SUPLEMENTARY MATERIAL

OPTIMAL DESIGNS FOR ESTIMATING A CHOICE HIERARCHY BY A GENERAL NESTED MULTINOMIAL LOGIT MODEL

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S.1. INFORMATION MATRIX OF 2 NESTS

The information matrix of the model depends in the parameters: it can be calculated as:

$$M_i = E\begin{bmatrix} \left(\frac{\partial l_i}{\partial \boldsymbol{\beta}}\right)\left(\frac{\partial l_i}{\partial \boldsymbol{\beta}}\right)' & \left(\frac{\partial l_i}{\partial \boldsymbol{\beta}}\right)\left(\frac{\partial l_i}{\partial \lambda_1}\right)' & \left(\frac{\partial l_i}{\partial \boldsymbol{\beta}}\right)\left(\frac{\partial l_i}{\partial \lambda_2}\right)' \\ \left(\frac{\partial l_i}{\partial \lambda_1}\right)\left(\frac{\partial l_i}{\partial \boldsymbol{\beta}}\right)' & \left(\frac{\partial l_i}{\partial \lambda_1}\right)\left(\frac{\partial l_i}{\partial \lambda_1}\right)' & \left(\frac{\partial l_i}{\partial \lambda_1}\right)\left(\frac{\partial l_i}{\partial \lambda_2}\right)' \\ \left(\frac{\partial l_i}{\partial \lambda_2}\right)\left(\frac{\partial l_i}{\partial \boldsymbol{\beta}}\right)' & \left(\frac{\partial l_i}{\partial \lambda_2}\right)\left(\frac{\partial l_i}{\partial \lambda_1}\right)' & \left(\frac{\partial l_i}{\partial \lambda_2}\right)\left(\frac{\partial l_i}{\partial \lambda_2}\right)' \end{bmatrix} \tag{1}$$

S.2 PARTIAL DERIVATIVES

There are three partial derivatives to calculate, however, in all of them there is the function $V_{i,1}$ and $V_{i,2}$. Thus, first we have to calculate the partial derivatives of these two functions.

$$\frac{\partial V_{i,1}}{\partial \boldsymbol{\beta}} = \frac{1}{\sum_{j\in\tau_1} e^{\boldsymbol{x}'_{ij}\boldsymbol{\beta}}}\sum_{j\in\tau_1} e^{\boldsymbol{x}'_{ij}\boldsymbol{\beta}}\boldsymbol{x_{ij}} = \sum_{j\in\tau_1}\frac{e^{\boldsymbol{x}'_{ij}\boldsymbol{\beta}}}{\sum_{j\in\tau_1} e^{\boldsymbol{x}'_{ij}\boldsymbol{\beta}}}p_{i,1}p_{i,1}^{-1}\boldsymbol{x_{ij}} = p_{i,1}^{-1}\sum_{j\in\tau_1} p_{ij}\boldsymbol{x_{ij}} \tag{2}$$

Analogously

$$\frac{\partial V_{i,2}}{\partial \boldsymbol{\beta}} = p_{i,2}^{-1}\sum_{j\in\tau_2} p_{ij}\boldsymbol{x_{ij}} \tag{3}$$

With the previous equations now is possible to calculate: $\frac{\partial l_i}{\partial \boldsymbol{\beta}}$. Así

$$\begin{aligned}
\frac{\partial l_i}{\partial \boldsymbol{\beta}} &= y_{i,1}\left(\lambda_1-1\right)p_{i,1}^{-1}\sum_{j\in\tau_1}p_{ij}\boldsymbol{x_{ij}}+\left(1-y_{i,1}\right)\left(\lambda_2-1\right)p_{i,2}^{-1}\sum_{j\in\tau_2}p_{ij}\boldsymbol{x_{ij}} \\
&- \\
&\frac{\lambda_1 e^{\lambda_1 V_{i,1}}p_{i,1}^{-1}\sum_{j\in\tau_1}p_{ij}\boldsymbol{x_{ij}}}{e^{\lambda_1 V_{i,1}}+e^{\lambda_2 V_{i,2}}}-\frac{\lambda_2 e^{\lambda_2 V_{i,2}}p_{i,2}^{-1}\sum_{j\in\tau_2}p_{ij}\boldsymbol{x_{ij}}}{e^{\lambda_1 V_{i,1}}+e^{\lambda_2 V_{i,2}}}+\sum_{j=1}^{J}y_{ij}\boldsymbol{x_{ij}} \\
&= y_{i,1}\left(\lambda_1-1\right)p_{i,1}^{-1}\sum_{j\in\tau_1}p_{ij}\boldsymbol{x_{ij}}+\left(1-y_{i,1}\right)\left(\lambda_2-1\right)p_{i,2}^{-1}\sum_{j\in\tau_2}p_{ij}\boldsymbol{x_{ij}} \\
&- \\
&\lambda_1\sum_{j\in\tau_1}p_{ij}\boldsymbol{x_{ij}}-\lambda_2\sum_{j\in\tau_2}p_{ij}\boldsymbol{x_{ij}}+\sum_{j=1}^{J}y_{ij}\boldsymbol{x_{ij}}
\end{aligned}$$

