT}\left\|T^{-1}\check{\mathbf{F}}'\mathbf{F}^0\right\| \left\|\widehat{\mathbf{\Sigma}}_{\mathbf{Q}}^{-1} - \widehat{\mathbf{\Sigma}}_{\mathbf{F}_u^+}^{-1}\right\| \left\|T^{-1}(\mathbf{F}^0)'\bar{\mathbf{U}}_m\right\| = O_p(N^{-1/2}) + O_p(T^{-1/2}),$$
$$\left\|\mathbf{A}_{NT,5,2}^{\mathbf{F}}\right\| \leq \sqrt{NT}\left\|T^{-1}\check{\mathbf{F}}'\mathbf{F}^0\right\| \left\|\widehat{\mathbf{\Sigma}}_{\mathbf{Q}}^{-1} - \widehat{\mathbf{\Sigma}}_{\mathbf{F}_u^+}^{-1}\right\| \left\|T^{-1}(\bar{\mathbf{U}}^0)'\bar{\mathbf{U}}_m\right\| = O_p(\sqrt{T}N^{-1/2}) + O_p(1),$$
$$\left\|\mathbf{A}_{NT,5,3}^{\mathbf{F}}\right\| \leq \sqrt{NT}\left\|T^{-1}\check{\mathbf{F}}'\bar{\mathbf{U}}^0\right\| \left\|\widehat{\mathbf{\Sigma}}_{\mathbf{Q}}^{-1} - \widehat{\mathbf{\Sigma}}_{\mathbf{F}_u^+}^{-1}\right\| \left\|T^{-1}(\mathbf{F}^0)'\bar{\mathbf{U}}_m\right\| = O_p((NT)^{-1/2}) + O_p(T^{-1}),$$
$$\left\|\mathbf{A}_{NT,5,4}^{\mathbf{F}}\right\| \leq \sqrt{NT}\left\|T^{-1}\check{\mathbf{F}}'\bar{\mathbf{U}}^0\right\| \left\|\widehat{\mathbf{\Sigma}}_{\mathbf{Q}}^{-1} - \widehat{\mathbf{\Sigma}}_{\mathbf{F}_u^+}^{-1}\right\| \left\|T^{-1}(\bar{\mathbf{U}}^0)'\bar{\mathbf{U}}_m\right\| = O_p(N^{-1/2}) + O_p(T^{-1/2}),$$

in which case the leading term is $\mathbf{A}_{NT,5,2}^{\mathbf{F}}$. Hence, imposing $T/N \to M < \infty$,

$$\mathbf{A}_{NT,5}^{\mathbf{F}} = \frac{1}{N}\sum_{i=1}^{N} \mathbf{S}_w' \ddot{\mathbf{P}}_i' T^{-1}\check{\mathbf{F}}'\mathbf{F}^0\sqrt{T}\left[\widehat{\mathbf{\Sigma}}_{\mathbf{Q}}^{-1} - \widehat{\mathbf{\Sigma}}_{\mathbf{F}_u^+}^{-1}\right] T^{-1}(\bar{\mathbf{U}}^0)'\sqrt{N}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i + O_p(N^{-1/2}) + O_p(T^{-1/2}),$$
$$= \left[\frac{1}{N}\sum_{i=1}^{N}\left(\boldsymbol{\eta}_i' \otimes \mathbf{S}_w'\ddot{\mathbf{P}}_i'\right)\right] vec\left(T^{-1}\check{\mathbf{F}}'\mathbf{F}^0\sqrt{T}\left[\widehat{\mathbf{\Sigma}}_{\mathbf{Q}}^{-1} - \widehat{\mathbf{\Sigma}}_{\mathbf{F}_u^+}^{-1}\right] T^{-1}(\bar{\mathbf{U}}^0)'\sqrt{N}\bar{\mathbf{U}}_m\right)$$
$$+ O_p(N^{-1/2}) + O_p(T^{-1/2}),$$
$$= \mathbf{\Sigma}_{\boldsymbol{\eta}} vec\left(T^{-1}\check{\mathbf{F}}'\mathbf{F}^0\sqrt{T}\left[\widehat{\mathbf{\Sigma}}_{\mathbf{Q}}^{-1} - \widehat{\mathbf{\Sigma}}_{\mathbf{F}_u^+}^{-1}\right] T^{-1}(\bar{\mathbf{U}}^0)'\sqrt{N}\bar{\mathbf{U}}_m\right) + O_p(N^{-1/2}) + O_p(T^{-1/2}).$$

Next, consider the term in the $vec(\cdot)$ operator. Substituting in eq.(D-21) of Lemma 13, which is

$$\sqrt{T}\left[\widehat{\mathbf{\Sigma}}_{\mathbf{Q}}^{-1} - \widehat{\mathbf{\Sigma}}_{\mathbf{F}_u^+}^{-1}\right] = -\begin{bmatrix} \mathbf{0}_{(1+K+m)\times(1+K+m)} & T^{-1/2}\widehat{\mathbf{\Sigma}}_{\mathbf{Fu}} \\ T^{-1/2}\widehat{\mathbf{\Sigma}}_{\mathbf{Fu}}' & \mathbf{0}_{(K-m)\times(K-m)} \end{bmatrix}$$
$$+ O_p\left(\frac{1}{\sqrt{N}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right) + O_p\left(\frac{\sqrt{T}}{N}\right),$$

and noting that $\mathbf{F}^0 = [\mathbf{F}^*, \mathbf{0}_{T\times(K-m)}]$ and $\bar{\mathbf{U}}^0 = [\bar{\mathbf{U}}^0_m, \bar{\mathbf{U}}^0_{-m}]$ gives (since $T/N \to M < \infty$)

$$\begin{aligned}\mathbf{A}^{\mathbf{F}}_{NT,5} &= -\boldsymbol{\Sigma}_{\boldsymbol{\eta}} vec\left((T^{-1}\check{\mathbf{F}}'\mathbf{F}^*)T^{-1/2}\widehat{\boldsymbol{\Sigma}}_{\mathbf{Fu}}T^{-1}(\bar{\mathbf{U}}^0_{-m})'\sqrt{N}\bar{\mathbf{U}}_m\right) + O_p(N^{-1/2}) + O_p(T^{-1/2}),\\ &= -\boldsymbol{\Sigma}_{\boldsymbol{\eta}} vec\left(\boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*}T^{-1/2}\widehat{\boldsymbol{\Sigma}}_{\mathbf{Fu}}\boldsymbol{\Sigma}_{\mathbf{u}^0_{-m}\mathbf{u}_m}\right) + O_p(N^{-1/2}) + O_p(T^{-1/2}),\end{aligned}$$

where $\boldsymbol{\Sigma}_{\mathbf{u}^0_{-m}\mathbf{u}_m} = \mathbf{S}'_{-m}\mathbf{N}'\mathbf{R}'\boldsymbol{\Sigma}_{\ddot{\mathbf{U}}}\mathbf{R}_0\mathbf{T}\mathbf{B}_m$ from Lemma 12 and we recall from Lemma 13 that $\widehat{\boldsymbol{\Sigma}}_{\mathbf{Fu}} = \boldsymbol{\Sigma}^{-1}_{\mathbf{F}^*}(\mathbf{F}^* + \bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}^0_{-m}\boldsymbol{\Sigma}^{-1}_{\mathbf{u}^0_{-m}}$. By definition then

$$\begin{aligned}\boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*}T^{-1/2}\widehat{\boldsymbol{\Sigma}}_{\mathbf{Fu}}\boldsymbol{\Sigma}_{\mathbf{u}^0_{-m}\mathbf{u}_m} &= \boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*}\boldsymbol{\Sigma}^{-1}_{\mathbf{F}^*}\left(T^{-1/2}(\mathbf{F}^*)'\bar{\mathbf{U}}^0_{-m}\right)\boldsymbol{\Sigma}^{-1}_{\mathbf{u}^0_{-m}}\boldsymbol{\Sigma}_{\mathbf{u}^0_{-m}\mathbf{u}_m},\\ &\quad + \boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*}\boldsymbol{\Sigma}^{-1}_{\mathbf{F}^*}\left(T^{-1/2}(\bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}^0_{-m}\right)\boldsymbol{\Sigma}^{-1}_{\mathbf{u}^0_{-m}}\boldsymbol{\Sigma}_{\mathbf{u}^0_{-m}\mathbf{u}_m},\end{aligned}$$

where employing again (B-18)-(B-20) we have $(\mathbf{F}^*)'\bar{\mathbf{U}}^0_{-m} = \mathbf{S}'_m\mathbf{N}'\mathbf{R}'\ddot{\mathbf{P}}'\check{\mathbf{F}}'\sqrt{N}\ddot{\mathbf{U}}\mathbf{R}\mathbf{N}\mathbf{S}_{-m}$ and $(\bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}^0_{-m} = \mathbf{S}'_m\mathbf{N}'\mathbf{R}'\ddot{\mathbf{U}}'\sqrt{N}\ddot{\mathbf{U}}\mathbf{R}\mathbf{N}\mathbf{S}_{-m}$. This gives

$$\begin{aligned}\boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*}T^{-1/2}\widehat{\boldsymbol{\Sigma}}_{\mathbf{Fu}}\boldsymbol{\Sigma}_{\mathbf{u}^0_{-m}\mathbf{u}_m} &= \boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*}\boldsymbol{\Sigma}^{-1}_{\mathbf{F}^*}\mathbf{S}'_m\mathbf{N}'\mathbf{R}'\ddot{\mathbf{P}}'\left(T^{-1/2}\check{\mathbf{F}}'\sqrt{N}\ddot{\mathbf{U}}\right)\mathbf{R}\mathbf{N}\mathbf{S}_{-m}\boldsymbol{\Sigma}^{-1}_{\mathbf{u}^0_{-m}}\boldsymbol{\Sigma}_{\mathbf{u}^0_{-m}\mathbf{u}_m},\\ &\quad + \boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*}\boldsymbol{\Sigma}^{-1}_{\mathbf{F}^*}\mathbf{S}'_m\mathbf{N}'\mathbf{R}'\left(T^{-1/2}\sqrt{N}\ddot{\mathbf{U}}'\ddot{\mathbf{U}}\right)\mathbf{R}\mathbf{N}\mathbf{S}_{-m}\boldsymbol{\Sigma}^{-1}_{\mathbf{u}^0_{-m}}\boldsymbol{\Sigma}_{\mathbf{u}^0_{-m}\mathbf{u}_m},\end{aligned}$$

and in turn once substituted in $\mathbf{A}^{\mathbf{F}}_{NT,5}$

$$\begin{aligned}\mathbf{A}^{\mathbf{F}}_{NT,5} = &-\boldsymbol{\Sigma}_{\boldsymbol{\eta}}\left[\boldsymbol{\Sigma}'_{\mathbf{u}^0_{-m}\mathbf{u}_m}\boldsymbol{\Sigma}^{-1}_{\mathbf{u}^0_{-m}}\mathbf{S}'_{-m}\mathbf{N}'\mathbf{R}' \otimes \boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*}\boldsymbol{\Sigma}^{-1}_{\mathbf{F}^*}\mathbf{S}'_m\mathbf{N}'\mathbf{R}'\mathbf{P}'\right] vec\left(T^{-1/2}\check{\mathbf{F}}'\sqrt{N}\ddot{\mathbf{U}}\right)\\ &- \boldsymbol{\Sigma}_{\boldsymbol{\eta}}\left[\boldsymbol{\Sigma}'_{\mathbf{u}^0_{-m}\mathbf{u}_m}\boldsymbol{\Sigma}^{-1}_{\mathbf{u}^0_{-m}}\mathbf{S}'_{-m}\mathbf{N}'\mathbf{R}' \otimes \boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*}\boldsymbol{\Sigma}^{-1}_{\mathbf{F}^*}\mathbf{S}'_m\mathbf{N}'\mathbf{R}'\right] vec\left(T^{-1/2}\sqrt{N}\ddot{\mathbf{U}}'\ddot{\mathbf{U}}\right)\\ &+ O_p(N^{-1/2}) + O_p(T^{-1/2}).\end{aligned}$$

Finally, since from Lemma 9 the second term in this expression is of order $O_p(\sqrt{T}N^{-1/2})$ it is clear that

$$\begin{aligned}\mathbf{A}^{\mathbf{F}}_{NT,5} = &-\boldsymbol{\Sigma}_{\boldsymbol{\eta}}\left[\boldsymbol{\Sigma}'_{\mathbf{u}^0_{-m}\mathbf{u}_m}\boldsymbol{\Sigma}^{-1}_{\mathbf{u}^0_{-m}}\mathbf{S}'_{-m}\mathbf{N}'\mathbf{R}' \otimes \boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*}\boldsymbol{\Sigma}^{-1}_{\mathbf{F}^*}\mathbf{S}'_m\mathbf{N}'\mathbf{R}'\mathbf{P}'\right] vec\left(T^{-1/2}\check{\mathbf{F}}'\sqrt{N}\ddot{\mathbf{U}}\right)\\ &- \sqrt{T}N^{-1/2}\mathbf{b}^{\mathbf{F}}_1 + O_p(N^{-1/2}) + O_p(T^{-1/2}),\end{aligned}$$

with $\mathbf{b}^{\mathbf{F}}_1 = \boldsymbol{\Sigma}_{\boldsymbol{\eta}}\left[\boldsymbol{\Sigma}'_{\mathbf{u}^0_{-m}\mathbf{u}_m}\boldsymbol{\Sigma}^{-1}_{\mathbf{u}^0_{-m}}\mathbf{S}'_{-m}\mathbf{N}'\mathbf{R}' \otimes \boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*}\boldsymbol{\Sigma}^{-1}_{\mathbf{F}^*}\mathbf{S}'_m\mathbf{N}'\mathbf{R}'\right] vec\,(\boldsymbol{\Sigma}_{\ddot{\mathbf{U}}})$.
In conclusion, combining the results gives, provided $T/N \to M < \infty$,

$$\mathbf{A}^{\mathbf{F}}_{NT} = \boldsymbol{\Psi}_{\mathbf{F}} vec\left(\frac{\check{\mathbf{F}}'\sqrt{N}\ddot{\mathbf{U}}}{\sqrt{T}}\right) + \sqrt{\frac{T}{N}}\left(\mathbf{b}^{\mathbf{F}}_0 - \mathbf{b}^{\mathbf{F}}_1\right) + O_p\left(\frac{1}{\sqrt{N}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right),$$

with

$$\begin{aligned}\mathbf{b}^{\mathbf{F}}_0 &= \boldsymbol{\Sigma}_{\boldsymbol{\eta}}\left[\mathbf{B}'_m\mathbf{T}'\mathbf{R}'_0 \otimes \boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*}\boldsymbol{\Sigma}^{-1}_{\mathbf{F}^*}\mathbf{S}'_m\mathbf{N}'\mathbf{R}'\right] vec\,(\boldsymbol{\Sigma}_{\ddot{\mathbf{U}}})\,,\\ \mathbf{b}^{\mathbf{F}}_1 &= \boldsymbol{\Sigma}_{\boldsymbol{\eta}}\left[\boldsymbol{\Sigma}'_{\mathbf{u}^0_{-m}\mathbf{u}_m}\boldsymbol{\Sigma}^{-1}_{\mathbf{u}^0_{-m}}\mathbf{S}'_{-m}\mathbf{N}'\mathbf{R}' \otimes \boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*}\boldsymbol{\Sigma}^{-1}_{\mathbf{F}^*}\mathbf{S}'_m\mathbf{N}'\mathbf{R}'\right] vec\,(\boldsymbol{\Sigma}_{\ddot{\mathbf{U}}})\,,\end{aligned}$$

$$
\begin{aligned}
\boldsymbol{\Psi}_{\mathbf{F}} &= -\mathbf{V}_{\mathbf{F},1} + \mathbf{V}_{\mathbf{F},2} + \mathbf{V}_{\mathbf{F},3} - \mathbf{V}_{\mathbf{F},4}, \\
\mathbf{V}_{\mathbf{F},1} &= \boldsymbol{\Sigma}_{\boldsymbol{\eta}} \left[\mathbf{B}_m' \mathbf{T}' \mathbf{R}_0' \otimes \mathbf{I}_{1+K^2 m(1+p^*)} \right], \\
\mathbf{V}_{\mathbf{F},2} &= \boldsymbol{\Sigma}_{\boldsymbol{\eta}} \left[\mathbf{B}_m' \mathbf{T}' \mathbf{R}_0' \otimes \boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*} \boldsymbol{\Sigma}_{\mathbf{F}^*}^{-1} \mathbf{S}_m' \mathbf{N}' \mathbf{R}' \mathbf{P}' \right], \\
\mathbf{V}_{\mathbf{F},3} &= \boldsymbol{\Sigma}_{\boldsymbol{\eta}} \left[\mathbf{B}_m' \mathbf{T}' \mathbf{R}_0' \boldsymbol{\Sigma}_{\ddot{\mathbf{U}}} \mathbf{R} \mathbf{N} \mathbf{S}_{-m} \boldsymbol{\Sigma}_{\mathbf{u}_{-m}^0}^{-1} \mathbf{S}_{-m}' \mathbf{N}' \mathbf{R}' \otimes \mathbf{I}_{1+k^2 m(1+p^*)} \right], \\
\mathbf{V}_{\mathbf{F},4} &= \boldsymbol{\Sigma}_{\boldsymbol{\eta}} \left[\boldsymbol{\Sigma}_{\mathbf{u}_{-m}^0 \mathbf{u}_m}' \boldsymbol{\Sigma}_{\mathbf{u}_{-m}^0}^{-1} \mathbf{S}_{-m}' \mathbf{N}' \mathbf{R}' \otimes \boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*} \boldsymbol{\Sigma}_{\mathbf{F}^*}^{-1} \mathbf{S}_m' \mathbf{N}' \mathbf{R}' \mathbf{P}' \right],
\end{aligned}
$$

which is the result stated in eq.(D-23) of the lemma.

