Supplemental Material: Derivation of the Governing Equations for Horizontal and Vertical Coupling of One- and Two-dimensional Open Channel Flow Models

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1 Introduction

Many textbooks of open-channel flow present the derivation of the one-dimensional (1D) and two-dimensional (2D) shallow water equations (e.g., Liggett, 1994). However, these equations do not include the terms arising at the interfaces between 1D and 2D model regions that describe the transfer of mass and momentum across them when they are coupled. For completeness, herein, the governing equations (mass and momentum) for 1D and 2D models with permeable interfaces are derived from the three-dimensional, incompressible Reynolds-Averaged Navier-Stokes (RANS) equations. The equations are formulated in a Cartesian coordinate system (x, y, z) with the corresponding velocity components (u_x, u_y, u_z). The x-axis is directed downvalley, the y-axis is directed across the valley from right to left, and the z-axis is directed vertically upward.

2 Conservation of mass

The mass conservation equation is presented in the sections below in 1D and 2D forms that can be coupled both horizontally and vertically.

2.1 Two-dimensional mass conservation equation

Integrating the three-dimensional mass conservation equation for an incompressible flow ($\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$) over the water depth h between elevations z_b and z_t (see Figs. 1 and 2) and applying the Leibniz rule, yields the 2D depth-averaged mass conservation equation:

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} - G|_{z_t} + G|_{z_b} = 0 \tag{1}$$

where q_x and q_y are the unit discharges in x- and y-direction, respectively, and G is the mass flux through the surfaces at the top z_t and bottom z_b

For the two coupling procedures introduced below the upper boundary of the 2D region is the water free surface. Using the kinematic condition for a water free surface, the mass flux through the upper surface can be written as:

$$G|_{z_t} = -\frac{\partial \zeta}{\partial t} \tag{2}$$

where ζ is the water surface elevation and t is time. The lower surface of the 2D region is a solid wall (floodplain ground surface) in case of horizontal coupling and can be either a solid wall or permeable interface (between 1D and 2D regions) in case of vertical coupling. The mass transfer through the lower, stationary surface (assumed to be horizontal below) is defined as:

$$G|_{z_b} = \begin{cases} -u_z|_{z_b} & \text{if permeable surface} \\ 0 & \text{if solid surface} \end{cases} \tag{3}$$

2.2 One-dimensional mass conservation equation

Integrating Eq. (1) across the channel between distances y_r and y_l (it is assumed that the channel centerline is aligned with the x-axis and this stream channel direction is denoted hereon as s) and applying the Leibniz rule, yields the 1D mass conservation equation:

$$\frac{\partial Q}{\partial s} - H|_{y_l} + H|_{y_r} - \int_{y_r}^{y_l} \left(G|_{z_t} - G|_{z_b} \right) \mathrm{d}y = 0 \tag{4}$$

where the flow rate is $Q = \int_{y_r}^{y_l} q_s \mathrm{d}y$. The mass flux H through the vertical interfaces defined at the position of the margins for bankfull conditions $y = y_l$ and $y = y_r$ is defined as (for brevity only shown for $y = y_l$):

$$H|_{y_l} = q_s|_{y_l} \frac{\partial y_l}{\partial s} - q_y|_{y_l} \tag{5}$$

In both coupling approaches the bottom surface of the 1D model region is a solid wall (channel bottom), that is $G|_{z_b} = 0$. In case the top surface is a free surface, the integral of $G|_{z_t}$ from Eq. (4) yields

$$-\int_{y_r}^{y_l} G|_{z_t} \mathrm{d}y = \int_{y_r}^{y_l} \frac{\partial \zeta}{\partial t} = \frac{\partial A}{\partial t} \tag{6}$$

where A is cross-sectional area of the flow. Further, it was assumed that the interfaces at y_l and y_r are dependent on time. For an upper free surface the 1D mass conservation equation then reads

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial s} - H|_{y_l} + H|_{y_r} = 0 \tag{7}$$

2.3 Horizontal coupling of 1D and 2D models

Figure 1 illustrates the horizontal coupling procedure. The 1D model region covers the inbank and central overbank flow region, while the 2D model region covers the left and right overbank flow regions. Notice that the two floodplain regions do not directly interact, since mass and momentum need to be transferred through the 1D region.

