SUPPLEMENTARY APPENDIX FOR 'SEMIPARAMETRIC ESTIMATION OF FIRST-PRICE AUCTION MODELS'*

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Supplementary Appendix C: Proof of Proposition 2

In this supplementary document we prove the following 3 steps that are used in the proof of Proposition 2 in our paper.

Step 1 : $\tilde{\Gamma} - \hat{\Gamma} = o_{as}(1)$.

Step 2 : $\tilde{\Lambda} - \hat{\Lambda} = o_{as}(1)$, which with Assumption A6-(iii) imply that $\tilde{\Lambda}^{-1} - \hat{\Lambda}^{-1} = o_{as}(1)$.

Step 3 : $\sqrt{L}(S_L(\theta_0) - \hat{S}_L(\theta_0)) = O_p(1) + o_p(1)$.

STEP 1

In this step we show that $\tilde{\Gamma} - \hat{\Gamma} = o_{as}(1)$. Since Ω is a positive definite matrix and

$$\tilde{\Gamma} - \hat{\Gamma} = \left(\frac{\partial S_L^\top}{\partial \theta}(\tilde{\theta}) - \frac{\partial \hat{S}_L^\top}{\partial \theta}(\hat{\theta}) \right) \Omega = \left(\frac{1}{L} \sum_{\ell=1}^{L} \frac{1}{I_\ell} \sum_{p=1}^{I_\ell} \left(m_3^\top(V_{p\ell}, Z_\ell; \tilde{\theta}) - m_3^\top(\hat{V}_{p\ell}, Z_\ell; \hat{\theta}) \right) \right) \Omega,$$

it is enough to show that the norm of the first term is $o_{as}(1)$. We use the notational convention that for the function $m(\cdot)$, let $m_k(\cdot)$ denote its partial derivative with respect to its kth argument. Adding and

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subtracting $m_3^\top(\hat{V}_{p\ell}, Z_\ell; \tilde{\theta})$ and using the triangle inequality we get

$$\begin{aligned}
&\left\| \frac{1}{L}\sum_{\ell=1}^{L}\frac{1}{I_\ell}\sum_{p=1}^{I_\ell} m_3^\top(V_{p\ell}, Z_\ell; \tilde{\theta}) - m_3^\top(\hat{V}_{p\ell}, Z_\ell; \hat{\theta}))\right\| \\
&= \left\| \frac{1}{L}\sum_{\ell=1}^{L}\frac{1}{I_\ell}\sum_{p=1}^{I_\ell}\Big[(m_3^\top(V_{p\ell}, Z_\ell; \tilde{\theta}) - m_3^\top(\hat{V}_{p\ell}, Z_\ell; \tilde{\theta})) + m_3^\top(\hat{V}_{p\ell}, Z_\ell; \tilde{\theta}) - m_3^\top(\hat{V}_{p\ell}, Z_\ell; \hat{\theta}))\Big]\right\| \\
&\leq \left\| \frac{1}{L}\sum_{\ell=1}^{L}\frac{1}{I_\ell}\sum_{p=1}^{I_\ell}\Big[(m_3^\top(V_{p\ell}, Z_\ell; \tilde{\theta}) - m_3^\top(\hat{V}_{p\ell}, Z_\ell; \tilde{\theta}))\Big]\right\| + \left\| \frac{1}{L}\sum_{\ell=1}^{L}\frac{1}{I_\ell}\sum_{p=1}^{I_\ell} m_3^\top(\hat{V}_{p\ell}, Z_\ell; \tilde{\theta}) - m_3^\top(\hat{V}_{p\ell}, Z_\ell; \hat{\theta}))\right\| \\
&\equiv \aleph + \beth. \qquad \text{(C.1)}
\end{aligned}$$

Similar to the proof of Proposition 1, let $\chi_{p\ell} \in \{0,1\}$ be an indicator that is equal to one if the bid $B_{p\ell}$ is in the interior and is equal to zero if the bid is close to either one of the two boundaries, i.e., $\chi_{p\ell} := \mathbb{1}(|B_{p\ell} - \underline{b}| \geq h_{2g}$ and $|B_{p\ell} - \overline{b}| \geq h_{2g})$. Applying the triangle inequality to $\aleph$ in (C.1) and using Assumption A6-(i) we get

$$\begin{aligned}
\aleph &\leq \frac{1}{L}\sum_{\ell=1}^{L}\frac{1}{I_\ell}\sum_{p=1}^{I_\ell}\left\|\Big[m_3^\top(V_{p\ell}, Z_\ell; \tilde{\theta}) - m_3^\top(\hat{V}_{p\ell}, Z_\ell; \tilde{\theta})\Big]\right\| \leq \frac{1}{L}\sum_{\ell=1}^{L}\frac{1}{I_\ell}\sum_{p=1}^{I_\ell} K_2(Z_\ell)|V_{p\ell} - \hat{V}_{p\ell}| \\
&= \frac{1}{L}\sum_{\ell=1}^{L}\frac{1}{I_\ell}\sum_{p=1}^{I_\ell} K_2(Z_\ell)|\hat{V}_{p\ell} - V_{p\ell}|\chi_{p\ell} + \frac{1}{L}\sum_{\ell=1}^{L}\frac{1}{I_\ell}\sum_{p=1}^{I_\ell} K_2(Z_\ell)|\hat{V}_{p\ell} - V_{p\ell}|(1-\chi_{p\ell}) \\
&\leq \frac{1}{L}\sum_{\ell=1}^{L}\frac{1}{I_\ell}\sum_{p=1}^{I_\ell} K_2(Z_\ell)\sup_{p\ell}|\hat{V}_{p\ell} - V_{p\ell}|\chi_{p\ell} + \frac{1}{L}\sum_{\ell=1}^{L}\frac{1}{I_\ell}\sum_{p=1}^{I_\ell} K_2(Z_\ell)|\hat{V}_{p\ell} - V_{p\ell}|(1-\chi_{p\ell}) \\
&= (\mathrm{E}[K_2(Z)] + o_{as}(1))\sup_{p\ell}|\hat{V}_{p\ell} - V_{p\ell}|\chi_{p\ell} + \frac{1}{L}\sum_{\ell=1}^{L}\frac{1}{I_\ell}\sum_{p=1}^{I_\ell} K_2(Z_\ell)|\hat{V}_{p\ell} - V_{p\ell}|(1-\chi_{p\ell}).
\end{aligned}$$

The first term in the RHS is $o_{as}(1)$ because $\hat{g}$ (and thus $\hat{V}_{p\ell}$) is uniformly consistent in the interior region of the support of bids (GPV(2000), Proposition 3). The second term can also be shown to be $o_{as}(1)$, by applying the same reasoning as in the proof of Proposition 1. Similarly, applying the triangle inequality to $\beth$ in (C.1) and Assumption A6-(ii) and $\tilde{\theta} \xrightarrow{a.s} \theta_0$ and $\hat{\theta} \xrightarrow{a.s} \theta_0$, we get

$$\beth \leq \frac{1}{L}\sum_{\ell=1}^{L}\frac{1}{I_\ell}\sum_{p=1}^{I_\ell}\left\|m_3^\top(\hat{V}_{p\ell}, Z_\ell; \tilde{\theta}) - m_3^\top(\hat{V}_{p\ell}, Z_\ell; \hat{\theta})\right\| \leq \frac{1}{L}\sum_{\ell=1}^{L}\frac{1}{I_\ell}\sum_{p=1}^{I_\ell} K_3(V_{p\ell}, Z_\ell)\sup\|\tilde{\theta} - \hat{\theta}\| = o_{as}(1).$$

STEP 2

We want to show $\tilde{\Lambda}^{-1} - \hat{\Lambda}^{-1} = o_{as}(1)$, for which it suffices to show $\tilde{\Lambda} - \hat{\Lambda} = o_{as}(1)$. Since $\tilde{\theta} \leq \overline{\theta} \leq \theta_0$, the triangle inequality and Assumption A6-(ii) give

$$\begin{aligned}\left\| \frac{\partial S_L}{\partial \theta^\top}(\overline{\theta}) - \frac{\partial S_L}{\partial \theta^\top}(\tilde{\theta}) \right\| &\leq \frac{1}{L}\sum_{\ell=1}^{L}\frac{1}{I_\ell}\sum_{p=1}^{I_\ell}\|m_3(V_{p\ell}, Z_\ell; \overline{\theta}) - m_3(V_{p\ell}, Z_\ell; \tilde{\theta})\| \leq \frac{1}{L}\sum_{\ell=1}^{L}\frac{1}{I_\ell}\sum_{p=1}^{I_\ell} K_3(V_{p\ell}, Z_\ell)\|\overline{\theta} - \tilde{\theta}\| \\ &= \{\mathrm{E}[K_3(V,Z)] + o_{as}(1)\}o_{as}(1) = o_{as}(1),\end{aligned}$$

where the last equality follows from $\tilde{\theta} \xrightarrow{a.s} \theta_0$, and hence

$$\frac{\partial S_L}{\partial \theta^\top}(\overline{\theta}) = \frac{\partial S_L}{\partial \theta^\top}(\tilde{\theta}) + o_{as}(1). \tag{C.2}$$

Similarly, since $\hat{\theta} \leq \overline{\theta}^* \leq \theta_0$ and $\hat{\theta} \xrightarrow{a.s} \theta_0$, we get

$$\frac{\partial \hat{S}_L}{\partial \theta^\top}(\overline{\theta}^*) = \frac{\partial \hat{S}_L}{\partial \theta^\top}(\hat{\theta}) + o_{as}(1). \tag{C.3}$$

Substituting (C.2) in (A.5), and (C.3) in (A.6), gives

$$\begin{aligned}\tilde{\Lambda} - \hat{\Lambda} &= \left[\frac{\partial S_L^\top}{\partial \theta}(\tilde{\theta})\Omega\left(\frac{\partial S_L}{\partial \theta^\top}(\tilde{\theta}) + o_{as}(1)\right)\right] - \left[\frac{\partial \hat{S}_L^\top}{\partial \theta}(\hat{\theta})\Omega\left(\frac{\partial \hat{S}_L}{\partial \theta^\top}(\hat{\theta}) + o_{as}(1)\right)\right] \\ &= \left(\frac{\partial S_L^\top}{\partial \theta}(\tilde{\theta})\Omega\frac{\partial S_L}{\partial \theta^\top}(\tilde{\theta})\right) - \left(\frac{\partial \hat{S}_L^\top}{\partial \theta}(\hat{\theta})\Omega\frac{\partial \hat{S}_L}{\partial \theta^\top}(\hat{\theta})\right) + o_{as}(1) \\ &= \underbrace{\left(\frac{\partial S_L^\top}{\partial \theta}(\tilde{\theta})\Omega - \frac{\partial \hat{S}_L^\top}{\partial \theta}(\hat{\theta})\Omega\right)}_{=\tilde{\Gamma}-\hat{\Gamma}=o_{as}(1)}\underbrace{\left(\frac{\partial S_L}{\partial \theta^\top}(\tilde{\theta}) + \frac{\partial \hat{S}_L}{\partial \theta^\top}(\hat{\theta})\right)}_{<\infty} + o_{as}(1) = o_{as}(1),\end{aligned} \tag{C.4}$$