Given that $V_{i,1}$ and $V_{i,2}$ are not dependent of λ_1 and λ_2, then the partial derivatives with respect to these variables are 0. Thus:

$$\frac{\partial l_i}{\partial \lambda_1}=y_{i,1}V_{i,1}-\frac{e^{\lambda_1 V_{i,1}}V_{i,1}}{e^{\lambda_1 V_{i,1}}+e^{\lambda_2 V_{i,2}}}=y_{i,1}V_{i,1}-p_{i,1}V_{i,1}=V_{i,1}\left(y_{i,1}-p_{i,1}\right) \tag{4}$$

analogously:

$$\frac{\partial l_i}{\partial \lambda_2}=V_{i,1}\left(1-y_{i,1}-p_{i,2}\right) \tag{5}$$

S.3 CALCULATION OF THE RESULTS OF THE PARTIAL FRACTIONS

With the expressions of the partial derivatives, it is possible to calculate every element that is part of the information matrix (Equation 1).

- $\left(\frac{\partial l_i}{\partial \boldsymbol{\beta}}\right)\left(\frac{\partial l_i}{\partial \boldsymbol{\beta}}\right)'$. The element $\left(\frac{\partial l_i}{\partial \boldsymbol{\beta}}\right)\left(\frac{\partial l_i}{\partial \boldsymbol{\beta}}\right)'$ Observe that $y_{i,1}^2=y_{i,1}$ and similarly $\left(1-y_{i,1}\right)^2=y_{i,2}^2=y_{i,2}$; in addition if $y_{i,1}=1$ then $1-y_{i,1}=0$ y viceversa.

Product between first element with the five addend:

$$y_{i,1}\left(\lambda_1-1\right)^2 p_{i,1}^{-2}\left(\sum_{j\in\tau_1} p_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j\in\tau_1} p_{ij}\boldsymbol{x'_{ij}}\right) \tag{6}$$

$$0 \tag{7}$$

$$-y_{i,1}\left(\lambda_1-1\right)\lambda_1 p_{i,1}^{-1}\left(\sum_{j\in\tau_1} p_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j\in\tau_1} p_{ij}\boldsymbol{x'_{ij}}\right) \tag{8}$$

$$-y_{i,1}\left(\lambda_1-1\right)\lambda_2 p_{i,1}^{-1}\left(\sum_{j\in\tau_1} p_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j\in\tau_2} p_{ij}\boldsymbol{x'_{ij}}\right) \tag{9}$$

$$\begin{gathered} y_{i,1}\left(\lambda_1-1\right)p_{i,1}^{-1}\left(\sum_{j\in\tau_1} p_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j\in\tau_1} y_{ij}\boldsymbol{x'_{ij}}\right) \\ + \\ y_{i,1}\left(\lambda_1-1\right)p_{i,1}^{-1}\left(\sum_{j\in\tau_1} p_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j\in\tau_2} y_{ij}\boldsymbol{x'_{ij}}\right) \end{gathered} \tag{10}$$

Product between second element with the five addend:

$$0 \tag{11}$$

$$\left(1-y_{i,1}\right)\left(\lambda_2-1\right)^2 p_{i,2}^{-2}\left(\sum_{j\in\tau_2} p_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j\in\tau_2} p_{ij}\boldsymbol{x'_{ij}}\right) \tag{12}$$

$$-\left(1-y_{i,1}\right)\left(\lambda_2-1\right)\lambda_1 p_{i,2}^{-1}\left(\sum_{j\in\tau_2} p_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j\in\tau_1} p_{ij}\boldsymbol{x'_{ij}}\right) \tag{13}$$

$$-\left(1-y_{i,1}\right)\left(\lambda_2-1\right)\lambda_2 p_{i,2}^{-1}\left(\sum_{j\in\tau_2} p_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j\in\tau_2} p_{ij}\boldsymbol{x'_{ij}}\right) \tag{14}$$

$$\begin{gathered} \left(1-y_{i,1}\right)\left(\lambda_2-1\right)p_{i,2}^{-1}\left(\sum_{j\in\tau_2} p_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j\in\tau_1} y_{ij}\boldsymbol{x'_{ij}}\right) \\ + \\ \left(1-y_{i,1}\right)\left(\lambda_2-1\right)p_{i,2}^{-1}\left(\sum_{j\in\tau_2} p_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j\in\tau_2} y_{ij}\boldsymbol{x'_{ij}}\right) \end{gathered} \tag{15}$$