Proof of Lemma 15

Let $\mathbf{A}^{\varepsilon} = \frac{1}{NT}\sum_{i=1}^{N}\mathbf{w}_i'\mathbf{M}\boldsymbol{\varepsilon}_i$ and note that given $\bar{\mathbf{w}} \subseteq \mathbf{Q}$ then $\mathbf{M}\bar{\mathbf{w}} = \mathbf{0}_{T\times k_w}$. Therefore, substituting in (B-13)

$$
\begin{aligned}
\mathbf{A}^{\varepsilon} &= \frac{1}{NT}\sum_{i=1}^{N}\left(\mathbf{w}_i' - \bar{\mathbf{w}}\right)\mathbf{M}\boldsymbol{\varepsilon}_i = \frac{1}{NT}\sum_{i=1}^{N}\left(\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\check{\mathbf{F}}' + \boldsymbol{\epsilon}_i' - \bar{\boldsymbol{\epsilon}}'\right)\mathbf{M}\boldsymbol{\varepsilon}_i,\\
&= \frac{1}{NT}\sum_{i=1}^{N}\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\check{\mathbf{F}}'\mathbf{M}\boldsymbol{\varepsilon}_i + \frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\epsilon}_i'\mathbf{M}\boldsymbol{\varepsilon}_i - \frac{1}{T}\bar{\boldsymbol{\epsilon}}'\mathbf{M}\bar{\boldsymbol{\varepsilon}},\\
&= \mathbf{A}_1^{\varepsilon} + \mathbf{A}_2^{\varepsilon} - \mathbf{A}_3^{\varepsilon},
\end{aligned}
\tag{D-39}
$$

with $\widetilde{\mathbf{P}}_i = \ddot{\mathbf{P}}_i - \ddot{\mathbf{P}}$ and obvious definitions for $\mathbf{A}_1^{\varepsilon}, \mathbf{A}_2^{\varepsilon}$ and $\mathbf{A}_3^{\varepsilon}$. We start with the first term and decompose it as

$$
\begin{aligned}
\mathbf{A}_1^{\varepsilon} &= \frac{1}{NT}\sum_{i=1}^{N}\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\check{\mathbf{F}}'\mathbf{M}_{\mathbf{F}}\boldsymbol{\varepsilon}_i + \frac{1}{NT}\sum_{i=1}^{N}\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\check{\mathbf{F}}'(\mathbf{M}_{\mathbf{F}} - \mathbf{M})\boldsymbol{\varepsilon}_i,\\
&= \mathbf{A}_{11}^{\varepsilon} + \mathbf{A}_{12}^{\varepsilon}.
\end{aligned}
$$

For the first term we find, writing it in full and substituting in (B-18), $\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^*} = T^{-1}(\mathbf{F}^*)'\mathbf{F}^*$ and $\widehat{\boldsymbol{\Sigma}}_{\check{\mathbf{F}}} = T^{-1}\check{\mathbf{F}}'\check{\mathbf{F}}$,

$$
\begin{aligned}
\mathbf{A}_{11}^{\varepsilon} &= \frac{1}{N}\sum_{i=1}^{N}\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\frac{\check{\mathbf{F}}'\boldsymbol{\varepsilon}_i}{T} - \frac{1}{N}\sum_{i=1}^{N}\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\frac{\check{\mathbf{F}}'\mathbf{F}^*}{T}\left(\frac{\mathbf{F}^{*\prime}\mathbf{F}^*}{T}\right)^{-1}\frac{\mathbf{F}^{*\prime}\boldsymbol{\varepsilon}_i}{T},\\
&= \frac{1}{N}\sum_{i=1}^{N}\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\left[\mathbf{I}_{1+K^2m(1+p^*)} - \widehat{\boldsymbol{\Sigma}}_{\check{\mathbf{F}}}\ddot{\mathbf{P}}\mathbf{R}\mathbf{N}\mathbf{S}_m\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^*}^{-1}\mathbf{S}_m'\mathbf{N}'\mathbf{R}'\ddot{\mathbf{P}}'\right]\frac{\check{\mathbf{F}}'\boldsymbol{\varepsilon}_i}{T},\\
&= \left[vec\left(\widehat{\mathbf{B}}^{\mathbf{F}}\right)'\otimes\mathbf{I}_{k_w}\right]vec\left[\frac{1}{N}\sum_{i=1}^{N}\left(\frac{\boldsymbol{\varepsilon}_i'\check{\mathbf{F}}}{T}\otimes\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right)\right],\\
&= \widehat{\boldsymbol{\Psi}}_{\boldsymbol{\epsilon}}vec\left[\frac{1}{N}\sum_{i=1}^{N}\left(\frac{\boldsymbol{\varepsilon}_i'\check{\mathbf{F}}}{T}\otimes\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right)\right],
\end{aligned}
$$

where $\widehat{\boldsymbol{\Psi}}_{\boldsymbol{\epsilon}} = \left[vec\left(\widehat{\mathbf{B}}^{\mathbf{F}}\right)'\otimes\mathbf{I}_{k_w}\right]$ and $\widehat{\mathbf{B}}^{\mathbf{F}} = \mathbf{I}_{1+K^2m(1+p^*)} - \widehat{\boldsymbol{\Sigma}}_{\check{\mathbf{F}}}\ddot{\mathbf{P}}\mathbf{R}\mathbf{N}\mathbf{S}_m\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^*}^{-1}\mathbf{S}_m'\mathbf{N}'\mathbf{R}'\ddot{\mathbf{P}}'$. From $\left\|\widehat{\mathbf{B}}^{\mathbf{F}}\right\| = O_p(1)$ by results in Lemma 10 and eq.(D-17) of Lemma 11 follows

$$
\|\mathbf{A}_{11}^{\varepsilon}\| \leq \left\|\widehat{\boldsymbol{\Psi}}_{\boldsymbol{\epsilon}}\right\|\left\|\frac{1}{N}\sum_{i=1}^{N}\left(T^{-1}\boldsymbol{\varepsilon}_i'\check{\mathbf{F}}\otimes\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right)\right\| = O_p\left(\frac{1}{\sqrt{NT}}\right).
$$

Next, for $\mathbf{A}_{12}^{\varepsilon}$ we use the decomposition in (D-1) and obtain

$$
\|\mathbf{A}_{12}^{\varepsilon}\| \leq \|\mathbf{A}_{121}^{\varepsilon}\| + \|\mathbf{A}_{122}^{\varepsilon}\| + \|\mathbf{A}_{123}^{\varepsilon}\| + \|\mathbf{A}_{124}^{\varepsilon}\| + \|\mathbf{A}_{125}^{\varepsilon}\| = O_p(N^{-1}T^{-1/2}) + O_p(N^{-1/2}T^{-1}),
$$

because, denoting $\widehat{\boldsymbol{\Sigma}}_{\mathbf{u}_{-m}^0} = T^{-1}(\bar{\mathbf{U}}_{-m}^0)'\bar{\mathbf{U}}_{-m}^0$ we have

$$
\|\mathbf{A}_{121}^{\varepsilon}\| \leq \left\|\frac{1}{N}\sum_{i=1}^{N}\left(\frac{\boldsymbol{\varepsilon}_i'\bar{\mathbf{U}}_{-m}^0}{T}\otimes\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right)\right\|\left\|\frac{\check{\mathbf{F}}'\bar{\mathbf{U}}_{-m}^0}{T}\right\|\left\|\widehat{\boldsymbol{\Sigma}}_{\mathbf{u}_{-m}^0}^{-1}\right\| = O_p\left(\frac{1}{N\sqrt{T}}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right),
$$

$$\|\mathbf{A}_{122}^{\varepsilon}\| \leq \left\| \frac{1}{N}\sum_{i=1}^{N}\left(\frac{\boldsymbol{\varepsilon}_i'\bar{\mathbf{U}}_m^0}{T}\otimes \mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right)\right\| \left\|\frac{\check{\mathbf{F}}'\bar{\mathbf{U}}_m^0}{T}\right\| \left\|\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^*}^{-1}\right\| = O_p\left(\frac{1}{N^2\sqrt{T}}\right) + O_p\left(\frac{1}{N^{3/2}T}\right),$$

$$\|\mathbf{A}_{123}^{\varepsilon}\| \leq \left\| \frac{1}{N}\sum_{i=1}^{N}\left(\frac{\boldsymbol{\varepsilon}_i'\mathbf{F}^*}{T}\otimes \mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right)\right\| \left\|\frac{\check{\mathbf{F}}'\bar{\mathbf{U}}_m^0}{T}\right\| \left\|\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^*}^{-1}\right\| = O_p\left(\frac{1}{NT}\right),$$

$$\|\mathbf{A}_{124}^{\varepsilon}\| \leq \left\| \frac{1}{N}\sum_{i=1}^{N}\left(\frac{\boldsymbol{\varepsilon}_i'\bar{\mathbf{U}}_m^0}{T}\otimes \mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right)\right\| \left\|\frac{\check{\mathbf{F}}'\mathbf{F}^*}{T}\right\| \left\|\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^*}^{-1}\right\| = O_p\left(\frac{1}{N\sqrt{T}}\right) + O_p\left(\frac{1}{N^{3/2}}\right),$$

$$\|\mathbf{A}_{125}^{\varepsilon}\| \leq \left\| \frac{1}{N}\sum_{i=1}^{N}\left(\frac{\boldsymbol{\varepsilon}_i'\mathbf{Q}_0}{T}\otimes \mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right)\right\| \left\|\frac{\check{\mathbf{F}}'\mathbf{Q}_0}{T}\right\| \left\|\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1} - \widehat{\boldsymbol{\Sigma}}_{\mathbf{F}_u^+}^{-1}\right\| = O_p\left(\frac{1}{N\sqrt{T}}\right) + O_p\left(\frac{1}{\sqrt{N}T}\right),$$

by the results in Lemmas 10, 11 and 13. Hence, we conclude that

$$\|\mathbf{A}_1^{\varepsilon}\| = O_p((NT)^{-1/2}), \tag{D-40}$$

and, defining $\mathbf{A}_{NT,1}^{\varepsilon} = \sqrt{NT}\mathbf{A}_1^{\varepsilon}$ also, since $\widehat{\mathbf{B}}^{\mathbf{F}} = \mathbf{B}^{\mathbf{F}} + O_p(N^{-1/2}) + O_p(T^{-1/2})$ by Lemma 12, with $\mathbf{B}^{\mathbf{F}} = \mathbf{I}_{1+K^2m(1+p^*)} - \boldsymbol{\Sigma}_{\check{\mathbf{F}}}\mathbf{PRNS}_m\boldsymbol{\Sigma}_{\mathbf{F}^*}^{-1}\mathbf{S}_m'\mathbf{N}'\mathbf{R}'\mathbf{P}'$ and $\boldsymbol{\Psi}_{\boldsymbol{\epsilon}} = \left[vec\left(\mathbf{B}^{\mathbf{F}}\right)' \otimes \mathbf{I}_{k_w}\right]$

$$\mathbf{A}_{NT,1}^{\varepsilon} = \boldsymbol{\Psi}_{\boldsymbol{\epsilon}} vec\left[\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\frac{\boldsymbol{\varepsilon}_i'\check{\mathbf{F}}}{\sqrt{T}}\otimes \mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right)\right] + O_p\left(\frac{1}{\sqrt{N}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right) + O_p\left(\frac{\sqrt{T}}{N}\right). \tag{D-41}$$

We take on $\mathbf{A}_2^{\varepsilon}$ next. Decomposing it as before returns

$$\begin{aligned}\mathbf{A}_2^{\varepsilon} &= \frac{1}{N}\sum_{i=1}^{N}\frac{\boldsymbol{\epsilon}_i'\boldsymbol{\varepsilon}_i}{T} - \frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\epsilon}_i'\mathbf{H}_{\mathbf{F}}\boldsymbol{\varepsilon}_i - \frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\epsilon}_i'(\mathbf{M}_{\mathbf{F}} - \mathbf{M})\boldsymbol{\varepsilon}_i,\\ &= \mathbf{A}_{21}^{\varepsilon} - \mathbf{A}_{22}^{\varepsilon} - \mathbf{A}_{23}^{\varepsilon}.\end{aligned} \tag{D-42}$$

Clearly, since by Ass.1 the elements of $\boldsymbol{\epsilon}_i$ and $\boldsymbol{\varepsilon}_i$ are contemporaneously uncorrelated

$$\|\mathbf{A}_{21}^{\varepsilon}\| = \left\|\frac{1}{N}\sum_{i=1}^{N}\frac{\boldsymbol{\epsilon}_i'\boldsymbol{\varepsilon}_i}{T}\right\| = O_p\left(\frac{1}{\sqrt{NT}}\right), \tag{D-43}$$

whereas for the second term, by (D-8)-(D-9) of Lemma 10,

$$\left\|T^{-1}\boldsymbol{\epsilon}_i'\mathbf{H}_{\mathbf{F}}\boldsymbol{\varepsilon}_i\right\| \leq \left\|\frac{\boldsymbol{\epsilon}_i'\mathbf{F}^*}{T}\right\| \left\|\left(\frac{(\mathbf{F}^*)'\mathbf{F}^*}{T}\right)^{-1}\right\| \left\|\frac{(\mathbf{F}^*)'\boldsymbol{\varepsilon}_i}{T}\right\| = O_p\left(\frac{1}{T}\right),$$

and therefore

$$\|\mathbf{A}_{22}^{\varepsilon}\| = O_p\left(T^{-1}\right).$$

Letting again $\mathbf{A}_{NT,22}^{\varepsilon} = \sqrt{NT}\mathbf{A}_{22}^{\varepsilon}$ it is clear that

$$\left\|\mathbf{A}_{NT,22}^{\varepsilon}\right\| = O_p\left(\sqrt{N}T^{-1/2}\right).$$