where boundedness of the term in the second-last equality follows from repeated use of the triangle inequality, the Lipschitz conditions from Assumptions A6-(i)-(ii), which imply $\sup_{\theta\in\Theta}\|m_3(V,Z;\theta)\| \leq K_6(V,Z)$ with $\mathrm{E}[K_6(V,Z)] < \infty$, and the fact that the bias $(\hat{V} - V)$ is bounded, all of which are shown below:

$$
\begin{aligned}
\left\| \left(\frac{\partial S_L}{\partial \theta^\top}(\tilde{\theta}) + \frac{\partial \hat{S}_L}{\partial \theta^\top}(\hat{\theta}) \right) \right\| &= \left\| \frac{1}{L} \sum_{\ell=1}^{L} \frac{1}{I_\ell} \sum_{p=1}^{I_\ell} [m_3(V_{p\ell}, Z_\ell; \tilde{\theta}) + m_3(\hat{V}_{p\ell}, Z_\ell; \hat{\theta})] \right\| \\
&\leq \frac{1}{L} \sum_{\ell=1}^{L} \frac{1}{I_\ell} \sum_{p=1}^{I_\ell} \| [m_3(V_{p\ell}, Z_\ell; \tilde{\theta}) + m_3(\hat{V}_{p\ell}, Z_\ell; \hat{\theta})] \| \leq \frac{1}{L} \sum_{\ell=1}^{L} \frac{1}{I_\ell} \sum_{p=1}^{I_\ell} \| m_3(V_{p\ell}, Z_\ell; \tilde{\theta}) \| + \frac{1}{L} \sum_{\ell=1}^{L} \frac{1}{I_\ell} \sum_{p=1}^{I_\ell} \| m_3(\hat{V}_{p\ell}, Z_\ell; \hat{\theta}) \| \\
&\leq \frac{1}{L} \sum_{\ell=1}^{L} \frac{1}{I_\ell} \sum_{p=1}^{I_\ell} \sup_{\theta \in \Theta} \| m_3(V_{p\ell}, Z_\ell; \theta) \| + \frac{1}{L} \sum_{\ell=1}^{L} \frac{1}{I_\ell} \sum_{p=1}^{I_\ell} \| m_3(\hat{V}_{p\ell}, Z_\ell; \hat{\theta}) - m_3(\hat{V}_{p\ell}, Z_\ell; \theta_0) + m_3(\hat{V}_{p\ell}, Z_\ell; \theta_0) \| \\
&\qquad\qquad (\because \text{ add and subtract } \; m_3(\hat{V}_{p\ell}, Z_\ell; \theta_0)) \\
&\leq \frac{1}{L} \sum_{\ell=1}^{L} \frac{1}{I_\ell} \sum_{p=1}^{I_\ell} K_6(V_{p\ell}, Z_\ell) + \frac{1}{L} \sum_{\ell=1}^{L} \frac{1}{I_\ell} \sum_{p=1}^{I_\ell} \| m_3(\hat{V}_{p\ell}, Z_\ell; \hat{\theta}) - m_3(\hat{V}_{p\ell}, Z_\ell; \theta_0) \| \\
&\quad + \frac{1}{L} \sum_{\ell=1}^{L} \frac{1}{I_\ell} \sum_{p=1}^{I_\ell} \| m_3(\hat{V}_{p\ell}, Z_\ell; \theta_0) - m_3(V_{p\ell}, Z_\ell; \theta_0) + m_3(V_{p\ell}, Z_\ell; \theta_0) \| \qquad (\because \text{ add and subtract } \; m_3(V_{p\ell}, Z_\ell; \theta_0)) \\
&\leq \{ \mathrm{E}[K_6(V, Z)] + o_{as}(1) \} + \frac{1}{L} \sum_{\ell=1}^{L} \frac{1}{I_\ell} \sum_{p=1}^{I_\ell} K_3(V_{p\ell}, Z_\ell) \| \hat{\theta} - \theta_0 \| + \frac{1}{L} \sum_{\ell=1}^{L} \frac{1}{I_\ell} \sum_{p=1}^{I_\ell} \| m_3(\hat{V}_{p\ell}, Z_\ell; \theta_0) - m_3(V_{p\ell}, Z_\ell; \theta_0) \| \\
&\quad + \frac{1}{L} \sum_{\ell=1}^{L} \frac{1}{I_\ell} \sum_{p=1}^{I_\ell} \| m_3(V_{p\ell}, Z_\ell; \theta_0) \| \\
&\leq \{ \mathrm{E}[K_6(V, Z)] + o_{as}(1) \} + \mathrm{E}[K_3(V, Z)] o_{as}(1) + \frac{1}{L} \sum_{\ell=1}^{L} \frac{1}{I_\ell} \sum_{p=1}^{I_\ell} K_2(Z_\ell) | \hat{V}_{p\ell} - V_{p\ell} | + \frac{1}{L} \sum_{\ell=1}^{L} \frac{1}{I_\ell} \sum_{p=1}^{I_\ell} \sup_{\theta \in \Theta} \| m_3(V_{p\ell}, Z_\ell; \theta) \| \\
&\leq \underbrace{\{ \mathrm{E}[K_6(V, Z)] + o_{as}(1) \}}_{<\infty} + o_{as}(1) + \underbrace{\frac{1}{L} \sum_{\ell=1}^{L} \frac{1}{I_\ell} \sum_{p=1}^{I_\ell} K_2(Z_\ell) | \hat{V}_{p\ell} - V_{p\ell} |}_{<\infty} + \underbrace{\{ \mathrm{E}[K_6(V, Z)] + o_{as}(1) \}}_{\infty} < \infty.
\end{aligned}
$$

STEP 3

Applying the mean value theorem to $m(\cdot)$, and adding and subtracting $m_1(V_{p\ell}, Z_\ell; \theta_0)(\hat{V}_{p\ell} - V_{p\ell})$, we get

$$
\begin{aligned}
m(V_{p\ell}, Z_\ell; \theta_0) - m(\hat{V}_{p\ell}, Z_\ell; \theta_0) &= m_1(V^*_{p\ell}, Z_\ell; \theta_0)(V_{p\ell} - \hat{V}_{p\ell}), \qquad V^*_{p\ell} \in (V_{p\ell}, \hat{V}_{p\ell}) \\
&= -m_1(V_{p\ell}, Z_\ell; \theta_0)(\hat{V}_{p\ell} - V_{p\ell}) + [m_1(V_{p\ell}, Z_\ell; \theta_0) - m_1(V^*_{p\ell}, Z_\ell; \theta_0)](\hat{V}_{p\ell} - V_{p\ell}).
\end{aligned}
$$

To show $\sqrt{L}(S_L(\theta_0) - \hat{S}_L(\theta_0)] = O_p(1) + o_p(1)$, we use the above equation to get

$$\begin{aligned}\sqrt{L}(S_L(\theta_0) - \hat{S}_L(\theta_0)) &= \sqrt{L}\frac{1}{L}\sum_{\ell=1}^{L}\frac{1}{I_\ell}\sum_{p=1}^{I_\ell}\Big[m(V_{p\ell}, Z_\ell;\theta_0) - m(\hat{V}_{p\ell}, Z_\ell;\theta_0)\Big] \\ &= \sqrt{L}\frac{1}{L}\sum_{\ell=1}^{L}\frac{1}{I_\ell}\sum_{p=1}^{I_\ell}\Big[m_1(V_{p\ell}, Z_\ell;\theta_0) - m_1(V^*_{p\ell}, Z_\ell;\theta_0)\Big](\hat{V}_{p\ell} - V_{p\ell}) \\ &\quad -\sqrt{L}\frac{1}{L}\sum_{\ell=1}^{L}\frac{1}{I_\ell}\sum_{p=1}^{I_\ell}\Big[m_1(V_{p\ell}, Z_\ell;\theta_0)(\hat{V}_{p\ell} - V_{p\ell})\Big] \equiv \gimel_0 + \gimel_1, \end{aligned} \tag{C.5}$$

and then in (Step 3.1) we show $\gimel_0 = o_{as}(1)$ and in (Step 3.2) we show $\gimel_1 = O_p(1) + o_p(1)$.

STEP 3.1

Here, we want to show $\gimel_0$ in (C.5) is $o_{as}(1)$. Let r_v be the rate of convergence of $\hat{V}$, $V^*_{p\ell} \in (\hat{V}_{p\ell}, V_{p\ell})$. Then

$$\begin{aligned}\|\gimel_0\| &\le \sqrt{L}\frac{1}{L}\sum_{\ell=1}^{L}\frac{1}{I_\ell}\sum_{p=1}^{I_\ell}\|m_1(V_{p\ell}, Z_\ell;\theta_0) - m_1(V^*_{p\ell}, Z_\ell;\theta_0)\||\hat{V}_{p\ell} - V_{p\ell}| \\ &\le \sqrt{L}\frac{1}{L}\sum_{\ell=1}^{L}\frac{1}{I_\ell}\sum_{p=1}^{I_\ell}K_4(Z_\ell)|V_{p\ell} - V^*_{p\ell}||\hat{V}_{p\ell} - V_{p\ell}| \le \sqrt{L}\frac{1}{L}\sum_{\ell=1}^{L}\frac{1}{I_\ell}\sum_{p=1}^{I_\ell}K_4(Z_\ell)(\hat{V}_{p\ell} - V_{p\ell})^2 \\ &= \sqrt{L}\frac{1}{L}\sum_{\ell=1}^{L}\frac{1}{I_\ell}\sum_{p=1}^{I_\ell}K_4(Z_\ell)\Big\{(\hat{V}_{p\ell} - V_{p\ell})^2 \times \chi_{p\ell} + (\hat{V}_{p\ell} - V_{p\ell})^2 \times (1 - \chi_{p\ell})\Big\} \\ &= \underbrace{\sqrt{L}\frac{1}{L}\sum_{\ell=1}^{L}\frac{1}{I_\ell}\sum_{p=1}^{I_\ell}K_4(Z_\ell)(\hat{V}_{p\ell} - V_{p\ell})^2 \times \chi_{p\ell}}_{:=\gimel_{00}} + \underbrace{\sqrt{L}\frac{1}{L}\sum_{\ell=1}^{L}\frac{1}{I_\ell}\sum_{p=1}^{I_\ell}K_4(Z_\ell)(\hat{V}_{p\ell} - V_{p\ell})^2 \times (1 - \chi_{p\ell})}_{:=\gimel_{01}}, \\ &\equiv \gimel_{00} + \gimel_{01}, \end{aligned} \tag{C.6}$$

where the second inequality follows from Assumption A6-(iv), and the third inequality follows from $\hat{V}_{p\ell} \le V^*_{p\ell} \le V_{p\ell}$. Next, we show that $\gimel_{00}$ in (C.6) is $o_{as}(1)$. We begin with the definition

$$\gimel_{00} \le \sqrt{L}\left\{\sup_{p,\ell}(\hat{V}_{p\ell} - V_{p\ell})^2 \times \chi_{p\ell}\right\}\frac{1}{L}\sum_{\ell=1}^{L}K_4(Z_\ell) = \sqrt{L}\left\{\sup_{p,\ell}(\hat{V}_{p\ell} - V_{p\ell})^2 \times \chi_{p\ell}\right\}O_{as}(1). \tag{C.7}$$

For our choice of bandwidths, $\frac{L^{1/4}}{r_v} \to 0$ because $\hat{\phi}(\cdot; Z)$ converges at the rate $\left(\frac{L}{\log(L)}\right)^{\frac{R+1}{2R+d+3}}$, which is faster than $\left(\frac{L}{\log(L)}\right)^{1/4}$ as $R > d+1$ by Assumption A2-(ii). The rate of convergence of $\hat{\phi}(\cdot; Z)$ follows from the fact that g has R+1 derivatives and from Stone [1982] we know that the optimal convergence rate is $\left(\frac{L}{\log(L)}\right)^{\frac{R+1}{2R+d+3}}$, when h_g satisfies Assumption A4.AN. Thus the term in the parenthesis in (C.7)

is $O_{as}\left(\frac{1}{r_v^2}\right)$. Using this rate in (C.7) gives $\gimel_{00} \leq \sqrt{L}O_{as}\left(\frac{1}{r_v^2}\right)O_{as}(1) \leq O_{as}\left(\frac{L^{1/4}}{r_v}\right)O_{as}(1) = o_{as}(1)$, as desired.