Product between third element with the five addend:

$$-y_{i,1}\left(\lambda_1-1\right)\lambda_1 p_{i,1}^{-1}\left(\sum_{j\in\tau_1}p_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j\in\tau_1}p_{ij}\boldsymbol{x'_{ij}}\right) \tag{16}$$

$$-\left(1-y_{i,1}\right)\left(\lambda_2-1\right)\lambda_1 p_{i,2}^{-1}\left(\sum_{j\in\tau_1}p_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j\in\tau_2}p_{ij}\boldsymbol{x'_{ij}}\right) \tag{17}$$

$$\lambda_1^2\left(\sum_{j\in\tau_1}p_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j\in\tau_1}p_{ij}\boldsymbol{x'_{ij}}\right) \tag{18}$$

$$\lambda_1\lambda_2\left(\sum_{j\in\tau_1}p_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j\in\tau_2}p_{ij}\boldsymbol{x'_{ij}}\right) \tag{19}$$

$$-\lambda_1\left(\sum_{j\in\tau_1}p_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j\in\tau_1}y_{ij}\boldsymbol{x'_{ij}}\right)-\lambda_1\left(\sum_{j\in\tau_1}p_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j\in\tau_2}y_{ij}\boldsymbol{x'_{ij}}\right) \tag{20}$$

Product between fourth element with the five addend:

$$-y_{i,1}\left(\lambda_1-1\right)\lambda_2 p_{i,1}^{-1}\left(\sum_{j\in\tau_2}p_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j\in\tau_1}p_{ij}\boldsymbol{x'_{ij}}\right) \tag{21}$$

$$-\left(1-y_{i,1}\right)\left(\lambda_2-1\right)\lambda_2 p_{i,2}^{-1}\left(\sum_{j\in\tau_2}p_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j\in\tau_2}p_{ij}\boldsymbol{x'_{ij}}\right) \tag{22}$$

$$\lambda_1\lambda_2\left(\sum_{j\in\tau_2}p_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j\in\tau_1}p_{ij}\boldsymbol{x'_{ij}}\right) \tag{23}$$

$$\lambda_2^2\left(\sum_{j\in\tau_2}p_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j\in\tau_2}p_{ij}\boldsymbol{x'_{ij}}\right) \tag{24}$$

$$-\lambda_2\left(\sum_{j\in\tau_2}p_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j\in\tau_1}y_{ij}\boldsymbol{x'_{ij}}\right)-\lambda_2\left(\sum_{j\in\tau_2}p_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j\in\tau_2}y_{ij}\boldsymbol{x'_{ij}}\right) \tag{25}$$

Product between fifth element with the five addend:

$$\begin{gathered} y_{i,1}\left(\lambda_1-1\right)p_{i,1}^{-1}\left(\sum\nolimits_{j\in\tau_1}y_{ij}\boldsymbol{x_{ij}}\right)\left(\sum\nolimits_{j\in\tau_1}p_{ij}\boldsymbol{x'_{ij}}\right) \\ + \\ y_{i,1}\left(\lambda_1-1\right)p_{i,1}^{-1}\left(\sum\nolimits_{j\in\tau_2}y_{ij}\boldsymbol{x_{ij}}\right)\left(\sum\nolimits_{j\in\tau_1}p_{ij}\boldsymbol{x'_{ij}}\right) \end{gathered} \tag{26}$$

$$\begin{gathered} \left(1-y_{i,1}\right)\left(\lambda_2-1\right)p_{i,2}^{-1}\left(\sum\nolimits_{j\in\tau_1}y_{ij}\boldsymbol{x_{ij}}\right)\left(\sum\nolimits_{j\in\tau_2}p_{ij}\boldsymbol{x'_{ij}}\right) \\ + \\ \left(1-y_{i,1}\right)\left(\lambda_2-1\right)p_{i,2}^{-1}\left(\sum\nolimits_{j\in\tau_2}y_{ij}\boldsymbol{x_{ij}}\right)\left(\sum\nolimits_{j\in\tau_2}p_{ij}\boldsymbol{x'_{ij}}\right) \end{gathered} \tag{27}$$