Equation (7) describes the 1D continuity equation for the 1D region, which is delimited by the channel walls (channel bottom and banks), interfaces with 2D model regions when flow is overbank, and a top free surface. The channel walls are assumed to be impermeable. The mass flux H between 1D and 2D regions is zero when the flow is inbank and equals the unit normal discharge through the 1D-2D interface, q_n, when flow is overbank, that is:

$$H = \begin{cases} -q_n & \text{if } \zeta > z_{fp} \\ 0 & \text{if } \zeta \leq z_{fp} \end{cases} \tag{8}$$

where z_{fp} is the floodplain elevation at the channel margin.

The 2D region is bounded by the water surface at the top, a solid wall at the bottom (floodplain ground surface), and a permeable vertical interface with the 1D region. The 2D mass conservation equation becomes the standard 2D shallow water equation (e.g., Eq. (2.79) in Wu (2008)).

2.4 Vertical coupling of 1D and 2D models

Figure 2 illustrates the vertical coupling approach. The 1D model region covers the inbank region, while the 2D model region overlies the 1D region and covers the entire overbank flow region (left, central, and right).

The 1D region is bounded by the bottom and banks and a top permeable interface with the 2D model region. Eq. (4) can then be written as:

$$\frac{\partial Q}{\partial s} = \int_{y_r}^{y_l} G|_{z_t} \mathrm{d}y \tag{9}$$

where the right-hand side of Eq. (9) is defined as:

$$-\int_{y_r}^{y_l} G|_{z_t} \mathrm{d}y = \begin{cases} \dfrac{\partial A}{\partial t} & \text{if } \zeta < z_{fp} \\ B_{z_t}\tilde{u}_z & \text{if } \zeta \geq z_{fp} \end{cases} \tag{10}$$

where B_{z_t} is the channel top width and $\tilde{u}_z$ is the mean vertical velocity at the interface between 1D and 2D model regions. The tilde denotes the average value over the width of the 1D-2D interface. If the interface is not horizontal, the normal velocity to it should be used instead.

The transfer of mass across the interface with the 2D region can be represented by a vertical column of water with length d. The mass conservation equation (the combination of Eqs 9 and 10) can then be written in a more generic form as:

$$B\frac{\partial \eta}{\partial t} + \frac{\partial Q}{\partial s} = 0 \tag{11}$$

where B is the channel width of the flow in the 1D region and

$$\eta = \begin{cases} \zeta & \text{if } \zeta < z_{fp} \\ \tilde{d} & \text{if } \zeta \geq z_{fp} \end{cases} \tag{12}$$

when the transfer of mass across the transversal direction of the 1D flow is approximately constant, $\tilde{d}$ becomes d. $\tilde{d}$ or d is the coupling term in the vertical approach and replaces h in the 1D equations, leaving the same number of unknowns as equations. Therefore, the vertical coupling approach computes $\tilde{d}$ or d during the same process of solving the set of equations.

The 2D model region is bounded by the water free surface, a wall on the floodplain ground surface, and a permeable interface above the channel. Eq. (1) is then written as:

$$\frac{\partial h}{\partial t} + \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + G|_{z_b} = 0 \tag{13}$$

where $G|_{z_b}$ is given by Eq. (3) in which the vertical velocity $u_z|_{z_b}$ can be replaced by $\partial\tilde{d}/\partial t$. Further, $G|_{z_t}$ was represented by Eq. (2) with ζ replaced by h.

3 Conservation of momentum

In the following analysis, shallow water hypothesis is assumed, that is, the horizontal length scale is much larger than the vertical length scale. The vertical momentum equation can then be simplified as $\partial p/\partial z = -\rho g$, which after integration yields the hydrostatic pressure distribution $p = \rho g(\zeta - z)$. Further, the Reynolds stresses τ_{xz} and τ_{yz} are much larger than τ_{xx}, τ_{xy}, and τ_{yy}. The normal Reynolds stresses are therefore omitted, however τ_{xy} is retained as it can be important near vertical walls.