Next, we show that $\gimel_{01}$ in (C.6) is also $o_{as}(1)$. For that we can use similar steps we have used above and in the proof of Proposition 1. Note that the bias $(\hat{V}_{p\ell} - V_{p\ell})$ is bounded by a constant multiplied by h_g^R, and the LLN implies that $\frac{1}{L}\sum_{\ell=1}^{L}\frac{1}{I_\ell}\sum_{p=1}^{I_\ell}K_4(Z_\ell) \times (1-\chi_{p\ell})$ converges to its expectation, which can be shown to be stochastically bounded by following the steps as in the proof of Proposition 1. These results together imply $\gimel_{01} \leq \sqrt{L}O_{as}(1)O_{as}(h_g^{2R})$, i.e., $\gimel_{01}$ is also stochastically bounded. Given that $d = 1$, Assumption A4.AN (ii) implies that $\sqrt{L}h_g^{2R} \to 0$, and from the definition of $O_{as}(\cdot)$ and $o_{as}(\cdot)$, we get $\gimel_{01} = o_{as}(1)$ as desired.

STEP 3.2

This step is long and broken into 11 sub-steps. **Step. 3.2.0** is the final conclusion of **Step. 3.2.** that uses results from **Step. 3.2.1-Step. 3.2.10**. We consider auctions with the same number of bidders, I.

Step. 3.2.0. Starting with the definition of $S_L(\theta_0)$ and $\hat{S}_L(\theta_0)$, from (C.5), we show the following

$$
\begin{aligned}
\sqrt{L}(S_L(\theta_0) - \hat{S}_L(\theta_0)) &= \underbrace{\sqrt{L}\frac{1}{L}\sum_{\ell=1}^{L}\frac{1}{I_\ell}\sum_{p=1}^{I_\ell}\left[m_1(V_{p\ell}, Z_\ell;\theta_0) - m_1(V_{p\ell}^*, Z_\ell;\theta_0)\right](\hat{V}_{p\ell} - V_{p\ell})}_{=o_p(1),\ \texttt{from Step 3.1}} \\
&\quad - \underbrace{\sqrt{L}\frac{1}{L}\sum_{\ell=1}^{L}\frac{1}{I_\ell}\sum_{p=1}^{I_\ell}\left[m_1(V_{p\ell}, Z_\ell;\theta_0)(\hat{V}_{p\ell} - V_{p\ell})\right]}_{=\gimel_1,\ \texttt{from (C.5)}} \\
&= o_p(1) + \gimel_{11} + \gimel_{12} \quad \texttt{see (C.12)} \\
&= o_p(1) + \gimel_{111} + \gimel_{112} + \underbrace{\gimel_{12}}_{=o(1),\ \texttt{see C.29, C.30, C.31}} \quad \texttt{see (C.13)} \\
&= o_p(1) + o(1) + \underbrace{\gimel_{111}}_{=o_p(1),\ \texttt{see (C.18), (C.19)}} + \gimel_{112} \\
&= o_p(1) + o(1) + \gimel_{112} \\
&= o_p(1) + o(1) + \sqrt{L}\frac{L(L-1)}{L^2}U_L, \quad \texttt{see (C.13)}
\end{aligned}
$$

where U_L is defined in (C.17). Now, adding and subtracting

$$\sqrt{L}\frac{L(L-1)}{L^2}\hat{U}_L \equiv \sqrt{L}\frac{L(L-1)}{L^2}\left(E[\mathbf{r}_L(Y_{p\ell},I)] + \frac{2}{L}\sum_{\{\ell\in L_I\}}\frac{1}{I}\sum_{p=1}^{I}[\mathbf{r}_L(Y_{p\ell},I) - E[\mathbf{r}_L(Y_{p\ell},I)]]\right)$$

to the RHS of the above equation, where $\mathbf{r}_L$ is defined in (C.20), we get

$$\begin{aligned}
\sqrt{L}(S_L(\theta_0) - \hat{S}_L(\theta_0)) &= o_p(1) + \underbrace{\sqrt{L}\frac{L(L-1)}{L^2}(U_L - \hat{U}_L)}_{=o_p(1),\ \texttt{see Step. 3.2.6.}} + \sqrt{L}\frac{L(L-1)}{L^2}\hat{U}_L \\
&= o_p(1) + \underbrace{\sqrt{L}\frac{L(L-1)}{L^2}\hat{U}_L}_{=o(1)+\Xi,\quad \texttt{see (C.24), (C.25)}} \\
&= o_p(1) + \Xi \\
&= o_p(1) - \frac{2}{\sqrt{L_I}}\sum_{\{\ell\in L_I\}}\frac{1}{I}\sum_{p=1}^{I}[r(Y_{p\ell},I) - \mathrm{E}r(Y,I)] = o_p(1) + O_p(1), \quad (\texttt{see (C.28)}).
\end{aligned}$$

In the remainder of this step, we do two things: derive all the steps used to obtain the expression above, and show that these terms are indeed $o_p(1) + O_p(1)$. Since from **Step 3.1** we already know that $\gimel_0$ in (C.5) is $o_p(1)$ (in fact we have shown that $\gimel_0 = o_{as}(1)$) what remains to be shown is $\gimel_1 = O_p(1) + o_p(1)$. To that end, we decompose $\gimel_1$ into the sum $\gimel_{11} + \gimel_{12}$ (defined later) and use the triangular inequality to bound these terms, and show that the sum is $O_p(1) + o_p(1)$. In other words **Step. 3.2.0** simply collects all the sub-steps **Step. 3.2.1**–**Step. 3.2.10** without the inequalities. Given the length of this step, and the nature of the problem we first give an outline of the steps involved.

To show that $\gimel_1$ in (C.5) is $O_p(1) + o_p(1)$, we proceed in the following steps:

$$
\begin{aligned}
& ||\gimel_1|| \leq \|\gimel_{11}\| + \|\gimel_{12}\| \quad (\text{See (C.12)}) - \textbf{Step. 3.2.1} \\
\leq\ & \|\gimel_{111}\| + \|\gimel_{112}\| + \|\gimel_{12}\| \quad (\because \gimel_{11} = \gimel_{111} + \gimel_{112}, \text{See (C.13)}) - \textbf{Step. 3.2.2} \\
=\ & \|\daleth_1 \times \daleth_2\| + \|\gimel_{112}\| + \|\gimel_{12}\| \quad (\because \gimel_{111} = \daleth_1 \times \daleth_2, \text{See (C.18)}) - \textbf{Step. 3.2.3} \\
=\ & o_p(1) + \|\gimel_{112}\| + \|\gimel_{12}\| \quad (\because \daleth_1 < \infty, \daleth_2 = o_p(1), \text{See (C.19)}) - \textbf{Step. 3.2.4} \\
\leq\ & o_p(1) + \|\gimel_{1121}\| + \|\gimel_{1122}\| + \|\gimel_{12}\| \quad (\because \gimel_{112} = \gimel_{1121} + \gimel_{1122}, \text{See (C.21)}) - \textbf{Step. 3.2.5} \\
\leq\ & o_p(1) + o_p(1) + \|\Im\| + \|\Xi\| + \|\gimel_{12}\| \quad (\because \gimel_{1121} = o_p(1), \text{See (C.22) and } \gimel_{1122} = \Im + \Xi, \text{See (C.24)}) - \textbf{Step. 3.2.6} \\
=\ & o_p(1) + \|\Xi\| + \|\gimel_{12}\| \quad (\because \Im = o(1), \text{See (C.25)}) - \textbf{Step. 3.2.7} \\
=\ & o_p(1) + O_p(1) + \|\gimel_{12}\| \quad (\because \Xi = O_p(1), \text{See (C.28)}) - \textbf{Step. 3.2.8} \\
=\ & o_p(1) + O_p(1) + \|\gimel_{121} \times \gimel_{122}\| \quad (\because \gimel_{12} = \gimel_{121} \times \gimel_{122}, \text{See (C.29)}) - \textbf{Step. 3.2.9} \\
=\ & o_p(1) + O_p(1) \quad (\because \gimel_{121} < \infty, \gimel_{122} = o(1) \Rightarrow \gimel_{12} = \gimel_{121} \times \gimel_{122} = o(1), \text{See (C.31)}) - \textbf{Step. 3.2.10}.
\end{aligned}
$$

Before proceeding, we note that equation (3) in the paper is numerically equivalent to the second line of each of the following equations. For the proof we will use this alternative expression for convenience.

$$
\begin{aligned}
\hat{G}(b|x,i) &= \frac{1}{h_G} \sum_{\{\ell \in L_i\}} \sum_{p=1}^{i} e_1^\top (X_{i,R+1}^\top W_x^G X_{i,R+1})^{-1} X_{R+1,\ell} K_G \left(\frac{X_\ell - x}{h_G} \right) \mathbb{1}(B_{p\ell} \leq b) \\
&= \frac{1}{L h_G} \frac{L}{n_i} \sum_{\{\ell \in L_i\}} \sum_{p=1}^{i} e_1^\top \left(\frac{X_{i,R+1}^\top W_x^G X_{i,R+1}}{n_i} \right)^{-1} X_{R+1,\ell} K_G \left(\frac{X_\ell - x}{h_G} \right) \mathbb{1}(B_{p\ell} \leq b) \\
\hat{g}(b|x,i) &= \frac{1}{h_{1g} h_{2g}} \sum_{\{\ell \in L_i\}} \sum_{p=1}^{i} e_1^\top (X_{i,R}^\top W_x^g X_{i,R})^{-1} X_{R,\ell} K_{1g} \left(\frac{X_\ell - x}{h_{1g}} \right) K_{2g} \left(\frac{B_{p\ell} - b}{h_{2g}} \right) \\
&= \frac{1}{L h_{1g} h_{2g}} \frac{L}{n_i} \sum_{\{\ell \in L_i\}} \sum_{p=1}^{i} e_1^\top \left(\frac{X_{i,R}^\top W_x^g X_{i,R}}{n_i} \right)^{-1} X_{R,\ell} \times K_{1g} \left(\frac{X_\ell - x}{h_{1g}} \right) K_{2g} \left(\frac{B_{p\ell} - b}{h_{2g}} \right) \qquad \text{(C.8)}
\end{aligned}
$$

We also need to introduce new notations as follows:

$$K_{G,h_G}(X_j - X_\ell) = \frac{1}{h_G} K_G\left(\frac{X_j - X_\ell}{h_G}\right); \tag{C.9}$$

$$K_{1g,h_g}(X_j - X_\ell) = \frac{1}{h_g} K_{1g}\left(\frac{X_j - X_\ell}{h_g}\right); \quad K_{2g,h_g}(B_{qj} - B_{p\ell}) = \frac{1}{h_g} K_{2g}\left(\frac{B_{qj} - B_{p\ell}}{h_g}\right); \tag{C.10}$$

$$\omega^G_{I,R+1,j} = e_1^\top \left(\frac{X^\top_{I,R+1} W^G_x X_{I,R+1}}{n_I}\right)^{-1} X_{R+1,j}; \quad \omega^g_{I,R,j} = e_1^\top \left(\frac{X^\top_{I,R} W^g_x X_{I,R}}{n_I}\right)^{-1} X_{R,j}. \tag{C.11}$$

where $n_I = IL_I$ and we recall that $L_I = \#\{\ell : I_\ell = I\}$ is the set of auctions with I bidders.