$$-\lambda_1\left(\sum_{j\in\tau_1}y_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j\in\tau_1}p_{ij}\boldsymbol{x'_{ij}}\right)-\lambda_1\left(\sum_{j\in\tau_2}y_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j\in\tau_1}p_{ij}\boldsymbol{x'_{ij}}\right) \tag{28}$$

$$-\lambda_2\left(\sum_{j\in\tau_1}y_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j\in\tau_2}p_{ij}\boldsymbol{x'_{ij}}\right)-\lambda_2\left(\sum_{j\in\tau_2}y_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j\in\tau_2}p_{ij}\boldsymbol{x'_{ij}}\right) \tag{29}$$

$$\left(\sum_{j=1}^{J}y_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j=1}^{J}y_{ij}\boldsymbol{x'_{ij}}\right) \tag{30}$$

Given that $E\left[y_{ij}\right]=p_{ij}$ will be classified according to this expression. Besides $y_{i,A}\sum_{\tau_a}y_{ij}=y_{ij}$ si $a=A$ and zero in other case. Thus:

Coefficients of $\left(\sum_{j\in\tau_1}p_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j\in\tau_1}p_{ij}\boldsymbol{x'_{ij}}\right)$

$$y_{i,1}\left(\lambda_1-1\right)^2p_{i,1}^{-2} \tag{31}$$

$$-2y_{i,1}\left(\lambda_1-1\right)\lambda_1p_{i,1}^{-1} \tag{32}$$

$$2\left(\lambda_1-1\right)p_{i,1}^{-1} \tag{33}$$

$$\lambda_1^2 \tag{34}$$

$$-2\lambda_1 \tag{35}$$

The previous equations cannot be reduced until the expected values are calculated. Thus, it is given that:

$$y_{i,1}\left(\lambda_1-1\right)^2p_{i,1}^{-2}-2y_{i,1}\left(\lambda_1-1\right)\lambda_1p_{i,1}^{-1}+2\left(\lambda_1-1\right)p_{i,1}^{-1}+\lambda_1^2--2\lambda_1 \tag{36}$$

Coefficients of $\left(\sum_{j\in\tau_2}p_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j\in\tau_2}p_{ij}\boldsymbol{x'_{ij}}\right)$

Note that coefficients are similar to $\left(\sum_{j\in\tau_1} p_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j\in\tau_1} p_{ij}\boldsymbol{x'_{ij}}\right)$. Therefore it is given that

$$y_{i,2}\left(\lambda_2-1\right)^2 p_{i,2}^{-2} \tag{37}$$

$$-2y_{i,2}\left(\lambda_2-1\right)\lambda_2 p_{i,2}^{-1} \tag{38}$$

$$2\left(\lambda_2-1\right)p_{i,2}^{-1} \tag{39}$$

$$\lambda_2^2 \tag{40}$$

$$-2\lambda_2 \tag{41}$$

Similarly, the previous equatios cannot be itemized until the calculation of the expected values, for which

$$y_{i,2}\left(\lambda_2-1\right)^2 p_{i,2}^{-2}-2y_{i,2}\left(\lambda_2-1\right)\lambda_2 p_{i,2}^{-1}+2\left(\lambda_2-1\right)p_{i,2}^{-1}+\lambda_2^2-2\lambda_2 \tag{42}$$

Coefficients of $\left(\sum_{j\in\tau_1} p_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j\in\tau_2} p_{ij}\boldsymbol{x'_{ij}}\right)$ The coefficients are:

$$-y_{i,1}\left(\lambda_1-1\right)\lambda_2 p_{i,1}^{-1} \tag{43}$$

$$y_{i,1}\left(\lambda_1-1\right)p_{i,1}^{-1} \tag{44}$$

$$-y_{i,2}\left(\lambda_2-1\right)\lambda_1 p_{i,2}^{-1} \tag{45}$$

$$y_{i,2}\left(\lambda_2-1\right)p_{i,2}^{-1} \tag{46}$$

$$\lambda_1\lambda_2 \tag{47}$$

$$-\lambda_1 \tag{48}$$

$$-\lambda_2 \tag{49}$$

Note that (43) and (46) are zero because of the term that is accompanying them. In this manner the previous coefficients can be expresse.d as follows:

$$-y_{i,1}\left(\lambda_1-1\right)\lambda_2 p_{i,1}^{-1}-y_{i,2}\left(\lambda_2-1\right)\lambda_1 p_{i,2}^{-1}+\lambda_1\lambda_2-\lambda_1-\lambda_2 \tag{50}$$