To evaluate $\mathbf{A}_{23}^{\varepsilon}$ we again split it into 5 key components

$$\|\mathbf{A}_{231}^{\varepsilon}\| \leq \left\|T^{-1}\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}_{-m}^0\right\| \left\|\widehat{\boldsymbol{\Sigma}}_{\mathbf{u}_{-m}^0}^{-1}\right\| \left\|T^{-1}(\bar{\mathbf{U}}_{-m}^0)'\boldsymbol{\varepsilon}_i\right\| = O_p(N^{-1}) + O_p(T^{-1}) + O_p((NT)^{-1/2}),$$
$$\|\mathbf{A}_{232}^{\varepsilon}\| \leq \left\|T^{-1}\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}_{m}^0\right\| \left\|\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^*}^{-1}\right\| \left\|T^{-1}(\bar{\mathbf{U}}_{m}^0)'\boldsymbol{\varepsilon}_i\right\| = O_p(N^{-2}) + O_p(N^{-3/2}T^{-1/2}) + O_p((NT)^{-1}),$$
$$\|\mathbf{A}_{233}^{\varepsilon}\| \leq \left\|T^{-1}\boldsymbol{\epsilon}_i'\mathbf{F}^*\right\| \left\|\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^*}^{-1}\right\| \left\|T^{-1}(\bar{\mathbf{U}}_{m}^0)'\boldsymbol{\varepsilon}_i\right\| = O_p(N^{-1}T^{-1/2}) + O_p(N^{-1/2}T^{-1}),$$
$$\|\mathbf{A}_{234}^{\varepsilon}\| \leq \left\|T^{-1}\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}_{m}^0\right\| \left\|\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^*}^{-1}\right\| \left\|T^{-1}(\mathbf{F}^*)'\boldsymbol{\varepsilon}_i\right\| = O_p(N^{-1}T^{-1/2}) + O_p(N^{-1/2}T^{-1}),$$
$$\|\mathbf{A}_{235}^{\varepsilon}\| \leq \left\|T^{-1}\boldsymbol{\epsilon}_i'\mathbf{Q}_0\right\| \left\|\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1} - \widehat{\boldsymbol{\Sigma}}_{\mathbf{F}_u^+}^{-1}\right\| \left\|T^{-1}\mathbf{Q}_0'\boldsymbol{\varepsilon}_i\right\| = O_p(T^{-1}) + O_p((NT)^{-1/2}),$$

which leads to

$$\|\mathbf{A}_{23}^{\varepsilon}\| = O_p(N^{-1}) + O_p(T^{-1}) + O_p((NT)^{-1/2}),$$

and therefore

$$\|\mathbf{A}_{2}^{\varepsilon}\| = O_p(N^{-1}) + O_p(T^{-1}) + O_p((NT)^{-1/2}). \tag{D-44}$$

Finally, for $\mathbf{A}_3^{\varepsilon}$ we find

$$\|\mathbf{A}_{3}^{\varepsilon}\| \leq \left\|T^{-1}\bar{\boldsymbol{\epsilon}}'\bar{\boldsymbol{\varepsilon}}\right\| + \left\|T^{-1}\bar{\boldsymbol{\epsilon}}'\mathbf{Q}_0(T^{-1}\mathbf{Q}_0'\mathbf{Q}_0)^{-1}\mathbf{Q}_0'\bar{\boldsymbol{\varepsilon}}\right\| = O_p(N^{-1}), \tag{D-45}$$

since $\|T^{-1}\bar{\boldsymbol{\epsilon}}'\bar{\boldsymbol{\varepsilon}}\| = O_p(T^{-1/2}N^{-1})$ due to $\bar{\boldsymbol{\epsilon}}$ and $\bar{\boldsymbol{\varepsilon}}$ being uncorrelated $O_p(N^{-1/2})$ variables, and because the norm of the final term can be decomposed in the following five components

$$\|\mathbf{A}_{31}^{\varepsilon}\| \leq \left\|T^{-1}\bar{\boldsymbol{\epsilon}}'\bar{\mathbf{U}}_{-m}^0\right\| \left\|[T^{-1}(\bar{\mathbf{U}}_{-m}^0)'\bar{\mathbf{U}}_{-m}^0]^{-1}\right\| \left\|T^{-1}(\bar{\mathbf{U}}_{-m}^0)'\bar{\boldsymbol{\varepsilon}}\right\| = O_p(N^{-1}),$$
$$\|\mathbf{A}_{32}^{\varepsilon}\| \leq \left\|T^{-1}\bar{\boldsymbol{\epsilon}}'\bar{\mathbf{U}}_{m}^0\right\| \left\|[T^{-1}(\mathbf{F}^*)'\mathbf{F}^*]^{-1}\right\| \left\|T^{-1}(\bar{\mathbf{U}}_{m}^0)'\bar{\boldsymbol{\varepsilon}}\right\| = O_p(N^{-2}),$$
$$\|\mathbf{A}_{33}^{\varepsilon}\| \leq \left\|T^{-1}\bar{\boldsymbol{\epsilon}}'\mathbf{F}^*\right\| \left\|[T^{-1}(\mathbf{F}^*)'\mathbf{F}^*]^{-1}\right\| \left\|T^{-1}(\bar{\mathbf{U}}_{m}^0)'\bar{\boldsymbol{\varepsilon}}\right\| = O_p(N^{-3/2}T^{-1/2}),$$
$$\|\mathbf{A}_{34}^{\varepsilon}\| \leq \left\|T^{-1}\bar{\boldsymbol{\epsilon}}'\bar{\mathbf{U}}_{m}^0\right\| \left\|[T^{-1}(\mathbf{F}^*)'\mathbf{F}^*]^{-1}\right\| \left\|T^{-1}(\mathbf{F}^*)'\bar{\boldsymbol{\varepsilon}}\right\| = O_p(N^{-3/2}T^{-1/2}),$$
$$\|\mathbf{A}_{35}^{\varepsilon}\| \leq \left\|T^{-1}\bar{\boldsymbol{\epsilon}}'\mathbf{Q}_0\right\| \left\|\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1} - \widehat{\boldsymbol{\Sigma}}_{\mathbf{F}_u^+}^{-1}\right\| \left\|T^{-1}\mathbf{Q}_0'\bar{\boldsymbol{\varepsilon}}\right\| = O_p(N^{-3/2}) + O_p(N^{-1}T^{-1/2}),$$

where we used the fact that the terms involving $\bar{\boldsymbol{\epsilon}}$ and $\bar{\boldsymbol{\varepsilon}}$ have the same order as those involving $\bar{\mathbf{U}}_m^0$ in Lemma 10. It will be convenient to also define $\mathbf{A}_{NT,3}^{\varepsilon} = \sqrt{NT}\mathbf{A}_3^{\varepsilon}$

$$\mathbf{A}_{NT,3}^{\varepsilon} = \sqrt{\frac{T}{N}}\left(\frac{\sqrt{N}\bar{\boldsymbol{\epsilon}}'\bar{\mathbf{U}}_{-m}^0}{T}\right)\left(\frac{(\bar{\mathbf{U}}_{-m}^0)'\bar{\mathbf{U}}_{-m}^0}{T}\right)^{-1}\left(\frac{(\bar{\mathbf{U}}_{-m}^0)'\sqrt{N}\bar{\boldsymbol{\varepsilon}}}{T}\right) + O_p\left(\frac{1}{\sqrt{N}}\right) + O_p\left(\frac{\sqrt{T}}{N}\right),$$

such that if $T/N \to M < \infty$, making use of (B-14),

$$\mathbf{A}_{NT,3}^{\varepsilon} = \sqrt{T}N^{-1/2}\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}\mathbf{U}_{-m}}\boldsymbol{\Sigma}_{\mathbf{u}_{-m}^0}^{-1}\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}\mathbf{U}_{-m}}' + O_p(T^{-1/2}) + O_p(N^{-1/2}), \tag{D-46}$$

with $\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}\mathbf{U}_{-m}} = \mathbf{S}_w'\boldsymbol{\Sigma}_{\ddot{\mathbf{U}}}\mathbf{RNS}_{-m}$, $\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}\mathbf{U}_{-m}} = E(\boldsymbol{\varepsilon}_i'\ddot{\mathbf{U}}_i/T)\mathbf{RNS}_{-m}$, $\boldsymbol{\Sigma}_{\mathbf{u}_{-m}^0} = \mathbf{S}_{-m}'\mathbf{N}'\mathbf{R}'\boldsymbol{\Sigma}_{\ddot{\mathbf{U}}}\mathbf{RNS}_{-m}$ and $\boldsymbol{\Sigma}_{\ddot{\mathbf{U}}} = E(\ddot{\mathbf{U}}_i'\ddot{\mathbf{U}}_i/T)$.

Combining (D-40)-(D-45) in (D-39) leads to the conclusion that

$$\|\mathbf{A}^{\varepsilon}\| = O_p(N^{-1}) + O_p(T^{-1}) + O_p((NT)^{-1/2}),$$

which is the result stated in the lemma. Letting $\mathbf{A}_{NT}^{\varepsilon} = \sqrt{NT}\mathbf{A}^{\varepsilon}$, the result above implies

$$\|\mathbf{A}_{NT}^{\varepsilon}\| = O_p(1) + O_p(\sqrt{T}N^{-1/2}) + O_p(\sqrt{N}T^{-1/2}).$$

Proof of Lemma 16

Consider $\widehat{\sigma}^2_\varepsilon(\cdot)$ defined in eq.(20) evaluated at $\boldsymbol{\delta}_0 \neq \boldsymbol{\delta}$, with $\boldsymbol{\delta} = [\rho, \boldsymbol{\beta}']'$ the true parameter vector. Suppose that $p^* \geq p$ and Ass.1-5 hold. We can then make use of (B-23) to get

$$\begin{aligned}\widehat{\sigma}^2_\varepsilon(\boldsymbol{\delta}_0) &= \frac{1}{N(T-c)}\sum_{i=1}^N \left\|\mathbf{M}\left(\mathbf{y}_i - \mathbf{w}_i\boldsymbol{\delta}_0\right)\right\|^2 = \frac{1}{N(T-c)}\sum_{i=1}^N \left\|\mathbf{M}\left(\mathbf{w}_i(\boldsymbol{\delta}-\boldsymbol{\delta}_0) + \mathbf{F}\boldsymbol{\gamma}_i + \boldsymbol{\varepsilon}_i\right)\right\|^2, \\ &= \frac{T}{T-c}\frac{1}{NT}\sum_{i=1}^N \left\|\mathbf{M}\left(\mathbf{w}_i(\boldsymbol{\delta}-\boldsymbol{\delta}_0) - \bar{\mathbf{U}}_m\boldsymbol{\gamma}_i + \boldsymbol{\varepsilon}_i\right)\right\|^2.\end{aligned}$$

For its components we find, denoting first $\widehat{\boldsymbol{\Sigma}}_{\boldsymbol{\gamma}} = \left[\frac{1}{N}\sum_{i=1}^N\left(\boldsymbol{\gamma}_i' \otimes \boldsymbol{\gamma}_i'\right)\right]$, with $\left\|\widehat{\boldsymbol{\Sigma}}_{\boldsymbol{\gamma}}\right\| = O_p(1)$ by Ass.3,

$$\left\|\frac{1}{NT}\sum_{i=1}^N \boldsymbol{\gamma}_i'\bar{\mathbf{U}}_m'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\gamma}_i\right\| \leq \left\|\widehat{\boldsymbol{\Sigma}}_{\boldsymbol{\gamma}}\right\|\left\|\frac{\bar{\mathbf{U}}_m'\bar{\mathbf{U}}_m}{T}\right\| + \left\|\widehat{\boldsymbol{\Sigma}}_{\boldsymbol{\gamma}}\right\|\left\|\frac{\bar{\mathbf{U}}_m'\mathbf{Q}_0}{T}\right\|\left\|\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1}\right\|\left\|\frac{\mathbf{Q}_0'\bar{\mathbf{U}}_m}{T}\right\| = O_p\left(\frac{1}{N}\right),$$

where we made use of Lemma 10 by noting that $\bar{\mathbf{U}}_m$ is by the definition above (B-16) a subset of $\bar{\mathbf{U}}_m^0$. Also, since for any $\|\boldsymbol{\delta}-\boldsymbol{\delta}_0\| < \infty$, by (D-22) of Lemma 14

$$\left\|\frac{1}{N}\sum_{i=1}^N(\boldsymbol{\delta}-\boldsymbol{\delta}_0)'\frac{\mathbf{w}_i'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\gamma}_i}{T}\right\| \leq \|\boldsymbol{\delta}-\boldsymbol{\delta}_0\|\left\|\frac{1}{N}\sum_{i=1}^N\frac{\mathbf{w}_i'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\gamma}_i}{T}\right\| = O_p\left(\frac{1}{N}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right),$$

and similarly by (D-24) of Lemma 15

$$\left\|\frac{1}{N}\sum_{i=1}^N(\boldsymbol{\delta}-\boldsymbol{\delta}_0)'\frac{\mathbf{w}_i'\mathbf{M}\boldsymbol{\varepsilon}_i}{T}\right\| \leq \|\boldsymbol{\delta}-\boldsymbol{\delta}_0\|\left\|\frac{1}{N}\sum_{i=1}^N\frac{\mathbf{w}_i'\mathbf{M}\boldsymbol{\varepsilon}_i}{T}\right\| = O_p\left(\frac{1}{N}\right) + O_p\left(\frac{1}{T}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right).$$

Also

$$\begin{aligned}\left\|\frac{1}{NT}\sum_{i=1}^N \boldsymbol{\varepsilon}_i'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\gamma}_i\right\| &\leq \left\|\frac{1}{N}\sum_{i=1}^N\frac{\boldsymbol{\varepsilon}_i'\bar{\mathbf{U}}_m}{T}\boldsymbol{\gamma}_i\right\| + \left\|\frac{1}{N}\sum_{i=1}^N\left(\boldsymbol{\gamma}_i'\otimes\frac{\boldsymbol{\varepsilon}_i'\mathbf{Q}_0}{T}\right)\right\|\left\|\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1}\right\|\left\|\frac{\mathbf{Q}_0'\bar{\mathbf{U}}_m}{T}\right\|, \\ &\leq \left\|\frac{1}{N}\sum_{i=1}^N\left(\boldsymbol{\gamma}_i'\otimes\frac{\boldsymbol{\varepsilon}_i'\bar{\mathbf{U}}_m}{T}\right)\right\|\|\mathbf{I}_m\| + \left\|\frac{1}{N}\sum_{i=1}^N\left(\boldsymbol{\gamma}_i'\otimes\frac{\boldsymbol{\varepsilon}_i'\mathbf{Q}_0}{T}\right)\right\|\left\|\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1}\right\|\left\|\frac{\mathbf{Q}_0'\bar{\mathbf{U}}_m}{T}\right\|,\end{aligned}$$