Step. 3.2.1. Without loss of generality, let us consider auctions with $I_\ell = I$ many bidders (see Remark 1 in the paper). From equation (2) in the paper and the definition of $\hat{V}$, we get

$$\begin{aligned}
\|\gimel_1\| &= \left\| \sqrt{L}\frac{1}{L}\sum_{\{\ell\in L_I\}} \frac{1}{I}\sum_{p=1}^{I} m_1(V_{p\ell}, X_\ell, I; \theta_0)(\hat{V}_{p\ell} - V_{p\ell}) \right\| \\
&= \left\| \sqrt{L}\frac{1}{L}\sum_{\{\ell\in L_I\}} \frac{1}{I}\sum_{p=1}^{I} m_1(V_{p\ell}, X_\ell, I; \theta_0)\frac{1}{I-1}\left[\frac{\hat{G}(B_{p\ell}|X_\ell, I)}{\hat{g}(B_{p\ell}|X_\ell, I)} - \frac{G_0(B_{p\ell}|X_\ell, I)}{g_0(B_{p\ell}|X_\ell, I)}\right] \right\|.
\end{aligned}$$

Applying the identity $\frac{\tilde{a}}{\tilde{b}} - \frac{a}{b} \equiv \frac{\tilde{a} - \frac{a}{b}\tilde{b}}{b} + \frac{a}{b}\frac{1}{\tilde{b}b}(\tilde{b} - b)^2 - \frac{1}{\tilde{b}b}(\tilde{a} - a)(\tilde{b} - b)$ to the expression $\left[\frac{\hat{G}(B_{p\ell}|X_\ell,I)}{\hat{g}(B_{p\ell}|X_\ell,I)} - \frac{G_0(B_{p\ell}|X_\ell,I)}{g_0(B_{p\ell}|X_\ell,I)}\right]$ gives

$$\begin{aligned}
\|\gimel_1\| &= \Bigg\| \sqrt{L}\frac{1}{L}\sum_{\{\ell\in L_I\}} \frac{1}{I(I-1)}\sum_{p=1}^{I} m_1(V_{p\ell}, X_\ell, I; \theta_0)\Bigg\{ \frac{\hat{G}(B_{p\ell}|X_\ell, I) - \frac{G_0(B_{p\ell}|X_\ell,I)}{g_0(B_{p\ell}|X_\ell,I)}\hat{g}(B_{p\ell}|X_\ell, I)}{g_0(B_{p\ell}|X_\ell, I)} \\
&\quad + \frac{G_0(B_{p\ell}|X_\ell, I)}{g_0(B_{p\ell}|X_\ell, I)}\frac{1}{\hat{g}(B_{p\ell}|X_\ell, I)g_0(B_{p\ell}|X_\ell, I)}\left[\hat{g}(B_{p\ell}|X_\ell, I) - g_0(B_{p\ell}|X_\ell, I)\right]^2 \\
&\quad - \frac{1}{\hat{g}(B_{p\ell}|X_\ell, I)g_0(B_{p\ell}|X_\ell, I)}\left[\hat{G}(B_{p\ell}|X_\ell, I) - G_0(B_{p\ell}|X_\ell, I)\right]\left[\hat{g}(B_{p\ell}|X_\ell, I) - g_0(B_{p\ell}|X_\ell, I)\right] \Bigg\}\Bigg\| \\
&\leq \left\| \sqrt{L}\frac{1}{L}\sum_{\{\ell\in L_I\}} \frac{1}{I(I-1)}\sum_{p=1}^{I} m_1(V_{p\ell}, X_\ell, I; \theta_0)\left(\frac{\hat{G}(B_{p\ell}|X_\ell, I) - \frac{G_0(B_{p\ell}|X_\ell,I)}{g_0(B_{p\ell}|X_\ell,I)}\hat{g}(B_{p\ell}|X_\ell, I)}{g_0(B_{p\ell}|X_\ell, I)}\right)\right\| \\
&\quad + \Bigg\| \sqrt{L}\frac{1}{L}\sum_{\{\ell\in L_I\}} \frac{1}{I(I-1)}\sum_{p=1}^{I} m_1(V_{p\ell}, X_\ell, I; \theta_0)\Bigg(\frac{G_0(B_{p\ell}|X_\ell, I)}{g_0(B_{p\ell}|X_\ell, I)}\frac{1}{\hat{g}(B_{p\ell}|X_\ell, I)g_0(B_{p\ell}|X_\ell, I)} \\
&\quad \times \left[\hat{g}(B_{p\ell}|X_\ell, I) - g_0(B_{p\ell}|X_\ell, I)\right]^2 - \frac{1}{\hat{g}(B_{p\ell}|X_\ell, I)g_0(B_{p\ell}|X_\ell, I)}\left[\hat{G}(B_{p\ell}|X_\ell, I) - G_0(B_{p\ell}|X_\ell, I)\right] \\
&\quad \left[\hat{g}(B_{p\ell}|X_\ell, I) - g_0(B_{p\ell}|X_\ell, I)\right]\Bigg)\Bigg\| \qquad (\because \text{triangle inequality}) \\
&\overset{def}{=} \|\gimel_{11}\| + \|\gimel_{12}\|.
\end{aligned} \tag{C.12}$$

Step. 3.2.2. Since we derive the result for auctions with fix number, I, of bidders, whenever there is no confusion we use L instead of L_I to denote the sample size. However, we use $\frac{1}{L}\sum_{\{\ell\in L_I\}}(\cdot)$ to denote $\frac{1}{L}\sum_{\ell=1}^{L}(\cdot)\mathbb{1}\{\ell : I_\ell = I\}$. Next we show that $\gimel_{11}$ in (C.12) is a double sum over all observations and can be written as a sum of the diagonal terms and of the off-diagonal terms. In particular we determine two terms R_L in (C.16), and U_L in (C.17), such that

$$\gimel_{11} = \sqrt{L}R_L + \sqrt{L}\frac{L(L-1)}{L^2}U_L \stackrel{def}{=} \gimel_{111} + \gimel_{112}. \tag{C.13}$$

Using the definition of the LPE estimators (C.8) and simplifying the term in the parenthesis, in the definition of $\gimel_{11}$ in (C.12), and using Equations (C.9) – (C.11), we get

$$\begin{aligned}
&\frac{1}{g_0(B_{p\ell}|X_\ell,I)}\left[\hat{G}(B_{p\ell}|X_\ell,I) - \frac{G_0(B_{p\ell}|X_\ell,I)}{g_0(B_{p\ell}|X_\ell,I)}\hat{g}(B_{p\ell}|X_\ell,I)\right] \\
= &\frac{1}{g_0(B_{p\ell}|X_\ell,I)}\left[\frac{1}{Lh_G}\frac{L}{n_I}\sum_{\{j\in L_I\}}\sum_{q=1}^{I}e_1^\top\left(\frac{X_{I,R+1}^\top W_x^G X_{I,R+1}}{n_I}\right)^{-1}X_{R+1,j}K_G\left(\frac{X_j-X_\ell}{h_G}\right)\mathbb{1}(B_{qj}\le B_{p\ell})\right. \\
&\left.-\frac{G_0(B_{p\ell}|X_\ell,I)}{g_0(B_{p\ell}|X_\ell,I)}\frac{1}{Lh_g^2}\frac{L}{n_I}\sum_{\{j\in L_I\}}\sum_{q=1}^{I}e_1^\top\left(\frac{X_{I,R}^\top W_x^g X_{I,R}}{n_I}\right)^{-1}X_{R,j}K_{1g}\left(\frac{X_j-X_\ell}{h_g}\right)K_{2g}\left(\frac{B_{qj}-B_{p\ell}}{h_g}\right)\right] \\
= &\frac{1}{g_0(B_{p\ell}|X_\ell,I)}\left[\frac{1}{L}\frac{L}{n_I}\sum_{\{j\in L_I\}}\sum_{q=1}^{I}\omega_{I,R+1,j}^G K_{G,h_G}(X_j-X_\ell)\mathbb{1}(B_{qj}\le B_{p\ell})\right. \\
&\left.-\frac{G_0(B_{p\ell}|X_\ell,I)}{g_0(B_{p\ell}|X_\ell,I)}\frac{1}{L}\frac{L}{n_I}\sum_{\{j\in L_I\}}\sum_{q=1}^{I}\omega_{I,R,j}^g K_{1g,h_g}(X_j-X_\ell)K_{2g,h_g}(B_{qj}-B_{p\ell})\right].
\end{aligned} \tag{C.14}$$

Substituting the first term in (C.12) with (C.14), we get

$$\begin{aligned}
\gimel_{11} = \sqrt{L}\Bigg(&\frac{1}{L^2}\frac{L}{n_I}\sum_{\{\ell\in L_I\}}\sum_{\{j\in L_I\}}\frac{1}{I(I-1)}\sum_{p=1}^{I}\sum_{q=1}^{I}\frac{m_1(V_{p\ell},X_\ell,I;\theta_0)}{g_0(B_{p\ell}|X_\ell,I)}\Bigg[\omega_{I,R+1,j}^G K_{G,h_G}(X_j-X_\ell)\mathbb{1}(B_{qj}\le B_{p\ell}) \\
&-\frac{G_0(B_{p\ell}|X_\ell,I)}{g_0(B_{p\ell}|X_\ell,I)}\omega_{I,R,j}^g K_{1g,h_g}(X_j-X_\ell)K_{2g,h_g}(B_{qj}-B_{p\ell})\Bigg]\Bigg).
\end{aligned} \tag{C.15}$$

The terms inside the parenthesis on the RHS of (C.15) can be written as a sum of two terms: diagonal terms (i.e., when $\ell = j, p = q$)

$$R_L = \frac{1}{L^2}\frac{L}{n_I}\sum_{\{\ell\in L_I\}}\frac{1}{I(I-1)}\sum_{p=1}^{I}\frac{m_1(V_{p\ell},X_\ell,I;\theta_0)}{g_0(B_{p\ell}|X_\ell,I)}\left[\omega_{I,R+1,j}^G K_{G,h_G}(0) - \frac{G_0(B_{p\ell}|X_\ell,I)}{g_0(B_{p\ell}|X_\ell,I)}\omega_{I,R,j}^g K_{1g,h_g}(0)K_{2g,h_g}(0)\right], \tag{C.16}$$

and off-diagonal terms (i.e., when $\ell \neq j$)

$$U_L = \frac{1}{L(L-1)}\frac{L}{n_I}\sum_{\{\ell\in L_I\}}\sum_{\substack{\{j\in L_I\\ j\neq\ell\}}}\frac{1}{I(I-1)}\sum_{p=1}^{I}\sum_{q=1}^{I}\frac{m_1(V_{p\ell},X_\ell,I;\theta_0)}{g_0(B_{p\ell}|X_\ell,I)}\Bigg[\omega^G_{I,R+1,j}K_{G,h_G}(X_j-X_\ell)$$
$$\times\mathbb{1}(B_{qj}\leq B_{p\ell})-\frac{G_0(B_{p\ell}|X_\ell,I)}{g_0(B_{p\ell}|X_\ell,I)}\omega^g_{I,R,j}K_{1g,h_g}(X_j-X_\ell)K_{2g,h_g}(B_{qj}-B_{p\ell})\Bigg]. \quad \text{(C.17)}$$

After multiplying (C.16) and (C.17) by $\sqrt{L}$ and $\sqrt{L}/L^2$, respectively, we get $\gimel_{11}=\gimel_{111}+\gimel_{112}$ in (C.13).