Coefficients of $\left(\sum_{j\in\tau_2} p_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j\in\tau_1} p_{ij}\boldsymbol{x'_{ij}}\right)$ The coefficients are:

$$-y_{i,2}\left(\lambda_2-1\right)\lambda_1 p_{i,2}^{-1} \tag{51}$$

$$y_{i,2}\left(\lambda_2-1\right)p_{i,2}^{-1} \tag{52}$$

$$-y_{i,1}\left(\lambda_1-1\right)\lambda_2 p_{i,1}^{-1} \tag{53}$$

$$y_{i,1}\left(\lambda_1-1\right)p_{i,1}^{-1} \tag{54}$$

$$\lambda_1\lambda_2 \tag{55}$$

$$-\lambda_1 \tag{56}$$

$$-\lambda_2 \tag{57}$$

Similarly (52) and (54) become zero because of the term that is accompanying them. Therefore, the previous coefficients can be expressed as follows:

$$-y_{i,2}\left(\lambda_2-1\right)\lambda_1 p_{i,2}^{-1}-y_{i,1}\left(\lambda_1-1\right)\lambda_2 p_{i,1}^{-1}+\lambda_1\lambda_2-\lambda_1-\lambda_2 \tag{58}$$

Note that the only element which is not itemized is the one given by Equation 30.

- $\left(\frac{\partial l_i}{\partial\boldsymbol{\beta}}\right)\left(\frac{\partial l_i}{\partial\lambda_1}\right)' y \left(\frac{\partial l_i}{\partial\boldsymbol{\beta}}\right)\left(\frac{\partial l_i}{\partial\lambda_2}\right)'$

$\left(\frac{\partial l_i}{\partial \boldsymbol{\beta}}\right)\left(\frac{\partial l_i}{\partial \lambda_1}\right)'$ can be expressed as the sum of the following 10 terms:

$$y_{i,1}\left(\lambda_1 - 1\right) p_{i,1}^{-1} V_{i,1} y_{i,1} \sum_{j\in\tau_1} p_{ij}\boldsymbol{x_{ij}} \tag{59}$$

$$-y_{i,1}\left(\lambda_1 - 1\right) p_{i,1}^{-1} V_{i,1} p_{i,1} \sum_{j\in\tau_1} p_{ij}\boldsymbol{x_{ij}} \tag{60}$$

$$y_{i,2}\left(\lambda_2 - 1\right) p_{i,2}^{-1} V_{i,1} y_{i,1} \sum_{j\in\tau_2} p_{ij}\boldsymbol{x_{ij}} \tag{61}$$

$$-y_{i,2}\left(\lambda_2 - 1\right) p_{i,2}^{-1} V_{i,1} p_{i,1} \sum_{j\in\tau_2} p_{ij}\boldsymbol{x_{ij}} \tag{62}$$

$$\lambda_1 V_{i,1} y_{i,1} \sum_{j\in\tau_1} p_{ij}\boldsymbol{x_{ij}} \tag{63}$$

$$-\lambda_1 V_{i,1} p_{i,1} \sum_{j\in\tau_1} p_{ij}\boldsymbol{x_{ij}} \tag{64}$$

$$\lambda_2 V_{i,1} y_{i,1} \sum_{j\in\tau_2} p_{ij}\boldsymbol{x_{ij}} \tag{65}$$

$$-\lambda_2 V_{i,1} p_{i,1} \sum_{j\in\tau_2} p_{ij}\boldsymbol{x_{ij}} \tag{66}$$

$$V_{i,1} y_{i,1} \sum_{j=1}^{J} y_{ij}\boldsymbol{x_{ij}} \tag{67}$$

$$-V_{i,1} p_{i,1} \sum_{j=1}^{J} y_{ij}\boldsymbol{x_{ij}} \tag{68}$$

Note that equations (59) y (60) can be reduced and factorized as:

$$\left(y_{i,1} - y_{i,1}p_{i,1}\right)\left(\lambda_1 - 1\right) p_{i,1}^{-1} V_{i,1} \sum_{j\in\tau_1} p_{ij}\boldsymbol{x_{ij}} \tag{69}$$