Letting $\gamma_{i,d}$ denote the element on row $d = 1,\ldots,m$ of $\boldsymbol{\gamma}_i$, the elements on columns $c(d-1)+1$ to cd of $\frac{1}{N}\sum_{i=1}^N\left(\boldsymbol{\gamma}_i'\otimes T^{-1}\boldsymbol{\varepsilon}_i'\mathbf{Q}_0\right)$ and columns $m(d-1)+1$ to md of $\frac{1}{N}\sum_{i=1}^N\left(\boldsymbol{\gamma}_i'\otimes T^{-1}\boldsymbol{\varepsilon}_i'\bar{\mathbf{U}}_m\right)$ are given by

$$\begin{aligned}&\frac{1}{N}\sum_{i=1}^N\gamma_{i,d}\frac{\boldsymbol{\varepsilon}_i'\mathbf{F}^0}{T} + \frac{1}{N}\sum_{i=1}^N\gamma_{i,d}\frac{\boldsymbol{\varepsilon}_i'\bar{\mathbf{U}}^0}{T} = \left[\frac{\bar{\mathbf{a}}_d'\mathbf{F}^*}{T}, \mathbf{0}_{1\times(K-m)}\right] + \left[\frac{\bar{\mathbf{a}}_d'\bar{\mathbf{U}}_m^0}{T}, \frac{\bar{\mathbf{a}}_d'\bar{\mathbf{U}}_{-m}^0}{T}\right], \\ &\frac{1}{N}\sum_{i=1}^N\gamma_{i,d}\frac{\boldsymbol{\varepsilon}_i'\bar{\mathbf{U}}_m}{T} = \frac{\bar{\mathbf{a}}_d'\bar{\mathbf{U}}_m}{T},\end{aligned}$$

respectively, with $\bar{\mathbf{a}}_d = \frac{1}{N}\sum_{i=1}^N\gamma_{i,d}\boldsymbol{\varepsilon}_i$ and where we note that $\|\bar{\mathbf{a}}_d\| = O_p(\sqrt{T}N^{-1/2})$ by the independence of $\gamma_{i,d}$ and $\boldsymbol{\varepsilon}_i$ from Ass.1 and 3. As such, with (B-18)-(B-20) and (B-22)

$$\left\|\frac{\bar{\mathbf{a}}_d'\mathbf{F}^*}{T}\right\| \leq \left\|\frac{\bar{\mathbf{a}}_d'\check{\mathbf{F}}}{T}\right\|\left\|\ddot{\mathbf{P}}\right\|\|\mathbf{R}\|\|\mathbf{N}\|\|\mathbf{S}_m\| = O_p\left(\frac{1}{\sqrt{NT}}\right),$$

$$\left\|\frac{\bar{\mathbf{a}}_d'\bar{\mathbf{U}}_m^0}{T}\right\| \leq \left\|\frac{\bar{\mathbf{a}}_d'\ddot{\mathbf{U}}}{T}\right\| \|\mathbf{R}\| \|\mathbf{N}\| \|\mathbf{S}_m\| = O_p\left(\frac{1}{N}\right),$$
$$\left\|\frac{\bar{\mathbf{a}}_d'\bar{\mathbf{U}}_{-m}^0}{T}\right\| \leq \sqrt{N}\left\|\frac{\bar{\mathbf{a}}_d'\ddot{\mathbf{U}}}{T}\right\| \|\mathbf{R}\| \|\mathbf{N}\| \|\mathbf{S}_{-m}\| = O_p\left(\frac{1}{\sqrt{N}}\right),$$
$$\left\|\frac{\bar{\mathbf{a}}_d'\bar{\mathbf{U}}_m}{T}\right\| \leq \left\|\frac{\bar{\mathbf{a}}_d'\ddot{\mathbf{U}}}{T}\right\| \|\mathbf{R}_0\| \|\mathbf{T}\| \|\mathbf{B}_m\| = O_p\left(\frac{1}{N}\right),$$

since $\left\|\frac{\bar{\mathbf{a}}_d'\check{\mathbf{F}}}{T}\right\| = O_p((NT)^{-1/2})$ by independence of $\bar{\mathbf{a}}_d$ and $\check{\mathbf{F}}$, and because $\left\|\frac{\bar{\mathbf{a}}_d'\ddot{\mathbf{U}}}{T}\right\| \leq T^{-1}\|\bar{\mathbf{a}}_d\| \left\|\ddot{\mathbf{U}}\right\| = O_p(N^{-1})$ since $\left\|\ddot{\mathbf{U}}\right\| = O_p(\sqrt{T}N^{-1/2})$. As such, $\left\|\frac{1}{N}\sum_{i=1}^N(\boldsymbol{\gamma}_i' \otimes T^{-1}\boldsymbol{\varepsilon}_i'\mathbf{Q}_0)\right\| = O_p(N^{-1/2})$ and $\left\|\frac{1}{N}\sum_{i=1}^N\left(\boldsymbol{\gamma}_i' \otimes T^{-1}\boldsymbol{\varepsilon}_i'\bar{\mathbf{U}}_m\right)\right\| = O_p(N^{-1})$. Thus, inserting also $\left\|T^{-1}\mathbf{Q}_0'\bar{\mathbf{U}}_m\right\| = O_p(N^{-1/2})$ by Lemma 10 gives

$$\left\|\frac{1}{NT}\sum_{i=1}^N \boldsymbol{\varepsilon}_i'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\gamma}_i\right\| = O_p\left(\frac{1}{N}\right), \tag{D-47}$$

and therefore,

$$\widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta}_0) = \frac{T}{T-c}(\boldsymbol{\delta}-\boldsymbol{\delta}_0)'\widehat{\boldsymbol{\Sigma}}(\boldsymbol{\delta}-\boldsymbol{\delta}_0) + \frac{T}{T-c}\frac{1}{NT}\sum_{i=1}^N \boldsymbol{\varepsilon}_i'\mathbf{M}\boldsymbol{\varepsilon}_i + O_p\left(\frac{1}{N}\right) + O_p\left(\frac{1}{T}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right).$$

The final term in this expression we can decompose as

$$\frac{1}{NT}\sum_{i=1}^N \boldsymbol{\varepsilon}_i'\mathbf{M}\boldsymbol{\varepsilon}_i = \frac{1}{N}\sum_{i=1}^N \frac{\boldsymbol{\varepsilon}_i'\boldsymbol{\varepsilon}_i}{T} - \frac{1}{NT}\sum_{i=1}^N \boldsymbol{\varepsilon}_i'\mathbf{H}\boldsymbol{\varepsilon}_i.$$

Consider the last term and recall from Lemma 10 that $\|T^{-1}\boldsymbol{\varepsilon}_i'\mathbf{Q}_0\| = O_p(T^{-1/2})$. Note that we can write with $\mathbf{q}_{0,t}$ denoting the $t-th$ row of $\mathbf{Q}_0$ and $\bar{\varepsilon}_{t,s} = \frac{1}{N}\sum_{i=1}^N \varepsilon_{it}\varepsilon_{is}$, with notably $\bar{\varepsilon}_{t,s} = O_p(N^{-1/2})$ for $s \neq t$ and $\bar{\varepsilon}_{t,t} = \sigma_\varepsilon^2 + O_p(N^{-1/2})$,

$$\begin{aligned}
\frac{1}{NT}\sum_{i=1}^N \boldsymbol{\varepsilon}_i'\mathbf{H}\boldsymbol{\varepsilon}_i &= \frac{1}{N}\sum_{i=1}^N \frac{\boldsymbol{\varepsilon}_i'\mathbf{Q}_0}{T}\left(\frac{\mathbf{Q}_0'\mathbf{Q}_0}{T}\right)^{-1}\frac{\mathbf{Q}_0'\boldsymbol{\varepsilon}_i}{T} = \frac{1}{NT^2}\sum_{i=1}^N \boldsymbol{\varepsilon}_i'\mathbf{Q}_0\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1}\mathbf{Q}_0'\boldsymbol{\varepsilon}_i, \\
&= \frac{1}{NT^2}\sum_{i=1}^N\sum_{t=1}^T\sum_{s=1}^T \varepsilon_{it}\varepsilon_{is}\mathbf{q}_{0,t}'\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1}\mathbf{q}_{0,s}, \\
&= \frac{1}{NT^2}\sum_{t=1}^T\sum_{s=1}^T \bar{\varepsilon}_{t,s}\mathbf{q}_{0,t}'\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1}\mathbf{q}_{0,s}, \\
&= \frac{1}{T^2}\sigma_\varepsilon^2 tr(\mathbf{Q}_0\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1}\mathbf{Q}_0') + \frac{1}{NT^2}\sum_{t=1}^T(\bar{\varepsilon}_{t,t}-\sigma_\varepsilon^2)\mathbf{q}_{0,t}'\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1}\mathbf{q}_{0,t} \\
&\quad + \frac{1}{NT^2}\sum_{t=1}^T\sum_{s\neq t}^T \bar{\varepsilon}_{t,s}\mathbf{q}_{0,t}'\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1}\mathbf{q}_{0,s},
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{T^2}\sigma_\varepsilon^2 Tc + O_p(N^{-1/2}T^{-1}), \\
&= \frac{c}{T}\sigma_\varepsilon^2 + O_p(N^{-1/2}T^{-1}),
\end{aligned}
$$

and also for the first term

$$
\begin{aligned}
\frac{1}{N}\sum_{i=1}^{N}\frac{\boldsymbol{\varepsilon}_i'\boldsymbol{\varepsilon}_i}{T} &= \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\varepsilon_{it}^2 = \sigma_\varepsilon^2 + \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}(\varepsilon_{it}^2 - \sigma_\varepsilon^2) = \sigma_\varepsilon^2 + \frac{1}{T}\sum_{t=1}^{T}(\bar{\varepsilon}_{t,t} - \sigma_\varepsilon^2), \\
&= \sigma_\varepsilon^2 + O_p((NT)^{-1/2}),
\end{aligned}
$$

which gives, combined into the expression above,

$$
\frac{T}{T-c}\left[\frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\varepsilon}_i'\mathbf{M}\boldsymbol{\varepsilon}_i\right] = \frac{T}{T-c}\left[\sigma_\varepsilon^2 - \frac{c}{T}\sigma_\varepsilon^2\right] + O_p\left(\frac{1}{\sqrt{NT}}\right) = \sigma_\varepsilon^2 + O_p\left(\frac{1}{\sqrt{NT}}\right). \quad \text{(D-48)}
$$

Finally, since $\left\|\widehat{\boldsymbol{\Sigma}}\right\| = O_p(1)$ and making use of $\frac{T}{T-c} \to 1$ we conclude that

$$
\widehat{\sigma}_\varepsilon^2\left(\boldsymbol{\delta}_0\right) = \sigma_\varepsilon^2 + (\boldsymbol{\delta} - \boldsymbol{\delta}_0)'\widehat{\boldsymbol{\Sigma}}(\boldsymbol{\delta} - \boldsymbol{\delta}_0) + O_p(N^{-1}) + O_p(T^{-1}) + O_p((NT)^{-1/2}),
$$

which is the first result stated in the lemma.

It remains to consider $\boldsymbol{\delta}_0 = \boldsymbol{\delta}$. Clearly, in this case

$$
\begin{aligned}
\widehat{\sigma}_\varepsilon^2\left(\boldsymbol{\delta}\right) &= \frac{T}{T-c}\frac{1}{NT}\sum_{i=1}^{N}\left\|\mathbf{M}(\boldsymbol{\varepsilon}_i - \bar{\mathbf{U}}_m\boldsymbol{\gamma}_i)\right\|^2, \\
&= \frac{T}{T-c}\left[\frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\varepsilon}_i'\mathbf{M}\boldsymbol{\varepsilon}_i - 2\frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\varepsilon}_i'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\gamma}_i + \frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\gamma}_i'\bar{\mathbf{U}}_m'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\gamma}_i\right],
\end{aligned}
$$

and therefore, substituting in earlier results such as (D-47) and (D-48)

$$
\widehat{\sigma}_\varepsilon^2\left(\boldsymbol{\delta}\right) = \sigma_\varepsilon^2 + O_p(N^{-1}) + O_p((NT)^{-1/2}).
$$

This proves the lemma.

Proof of Lemma 17

Consider $\mathbf{A}^c = \mathbf{A}^\varepsilon + \frac{1}{T}\widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta})\boldsymbol{\upsilon}$ evaluated at $\boldsymbol{\delta}_0 = \boldsymbol{\delta}$, where $\boldsymbol{\upsilon}$ denotes $\boldsymbol{\upsilon}(\rho_0, \mathbf{H})$ evaluated at $\rho_0 = \rho$. Making use of the notation introduced in Lemma 15, specifically (D-39), we can decompose it as follows

$$
\begin{aligned}
\mathbf{A}^c &= \mathbf{A}^\varepsilon + \frac{1}{T}\widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta})\boldsymbol{\upsilon}, \\
&= \mathbf{A}_1^\varepsilon + \mathbf{A}_2^\varepsilon - \mathbf{A}_3^\varepsilon + \frac{1}{T}\widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta})\boldsymbol{\upsilon},
\end{aligned}
$$

$$= \mathbf{A}_1^{\varepsilon} + \mathbf{A}_{21}^{\varepsilon} - \mathbf{A}_3^{\varepsilon} - \frac{1}{NT}\sum_{i=1}^{N} \boldsymbol{\epsilon}_i' \mathbf{H} \boldsymbol{\varepsilon}_i + \frac{1}{T}\widehat{\sigma}_{\varepsilon}^2(\boldsymbol{\delta})\boldsymbol{\upsilon},$$
$$= \mathbf{A}_1^{\varepsilon} + \mathbf{A}_{21}^{\varepsilon} - \mathbf{A}_3^{\varepsilon} - \mathbf{A}_0^{c},$$

where in the final equality we substituted in $\mathbf{A}_{21}^{\varepsilon} = \frac{1}{NT}\sum_{i=1}^{N} \boldsymbol{\epsilon}_i' \boldsymbol{\varepsilon}_i$ of (D-42) and defined

$$\mathbf{A}_0^{c} = \frac{1}{NT}\sum_{i=1}^{N} \boldsymbol{\epsilon}_i' \mathbf{H} \boldsymbol{\varepsilon}_i - \frac{1}{T}\widehat{\sigma}_{\varepsilon}^2(\boldsymbol{\delta})\boldsymbol{\upsilon}.$$

For this term we can write

$$\mathbf{A}_0^{c} = \frac{1}{NT}\sum_{i=1}^{N} \left[\boldsymbol{\epsilon}_i' \mathbf{H} \boldsymbol{\varepsilon}_i - \sigma_{\varepsilon}^2 \boldsymbol{\upsilon}\right] - \frac{1}{T}\left[\widehat{\sigma}_{\varepsilon}^2(\boldsymbol{\delta}) - \sigma_{\varepsilon}^2\right]\boldsymbol{\upsilon}.$$

Recall that $T^{-1}\boldsymbol{\epsilon}_i'\mathbf{H}\boldsymbol{\varepsilon}_i = T^{-1}\boldsymbol{\epsilon}_i'\mathbf{Q}_0\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1}T^{-1}\mathbf{Q}_0'\boldsymbol{\varepsilon}_i$ and that from Lemma 10 $\|T^{-1}\boldsymbol{\epsilon}_i'\mathbf{Q}_0\|$ and $\|T^{-1}\boldsymbol{\varepsilon}_i'\mathbf{Q}_0\|$ are $O_p(T^{-1/2})$. Also denote with $h_{t,s}$ the element on row t and column s of $\mathbf{H}$ and $\bar{\boldsymbol{\epsilon}}_{t,s} = \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\epsilon}_{it}\varepsilon_{is}$ such that $\left[\bar{\boldsymbol{\epsilon}}_{t,s} - \sigma_{\varepsilon}^2\mathbf{q}_1\rho^{t-1-s}\mathbb{1}_{(t-1\geq s)}\right] = O_p(N^{-1/2})$ for all t and s, where $\mathbb{1}_a$ denotes the indicator function that returns one if the condition a is true, and zero otherwise. This gives

$$\begin{aligned}\frac{1}{NT}\sum_{i=1}^{N}\left[\boldsymbol{\epsilon}_i'\mathbf{H}\boldsymbol{\varepsilon}_i - \sigma_{\varepsilon}^2\boldsymbol{\upsilon}\right] &= \frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T} h_{t,s}\frac{1}{N}\sum_{i=1}^{N}\left[\boldsymbol{\epsilon}_{it}\varepsilon_{is} - \sigma_{\varepsilon}^2\mathbf{q}_1\rho^{t-1-s}\mathbb{1}_{(t-1\geq s)}\right],\\ &= \frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T} h_{t,s}\left[\bar{\boldsymbol{\epsilon}}_{t,s} - \sigma_{\varepsilon}^2\mathbf{q}_1\rho^{t-1-s}\mathbb{1}_{(t-1\geq s)}\right] = O_p(N^{-1/2}T^{-1}).\end{aligned}$$

Second, note that the function $\boldsymbol{\upsilon} = \upsilon(\rho, \mathbf{H})\mathbf{q}_1 = tr(\mathbf{H}\mathbf{L}\mathbf{J}^{-1}(\rho))\mathbf{q}_1$ calculates the sum of the lower triangular elements of $\mathbf{H}$ weighted by the columns of $\mathbf{J}^{-1}(\rho)$, with $\mathbf{J}(\rho)$ a $T\times T$ matrix with ones on the main diagonal, $-\rho$ on the first lower sub-diagonal, and zeros on all other entries, and $\mathbf{L}$ the $T \times T$ lag operator with ones on the first lower sub-diagonal and zeros on all other entries. We then have that $\|\boldsymbol{\upsilon}\| = O_p(1)$ since $\rho < 1$ under Ass.5 such that each column of the weighting matrix $\mathbf{J}^{-1}(\rho)$ contains an exponentially decaying sequence and its row and column norms are bounded by a finite constant which is independent of T.