Step. 3.2.3. Here we show that $\gimel_{111}=\daleth_1\times\daleth_2$. Applying Cauchy-Schwarz inequality to (C.16) gives

$$\|\gimel_{111}\|=\sqrt{L}\|R_L\|\leq\left(\frac{1}{L}\frac{L}{n_I}\sum_{\{\ell\in L_I\}}\frac{1}{I(I-1)}\sum_{p=1}^{I}\|m_1(V_{p\ell},X_\ell,I;\theta_0)\|^2\right)^{\frac{1}{2}}$$
$$\times\ \sqrt{L}\left(\frac{1}{L}\frac{L}{n_I}\sum_{\{\ell\in L_I\}}\frac{1}{I(I-1)}\sum_{p=1}^{I}\frac{1}{g_0(B_{p\ell}|X_\ell,I)^2}\left[\omega^G_{I,R+1,j}\frac{K_{G,h_G}(0)}{L}-\frac{G_0(B_{p\ell}|X_\ell,I)}{g_0(B_{p\ell}|X_\ell,I)}\omega^g_{I,R,j}\frac{K_{1g,h_g}(0)K_{2g,h_g}(0)}{L}\right]^2\right)^{\frac{1}{2}}$$
$$\stackrel{def}{=}\ \daleth_1\times\daleth_2. \quad \text{(C.18)}$$

In the next sub-step we show that $\gimel_{111}=o_p(1)$ by showing that $\daleth_1^2<\infty$ and $\daleth_2=o_p(1)$.

Step. 3.2.4. From Assumption A6-(v), $0<(1/(I-1))\leq 1$ for each $I\in\mathcal{I}$ and $L/n_I=1/I<1$:

$$\daleth_1^2 = \frac{1}{L}\frac{L}{n_I}\sum_{\{\ell\in L_I\}}\frac{1}{I(I-1)}\sum_{p=1}^{I}\|m_1(V_{p\ell},X_\ell,I;\theta_0)\|^2\leq\frac{L}{n_I}\frac{1}{L}\sum_{\{\ell\in L_I\}}\frac{1}{I}\sum_{p=1}^{I}\frac{1}{(I-1)}\sup_{\theta\in\Theta}\|m_1(V_{p\ell},X_\ell,I;\theta)\|^2$$
$$\leq\frac{1}{L}\sum_{\{\ell\in L_I\}}\frac{1}{I}\sum_{p=1}^{I}K_5(V_{p\ell},X_\ell,I)^2=\mathrm{E}[K_5(V,X,I)^2]+o_p(1)<\infty.$$

Now we show that $\daleth_2=o_p(1)$. For that we begin with the definition

$$\daleth_2 = \sqrt{L}\left(\frac{1}{L}\frac{L}{n_I}\sum_{\{\ell\in L_I\}}\frac{1}{I(I-1)}\sum_{p=1}^{I}\frac{1}{g_0(B_{p\ell}|X_\ell,I)^2}\left[\omega^G_{I,R+1,j}\frac{K_G(0)}{Lh_G}-\frac{G_0(B_{p\ell}|X_\ell,I)}{g_0(B_{p\ell}|X_\ell,I)}\omega^g_{I,R,j}\frac{K_{1g,h_g}(0)K_{2g,h_g}(0)}{Lh_g^2}\right]^2\right)^{\frac{1}{2}}$$
$$=\sqrt{L}\left(\frac{1}{L}\frac{L}{n_I}\sum_{\{\ell\in L_I\}}\frac{1}{I(I-1)}\sum_{p=1}^{I}\underbrace{\frac{1}{g_0(B_{p\ell}|X_\ell,I)^2}}_{\leq\kappa_1^2<\infty}\left[O_p(1)O_p\left(\frac{1}{Lh_G}\right)-\underbrace{\frac{G_0(B_{p\ell}|X_\ell,I)}{g_0(B_{p\ell}|X_\ell,I)}}_{\leq\kappa_2<\infty}O_p(1)O_p\left(\frac{1}{Lh_g^2}\right)\right]^2\right)^{\frac{1}{2}}$$
$$<\sqrt{L}\kappa_1\left[O_p\left(\frac{1}{Lh_G}\right)-\kappa_2O_p\left(\frac{1}{Lh_g^2}\right)\right]=\kappa_1\left[O_p\left(\frac{1}{\sqrt{L}h_G}\right)-\kappa_2O_p\left(\frac{1}{\sqrt{L}h_g^2}\right)\right].$$

where the second equality follows from observing that

$$\omega^G_{I,R+1,j} = e_1^\top \left(\frac{X^\top_{I,R+1} W^G_x X_{I,R+1}}{n_I} \right)^{-1} X_{R+1,j} = e_1^\top \left[\frac{1}{n_I h_G} \sum_{\iota=1}^{n_I} \mathbf{x}_\iota^\top \mathbf{x}_\iota K_G \left(\frac{x_\iota - x_j}{h_g} \right) \right]^{-1} e_1 = O_p(1);$$

and, following the same arguments, $\omega^g_{I,R,j} = O_p(1)$; $K_G(0)/Lh_G = O_p(1/Lh_G)$; and, lastly, that $(K_{1g,h_g}(0)K_{2g,h_g}(0))/Lh_g^2 = O_p(1/Lh_g^2)$. From Assumption A4. AN (i) and (ii) $\log L/(\sqrt{L}h_G) \to 0$ and $\log L/(\sqrt{L}h_g^2) \to 0$ as $L \to \infty$, which imply $\sqrt{L}h_G \to \infty$ and $\sqrt{L}h_g^2 \to \infty$, respectively. Thus

$$\gimel_2 < \kappa_1 [o_p(1) - \kappa_2 o_p(1)] = o_p(1). \tag{C.19}$$

Step. 3.2.5. Next we show $\beth_{112}$ (C.13) is equal to the sum $\beth_{1121} + \beth_{1122}$. Let

$$\begin{aligned} p_L\left((B_{p\ell}, X_\ell, I_\ell), (B_{qj}, X_j, I_j)\right) &\overset{def}{=} \frac{m_1(V_{p\ell}, X_\ell, I_\ell; \theta_0) K^{**}(B_{p\ell}, B_{qj}, X_\ell, X_j, I_\ell, I_j)}{2} \\ &\quad + \frac{m_1(V_{qj}, X_j, I_j; \theta_0) K^{**}(B_{qj}, B_{p\ell}, X_j, X_\ell, I_j, I_\ell)}{2}, \end{aligned}$$

be a symmetric function where

$$\begin{aligned} K^{**}(B_{p\ell}, B_{qj}, X_\ell, X_j, I_\ell, I_j) &= \frac{L}{n_{I_\ell}(I_\ell - 1)} \frac{1}{g_0(B_{p\ell}|X_\ell, I_\ell)} \Bigg[\omega^G_{I,R+1,j} K_{G,h_G}(X_j - X_\ell) \\ &\times \mathbb{1}(B_{qj} \le B_{p\ell}) - \frac{G_0(B_{p\ell}|X_\ell, I_\ell)}{g_0(B_{p\ell}|X_\ell, I_\ell)} \omega^g_{I,R,j} K_{1g,h_g}(X_j - X_\ell) K_{2g,h_g}(B_{qj} - B_{p\ell}) \Bigg] + \frac{L}{n_{I_j}(I_j - 1)} \frac{1}{g_0(B_{qj}|X_j, I_j)} \\ &\times \left[\omega^G_{I,R+1,j} K_{G,h_G}(X_\ell - X_j) \mathbb{1}(B_{p\ell} \le B_{qj}) - \frac{G_0(B_{qj}|X_j, I_j)}{g_0(B_{qj}|X_j, I_j)} \omega^g_{I,R,j} K_{1g,h_g}(X_\ell - X_j) K_{2g,h_g}(B_{p\ell} - B_{qj}) \right]. \end{aligned}$$

Let $Y_{p\ell} \equiv (B_{p\ell}, X_\ell)$. Since we focus on auctions with I bidders, we evaluate $p_L(\cdot,\cdot)$ at $I_j = I_\ell = I$, and define

$$\mathbf{r}_L(Y_{p\ell}, I) = E[p_L((Y_{p\ell}, I), (Y_{qj}, I))], \quad j \neq \ell; \quad \mathrm{E}[\mathbf{r}_L(Y_{p\ell}, I)] = \mathrm{E}[p_L((Y_{p\ell}, I), (Y_{qj}, I))] \tag{C.20}$$

$$\therefore \hat{U}_L = E[\mathbf{r}_L(Y_{p\ell}, I)] + \frac{2}{L} \sum_{\{\ell \in L_I\}} \frac{1}{I} \sum_{p=1}^{I} [\mathbf{r}_L(Y_{p\ell}, I) - E[\mathbf{r}_L(Y_{p\ell}, I)]].$$

is the projection. Adding and subtracting $\frac{L(L-1)}{L^2}\sqrt{L}\hat{U}_L$ from $\beth_{112}$ defined in (C.13) we get

$$\beth_{112} = \frac{L(L-1)}{L^2}\sqrt{L}(U_L - \hat{U}_L) + \frac{L(L-1)}{L^2}\sqrt{L}\hat{U}_L \overset{def}{=} \beth_{1121} + \beth_{1122}. \tag{C.21}$$