Remember that $y_{i,1}\sum_{j\in\tau_1} y_{ij}\boldsymbol{x_{ij}} = \sum_{j\in\tau_1} y_{ij}\boldsymbol{x_{ij}}$, in this manner equations (67) y (68) and can be extended as follows:

$$\left(1 - p_{i,1}\right) V_{i,1} \sum_{j\in\tau_1} y_{ij}\boldsymbol{x_{ij}} + V_{i,1} y_{i,1} \sum_{j\in\tau_2} y_{ij}\boldsymbol{x_{ij}} - V_{i,1} p_{i,1} \sum_{j\in\tau_2} y_{ij}\boldsymbol{x_{ij}} \tag{70}$$

Finally

$$\left(\frac{\partial l_i}{\partial \boldsymbol{\beta}}\right)\left(\frac{\partial l_i}{\partial \lambda_1}\right)' = (69) + (61) + (62) + (63) + (64) + (65) + (66) + (70) \tag{71}$$

Similarly $\left(\frac{\partial l_i}{\partial \boldsymbol{\beta}}\right)\left(\frac{\partial l_i}{\partial \lambda_2}\right)'$ can be expressed as the sum of the following 10 terms:

$$y_{i,1}\left(\lambda_1 - 1\right) p_{i,1}^{-1} V_{i,2} y_{i,2} \sum_{j\in\tau_1} p_{ij}\boldsymbol{x_{ij}} \tag{72}$$

$$-y_{i,1}\left(\lambda_1 - 1\right) p_{i,1}^{-1} V_{i,2} p_{i,2} \sum_{j\in\tau_1} p_{ij}\boldsymbol{x_{ij}} \tag{73}$$

$$y_{i,2}\left(\lambda_2 - 1\right) p_{i,2}^{-1} V_{i,2} y_{i,2} \sum_{j\in\tau_2} p_{ij}\boldsymbol{x_{ij}} \tag{74}$$

$$-y_{i,2}\left(\lambda_2 - 1\right) p_{i,2}^{-1} V_{i,2} p_{i,2} \sum_{j\in\tau_2} p_{ij}\boldsymbol{x_{ij}} \tag{75}$$

$$\lambda_1 V_{i,2} y_{i,2} \sum_{j\in\tau_1} p_{ij}\boldsymbol{x_{ij}} \tag{76}$$

$$-\lambda_1 V_{i,2} p_{i,2} \sum_{j\in\tau_1} p_{ij}\boldsymbol{x_{ij}} \tag{77}$$

$$\lambda_2 V_{i,2} y_{i,2} \sum_{j\in\tau_2} p_{ij}\boldsymbol{x_{ij}} \tag{78}$$

$$-\lambda_2 V_{i,2} p_{i,2} \sum_{j\in\tau_2} p_{ij}\boldsymbol{x_{ij}} \tag{79}$$

$$V_{i,2} y_{i,2} \sum_{j=1}^{J} y_{ij}\boldsymbol{x_{ij}} \tag{80}$$

$$-V_{i,2} p_{i,2} \sum_{j=1}^{J} y_{ij}\boldsymbol{x_{ij}} \tag{81}$$

So (74) and (75) can be reduced and factorized as:

$$\left(y_{i,2} - y_{i,2}p_{i,2}\right)\left(\lambda_2 - 1\right) p_{i,2}^{-1} V_{i,2} \sum_{j\in\tau_2} p_{ij}\boldsymbol{x_{ij}} \tag{82}$$

Remember that $y_{i,2}\sum_{j\in\tau_2} y_{ij}\boldsymbol{x_{ij}} = \sum_{j\in\tau_2} y_{ij}\boldsymbol{x_{ij}}$ In this way, (80) and (81) can be expressed as:

$$\left(1 - p_{i,2}\right) V_{i,2} \sum_{j\in\tau_2} y_{ij}\boldsymbol{x_{ij}} + V_{i,2} y_{i,2} \sum_{j\in\tau_1} y_{ij}\boldsymbol{x_{ij}} - V_{i,2} p_{i,2} \sum_{j\in\tau_1} y_{ij}\boldsymbol{x_{ij}} \tag{83}$$

Therefore

$$\left(\frac{\partial l_i}{\partial \boldsymbol{\beta}}\right)\left(\frac{\partial l_i}{\partial \lambda_2}\right)' = (72) + (73) + (82) + (76) + (77) + (78) + (79) + (83) \tag{84}$$