Therefore, also substituting in $\|\widehat{\sigma}_{\varepsilon}^2(\boldsymbol{\delta}) - \sigma_{\varepsilon}^2\| = O_p(N^{-1}) + O_p((NT)^{-1/2})$ by (D-27) of Lemma 16 gives

$$\|\mathbf{A}_0^{c}\| \leq \left\|\frac{1}{NT}\sum_{i=1}^{N}\left[\boldsymbol{\epsilon}_i'\mathbf{H}\boldsymbol{\varepsilon}_i - \sigma_{\varepsilon}^2\boldsymbol{\upsilon}\right]\right\| + \frac{1}{T}\left\|\widehat{\sigma}_{\varepsilon}^2(\boldsymbol{\delta}) - \sigma_{\varepsilon}^2\right\|\|\boldsymbol{\upsilon}\| = O_p(N^{-1/2}T^{-1}).$$

such that from the respective results in eqs.(D-40), (D-43) and (D-45) of the proof for Lemma 15 follows

$$\|\mathbf{A}^{c}\| \leq \|\mathbf{A}_1^{\varepsilon}\| + \|\mathbf{A}_{21}^{\varepsilon}\| + \|\mathbf{A}_3^{\varepsilon}\| + \|\mathbf{A}_0^{c}\| = O_p(N^{-1}) + O_p((NT)^{-1/2}).$$

Also, letting $\mathbf{A}_{NT}^{c} = \sqrt{NT}\mathbf{A}^{c}$ and imposing that $T/N \to M < \infty$ yields

$$\mathbf{A}_{NT}^{c} = \mathbf{A}_{NT,1}^{\varepsilon} + \mathbf{A}_{NT,21}^{\varepsilon} - \mathbf{A}_{NT,3}^{\varepsilon} + O_p(T^{-1/2}),$$

with $\mathbf{A}^{\varepsilon}_{NT,1}$ and $\mathbf{A}^{\varepsilon}_{NT,3}$ defined in (D-41) and (D-46), respectively, and where $\mathbf{A}^{\varepsilon}_{NT,21} = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{\boldsymbol{\epsilon}_i'\boldsymbol{\varepsilon}_i}{\sqrt{T}}$. Substituting in the respective definitions gives the result stated in the lemma.

Next, consider the moment vector evaluated at any $\boldsymbol{\delta}_0 \neq \boldsymbol{\delta}$ such that $\|\boldsymbol{\delta}-\boldsymbol{\delta}_0\| < \infty$,

$$\widetilde{\mathbf{A}}^c(\boldsymbol{\delta}_0) = \mathbf{A}^{\varepsilon} + T^{-1}\widehat{\sigma}^2_{\varepsilon}(\boldsymbol{\delta}_0)\boldsymbol{\upsilon}(\rho_0) = \mathbf{A}^{\varepsilon}_1 + \mathbf{A}^{\varepsilon}_{21} - \mathbf{A}^{\varepsilon}_3 - \widetilde{\mathbf{A}}^c_0(\boldsymbol{\delta}_0),$$

with $\widetilde{\mathbf{A}}^c_0(\boldsymbol{\delta}_0) = \frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\epsilon}_i'\mathbf{H}\boldsymbol{\varepsilon}_i - \frac{1}{T}\widehat{\sigma}^2_{\varepsilon}(\boldsymbol{\delta}_0)\boldsymbol{\upsilon}(\rho_0)$ and $\boldsymbol{\upsilon}(\rho_0) = \upsilon(\rho_0,\mathbf{H})\mathbf{q}_1$, and, as before by eqs.(D-40), (D-43) and (D-45)

$$\widetilde{\mathbf{A}}^c(\boldsymbol{\delta}_0) = -\widetilde{\mathbf{A}}^c_0(\boldsymbol{\delta}_0) + O_p(N^{-1}) + O_p((NT)^{-1/2}).$$

We get using the same steps as above and substituting in earlier results

$$\begin{aligned}\widetilde{\mathbf{A}}^c_0(\boldsymbol{\delta}_0) &= \frac{1}{NT}\sum_{i=1}^{N}\left[\boldsymbol{\epsilon}_i'\mathbf{H}\boldsymbol{\varepsilon}_i - \sigma^2_{\varepsilon}\boldsymbol{\upsilon}\right] - \frac{1}{T}\left[\widehat{\sigma}^2_{\varepsilon}(\boldsymbol{\delta}_0) - \sigma^2_{\varepsilon}\right]\boldsymbol{\upsilon}(\rho_0) - \frac{1}{T}\sigma^2_{\varepsilon}\left[\boldsymbol{\upsilon}(\rho_0) - \boldsymbol{\upsilon}\right],\\ &= -\frac{1}{T}\left[\widehat{\sigma}^2_{\varepsilon}(\boldsymbol{\delta}_0) - \sigma^2_{\varepsilon}\right]\boldsymbol{\upsilon}(\rho_0) - \frac{1}{T}\sigma^2_{\varepsilon}\left[\boldsymbol{\upsilon}(\rho_0) - \boldsymbol{\upsilon}\right] + O_p(N^{-1/2}T^{-1}).\end{aligned}$$

In turn, substituting in (D-26) of Lemma 16 returns

$$\widetilde{\mathbf{A}}^c_0(\boldsymbol{\delta}_0) = -\frac{1}{T}(\boldsymbol{\delta}-\boldsymbol{\delta}_0)'\widehat{\boldsymbol{\Sigma}}(\boldsymbol{\delta}-\boldsymbol{\delta}_0)\boldsymbol{\upsilon}(\rho_0) - \frac{1}{T}\sigma^2_{\varepsilon}\left[\boldsymbol{\upsilon}(\rho_0) - \boldsymbol{\upsilon}\right] + O_p(N^{-1/2}T^{-1}),$$

and therefore

$$\widetilde{\mathbf{A}}^c(\boldsymbol{\delta}_0) = \frac{1}{T}(\boldsymbol{\delta}-\boldsymbol{\delta}_0)'\widehat{\boldsymbol{\Sigma}}(\boldsymbol{\delta}-\boldsymbol{\delta}_0)\boldsymbol{\upsilon}(\rho_0) + \frac{1}{T}\sigma^2_{\varepsilon}\left[\boldsymbol{\upsilon}(\rho_0) - \boldsymbol{\upsilon}\right] + O_p(N^{-1}) + O_p((NT)^{-1/2}),$$

which ends the proof.

Proof of Lemma 18

Consider, since $\mathbf{M}\bar{\mathbf{w}} = \mathbf{0}$,

$$\begin{aligned}\widehat{\boldsymbol{\Sigma}} &= \frac{1}{NT}\sum_{i=1}^{N}\mathbf{w}_i'\mathbf{M}\mathbf{w}_i = \frac{1}{NT}\sum_{i=1}^{N}\left(\mathbf{w}_i - \bar{\mathbf{w}}\right)'\mathbf{M}\left(\mathbf{w}_i - \bar{\mathbf{w}}\right),\\ &= \frac{1}{NT}\sum_{i=1}^{N}\left(\check{\mathbf{F}}\widetilde{\mathbf{P}}_i\mathbf{S}_w + \boldsymbol{\epsilon}_i - \bar{\boldsymbol{\epsilon}}\right)'\mathbf{M}\left(\check{\mathbf{F}}\widetilde{\mathbf{P}}_i\mathbf{S}_w + \boldsymbol{\epsilon}_i - \bar{\boldsymbol{\epsilon}}\right),\end{aligned}$$

where noting that $\boldsymbol{\epsilon}_i = \ddot{\mathbf{U}}_i\mathbf{S}_w$ it is easily seen from Lemmas 10 and 11

$$\left\|\frac{1}{NT}\sum_{i=1}^{N}\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\check{\mathbf{F}}'\mathbf{M}\bar{\boldsymbol{\epsilon}}\right\| = O_p\left(\frac{1}{N}\right), \left\|\frac{1}{NT}\sum_{i=1}^{N}\bar{\boldsymbol{\epsilon}}'\mathbf{M}\bar{\boldsymbol{\epsilon}}\right\| = O_p\left(\frac{1}{N}\right), \left\|\frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\epsilon}_i'\mathbf{M}\bar{\boldsymbol{\epsilon}}\right\| = O_p\left(\frac{1}{N}\right).$$

Also, from (D-40)

$$\left\|\frac{1}{NT}\sum_{i=1}^{N}\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\check{\mathbf{F}}'\mathbf{M}\boldsymbol{\epsilon}_i\right\| = O_p\left(\frac{1}{\sqrt{NT}}\right),$$

and making use of Lemma 10 and Ass.1 and 5,

$$\begin{aligned}\frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\epsilon}_i'\mathbf{M}\boldsymbol{\epsilon}_i &= \frac{1}{N}\sum_{i=1}^{N}\frac{\boldsymbol{\epsilon}_i'\boldsymbol{\epsilon}_i}{T} - \frac{1}{N}\sum_{i=1}^{N}\frac{\boldsymbol{\epsilon}_i'\mathbf{Q}_0}{T}\left(\frac{\mathbf{Q}_0'\mathbf{Q}_0}{T}\right)^{-1}\frac{\mathbf{Q}_0'\boldsymbol{\epsilon}_i}{T} = \frac{1}{N}\sum_{i=1}^{N}\frac{\boldsymbol{\epsilon}_i'\boldsymbol{\epsilon}_i}{T} + O_p\left(\frac{1}{T}\right),\\ &= \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}} + O_p(T^{-1}) + O_p((NT)^{-1/2}),\end{aligned}$$

with $\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}} = E(\boldsymbol{\epsilon}_i'\boldsymbol{\epsilon}_i/T)$. Next up is

$$\frac{1}{NT}\sum_{i=1}^{N}\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\check{\mathbf{F}}'\mathbf{M}\check{\mathbf{F}}\widetilde{\mathbf{P}}_i\mathbf{S}_w = \frac{1}{NT}\sum_{i=1}^{N}\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\check{\mathbf{F}}'\mathbf{M}_{\mathbf{F}}\check{\mathbf{F}}\widetilde{\mathbf{P}}_i\mathbf{S}_w - \frac{1}{NT}\sum_{i=1}^{N}\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\check{\mathbf{F}}'(\mathbf{M}_{\mathbf{F}}-\mathbf{M})\check{\mathbf{F}}\widetilde{\mathbf{P}}_i\mathbf{S}_w,$$

where for the second term, defining $\widehat{\boldsymbol{\Sigma}}_{\widetilde{\mathbf{P}}} = \left[\frac{1}{N}\sum_{i=1}^{N}\left(\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\otimes\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right)\right]$,

$$\left\|\frac{1}{NT}\sum_{i=1}^{N}\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\check{\mathbf{F}}'(\mathbf{M}_{\mathbf{F}}-\mathbf{M})\check{\mathbf{F}}\widetilde{\mathbf{P}}_i\mathbf{S}_w\right\| \leq \left\|\widehat{\boldsymbol{\Sigma}}_{\widetilde{\mathbf{P}}}\right\|\left\|\check{\mathbf{F}}'(\mathbf{M}_{\mathbf{F}}-\mathbf{M})\check{\mathbf{F}}\right\|,$$

for which $\left\|\widehat{\boldsymbol{\Sigma}}_{\widetilde{\mathbf{P}}}\right\| = O_p(1)$ by Ass.3 and the norm in the end can be decomposed into 5 parts by (D-1). Using the shorthand $\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}} = [T^{-1}\mathbf{Q}_0'\mathbf{Q}_0]$, $\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^*} = [T^{-1}(\mathbf{F}^*)'\mathbf{F}^*]$ and $\widehat{\boldsymbol{\Sigma}}_{\mathbf{u}_{-m}^0} = [T^{-1}(\bar{\mathbf{U}}_{-m}^0)'\bar{\mathbf{U}}_{-m}^0]$ we get for each respective component

$$\begin{aligned}\|\mathbf{K}_1\| &\leq \left\|\frac{\check{\mathbf{F}}'\bar{\mathbf{U}}_{-m}^0}{T}\right\|\left\|\widehat{\boldsymbol{\Sigma}}_{\mathbf{u}_{-m}^0}^{-1}\right\|\left\|\frac{(\bar{\mathbf{U}}_{-m}^0)'\check{\mathbf{F}}}{T}\right\| = O_p\left(\frac{1}{T}\right),\\ \|\mathbf{K}_2\| &\leq \left\|\frac{\check{\mathbf{F}}'\bar{\mathbf{U}}_m^0}{T}\right\|\left\|\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^*}^{-1}\right\|\left\|\frac{(\bar{\mathbf{U}}_m^0)'\check{\mathbf{F}}}{T}\right\| = O_p\left(\frac{1}{NT}\right),\\ \|\mathbf{K}_3\| &\leq \left\|\frac{\check{\mathbf{F}}'\mathbf{F}^*}{T}\right\|\left\|\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^*}^{-1}\right\|\left\|\frac{(\bar{\mathbf{U}}_m^0)'\check{\mathbf{F}}}{T}\right\| = O_p\left(\frac{1}{\sqrt{NT}}\right),\\ \|\mathbf{K}_4\| &\leq \left\|\frac{\check{\mathbf{F}}'\bar{\mathbf{U}}_m^0}{T}\right\|\left\|\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^*}^{-1}\right\|\left\|\frac{(\mathbf{F}^*)'\check{\mathbf{F}}}{T}\right\| = O_p\left(\frac{1}{\sqrt{NT}}\right),\\ \|\mathbf{K}_5\| &\leq \left\|\frac{\check{\mathbf{F}}'\mathbf{Q}_0}{T}\right\|\left\|\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1}-\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}_u^+}^{-1}\right\|\left\|\frac{\mathbf{Q}_0'\check{\mathbf{F}}}{T}\right\| = O_p\left(\frac{1}{\sqrt{N}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right),\end{aligned}$$

which makes use of (B-18)-(B-20) and Lemmas 9, 10 and 13. As such, $\left\|\check{\mathbf{F}}'(\mathbf{M}_{\mathbf{F}}-\mathbf{M})\check{\mathbf{F}}\right\| = O_p(N^{-1/2}) + O_p(T^{-1/2})$ and

$$\frac{1}{NT}\sum_{i=1}^{N}\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\check{\mathbf{F}}'\mathbf{M}\check{\mathbf{F}}\widetilde{\mathbf{P}}_i\mathbf{S}_w = \frac{1}{NT}\sum_{i=1}^{N}\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\check{\mathbf{F}}'\mathbf{M}_{\mathbf{F}}\check{\mathbf{F}}\widetilde{\mathbf{P}}_i\mathbf{S}_w + O_p\left(\frac{1}{\sqrt{N}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right).$$

Here we have, recalling $\widehat{\boldsymbol{\Sigma}}_{\check{\mathbf{F}}} = T^{-1}\check{\mathbf{F}}'\check{\mathbf{F}}$ and using (B-18) and Lemma 12,

$$\frac{1}{NT}\sum_{i=1}^{N}\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\check{\mathbf{F}}'\mathbf{M}_{\mathbf{F}}\check{\mathbf{F}}\widetilde{\mathbf{P}}_i\mathbf{S}_w$$

$$
\begin{aligned}
&= \frac{1}{N}\sum_{i=1}^{N} \mathbf{S}_w' \widetilde{\mathbf{P}}_i' \widehat{\mathbf{\Sigma}}_{\check{\mathbf{F}}} \widetilde{\mathbf{P}}_i \mathbf{S}_w - \frac{1}{N}\sum_{i=1}^{N} \mathbf{S}_w' \widetilde{\mathbf{P}}_i' \widehat{\mathbf{\Sigma}}_{\check{\mathbf{F}}} \ddot{\mathbf{P}} \mathbf{R} \mathbf{N} \mathbf{S}_m \widehat{\mathbf{\Sigma}}_{\mathbf{F}^*}^{-1} \mathbf{S}_m' \mathbf{N}' \mathbf{R}' \ddot{\mathbf{P}}' \widehat{\mathbf{\Sigma}}_{\check{\mathbf{F}}} \widetilde{\mathbf{P}}_i \mathbf{S}_w, \\
&= \frac{1}{N}\sum_{i=1}^{N} \mathbf{S}_w' \widetilde{\mathbf{P}}_i' \mathbf{V}^{\mathbf{F}} \widetilde{\mathbf{P}}_i \mathbf{S}_w + O_p(N^{-1/2}) + O_p(T^{-1/2}), \\
&= (vec(\mathbf{I}_{k_w})' \otimes \mathbf{I}_{k_w}) \left(\mathbf{I}_{k_w} \otimes \mathbf{\Sigma}_{\widetilde{\mathbf{P}}} vec(\mathbf{V}^{\mathbf{F}})\right) + O_p(N^{-1/2}) + O_p(T^{-1/2}),
\end{aligned}
$$

where $\mathbf{V}^{\mathbf{F}} = \mathbf{\Sigma}_{\check{\mathbf{F}}} - \mathbf{\Sigma}_{\check{\mathbf{F}}} \mathbf{P} \mathbf{R} \mathbf{N} \mathbf{S}_m \mathbf{\Sigma}_{\mathbf{F}^*}^{-1} \mathbf{S}_m' \mathbf{N}' \mathbf{R}' \mathbf{P}' \mathbf{\Sigma}_{\check{\mathbf{F}}}$ and $\mathbf{\Sigma}_{\widetilde{\mathbf{P}}} = E(\mathbf{S}_w' \widetilde{\mathbf{P}}_i' \otimes \mathbf{S}_w' \widetilde{\mathbf{P}}_i')$.