Step. 3.2.6. Next we show that $\gimel_{1121}$ defined in (C.21) is $o_p(1)$. To show that we use the U-statistic arguments in [Powell, Stock, and Stoker, 1989, Lemma 3.1]. In particular, using the definition of $K^{**}(\cdot)$ and $p_L(\cdot,\cdot)$ we can re-write U_L in (C.17) as a U-statistic:

$$
\begin{aligned}
U_L &= \frac{1}{L(L-1)}\frac{L}{n_I}\sum_{\{\ell\in L_I\}}\sum_{\substack{\{j\in L_I\\ j\neq\ell\}}}\frac{1}{I(I-1)}\sum_{p=1}^{I}\sum_{q=1}^{I}\frac{m_1(V_{p\ell},X_\ell,I;\theta_0)}{g_0(B_{p\ell}|X_\ell,I)} \\
&\quad \left[\omega^G_{I,R+1,j}K_{G,h_G}(X_j-X_\ell)\mathbb{1}(B_{qj}\le B_{p\ell})-\frac{G_0(B_{p\ell}|X_\ell,I)}{g_0(B_{p\ell}|X_\ell,I)}\omega^g_{I,R,j}K_{1g,h_g}(X_j-X_\ell)K_{2g,h_g}(B_{qj}-B_{p\ell})\right] \\
&= \frac{1}{L(L-1)}\sum_{\{\ell\in L_I-1\}}\sum_{\substack{\{j\in L_I\\ j=\ell+1\}}}\frac{1}{I}\sum_{p=1}^{I}\sum_{q=1}^{I}\left\{\frac{L}{n_I(I-1)}\frac{m_1(V_{p\ell},X_\ell,I;\theta_0)}{g_0(B_{p\ell}|X_\ell,I)}\right. \\
&\quad \left.\left[\omega^G_{I,R+1,j}K_{G,h_G}(X_j-X_\ell)\mathbb{1}(B_{qj}\le B_{p\ell})-\frac{G_0(B_{p\ell}|X_\ell,I)}{g_0(B_{p\ell}|X_\ell,I)}\omega^g_{I,R,j}K_{1g,h_g}(X_j-X_\ell)K_{2g,h_g}(B_{qj}-B_{p\ell})\right]\right\}, \\
&= \frac{2}{L(L-1)}\sum_{\{\ell\in L_I-1\}}\sum_{\substack{\{j\in L_I\\ j=\ell+1\}}}\frac{1}{I}\sum_{p=1}^{I}\sum_{q=1}^{I}p_L\left((B_{p\ell},X_\ell,I),(B_{qj},X_j,I)\right).
\end{aligned}
$$

We know from Powell, Stock, and Stoker [1989] that if $\mathrm{E}[\|p_L((Y_{p\ell},I),(Y_{qj},I))\|^2]=o(L)$ then $\sqrt{L}(U_L-\hat{U}_L)=o_p(1)$. In fact, it is sufficient to show that $\mathrm{E}[\|p_L((Y_{p\ell},I),(Y_{qj},I))\|^2|I]=o(L)$, which together with $\lim_{L\to\infty}\frac{L(L-1)}{L^2}=1$ imply $\mathrm{E}[\|p_L((Y_{p\ell},I),(Y_{qj},I))\|^2]=o(L)$ and hence

$$
\gimel_{1121}=\frac{L(L-1)}{L^2}\sqrt{L}(U_L-\hat{U}_L)=o_p(1). \tag{C.22}
$$

Henceforth, whenever possible, with a slight abuse of notation we use $g_0(Y_{p\ell}, I)(:= g_0(B_{p\ell}|X_\ell, I)f_Z(X, I))$ to denote the joint density of (B, X, I). Using the definition of $p_L(\cdot,\cdot)$ we get

$$
\begin{aligned}
&\mathrm{E}[\|p_L((Y_{p\ell}, I), (Y_{qj}, I))\|^2|I] = \int \|p_L((Y_{p\ell}, I), (Y_{qj}, I))\|^2 g_0(Y_{p\ell}|I)g_0(Y_{qj}|I)dY_{p\ell}dY_{qj} \\
= \;& \frac{1}{4}\int \Bigg\| \frac{L}{n_I(I-1)} \frac{m_1(V_{p\ell}, X_\ell, I;\theta_0)}{g_0(B_{p\ell}|X_\ell, I)} \Bigg[\frac{1}{h_G}\omega^G_{I,R+1,j} K_G\left(\frac{X_j - X_\ell}{h_G}\right) \mathbb{1}(B_{qj} \le B_{p\ell}) \\
& - \frac{G_0(B_{p\ell}|X_\ell, I)}{g_0(B_{p\ell}|X_\ell, I)} \frac{1}{h_g^2}\omega^g_{I,R,j} K_{1g}\left(\frac{X_j - X_\ell}{h_g}\right) K_{2g}\left(\frac{B_{qj} - B_{p\ell}}{h_g}\right)\Bigg] \\
& + \frac{L}{n_I(I-1)} \frac{m_1(V_{qj}, X_j, I;\theta_0)}{g_0(B_{qj}|X_j, I)} \Bigg[\frac{1}{h_G}\omega^G_{I,R+1,j} K_G\left(\frac{X_\ell - X_j}{h_G}\right) \mathbb{1}(B_{p\ell} \le B_{qj}) \\
& - \frac{G_0(B_{qj}|X_j, I)}{g_0(B_{qj}|X_j, I)} \frac{1}{h_g^2}\omega^g_{I,R,j} K_{1g}\left(\frac{X_\ell - X_j}{h_g}\right) K_{2g}\left(\frac{B_{p\ell} - B_{qj}}{h_g}\right)\Bigg]\Bigg\|^2 g_0(Y_{p\ell}|I)g_0(Y_{qj}|I)dY_{p\ell}dY_{qj}. \\
= \;& \frac{1}{4}\int \Bigg\{ \Bigg\| \Bigg[\frac{L}{n_I(I-1)} \frac{m_1(V_{p\ell}, X_\ell, I;\theta_0)}{g_0(B_{p\ell}|X_\ell, I)} \frac{1}{h_G}\omega^G_{I,R+1,j} K_G\left(\frac{X_j - X_\ell}{h_G}\right) \mathbb{1}(B_{qj} \le B_{p\ell}) \\
& - \frac{L}{n_I(I-1)} \frac{m_1(V_{qj}, X_j, I;\theta_0)}{g_0(B_{qj}|X_j, I)} \frac{1}{h_G}\omega^G_{I,R+1,j} K_G\left(\frac{X_\ell - X_j}{h_G}\right) \mathbb{1}(B_{p\ell} \le B_{qj})\Bigg] \\
& + \Bigg[\frac{L}{n_I(I-1)} \frac{m_1(V_{p\ell}, X_\ell, I;\theta_0)}{g_0(B_{p\ell}|X_\ell, I)} \frac{G_0(B_{p\ell}|X_\ell, I)}{g_0(B_{p\ell}|X_\ell, I)} \frac{1}{h_g^2}\omega^g_{I,R,j} K_{1g}\left(\frac{X_j - X_\ell}{h_g}\right) K_{2g}\left(\frac{B_{qj} - B_{p\ell}}{h_g}\right) \\
& - \frac{L}{n_I(I-1)} \frac{m_1(V_{qj}, X_j, I;\theta_0)}{g_0(B_{qj}|X_j, I)} \frac{G_0(B_{qj}|X_j, I)}{g_0(B_{qj}|X_j, I)} \frac{1}{h_g^2}\omega^g_{I,R,j} K_{1g}\left(\frac{X_\ell - X_j}{h_g}\right) K_{2g}\left(\frac{B_{p\ell} - B_{qj}}{h_g}\right)\Bigg]\Bigg\|^2\Bigg\} \\
& g_0(Y_{p\ell}|I)g_0(Y_{qj}|I)dY_{p\ell}dY_{qj} \\
= \;& \frac{1}{4}\int \Bigg\{ \Bigg\| \frac{1}{h_G}\frac{L}{n_I} \Bigg[\frac{1}{(I-1)} \frac{m_1(V_{p\ell}, X_\ell, I;\theta_0)}{g_0(B_{p\ell}|X_\ell, I)} \omega^G_{I,R+1,j} K_G\left(\frac{X_j - X_\ell}{h_G}\right) \mathbb{1}(B_{qj} \le B_{p\ell}) \\
& - \frac{1}{(I-1)} \frac{m_1(V_{qj}, X_j, I;\theta_0)}{g_0(B_{qj}|X_j, I)} \omega^G_{I,R+1,j} K_G\left(\frac{X_\ell - X_j}{h_G}\right) \mathbb{1}(B_{p\ell} \le B_{qj})\Bigg] \\
& + \frac{1}{h_g^2}\frac{L}{n_I} \Bigg[\frac{1}{(I-1)} \frac{m_1(V_{p\ell}, X_\ell, I;\theta_0)}{g_0(B_{p\ell}|X_\ell, I)} \frac{G_0(B_{p\ell}|X_\ell, I)}{g_0(B_{p\ell}|X_\ell, I)} \omega^g_{I,R,j} K_{1g}\left(\frac{X_j - X_\ell}{h_g}\right) K_{2g}\left(\frac{B_{qj} - B_{p\ell}}{h_g}\right) \\
& - \frac{1}{I-1} \frac{m_1(V_{qj}, X_j, I;\theta_0)}{g_0(B_{qj}|X_j, I)} \frac{G_0(B_{qj}|X_j, I)}{g_0(B_{qj}|X_j, I)} \omega^g_{I,R,j} K_{1g}\left(\frac{X_\ell - X_j}{h_g}\right) K_{2g}\left(\frac{B_{p\ell} - B_{qj}}{h_g}\right)\Bigg]\Bigg\|^2\Bigg\} \\
& g_0(Y_{p\ell}|I)g_0(Y_{qj}|I)dY_{p\ell}dY_{qj}.
\end{aligned}
$$

Then using $(a+b)^2 \leq 2(a^2+b^2)$ and the triangle inequality we get

$$
\begin{aligned}
& \mathrm{E}[\|p_L((Y_{p\ell}, I), (Y_{qj}, I))\|^2 | I] \\
\leq\ & \frac{1}{2}\int \Bigg\| \frac{L}{n_I}\frac{1}{h_G}\Bigg[\frac{1}{(I-1)}\frac{m_1(V_{p\ell}, X_\ell, I; \theta_0)}{g_0(B_{p\ell}|X_\ell, I)}\omega^G_{I,R+1,j}K_G\left(\frac{X_j - X_\ell}{h_G}\right)\mathbb{1}(B_{qj} \leq B_{p\ell}) \\
& -\frac{1}{(I-1)}\frac{m_1(V_{qj}, X_j, I; \theta_0)}{g_0(B_{qj}|X_j, I)}\omega^G_{I,R+1,j}K_G\left(\frac{X_\ell - X_j}{h_G}\right)\mathbb{1}(B_{p\ell} \leq B_{qj})\Bigg]\Bigg\|^2 g_0(Y_{p\ell}|I)g_0(Y_{qj}|I)dY_{p\ell}dY_{qj} \\
& +\frac{1}{2}\int \Bigg\| \frac{L}{n_I}\frac{1}{h_g^2}\Bigg[\frac{1}{(I-1)}\frac{m_1(V_{p\ell}, X_\ell, I; \theta_0)}{g_0(B_{p\ell}|X_\ell, I)}\frac{G_0(B_{p\ell}|X_\ell, I)}{g_0(B_{p\ell}|X_\ell, I)}\omega^g_{I,R,j}K_{1g}\left(\frac{X_j - X_\ell}{h_g}\right)K_{2g}\left(\frac{B_{qj} - B_{p\ell}}{h_g}\right) \\
& -\frac{1}{(I-1)}\frac{m_1(V_{qj}, X_j, I; \theta_0)}{g_0(B_{qj}|X_j, I)}\frac{G_0(B_{qj}|X_j, I)}{g_0(B_{qj}|X_j, I)}\omega^g_{I,R,j}K_{1g}\left(\frac{X_\ell - X_j}{h_g}\right)K_{2g}\left(\frac{B_{p\ell} - B_{qj}}{h_g}\right)\Bigg]\Bigg\|^2 \\
& g_0(Y_{p\ell}|I)g_0(Y_{qj}|I)dY_{p\ell}dY_{qj} \stackrel{def}{=} J_1 + J_2. \qquad \text{(C.23)}
\end{aligned}
$$