- $\left(\frac{\partial l_i}{\partial \lambda_1}\right)\left(\frac{\partial l_i}{\partial \lambda_1}\right)' \; y \left(\frac{\partial l_i}{\partial \lambda_2}\right)\left(\frac{\partial l_i}{\partial \lambda_2}\right)'$. For the calculation of this product note that

$$\left(\frac{\partial l_i}{\partial \lambda_1}\right)' = \left(\frac{\partial l_i}{\partial \lambda_1}\right) \quad \text{y} \quad \left(\frac{\partial l_i}{\partial \lambda_2}\right)' = \left(\frac{\partial l_i}{\partial \lambda_2}\right) \tag{85}$$

  By calculating the corresponding product it is given

$$\left(\frac{\partial l_i}{\partial \lambda_1}\right)\left(\frac{\partial l_i}{\partial \lambda_1}\right)' = V_{i,1}^2 \left(y_{i,1} - p_{i,1}\right)^2 = V_{i,1}\left(y_{i,1} - 2y_{i,1}p_{i,1} + p_{i,1}^2\right) \tag{86}$$

  Analogously

$$\left(\frac{\partial l_i}{\partial \lambda_2}\right)\left(\frac{\partial l_i}{\partial \lambda_2}\right)' = V_{i,2}\left(y_{i,2} - 2y_{i,2}p_{i,2} + p_{i,2}^2\right) \tag{87}$$

- $\left(\frac{\partial l_i}{\partial \lambda_1}\right)\left(\frac{\partial l_i}{\partial \lambda_2}\right)' \; y \left(\frac{\partial l_i}{\partial \lambda_2}\right)\left(\frac{\partial l_i}{\partial \lambda_1}\right)'$. The calculation of this product is more simple, because when (4) with (5) are multiplied it is given:

$$\begin{aligned} \left(\frac{\partial l_i}{\partial \lambda_1}\right)\left(\frac{\partial l_i}{\partial \lambda_2}\right)' &= \left(\frac{\partial l_i}{\partial \lambda_2}\right)\left(\frac{\partial l_i}{\partial \lambda_1}\right)' \\ &= V_{i,1}V_{i,2}\left(y_{i,1} - p_{i,1}\right)\left(1 - y_{i,1} - p_{i,2}\right) \end{aligned} \tag{88}$$

S.4 CALCULATION OF EXPECTED VALUES

Calculating the expected values has to be done on each one of the products of the partial derivatives calculated in the previous section. In order to do this, it is given that $p_{i,1} = 1 - p_{i,2}$, $E\left[y_{ij}\right] = p_{ij}$, $E\left[y_{i,1}\right] = p_{i,1}$ y $E\left[y_{i,2}\right] = p_{i,2}$.

- $E\left[\left(\frac{\partial l_i}{\partial \boldsymbol{\beta}}\right)\left(\frac{\partial l_i}{\partial \boldsymbol{\beta}}\right)'\right]$. The expected value of this item is divided on the sum of the individual expected values of the terms that have associated the coefficients described by (30), (36), (50), (58) and (42). In this manner it is given that:

  $E\left[((30))\right]$

$$E\left[\left(\sum_{j=1}^{J} y_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j=1}^{J} y_{ij}\boldsymbol{x}'_{\boldsymbol{ij}}\right)\right] = E\left[\sum_{j=1}^{J} y_{ij}\boldsymbol{x_{ij}}\boldsymbol{x}'_{\boldsymbol{ij}}\right] \tag{89}$$

$$= \sum_{j=1}^{J} p_{ij}\boldsymbol{x_{ij}}\boldsymbol{x}'_{\boldsymbol{ij}} \tag{90}$$

$E\left[((36))\left(\sum_{j\in\tau_1} p_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j\in\tau_1} p_{ij}\boldsymbol{x'_{ij}}\right)\right]$

$$\left[p_{i,1}^{-1}\left(\lambda_1^2\left(1-p_{i,1}\right)-1\right)\right]\left(\sum_{j\in\tau_1} p_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j\in\tau_1} p_{ij}\boldsymbol{x'_{ij}}\right) \tag{91}$$

$E\left[((42))\left(\sum_{j\in\tau_2} p_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j\in\tau_2} p_{ij}\boldsymbol{x'_{ij}}\right)\right]$

$$\left[p_{i,2}^{-1}\left(\lambda_2^2\left(1-p_{i,2}\right)-1\right)\right]\left(\sum_{j\in\tau_2} p_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j\in\tau_2} p_{ij}\boldsymbol{x'_{ij}}\right) \tag{92}$$