In conclusion, we have as $(N, T) \to \infty$ that

$$
\widehat{\mathbf{\Sigma}} \longrightarrow^{p} \mathbf{\Sigma}_{\check{\mathbf{F}}\mathbf{P}} + \mathbf{\Sigma}_{\boldsymbol{\epsilon}},
$$

with $\mathbf{\Sigma}_{\check{\mathbf{F}}\mathbf{P}} = (vec(\mathbf{I}_{k_w})' \otimes \mathbf{I}_{k_w}) \left(\mathbf{I}_{k_w} \otimes \mathbf{\Sigma}_{\widetilde{\mathbf{P}}} vec(\mathbf{V}^{\mathbf{F}})\right)$.

D.4 Proof of theorems

D.4.1 Proof of Theorem 2

Consider that the CCEPbc estimator in eq.(21) is equivalent to

$$\widehat{\boldsymbol{\delta}}_{bc} = \underset{\boldsymbol{\delta}_0 \in \chi}{\arg\min} \frac{1}{2} \left\| \boldsymbol{\varphi}(\boldsymbol{\delta}_0) \right\|^2 , \tag{D-49}$$

with $\boldsymbol{\varphi}(\boldsymbol{\delta}_0)$ given by

$$\boldsymbol{\varphi}(\boldsymbol{\delta}_0) = \frac{1}{NT} \sum_{i=1}^{N} \mathbf{w}_i' \mathbf{M} \mathbf{y}_i - \widehat{\boldsymbol{\Sigma}} \boldsymbol{\delta}_0 + \frac{1}{T} \widehat{\sigma}_{\varepsilon}^2(\boldsymbol{\delta}_0) \boldsymbol{\upsilon}(\rho_0),$$

and we will use $\boldsymbol{\upsilon}(\rho_0) = \upsilon(\rho_0, \mathbf{H})\mathbf{q}_1$. With eq.(6) the latter can be reformulated as

$$\begin{aligned}
\boldsymbol{\varphi}(\boldsymbol{\delta}_0) &= \widehat{\boldsymbol{\Sigma}} \left(\boldsymbol{\delta} - \boldsymbol{\delta}_0\right) + \frac{1}{NT} \sum_{i=1}^{N} \mathbf{w}_i' \mathbf{M} (\mathbf{F} \boldsymbol{\gamma}_i + \boldsymbol{\varepsilon}_i) + \frac{1}{T} \widehat{\sigma}_{\varepsilon}^2(\boldsymbol{\delta}_0) \boldsymbol{\upsilon}(\rho_0), \\
&= \widehat{\boldsymbol{\Sigma}} \left(\boldsymbol{\delta} - \boldsymbol{\delta}_0\right) + \mathbf{A}^{\mathbf{F}} + \widetilde{\mathbf{A}}^c(\rho_0),
\end{aligned}$$

where $\mathbf{A}^{\mathbf{F}} = \frac{1}{NT} \sum_{i=1}^{N} \mathbf{w}_i' \mathbf{M} \mathbf{F} \boldsymbol{\gamma}_i$ and $\widetilde{\mathbf{A}}^c(\rho_0) = \frac{1}{NT} \sum_{i=1}^{N} \mathbf{w}_i' \mathbf{M} \boldsymbol{\varepsilon}_i + \frac{1}{T} \widehat{\sigma}_{\varepsilon}^2(\boldsymbol{\delta}_0) \boldsymbol{\upsilon}(\rho_0)$. Under the assumption that χ is compact such that $\|\boldsymbol{\delta} - \boldsymbol{\delta}_0\| < \infty$ it follows from Lemmas 14 and 17

$$\boldsymbol{\varphi}(\boldsymbol{\delta}_0) = \widehat{\boldsymbol{\Sigma}} \left(\boldsymbol{\delta} - \boldsymbol{\delta}_0\right) + \frac{1}{T} (\boldsymbol{\delta} - \boldsymbol{\delta}_0)' \widehat{\boldsymbol{\Sigma}} (\boldsymbol{\delta} - \boldsymbol{\delta}_0) \boldsymbol{\upsilon}(\rho_0) + \frac{1}{T} \sigma_{\varepsilon}^2 \left[\boldsymbol{\upsilon}(\rho_0) - \boldsymbol{\upsilon}\right] + O_p\left(\frac{1}{N}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right),$$

where from here onward we omit the functional dependence of $\boldsymbol{\upsilon}(\cdot)$ when it is evaluated at the population parameter ρ. Also inserting Lemma 18 gives

$$\boldsymbol{\varphi}(\boldsymbol{\delta}_0) = \dot{\boldsymbol{\Sigma}} \left(\boldsymbol{\delta} - \boldsymbol{\delta}_0\right) + \frac{1}{T} (\boldsymbol{\delta} - \boldsymbol{\delta}_0)' \dot{\boldsymbol{\Sigma}} (\boldsymbol{\delta} - \boldsymbol{\delta}_0) \boldsymbol{\upsilon}(\rho_0) + \frac{1}{T} \sigma_{\varepsilon}^2 \left[\boldsymbol{\upsilon}(\rho_0) - \boldsymbol{\upsilon}\right] + o_p(1).$$

Note that $\|\boldsymbol{\upsilon}\| = O_p(1)$ since $|\rho| < 1$ by Ass.5, and if χ in eq.(D-49) is compact and accordingly restricted then $|\rho_0| < 1$ and therefore $\|\boldsymbol{\upsilon}(\rho_0)\| = O_p(1)$. Since also $\left\|\dot{\boldsymbol{\Sigma}}\right\| = O(1)$ it follows that as $(N, T) \to \infty$

$$\boldsymbol{\varphi}(\boldsymbol{\delta}_0) = \dot{\boldsymbol{\Sigma}} \left(\boldsymbol{\delta} - \boldsymbol{\delta}_0\right) + o_p(1),$$

for which the solution in (D-49) is clearly unique at $\boldsymbol{\delta}_0 = \boldsymbol{\delta}$ and therefore

$$\widehat{\boldsymbol{\delta}}_{bc} \longrightarrow^p \boldsymbol{\delta}, \tag{D-50}$$

as $(N, T) \to \infty$.

Define next the following vector evaluated at $\boldsymbol{\delta}_0 = \boldsymbol{\delta}$,

$$\boldsymbol{\psi}_{NT} = \sqrt{NT} \boldsymbol{\varphi}(\boldsymbol{\delta}) = \mathbf{A}_{NT}^{\mathbf{F}} + \mathbf{A}_{NT}^c,$$

with $\mathbf{A}_{NT}^{\mathbf{F}} = \sqrt{NT}\mathbf{A}^{\mathbf{F}}$, $\mathbf{A}_{NT}^{c} = \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{w}_i'\mathbf{M}\boldsymbol{\varepsilon}_i + T^{-1/2}\sqrt{N}\widehat{\sigma}_{\varepsilon}^2(\boldsymbol{\delta})\boldsymbol{\upsilon}$. Assuming that $T/N \to M < \infty$ and combining in this expression Lemmas 14 and 17 gives

$$\boldsymbol{\psi}_{NT} = \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\boldsymbol{\epsilon}_i'\boldsymbol{\varepsilon}_i + \boldsymbol{\Psi}_{\mathbf{F}}vec\left(\frac{\check{\mathbf{F}}'\sqrt{N}\ddot{\mathbf{U}}}{\sqrt{T}}\right) + \boldsymbol{\Psi}_{\boldsymbol{\epsilon}}vec\left[\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\frac{\boldsymbol{\varepsilon}_i'\check{\mathbf{F}}}{\sqrt{T}}\otimes\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right)\right]$$
$$+\sqrt{\frac{T}{N}}\left(\mathbf{b}_0^{\mathbf{F}} - \mathbf{b}_1^{\mathbf{F}} - \mathbf{b}^{\mathbf{U}}\right) + O_p\left(\frac{1}{\sqrt{N}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right),$$

where $\boldsymbol{\Psi}_{\mathbf{F}}, \mathbf{b}_0^{\mathbf{F}}$ and $\mathbf{b}_1^{\mathbf{F}}$ are fixed finite matrices defined below eq.(D-23) and similarly for $\boldsymbol{\Psi}_{\boldsymbol{\epsilon}}$ and $\mathbf{b}^{\mathbf{U}}$, which are stated below (D-29).
Then, recalling that the typical element of $\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\frac{\boldsymbol{\varepsilon}_i'\check{\mathbf{F}}}{\sqrt{T}}\otimes\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right)$ is given by $\frac{\sqrt{N}\bar{\mathbf{a}}_{r,s}'\check{\mathbf{F}}}{\sqrt{T}}$, with $\bar{\mathbf{a}}_{r,s} = \frac{1}{N}\sum_{i=1}^{N}\widetilde{p}_{i,r,s}\boldsymbol{\varepsilon}_i$ and $\widetilde{p}_{i,r,s}$ denoting row r and column s of $\mathbf{S}_w'\widetilde{\mathbf{P}}_i'$, and that $\widetilde{p}_{i,r,s}$, $\boldsymbol{\varepsilon}_i$ and $\check{\mathbf{F}}$ are independent over all i and t, we have given the moment restrictions in Ass.1-3 by a CLT for independent stationary variables as $(N,T) \to \infty$

$$\boldsymbol{\xi}_1 = vec\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\frac{\boldsymbol{\varepsilon}_i'\check{\mathbf{F}}}{\sqrt{T}}\otimes\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right)\right) \xrightarrow{d} \mathbf{n}^{\varepsilon\eta} \overset{d}{=} \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_{\check{\mathbf{F}}\boldsymbol{\varepsilon}}),$$

with $\boldsymbol{\Sigma}_{\check{\mathbf{F}}\boldsymbol{\varepsilon}} = \frac{1}{T}E\left[vec\left(\boldsymbol{\varepsilon}_i'\check{\mathbf{F}}\otimes\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right)vec\left(\boldsymbol{\varepsilon}_i'\check{\mathbf{F}}\otimes\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right)'\right]$. Also

$$\boldsymbol{\xi}_2 = vec\left(\frac{\check{\mathbf{F}}'\sqrt{N}\ddot{\mathbf{U}}}{\sqrt{T}}\right) \xrightarrow{d} \mathbf{n}^{fu} \overset{d}{=} \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{u}}),$$

where $\boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{u}} = \frac{1}{T}E\left[vec\left(\check{\mathbf{F}}'\ddot{\mathbf{U}}_i\right)vec\left(\check{\mathbf{F}}'\ddot{\mathbf{U}}_i\right)'\right]$ and finally

$$\boldsymbol{\xi}_3 = \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\boldsymbol{\epsilon}_i'\boldsymbol{\varepsilon}_i \xrightarrow{d} \mathbf{n}^{\epsilon\varepsilon} \overset{d}{=} \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}\boldsymbol{\varepsilon}}),$$

with $\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}\boldsymbol{\varepsilon}} = \frac{1}{T}E\left[\boldsymbol{\epsilon}_i'\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_i'\boldsymbol{\epsilon}_i'\right]$.

Let $\xi_{1,l}$ be the element on the $l-$th row of $\boldsymbol{\xi}_1$, and similarly for vectors $\boldsymbol{\xi}_2$ and $\boldsymbol{\xi}_3$. Then we have for any l and s

$$Cov(\xi_{1,l}, \xi_{2,s}) = 0, \qquad Cov(\xi_{1,l}, \xi_{3,s}) = 0, \qquad Cov(\xi_{2,l}, \xi_{3,s}) = 0,$$

where the first two statements hold by $E(\widetilde{\mathbf{P}}_i) = \mathbf{0}$ and the independence of $\widetilde{\mathbf{P}}_i$ from $\check{\mathbf{F}}, \boldsymbol{\epsilon}_i$ and $\boldsymbol{\varepsilon}_i$ for all i and t by Ass.3, and the last result holds since $E(\check{\mathbf{F}}) = \mathbf{0}$ from Lemma 1 and the independence of $\check{\mathbf{F}}$ from $\boldsymbol{\epsilon}_i$ and $\boldsymbol{\varepsilon}_i$ by Ass.2. The three normals $\mathbf{n}^{\varepsilon\eta}$, $\mathbf{n}^{fu}$ and $\mathbf{n}^{\epsilon\varepsilon}$ are therefore independent, and as $(N,T) \to \infty$ such that $T/N \to M < \infty$ follows

$$\boldsymbol{\psi}_{NT} \xrightarrow{d} \mathcal{N}\left(\sqrt{T}N^{-1/2}\mathbf{b}_0, \boldsymbol{\Phi}\right), \tag{D-51}$$

where

$$\mathbf{b}_0 = \mathbf{b}_0^{\mathbf{F}} - \mathbf{b}_1^{\mathbf{F}} - \mathbf{b}^{\mathbf{U}}, \tag{D-52}$$

$$\mathbf{\Phi} = \mathbf{\Sigma}_{\boldsymbol{\epsilon}\boldsymbol{\varepsilon}} + \mathbf{\Psi}_{\mathbf{F}}\mathbf{\Sigma}_{\check{\mathbf{F}}\mathbf{u}}\mathbf{\Psi}_{\mathbf{F}}' + \mathbf{\Psi}_{\boldsymbol{\epsilon}}\mathbf{\Sigma}_{\check{\mathbf{F}}\boldsymbol{\varepsilon}}\mathbf{\Psi}_{\boldsymbol{\epsilon}}'. \tag{D-53}$$

Next, recall from Section A.1 that the Jacobian for the CCEPbc estimator in (D-49) evaluated at $\boldsymbol{\delta}_0$ is given by

$$\mathbf{J}_{\mathrm{a}}(\boldsymbol{\delta}_0) = \frac{1}{T}\left[(\boldsymbol{\upsilon}(\rho_0) \otimes \dot{\boldsymbol{\sigma}}') + \left(\widehat{\sigma}_{\varepsilon}^2(\boldsymbol{\delta}_0)\mathbf{q}_1 \otimes \dot{\boldsymbol{\upsilon}}'\right)\right] - \widehat{\mathbf{\Sigma}},$$

with

$$\dot{\boldsymbol{\sigma}} = 2\frac{T}{T-c}\widehat{\mathbf{\Sigma}}\left(\boldsymbol{\delta}_0 - \widehat{\boldsymbol{\delta}}\right), \qquad \dot{\boldsymbol{\upsilon}} = \left(\sum_{t=1}^{T-1}(t-1)\rho_0^{t-2}\sum_{s=t+1}^{T} h_{s,s-t}\right)\mathbf{q}_1.$$