Next we show that J_1 and J_2 are both $o(L)$. Consider J_1 first. Using $V_{p\ell} = \xi(B_{p\ell}, X_\ell, I)$ and from the change of variable $u = \dfrac{Y_{p\ell} - Y_{qj}}{h_G} \equiv \left(\dfrac{B_{p\ell} - B_{qj}}{h_G}, \dfrac{X_\ell - X_j}{h_G}\right) = (u_1, u_2)$, we get

$$
\begin{aligned}
J_1 =\ & \frac{1}{2h_G^2}\int\left(\frac{L}{n_I}\frac{1}{(I-1)}\right)^2\Bigg\|\frac{m_1(\xi(B_{p\ell}, X_\ell, I), X_\ell, I; \theta_0)}{g_0(B_{p\ell}|X_\ell, I)}\omega^G_{I,R+1,j}K_G\left(\frac{X_j - X_\ell}{h_G}\right)\mathbb{1}(B_{qj} \leq B_{p\ell}) \\
& -\frac{m_1(V_{qj}, X_j, I; \theta_0)}{g_0(B_{qj}|X_j, I)}\omega^G_{I,R+1,j}K_G\left(\frac{X_\ell - X_j}{h_G}\right)\mathbb{1}(B_{p\ell} \leq B_{qj})\Bigg\|^2 g_0(Y_{p\ell}|I)g_0(Y_{qj}|I)dY_{p\ell}dY_{qj} \\
=\ & \frac{1}{2}\int\left(\frac{L}{n_I}\frac{1}{(I-1)}\right)^2\Bigg\|\frac{m_1(\xi(uh_G + Y_{qj}, I), u_2h_G + X_j, I; \theta_0)}{g_0(u_1h_G + B_{qj}|u_2h_G + X_j, I)}\omega^G_{I,R+1,j}K_G(-u_2)\mathbb{1}(B_{qj} \leq u_1h_G + B_{qj}) \\
& -\frac{m_1(V_{qj}, X_j, I; \theta_0)}{g_0(B_{qj}|X_j, I)}\omega^G_{I,R+1,j}K_G(u_2)\mathbb{1}(u_1h_G + B_{qj} \leq B_{qj})\Bigg\|^2 g_0(uh_G + Y_{qj}|I)g_0(Y_{qj}|I)dudY_{qj} \\
\leq\ & \int\Bigg[\Bigg\|\frac{m_1(\xi(uh_G + Y_{qj}, I), u_2h_G + X_j, I; \theta_0)}{g_0(u_1h_G + B_{qj}|u_2h_G + X_j, I)}\omega^G_{I,R+1,j}K_G(-u_2)\mathbb{1}(B_{qj} \leq u_1h_G + B_{qj})\Bigg\|^2 \\
& \qquad g_0(uh_G + Y_{qj}|I)g_0(Y_{qj}|I)dudY_{qj} \\
& +\int\Bigg\|\frac{m_1(V_{qj}, X_j, I; \theta_0)}{g_0(B_{qj}|X_j, I)}\omega^G_{I,R+1,j}K_G(u_2)\mathbb{1}(u_1h_G + B_{qj} \leq B_{qj})\Bigg\|^2\Bigg]g_0(uh_G + Y_{qj}|I)g_0(Y_{qj}|I)dudY_{qj} \\
\stackrel{def}{=}\ & J_{11} + J_{12},
\end{aligned}
$$

where the inequality follows from using the fact that $\left(\frac{L}{n_I}\frac{1}{(I-1)}\right)^2 < 1$ and the parallelogram law $\|a-b\|^2 + \|a+b\|^2 = 2(\|a\|^2 + \|b\|^2)$, which in turn implies that $\|a-b\|^2 \leq 2(\|a\|^2 + \|b\|^2)$. Next, we

consider J_{11}:

$$\begin{aligned} J_{11} &= \int \left\| m_1(\xi(uh_G + Y_{qj}, I), u_2 h_G + X_j, I; \theta_0) \omega^G_{I,R+1,j} K_G(-u_2) \mathbb{1}(B_{qj} \leq u_1 h_G + B_{qj}) \right\|^2 \\ &\quad \underbrace{\left(\frac{g_0(uh_G + Y_{qj}|I)}{g_0(u_1 h_G + B_{qj}|u_2 h_G + X_j, I)^2} \right)}_{\leq \overline{\kappa} < \infty} g_0(Y_{qj}|I) du dY_{qj} \\ &\leq \overline{\kappa} \int \|m_1(\xi(uh_G + Y_{qj}, I), u_2 h_G + X_j, I; \theta_0)\|^2 \left\| \omega^G_{I,R+1,j} K_G(-u_2) \mathbb{1}(B_{qj} \leq u_1 h_G + B_{qj}) \right\|^2 g_0(Y_{qj}|I) du dY_{qj}, \end{aligned}$$

where $\overline{\kappa}$ is a constant and is bounded because the densities are bounded away from zero, and the last inequality follows from Cauchy-Schwartz inequality. By the Lebesgue Dominated Convergence (LDC) Theorem and Assumptions A3 and A6-(v), as $L \to \infty$ the bandwidth $h_G \to 0$, and the first term converges to $m_1(V_{qj}, X_j, I; \theta_0)$ and because $\omega^G_{I,R+1} = O_p(1)$ and K_G is symmetric we get

$$J_{11} \leq \overline{\kappa} \int \|m_1(V_{qj}, X_j, I; \theta_0)\|^2 \left\| O_p(1) K_G(u_2) \right\|^2 g_0(Y_{qj}|I) du_2 dY_{qj} < \infty.$$

which in turn means $J_{11} = o(L)$. Applying similar lines of reasoning to J_{12} (change of variable, parallelogram law, bounded densities, and vanishing bandwidth) and Assumption A4.AN -(ii), we can show that $J_{12} = o(L)$. The same logic applies to J_2. Thus, $J_1 + J_2 = J_{11} + J_{12} + J_{21} + J_{22} = o(L)$, and $\gimel_{1121} = o_p(1)$.

Next, we show that $\gimel_{1122}$ in (C.21) can be written as $\Im + \Xi$. But this follows from the definition as

$$\begin{aligned} \gimel_{1122} &= \frac{L(L-1)}{L^2} \sqrt{L} \hat{\mathcal{U}}_L = \frac{L(L-1)}{L^2} \sqrt{L} \left\{ \mathrm{E}\mathbf{r}_L(Y_{p\ell}, I) + \frac{2}{L} \sum_{\{\ell \in L_I\}} \frac{1}{I} \sum_{p=1}^{I} [\mathbf{r}_L(Y_{p\ell}, I) - \mathrm{E}\mathbf{r}_L(Y_{p\ell}, I)] \right\} \\ &= \frac{L(L-1)}{L^2} \sqrt{L} \left\{ E[p_L((Y_{p\ell}, I), (Y_{qj}, I))] \right\} + \frac{L(L-1)}{L^2} \sqrt{L} \frac{2}{L} \sum_{\{\ell \in L_I\}} \frac{1}{I} \sum_{p=1}^{I} [\mathbf{r}_L(Y_{p\ell}, I) - \mathrm{E}\mathbf{r}_L(Y_{p\ell}, I)] \\ &\overset{def}{=} \Im + \Xi. \end{aligned} \tag{C.24}$$

Step. 3.2.7. Here we show that $\Im = o(1)$. Note that

$$\begin{aligned} \|\Im\| &= \frac{L(L-1)}{L^2} \sqrt{L} \|\mathrm{E}[p_L(Y_{p\ell}, I), (Y_{qj}, I))]\| \leq \frac{L(L-1)}{L^2} \sqrt{L} \left(\left(\mathrm{E}\|p_L(Y_{p\ell}, I), (Y_{qj}, I))\| \right)^2 \right)^{1/2} \\ &\qquad (\because \|\mathrm{E}(\cdot)\| \leq \mathrm{E}\|(\cdot)\|) \\ &\leq \frac{L(L-1)}{L^2} \sqrt{L} \left(\mathrm{E}\left[\|p_L(Y_{p\ell}, I), (Y_{qj}, I))\|^2 \right] \right)^{1/2} \qquad (\because \text{Jensen's inequality}) \\ &= \frac{L(L-1)}{L^2} \sqrt{L} (o(L))^{1/2} = o(1). \qquad (\because \textbf{Step. 3.2.6.}) \end{aligned} \tag{C.25}$$

Step. 3.2.8. Here, we show that Ξ in (C.24) is $O_p(1)$. From the definition of $\mathbf{r}_L(Y_{p\ell}, I), j \neq \ell$ (C.20). Consider $(Y_{p\ell}, I) = (y, i)$,

$$
\begin{aligned}
\mathbf{r}_L(y,i) &= \mathbf{r}_L(b,x,i) = \mathrm{E}[p_L((b,x,i),(B_{qj},X_j,I))|(y,i)] = \int p_L((b,x,i),(B_{qj},X_j,I))g_0(Y_{qj}|i)dY_{qj} \\
&= \frac{1}{2i}\frac{L_i}{n_i}\frac{1}{i-1}\Bigg\{\int\Bigg[\overbrace{\left(\frac{m_1(v,x,i;\theta_0)}{g_0(b|x,i)}\right)}^{:=\tilde{N}(y,i)}\frac{1}{h_G}\omega^G_{i,R+1,j}K_G\left(\frac{X_j-x}{h_G}\right)\mathbb{1}(B_{qj}\leq b) \\
&\quad + \underbrace{\left(\frac{m_1(V_{qj},X_j,I;\theta_0)}{g_0(B_{qj}|X_j,I)}\right)}_{:=\tilde{N}(Y_{qj},I)}\frac{1}{h_G}\omega^G_{I,R+1,j}K_G\left(\frac{x-X_j}{h_G}\right)\mathbb{1}(b\leq B_{qj})\Bigg]g_0(Y_{qj}|I)dY_{qj}\Bigg\} \\
&\quad -\frac{1}{2}\frac{L_i}{n_i}\frac{1}{i-1}\Bigg\{\int\Bigg[\overbrace{\frac{m_1(v,x,i;\theta_0)}{g_0(b|x,i)}\frac{G_0(b|x,i)}{g_0(b|x,i)}}^{:=N(y,i)}\frac{1}{h_g^2}\omega^g_{i,R,j}K_{1g}\left(\frac{X_j-x}{h_g}\right)K_{2g}\left(\frac{B_{qj}-b}{h_g}\right) \\
&\quad + \underbrace{\frac{m_1(V_{qj},X_j,I;\theta_0)}{g_0(B_{qj}|X_j)}\frac{G_0(B_{qj}|X_j,I)}{g_0(B_{qj}|X_j,i)}}_{:=N(Y_{qj},I)}\frac{1}{h_g^2}\omega^g_{I,R,j}K_{1g}\left(\frac{x-X_j}{h_g}\right)K_{2g}\left(\frac{b-B_{qj}}{h_g}\right)\Bigg]g_0(Y_{qj}|I)dY_{qj}\Bigg\} \\
&= \frac{1}{2i}\frac{L_i}{n_i}\frac{1}{i-1}\Bigg\{\int h_G\left[\tilde{N}(y,i)\omega^G_{i,R+1}K_G(u_2)\mathbb{1}(u_1h_G+b\leq b)+\tilde{N}(uh_G+y,i)\omega^G_{i,R+1}K_G(-u_2)\mathbb{1}(b\leq u_1h_G+b)\right] \\
&\quad g_0(uh_G+y|i)du\Bigg\} - \frac{1}{2I}\frac{L_I}{n_I}\frac{1}{I-1}\sum_{q=1}^{I}\Bigg\{\int\Bigg[N(y,i)\omega^g_{i,R,j}K_{1,g}(\tilde{u}_2)K_{2,g}(\tilde{u}_1) \\
&\quad +N(\tilde{u}h_g+y,i)\omega^g_{i,R,j}K_{1g}(-\tilde{u}_2)K_{2g}(-\tilde{u}_1)\Bigg]g_0(\tilde{u}h_g+y|i)d\tilde{u}\Bigg\},
\end{aligned}
\tag{C.26}
$$