3.1 Two-dimensional momentum conservation equation

Integrating the momentum equations in horizontal direction (for example, Eq. (4.15) in Kundu (2010) for $i = 1, 2$) over the water depth h between elevations z_b and z_t, applying the Leibniz rule, and assuming the shallow water hypothesis, yields the following 2D depth-averaged momentum equations in x- and y-direction:

$$\frac{\partial q_x}{\partial t} + \frac{\partial(\beta_{xx}\bar{u}_x q_x)}{\partial x} + \frac{\partial(\beta_{xy}\bar{u}_y q_x)}{\partial y} + gh\frac{\partial \zeta}{\partial x} - u_x|_{z_t} G|_{z_t} + u_x|_{z_b} G|_{z_b} = \frac{1}{\rho}\left[\frac{\partial h\bar{\tau}_{xy}}{\partial y} + \tau_x|_{z_t} - \tau_x|_{z_b}\right] \tag{14}$$

$$\frac{\partial q_y}{\partial t} + \frac{\partial(\beta_{xy}\bar{u}_x q_y)}{\partial x} + \frac{\partial(\beta_{yy}\bar{u}_y q_y)}{\partial y} + gh\frac{\partial \zeta}{\partial y} - u_y|_{z_t} G|_{z_t} + u_y|_{z_b} G|_{z_b} = \frac{1}{\rho}\left[\frac{\partial h\bar{\tau}_{xy}}{\partial x} + \tau_y|_{z_t} - \tau_y|_{z_b}\right] \tag{15}$$

where the overbar denotes depth-averaged variables, β is the momentum correction factor (assumed equal to one hereafter), and τ_x and τ_y are shear stresses acting on the top and bottom surface in x- and y-direction, respectively.

The upper surface ($z = z_t$) in the 2D computational region is the water free surface and is assumed frictionless, that is $\tau_i|_{z_t} = 0$. The lower surface can be either the interface between the 1D and 2D computational regions (in case of vertical coupling) or is the the floodplain ground surface. In the latter case the bottom shear stress is modeled using the friction slope S_f, that is

$$\tau_i|_{z_b} = \rho g h S_{f_i} \tag{16}$$

3.2 One-dimensional momentum conservation equation

Assuming the channel is aligned with the x-axis and denoting this stream channel direction by s, the 1D momentum conservation equation is obtained by integrating Eq. (14) across the channel between lateral distances y_r and y_l. Applying the Leibniz rule, substituting Eq. (16), assuming a solid channel bed ($G|_{z_b} = 0$) and rearranging terms yields:

$$\frac{\partial Q}{\partial t} + \frac{\partial UQ}{\partial s} + gA\left(\frac{\partial \tilde{\zeta}}{\partial s} + S_f\right) - (\bar{u}_s H)|_{y_l} + (\bar{u}_s H)|_{y_r} - (B\tilde{u}_s\tilde{G})|_{z_t} - \frac{1}{\rho}\left[(h\bar{\tau}_s)|_{y_l} - (h\bar{\tau}_s)|_{y_r} + (B\tilde{\tau}_s)|_{z_t}\right] = 0 \tag{17}$$

where U is the cross-sectional average velocity weighted by the unit discharge and $\tilde{\tau}_s = \int_{y_r}^{y_l} \tau_s \mathrm{d}y/B$ is the average streamwise shear stress. It was assumed that cross stream variations of the term $u_z G$ at the upper surface z_t are negligible and variables can be represented by their laterally-averaged values. The shear stresses acting on the solid margins y_l and y_r are incorporated into the friction slope S_f.

3.3 Horizontal coupling of 1D and 2D models

In the horizontal coupling method, the 1D model region has at the top the water free surface. The 1D momentum conservation equation (Eq. 17) reduces to:

$$\frac{\partial Q}{\partial t} + \frac{\partial UQ}{\partial s} + gA\left(\frac{\partial \zeta}{\partial s} + S_f\right) - (\bar{u}_s H)|_{y_r}^{y_l} - \frac{1}{\rho}(h\bar{\tau}_s)|_{y_r}^{y_l} = 0 \tag{18}$$

where the tilde on the water surface elevation has been dropped as it is assumed to be horizontal in the transverse direction of the 1D model region (note, the 3D numerical model presented in the main paper confirms this assumption for the evaluated compound meandering channel), and the mass flux H is defined by Eq. (8). The last two terms in Eq. (18) account for the transfer of momentum due to mass crossing the 1D-2D interface and to the shear stress acting on the same interface.