Consider then that as $(N,T) \to \infty$, $\widehat{\mathbf{\Sigma}} \to^p \dot{\mathbf{\Sigma}}$ by Lemma 18, $\widehat{\sigma}_{\varepsilon}^2(\boldsymbol{\delta}) \to^p \sigma_{\varepsilon}^2$ by Lemma 16 and $\|\dot{\boldsymbol{\upsilon}}\| = O_p(1)$. Also, from Lemmas 14, 15 and 18 follows $\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta} = \widehat{\mathbf{\Sigma}}^{-1}(\mathbf{A}^{\varepsilon} + \mathbf{A}^{\mathbf{F}}) \to^p \mathbf{0}_{k_w \times 1}$. Hence, evaluated at $\boldsymbol{\delta}_0 = \boldsymbol{\delta}$

$$\mathbf{\Delta} = \operatorname*{plim}_{(N,T)\to\infty} \mathbf{J}_{\mathrm{a}}(\boldsymbol{\delta}) = -\dot{\mathbf{\Sigma}}. \tag{D-54}$$

As such, with (D-50) and (D-51) we have using standard arguments as in Newey and McFadden (1994), as $(N,T) \to \infty$ such that $T/N \to M < \infty$,

$$\sqrt{NT}(\widehat{\boldsymbol{\delta}}_{bc} - \boldsymbol{\delta}) \xrightarrow{d} -(\mathbf{\Delta}'\mathbf{\Delta})^{-1}\mathbf{\Delta}'\boldsymbol{\psi}_{NT},$$

which implies, given (D-51),

$$\sqrt{NT}\left(\widehat{\boldsymbol{\delta}}_{bc} - \boldsymbol{\delta}\right) \xrightarrow{d} \mathcal{N}\left(-\sqrt{T}N^{-1/2}(\mathbf{\Delta}'\mathbf{\Delta})^{-1}\mathbf{\Delta}'\mathbf{b}_0, (\mathbf{\Delta}'\mathbf{\Delta})^{-1}\mathbf{\Delta}'\mathbf{\Phi}\mathbf{\Delta}(\mathbf{\Delta}'\mathbf{\Delta})^{-1}\right),$$

and in turn, since $\mathbf{\Delta} = -\dot{\mathbf{\Sigma}}$ such that $(\mathbf{\Delta}'\mathbf{\Delta})^{-1}\mathbf{\Delta}' = -\dot{\mathbf{\Sigma}}^{-1}$,

$$\sqrt{NT}\left(\widehat{\boldsymbol{\delta}}_{bc} - \boldsymbol{\delta}\right) \xrightarrow{d} \mathcal{N}\left(\sqrt{\kappa}\mathbf{b}, \dot{\mathbf{\Sigma}}^{-1}\mathbf{\Phi}\dot{\mathbf{\Sigma}}^{-1}\right), \tag{D-55}$$

where $\mathbf{b} = \dot{\mathbf{\Sigma}}^{-1}\mathbf{b}_0$ and we denote $\kappa = T/N$. Letting next $\kappa \to 0$ gives

$$\sqrt{NT}\left(\widehat{\boldsymbol{\delta}}_{bc} - \boldsymbol{\delta}\right) \xrightarrow{d} \mathcal{N}\left(\mathbf{0}_{k_w \times 1}, \dot{\mathbf{\Sigma}}^{-1}\mathbf{\Phi}\dot{\mathbf{\Sigma}}^{-1}\right),$$

which is the result reported in the theorem.

E Additional simulation tables

Table E-1: Monte Carlo results for ρ and β : baseline design with $\rho = 0.4$

		bias				*rmse*				$size_b$			
Estimator	(N,T)	10	20	30	50	10	20	30	50	10	20	30	50
Results for $\widehat{\rho}$													
CCEP	25	-0.198	-0.091	-0.058	-0.035	0.222	0.102	0.067	0.042	0.61	0.55	0.43	0.32
	100	-0.201	-0.093	-0.061	-0.036	0.216	0.098	0.064	0.038	0.92	0.97	0.94	0.84
	500	-0.199	-0.095	-0.061	-0.036	0.213	0.097	0.062	0.037	0.99	1.00	1.00	1.00
	5000	-0.200	-0.094	-0.062	-0.036	0.215	0.096	0.062	0.036	1.00	1.00	1.00	1.00
CCEPbc	25	-0.001	-0.001	0.001	0.000	0.093	0.045	0.033	0.024	0.04	0.06	0.07	0.06
	100	0.000	-0.001	0.000	0.000	0.043	0.022	0.016	0.012	0.04	0.05	0.06	0.05
	500	0.001	0.000	0.000	0.000	0.020	0.010	0.007	0.005	0.04	0.04	0.04	0.04
	5000	0.000	0.000	0.000	0.000	0.006	0.003	0.002	0.002	0.03	0.05	0.05	0.05
CCEPjk	25	0.070	0.015	0.008	0.003	0.267	0.069	0.042	0.028	0.41	0.20	0.13	0.09
	100	0.075	0.015	0.007	0.003	0.233	0.049	0.025	0.014	0.63	0.36	0.19	0.09
	500	0.077	0.017	0.008	0.003	0.228	0.043	0.019	0.008	0.75	0.64	0.44	0.18
	5000	0.079	0.016	0.009	0.003	0.220	0.040	0.017	0.006	0.82	0.80	0.75	0.58
FLSbc	25	-0.085	-0.018	-0.007	-0.003	0.130	0.052	0.037	0.026	0.10	0.03	0.02	0.02
	100	-0.105	-0.026	-0.012	-0.005	0.114	0.034	0.020	0.012	0.60	0.20	0.10	0.06
	500	-0.110	-0.026	-0.012	-0.005	0.112	0.029	0.014	0.007	0.96	0.72	0.39	0.14
	5000	-0.109	-0.026	-0.012	-0.005	0.110	0.028	0.012	0.005	1.00	1.00	0.99	0.77
Results for $\widehat{\beta}$													
CCEP	25	-0.033	-0.010	-0.005	-0.002	0.086	0.048	0.036	0.028	0.07	0.06	0.06	0.06
	100	-0.033	-0.008	-0.004	-0.001	0.055	0.025	0.018	0.014	0.15	0.06	0.06	0.06
	500	-0.033	-0.008	-0.003	-0.001	0.042	0.014	0.009	0.006	0.40	0.13	0.08	0.06
	5000	-0.032	-0.008	-0.004	-0.001	0.040	0.009	0.004	0.002	0.77	0.60	0.27	0.11
CCEPbc	25	0.000	-0.002	-0.002	0.000	0.080	0.047	0.037	0.028	0.04	0.06	0.06	0.05
	100	-0.001	0.000	-0.001	0.000	0.038	0.023	0.018	0.014	0.04	0.05	0.05	0.06
	500	0.000	0.000	0.000	0.000	0.017	0.010	0.008	0.006	0.04	0.05	0.06	0.05
	5000	0.000	0.000	0.000	0.000	0.005	0.003	0.003	0.002	0.03	0.06	0.05	0.05
CCEPjk	25	0.087	0.016	0.006	0.002	0.185	0.060	0.041	0.030	0.35	0.11	0.09	0.07
	100	0.083	0.018	0.008	0.003	0.134	0.035	0.021	0.015	0.54	0.20	0.09	0.07
	500	0.081	0.017	0.008	0.003	0.123	0.025	0.013	0.007	0.74	0.42	0.20	0.09
	5000	0.081	0.018	0.008	0.003	0.119	0.022	0.009	0.003	0.88	0.85	0.76	0.31
FLSbc	25	-0.002	0.009	0.004	0.004	0.085	0.057	0.043	0.032	0.04	0.04	0.03	0.01
	100	-0.016	-0.001	0.000	0.001	0.044	0.024	0.018	0.014	0.08	0.04	0.04	0.04
	500	-0.022	-0.003	-0.001	0.000	0.029	0.011	0.008	0.006	0.32	0.06	0.06	0.05
	5000	-0.021	-0.003	-0.001	0.000	0.025	0.005	0.003	0.002	0.85	0.21	0.08	0.06

Note: See Table 1, but with $\rho = 0.4$ and $\beta = 0.6$

Table E-2: Monte Carlo results for ρ : number and strength of factors ($N = 25$)

	bias	*rmse*	$size_b$	*bias*	*rmse*	$size_b$	*bias*	*rmse*	$size_b$	*bias*	*rmse*	$size_b$	*bias*	*rmse*	$size_b$	*bias*	*rmse*	$size_b$
	one factor																	
	$RI = 1$									$RI = 3$								
	$T = 10$			$T = 20$			$T = 30$			$T = 10$			$T = 20$			$T = 30$		
CCEP	−0.385	0.417	0.90	−0.176	0.188	0.92	−0.109	0.118	0.90	−0.393	0.424	0.92	−0.179	0.191	0.94	−0.111	0.120	0.90
CCEPbc	−0.004	0.151	0.06	0.000	0.064	0.08	0.000	0.038	0.06	0.002	0.153	0.08	0.000	0.063	0.08	0.000	0.038	0.06
CCEPjk	0.027	0.358	0.40	0.037	0.124	0.31	0.028	0.074	0.30	0.017	0.349	0.40	0.039	0.124	0.32	0.030	0.076	0.31
CCEP(+g)	−0.391	0.424	0.83	−0.177	0.190	0.91	−0.110	0.119	0.89	−0.397	0.431	0.84	−0.180	0.193	0.92	−0.112	0.121	0.89
CCEPbc(+g)	−0.003	0.169	0.04	0.001	0.065	0.07	0.000	0.039	0.06	0.002	0.166	0.05	0.001	0.066	0.07	0.000	0.039	0.06
CCEPjk(+g)	-	-	-	0.041	0.133	0.13	0.027	0.077	0.14	-	-	-	0.045	0.133	0.15	0.029	0.078	0.16
FLSbc	−0.261	0.276	0.37	−0.067	0.089	0.04	−0.029	0.054	0.04	−0.233	0.260	0.32	−0.058	0.084	0.03	−0.028	0.054	0.05
	two factors																	
	$RI = 1$									$RI = 3$								
	$T = 10$			$T = 20$			$T = 30$			$T = 10$			$T = 20$			$T = 30$		
CCEP	−0.386	0.417	0.89	−0.176	0.188	0.93	−0.112	0.119	0.89	−0.380	0.412	0.89	−0.161	0.175	0.88	−0.095	0.105	0.78
CCEPbc	0.002	0.155	0.06	−0.001	0.063	0.08	−0.001	0.038	0.05	0.014	0.161	0.08	0.019	0.073	0.12	0.017	0.046	0.09
CCEPjk	0.031	0.356	0.41	0.035	0.123	0.31	0.027	0.074	0.29	0.063	0.359	0.41	0.052	0.139	0.37	0.045	0.086	0.38
CCEP(+g)	−0.424	0.459	0.87	−0.196	0.209	0.94	−0.124	0.132	0.91	−0.401	0.440	0.86	−0.178	0.191	0.91	−0.109	0.117	0.86
CCEPbc(+g)	−0.006	0.173	0.05	−0.001	0.067	0.07	−0.001	0.041	0.06	0.006	0.169	0.05	0.008	0.068	0.08	0.006	0.039	0.05
CCEPjk(+g)	-	-	-	0.033	0.136	0.13	0.026	0.079	0.15	-	-	-	0.042	0.141	0.16	0.039	0.083	0.18
FLSbc	−0.527	0.531	0.85	−0.169	0.184	0.22	−0.063	0.088	0.07	−0.506	0.510	0.81	−0.141	0.164	0.14	−0.053	0.079	0.04

Note: see Table 2 but with $N = 25$.

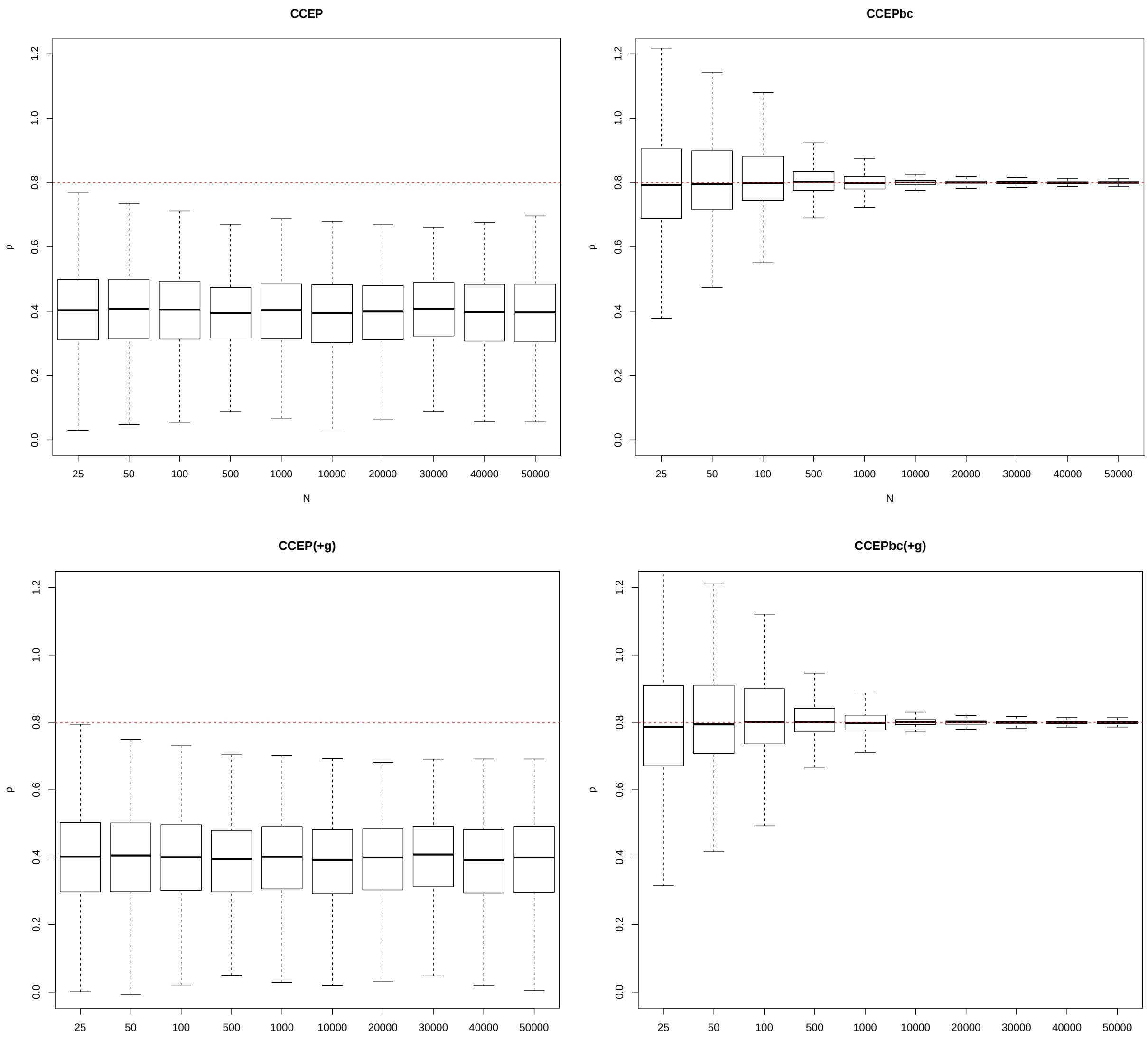
Table E-3: Monte Carlo results for ρ : dynamics in $\mathbf{z}_{it}$ with strong factors ($N = 25$)

	bias	*rmse*	$size_b$	*bias*	*rmse*	$size_b$	*bias*	*rmse*	$size_b$	*bias*	*rmse*	$size_b$
	one factor											
	$T = 10$			$T = 20$			$T = 30$			$T = 50$		
CCEP_p_0(+g)	−0.530	0.563	0.90	−0.236	0.251	0.95	−0.140	0.149	0.94	−0.076	0.081	0.90
CCEP_p_1(+g)	−0.666	0.702	0.81	−0.261	0.279	0.93	−0.148	0.159	0.93	−0.078	0.084	0.89
CCEP_p_T(+g)	-	-	-	−0.321	0.343	0.89	−0.196	0.210	0.88	−0.090	0.096	0.88
CCEPbc_p_0(+g)	−0.014	0.193	0.04	−0.006	0.073	0.06	−0.002	0.040	0.05	−0.002	0.023	0.06
CCEPbc_p_1(+g)	−0.033	0.265	0.03	−0.002	0.084	0.05	−0.001	0.044	0.04	0.000	0.023	0.05
CCEPbc_p_T(+g)	-	-	-	−0.006	0.106	0.03	−0.004	0.061	0.05	−0.001	0.026	0.04
CCEPjk_p_1(+g)	-	-	-	0.123	0.244	0.13	0.084	0.139	0.21	0.034	0.058	0.17
FLSbc	−0.254	0.270	0.47	−0.057	0.081	0.06	−0.026	0.048	0.04	−0.011	0.029	0.04
	two factors											
	$T = 10$			$T = 20$			$T = 30$			$T = 50$		
CCEP_p_0(+g)	−0.560	0.590	0.93	−0.252	0.268	0.97	−0.156	0.164	0.97	−0.085	0.090	0.95
CCEP_p_1(+g)	−0.720	0.750	0.86	−0.294	0.310	0.96	−0.169	0.177	0.96	−0.086	0.091	0.93
CCEP_p_T(+g)	-	-	-	−0.364	0.383	0.94	−0.225	0.239	0.94	−0.101	0.106	0.94
CCEPbc_p_0(+g)	−0.021	0.204	0.05	−0.015	0.075	0.07	−0.010	0.043	0.06	−0.007	0.024	0.06
CCEPbc_p_1(+g)	−0.023	0.275	0.03	−0.005	0.090	0.07	−0.002	0.045	0.05	−0.001	0.024	0.05
CCEPbc_p_T(+g)	-	-	-	−0.009	0.109	0.05	0.000	0.065	0.05	0.000	0.027	0.04
CCEPjk_p_1(+g)	-	-	-	0.120	0.240	0.14	0.090	0.149	0.24	0.044	0.064	0.22
FLSbc	−0.524	0.526	0.89	−0.138	0.162	0.19	−0.047	0.072	0.06	−0.012	0.033	0.03

Note: see Table 4 but with $N = 25$.