$E\left[((50))\left(\sum_{j\in\tau_1} p_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j\in\tau_2} p_{ij}\boldsymbol{x'_{ij}}\right)\right]$

$$-\left[\lambda_1\lambda_2\right]\left(\sum_{j\in\tau_1} p_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j\in\tau_2} p_{ij}\boldsymbol{x'_{ij}}\right) \tag{93}$$

$E\left[((58))\left(\sum_{j\in\tau_2} p_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j\in\tau_1} p_{ij}\boldsymbol{x'_{ij}}\right)\right]$

$$-\left[\lambda_1\lambda_2\right]\left(\sum_{j\in\tau_2} p_{ij}\boldsymbol{x_{ij}}\right)\left(\sum_{j\in\tau_1} p_{ij}\boldsymbol{x'_{ij}}\right) \tag{94}$$

Therefore it is obtained the expression

$$E\left[\left(\frac{\partial l_i}{\partial \boldsymbol{\beta}}\right)\left(\frac{\partial l_i}{\partial \boldsymbol{\beta}}\right)'\right] = (90)+(91)+(92)+(93)+(94) \tag{95}$$

- $E\left[\left(\frac{\partial l_i}{\partial \boldsymbol{\beta}}\right)\left(\frac{\partial l_i}{\partial \lambda_1}\right)'\right]$. Note that most of the elements of Equation 71 to calculate its expected value become 0, with the exception of the expected value of the terms given by (69) and (70). In this manner it is given that:

$$E\left[\left(\frac{\partial l_i}{\partial \boldsymbol{\beta}}\right)\left(\frac{\partial l_i}{\partial \lambda_1}\right)'\right] = \begin{array}{c}\left(p_{i,c}-p_{i,1}^2\right)\left(\lambda_1-1\right)p_{i,1}^{-1}V_{i,1}\sum_{j\in\tau_1} p_{ij}\boldsymbol{x_{ij}} \\ + \\ \left(1-p_{i,1}\right)V_{i,1}\sum_{j\in\tau_1} y_{ij}\boldsymbol{x_{ij}}\end{array} \tag{96}$$

But Equation 96 can be simplified to:

$$E\left[\left(\frac{\partial l_i}{\partial \boldsymbol{\beta}}\right)\left(\frac{\partial l_i}{\partial \lambda_1}\right)'\right] = \left(1-p_{i,1}\right)\lambda_1 V_{i,1}\sum_{j\in\tau_1} p_{ij}\boldsymbol{x_{ij}} \tag{97}$$

- $E\left[\left(\frac{\partial l_i}{\partial \boldsymbol{\beta}}\right)\left(\frac{\partial l_i}{\partial \lambda_2}\right)'\right]$. This element is analogous to the previous case, for which

$$E\left[\left(\frac{\partial l_i}{\partial \boldsymbol{\beta}}\right)\left(\frac{\partial l_i}{\partial \lambda_2}\right)'\right] = (1 - p_{i,2})\,\lambda_2 V_{i,2} \sum_{j \in \tau_2} p_{ij} \boldsymbol{x_{ij}} \tag{98}$$

- $E\left[\left(\frac{\partial l_i}{\partial \lambda_1}\right)\left(\frac{\partial l_i}{\partial \lambda_1}\right)'\right]$, $E\left[\left(\frac{\partial l_i}{\partial \lambda_2}\right)\left(\frac{\partial l_i}{\partial \lambda_2}\right)'\right]$ y $E\left[\left(\frac{\partial l_i}{\partial \lambda_1}\right)\left(\frac{\partial l_i}{\partial \lambda_2}\right)'\right]$. These expected values are easier to estimate, for these cases it is given that

$$E\left[\left(\frac{\partial l_i}{\partial \lambda_1}\right)\left(\frac{\partial l_i}{\partial \lambda_1}\right)'\right] = \left(p_{i,1} - p_{i,1}^2\right) V_{i,1}^2 = p_{i,1} p_{i,2} V_{i,1}^2 \tag{99}$$

$$E\left[\left(\frac{\partial l_i}{\partial \lambda_2}\right)\left(\frac{\partial l_i}{\partial \lambda_2}\right)'\right] = \left(p_{i,2} - p_{i,2}^2\right) V_{i,2}^2 = p_{i,1} p_{i,2} V_{i,2}^2 \tag{100}$$

$$E\left[\left(\frac{\partial l_i}{\partial \lambda_1}\right)\left(\frac{\partial l_i}{\partial \lambda_2}\right)'\right] = E\left[\left(\frac{\partial l_i}{\partial \lambda_2}\right)\left(\frac{\partial l_i}{\partial \lambda_1}\right)'\right] = 0 \tag{101}$$