The 2D model region is bounded vertically by the free water surface and by the wall represented by the floodplain ground surface. The 2D momentum equation in the x-direction (Eq. 14) reads:

$$\frac{\partial q_x}{\partial t} + \frac{\partial \bar{u}_x q_x}{\partial x} + \frac{\partial \bar{u}_y q_x}{\partial y} + gh\left(\frac{\partial \zeta}{\partial x} + S_{f_x}\right) = \frac{1}{\rho}\frac{\partial h\bar{\tau}_{xy}}{\partial y} \tag{19}$$

A similar equation can be developed for the y-direction. The local momentum flux, $\bar{u}_s H + h\bar{\tau}_s/\rho$ at the interface with the 1D model region is estimated from the 2D solution at the boundary between 1D and 2D domains.

3.4 Vertical coupling of 1D and 2D models

For the vertical coupling approach, the 1D model region is bounded by the channel bottom and banks and either the water free surface when the flow is inbank or a permeable interface with the overlying 2D model region when the flow is overbank. The 1D momentum conservation equation (Eq. 17) reads

$$\frac{\partial Q}{\partial t} + \frac{\partial UQ}{\partial s} + gA\left(\frac{\partial \zeta}{\partial s} + S_f\right) - (B\tilde{u}_s\tilde{G})|_{z_t} - \frac{1}{\rho}\left[(B\tilde{\tau}_s)|_{z_t}\right] = 0 \tag{20}$$

The last two terms in Eq. (20) denote momentum transfer through a surface at $z = z_t$. For a free surface these terms disappear and the classic 1D St. Venant equation is obtained.

The 2D model region is bounded vertically by the water free surface, a solid bottom surface on the floodplain, and a permeable bottom surface above the channel. The momentum conservation equations in x-direction reads:

$$\frac{\partial q_x}{\partial t} + \frac{\partial \bar{u}_x q_x}{\partial x} + \frac{\partial \bar{u}_y q_x}{\partial y} + gh\frac{\partial \zeta}{\partial x} + (u_x G)|z_b = \frac{1}{\rho}\left[\frac{\partial h \bar{\tau}_{xy}}{\partial y} - \tau_x|_{z_b}\right] \tag{21}$$

A similar equation can be develop for the y-direction. The last term on both, left- and right-hand side in the above equation denotes the transfer of momentum through the bottom surface. The flux $G|_{z_b}$ is given by Eq. (3) with the vertical velocity $u_z|_{z_b}$ replaced by $\partial d/\partial t$. On the floodplain ground surface the boundary shear stress is calculated using a friction slope given by Eq. (16).

4 Notation

α_s, α_n = denotes the components of the dummy variable α in the streamwise and transverse directions, respectively
$\tilde{\alpha}, \bar{\alpha}$ = denotes the width-averaged, and depth-averaged value of the dummy variable α, respectively
β_{ij} momentum correction factor $\beta_{ij} = \int_{z_b}^{z_t} u_i u_j dz / h\bar{u}_i\bar{u}_j$ (-)
ϵ = coefficient ϵ that is zero above the floodplain and one above the 1D region
η = representation of the water free surface whether the flow is inbank or overbank (m)
τ_{ij} = Reynolds stress tensor, i and j stand for x, y, z (Pa)
τ_x, τ_y = shear stresses acting on the top and bottom surfaces in x- and y-direction (Pa)
ρ = density of the water (kg m^{-3})
ζ = water surface elevation (m)
A = transversal area to the flow in the in-channel region (m^2)
B = channel top width (m)
d = mass flux per unit area an density in the interface of 1D-2D regions for vertical coupling (m)
g = gravity acceleration (m s^{-2})
$G|_{z_t}, G|_{z_b}$ = flow discharge per unit area in the top and bottom surfaces (m s^{-1})
$H|_{y_l}, H|_{y_r}$ = flow discharge per unit area in the vertical interfaces bordering the in-channel flow (ms^{-1})
h = water depth (m)
M = momentum transference in the interface of 1D-2D regions, per unit streamwise length and density (m^3 s^{-2})
M_a, M_d = advective and diffusive components of M, respectively (m^{3}s^{-2})
p = pressure (Pa)
Q = flow discharge within the channel (m^3 s^{-1})
q_x, q_y = unit flow discharge in x and y (m^2 s^{-1})
S_f = friction slope (-)
U = cross-section average velocity in the 1D region weighted by unit discharge $U = \int_{y_r}^{y_l} \bar{u_x} q_x \mathrm{d}y/Q$ (m s^{-1})
u_x, u_y, u_z = velocities in x, y and z, respectively (m)
z_{fp} = floodplain elevation at the top river banks (m)
z_t, z_b = position (elevation) of the top and bottom, respectively, for the 1D and 2D regions (m)