where the last equality uses the change of variable: $u = \dfrac{Y_{qj}-y}{h_G} \equiv \left(\dfrac{B_{qj}-b}{h_G}, \dfrac{X_j-x}{h_G}\right) = (u_1, u_2)$ and $\tilde{u} = \dfrac{Y_{qj}-y}{h_g} \equiv \left(\dfrac{B_{qj}-b}{h_g}, \dfrac{X_j-x}{h_g}\right) = (\tilde{u}_1, \tilde{u}_2)$. Consider the term $\omega^g_{i,R,j}$ in $\mathbf{r}_L(y,i)$. As $h = (h_G, h_g) \to 0$

$$
\begin{aligned}
\omega^g_{i,R,j} &= e_1^\top\left[\frac{1}{n_ih_g}\sum_{\iota=1}^{n_i}\mathbf{x}_\cdot\mathbf{x}_\cdot^\top K_{1g}\left(\frac{X_\iota-x}{h_g}\right)\right]^{-1}[1 \quad (X_j-x)\ldots(X_j-x)^{R-1}]^\top \\
&= e_1^\top\left[\frac{1}{n_ih_g}\sum_{\iota=1}^{n_i}\mathbf{x}_\cdot\mathbf{x}_\cdot^\top K_{1g}\left(\frac{X_\iota-x}{h_g}\right)\right]^{-1}[1 \quad (-\tilde{u}_2h_g)\ldots(-\tilde{u}_2h_g)^{R-1}]^\top \xrightarrow{p} \frac{1}{f_X(x,i)};
\end{aligned}
$$

the limit follows from: (i) $[1 \quad (-\tilde{u}_2h_g)\ldots(-\tilde{u}_2h_g)^{R-1}]^\top \xrightarrow{p} [1 \quad 0\cdots0]^\top = e_1$; (ii) $\left[\frac{1}{n_ih_g}\sum_{\iota=1}^{n_i}(\cdot)\right]^{-1} \xrightarrow{p} [\mathrm{E}(\cdot)]^{-1}$; and (iii) $e_1^\top Ae_1 = a_{11}$ where A is a matrix with typical element (a_{ij}). Using this limit in $\mathbf{r}_L(y,i)$

and making the change of variable, we get:

$$\begin{aligned}
\mathbf{r}_L(y,i) &\longrightarrow -\frac{1}{2}\frac{L_i}{n_i}\frac{1}{i-1}N(y,i)f_X(x,i)^{-1}g_0(y|i)\int K_{1,g}(\tilde{u}_2)K_{2,g}(\tilde{u}_1)d\tilde{u} \\
&\quad -\frac{1}{2}\frac{L_i}{n_i}\frac{1}{i-1}N(y,i)f_X(x,i)^{-1}g_0(y|i)\int K_{1g}\left(-\tilde{u}_2\right)K_{2g}\left(-\tilde{u}_1\right)d\tilde{u} \\
&= -\frac{1}{i(i-1)}N(y,i)f_X(x,i)^{-1}g_0(y|i) \quad (\because \int K_{1g}(u_2)K_{2g}(u_1)du = 1, K(-a) = K(a)) \\
&\stackrel{def}{=} -r(y,i). \qquad \text{(C.27)}
\end{aligned}$$

Therefore $\mathbf{r}_L(y,i) = -r(y,i) + o_p(1)$. Using this in the definition of Ξ in (C.24), we get

$$\begin{aligned}
\Xi &= \left(1-\frac{1}{L}\right)\frac{2}{\sqrt{L}}\sum_{\ell\in L_I}\frac{1}{I}\sum_{p=1}^{I}[\mathbf{r}_L(Y_{p\ell},I) - \mathrm{E}\mathbf{r}_L(Y,I)] \\
&= -\frac{2}{\sqrt{L}}\sum_{\ell\in L_I}\frac{1}{I}\sum_{p=1}^{I}\left\{r(Y_{p\ell},I) - \mathrm{E}[r(Y,I)]\right\} + o_p(1) = O_p(1). \qquad \text{(C.28)}
\end{aligned}$$

Step. 3.2.9. Next, we apply the Cauchy-Schwarz inequality to $\gimel_{12}$ defined in (C.12) to get

$$\begin{aligned}
\|\gimel_{12}\| &\leq \left(\frac{1}{L}\sum_{\ell\in L_I}\frac{1}{I(I-1)}\sum_{p=1}^{i}\|m_1(V_{p\ell},Z_\ell;\theta_0)\|^2\right)^{\frac{1}{2}} \times \left(\frac{1}{L}\sum_{\ell\in L_I}\frac{1}{I(I-1)}\right. \\
&\quad \sum_{p=1}^{I}L\left[\frac{1}{\hat{g}(B_{p\ell}|Z_\ell)g_0(B_{p\ell}|Z_\ell)}\left(\frac{G_0(B_{p\ell}|Z_\ell)}{g_0(B_{p\ell}|Z_\ell)}\left[\hat{g}(B_{p\ell}|Z_\ell) - g_0(B_{p\ell}|Z_\ell)\right]\right.\right. \\
&\quad \left.\left.\left. - \left[\hat{G}(B_{p\ell}|Z_\ell) - G_0(B_{p\ell}|Z_\ell)\right]\left[\hat{g}(B_{p\ell}|Z_\ell) - g_0(B_{p\ell}|Z_\ell)\right]\right)\right]^2\right)^{\frac{1}{2}} \stackrel{def}{=} \gimel_{121} \times \gimel_{122}. \qquad \text{(C.29)}
\end{aligned}$$

Step. 3.2.10. We want to show that $\gimel_{12}$ in (C.12) is $o_p(1)$. For that we show that $\gimel_{121}^2 < \infty$ and $\gimel_{122} = o_p(1)$. Applying Assumption A6-(v) and $0 < 1/(I-1) \leq 1$ in the definition of $\gimel_{12}$ we get

$$\begin{aligned}
\gimel_{121}^2 &= \frac{1}{L}\sum_{\ell\in L_I}\frac{1}{I(I-1)}\sum_{p=1}^{I}\|m_1(V_{p\ell},Z_\ell;\theta_0)\|^2 \leq \frac{1}{L}\sum_{\ell\in L_I}\frac{1}{I(I-1)}\sum_{p=1}^{I}\sup_{\theta\in\Theta}\|m_1(V_{p\ell},Z_\ell;\theta)\|^2 \\
&\leq \frac{1}{L}\sum_{\ell\in L_I}\frac{1}{I(I-1)}\sum_{p=1}^{I}K_5(V_{p\ell},Z_\ell)^2 \leq \frac{1}{L}\sum_{\ell\in L_I}\frac{1}{I}\sum_{p=1}^{I}K_5(V_{p\ell},Z_\ell)^2 = \mathrm{E}[K_5(V,Z)^2] + o_{as}(1) < \infty.
\end{aligned}$$

Next from the definition of $\gimel_{122}$ we get

$$\gimel_{122} = \left\{ \frac{1}{L} \sum_{\ell \in L_I} \frac{1}{I(I-1)} \sum_{p=1}^{I} L \left[\underbrace{\frac{1}{\hat{g}(B_{p\ell}|Z_\ell) g_0(B_{p\ell}|Z_\ell)}}_{\leq \kappa_1 < \infty} \left(\underbrace{\frac{G_0(B_{p\ell}|Z_\ell)}{g_0(B_{p\ell}|Z_\ell)}}_{\leq \kappa_2 < \infty} \left[\hat{g}(B_{p\ell}|Z_\ell) - g_0(B_{p\ell}|Z_\ell) \right]^2 \right.\right.\right.$$

$$\left.\left.\left. - \left[\hat{G}(B_{p\ell}|Z_\ell) - G_0(B_{p\ell}|Z_\ell) \right] \left[\hat{g}(B_{p\ell}|Z_\ell) - g_0(B_{p\ell}|Z_\ell) \right] \right) \right]^2 \right\}^{\frac{1}{2}}. \tag{C.30}$$

Using the convergence rate for g and G we get

$$\begin{aligned} & \left| \hat{g}(B_{p\ell}|Z_\ell) - g_0(B_{p\ell}|Z_\ell) \right|^2 = O_{as} \left(h_{1g}^2 + h_{2g}^2 + \frac{\log L}{L h_{1g} h_{2g}} \right) := O_{as} \left(1/r_g^2 \right); \textit{and} \\ & \left| \hat{G}(B_{p\ell}|Z_\ell) - G_0(B_{p\ell}|Z_\ell) \right| \left| \hat{g}(B_{p\ell}|Z_\ell) - g_0(B_{p\ell}|Z_\ell) \right| \\ = \; & O_{as} \left(h_G + \sqrt{\frac{\log L}{L h_G}} \right) O_{as} \left(h_{1g} + h_{2g} + \sqrt{\frac{\log L}{L h_{1g} h_{2g}}} \right) := O_{as} \left(1/r_G \right) O_{as} \left(1/r_g \right), \end{aligned}$$

which when applied to (C.30) give

$$\begin{aligned} \gimel_{122} \; < \; & \sqrt{L} \left\{ \frac{1}{L} \sum_{\ell \in L_I} \frac{1}{I(I-1)} \sum_{p=1}^{i} \left[\kappa_1 \left(\kappa_2 \times O_{as}(1/r_g^2) - O_{as}(1/r_G) \times O_{as}(1/r_g) \right) \right]^2 \right\}^{\frac{1}{2}} \\ \leq \; & \sqrt{L} \kappa_1 \left[\kappa_2 \times O_{as}(1/r_g^2) - O_{as}(1/r_G) \times O_{as}(1/r_g) \right] \\ = \; & \kappa_1 \left[\kappa_2 \times O_{as}(\sqrt{L}/{r_g}^2) - O_{as}(\sqrt{L}/r_G r_g) \right] = o_{as}(1), \end{aligned} \tag{C.31}$$

where last equality follows from A4.AN (i) and (ii). In particular, for r_g, we have $(\sqrt{L}/{r_g}^2) \to 0$ and $(\sqrt{L}/r_G r_g) \to 0$ since $\sqrt{L} h_{1g}^2 \to 0$ and $\frac{\log L}{\sqrt{L} h_{1g} h_{2g}} \to 0$; similarly for r_G. Thus, $\gimel_{122} = o_p(1)$ as well, as desired.

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