F Additional figures

Figure F-1: Monte Carlo results for ρ : Boxplots for CCEP and CCEPbc estimators over N for one normal factor ($m = 1$, $RI = 1$) with $T = 10$

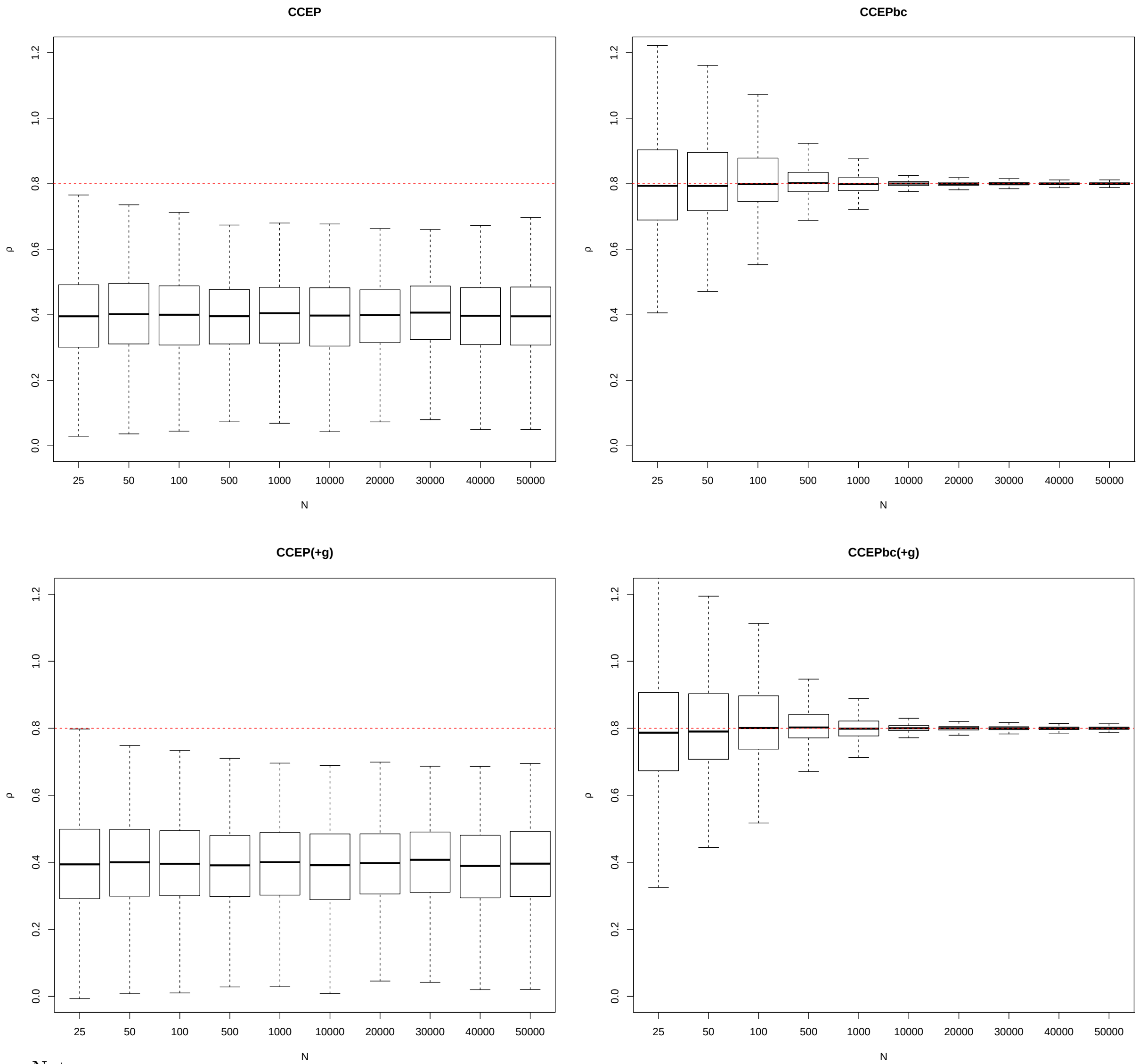


Notes:

(i) Reported are simulation results for estimating ρ in the baseline case for $T = 10$ and $N = 25, 50, 100, ..., 50.000$ (see notes Table 1). The CCEP estimators with a (+g) suffix (lower panel) make use of the $\overline{g}_t$ variable to project out the factors.

(ii) Dotted red lines indicate the population parameter value ($\rho = 0.8$). The boxplot 'whiskers' extend to the most extreme data point which is no more than 1.5 times the interquartile range from the box.

Figure F-2: Monte Carlo results for ρ : Boxplots for CCEP and CCEPbc estimators over N for one strong factor ($m = 1$, $RI = 3$) with $T = 10$.

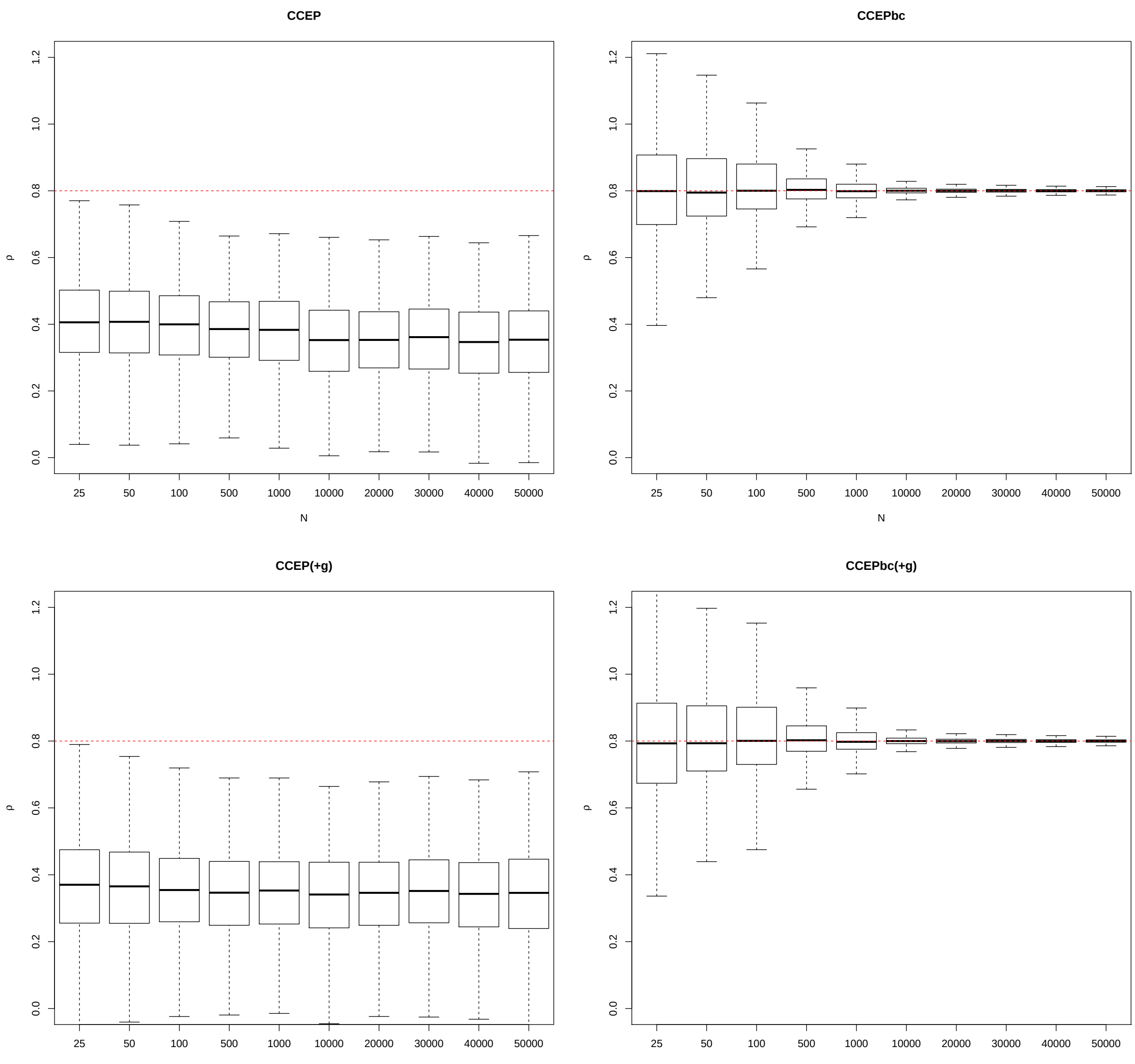


Notes:

(i) Reported are simulation results for estimating $\rho = 0.8$ with $m = 1$ and $RI = 3$ for $N = 25, 50, 100, ..., 50.000$. The CCEP estimators with a (+g) suffix (lower panel) make use of the $\overline{g}_t$ variable to project out the factors.

(ii) Dotted red lines indicate the population parameter value ($\rho = 0.8$). The boxplot 'whiskers' extend to the most extreme data point which is no more than 1.5 times the interquartile range from the box.

Figure F-3: Monte Carlo results for ρ : Boxplots for CCEP and CCEPbc estimators over N for two normal factors ($m = 2$, $RI = 1$) with $T = 10$.

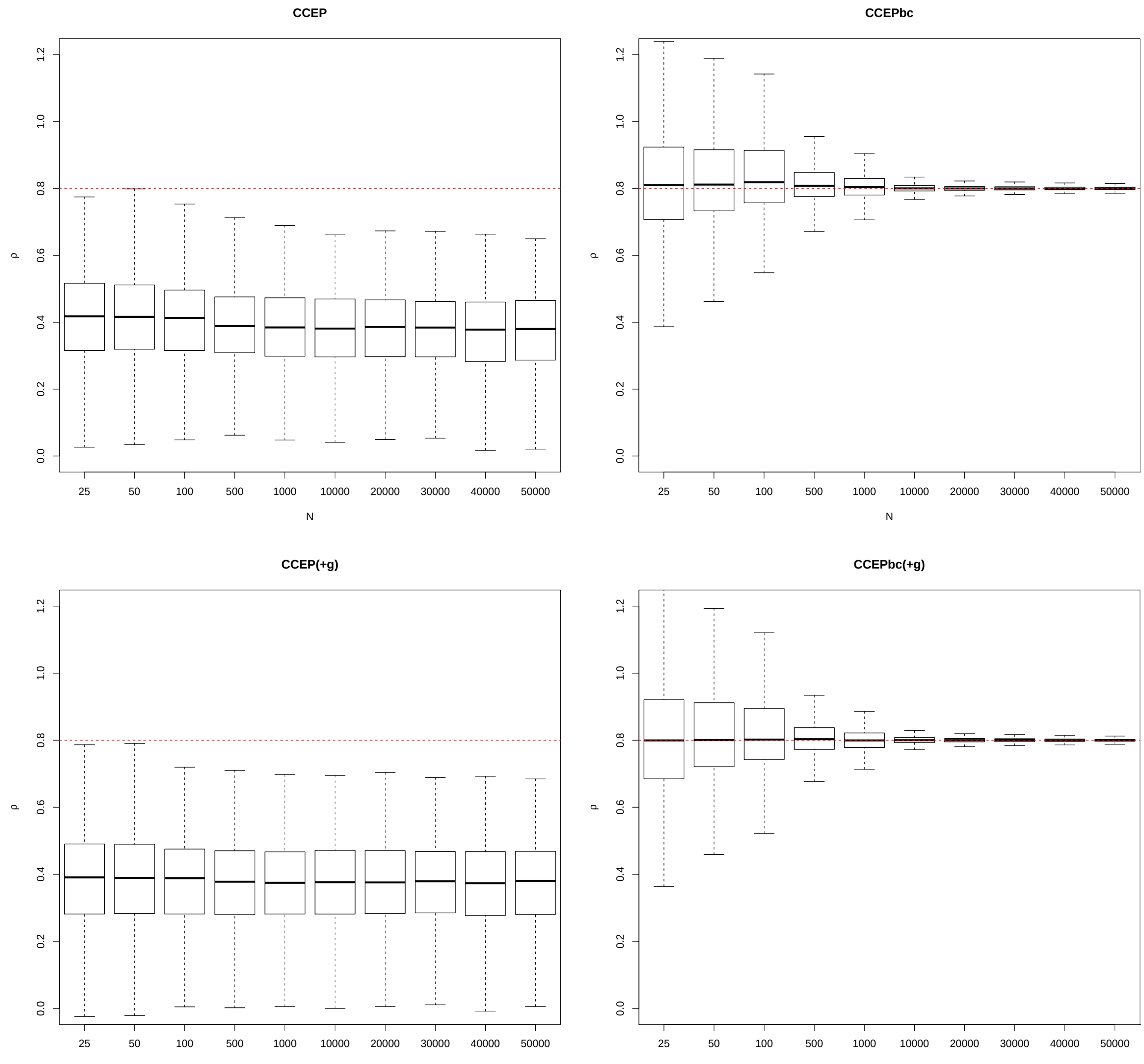


Notes:

(i) Reported are simulation results for estimating $\rho = 0.8$ with $m = 2$ and $RI = 1$ for $N = 25, 50, 100, ..., 50.000$. The CCEP estimators with a (+g) suffix (lower panel) make use of the $\overline{g}_t$ variable to project out the factors.

(ii) Dotted red lines indicate the population parameter value ($\rho = 0.8$). The boxplot 'whiskers' extend to the most extreme data point which is no more than 1.5 times the interquartile range from the box.

Figure F-4: Monte Carlo results for ρ: Boxplots for CCEP and CCEPbc estimators over N for two strong factors ($m = 2$, $RI = 3$) with $T = 10$.



Notes:

(i) Reported are simulation results for estimating $\rho = 0.8$ with $m = 2$ and $RI = 3$ for $N = 25, 50, 100, ..., 50.000$. The CCEP estimators with a (+g) suffix (lower panel) make use of the $\overline{g}_t$ variable to project out the factors.

(ii) Dotted red lines indicate the population parameter value ($\rho = 0.8$). The boxplot 'whiskers' extend to the most extreme data point which is no more than 1.5 times the interquartile range from the box.