References

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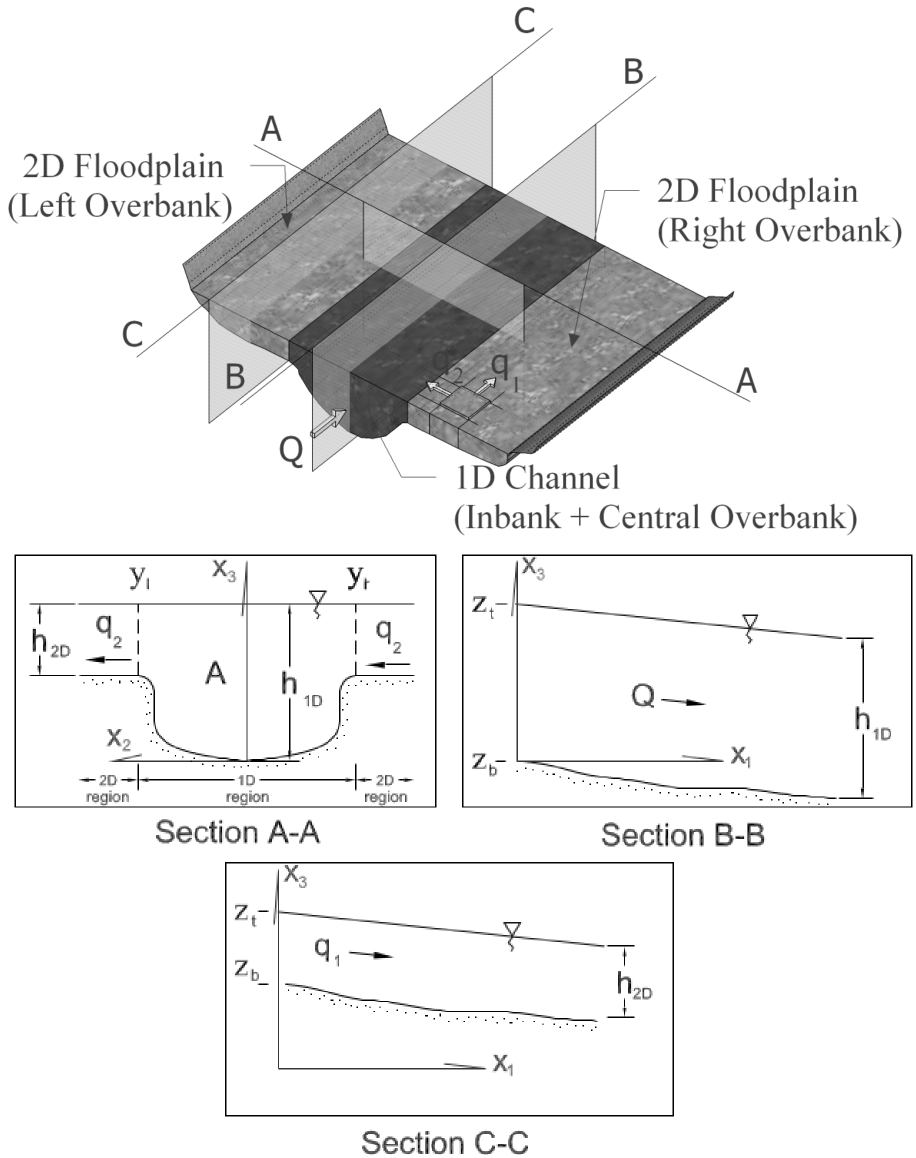

Figure 1: Horizontal coupling method. Top figure: three-dimensional view of the coupling of 1D (inbank+central overbank) and 2D (left+right overbanks) flows. Bottom figures: A-A: transversal cross section, B-B: longitudinal profile along the main channel, and C-C: longitudinal profile along the floodplain channel. Notice that the left floodplain (overbank) is separated from the right floodplain (overbank) region by the 1D region.

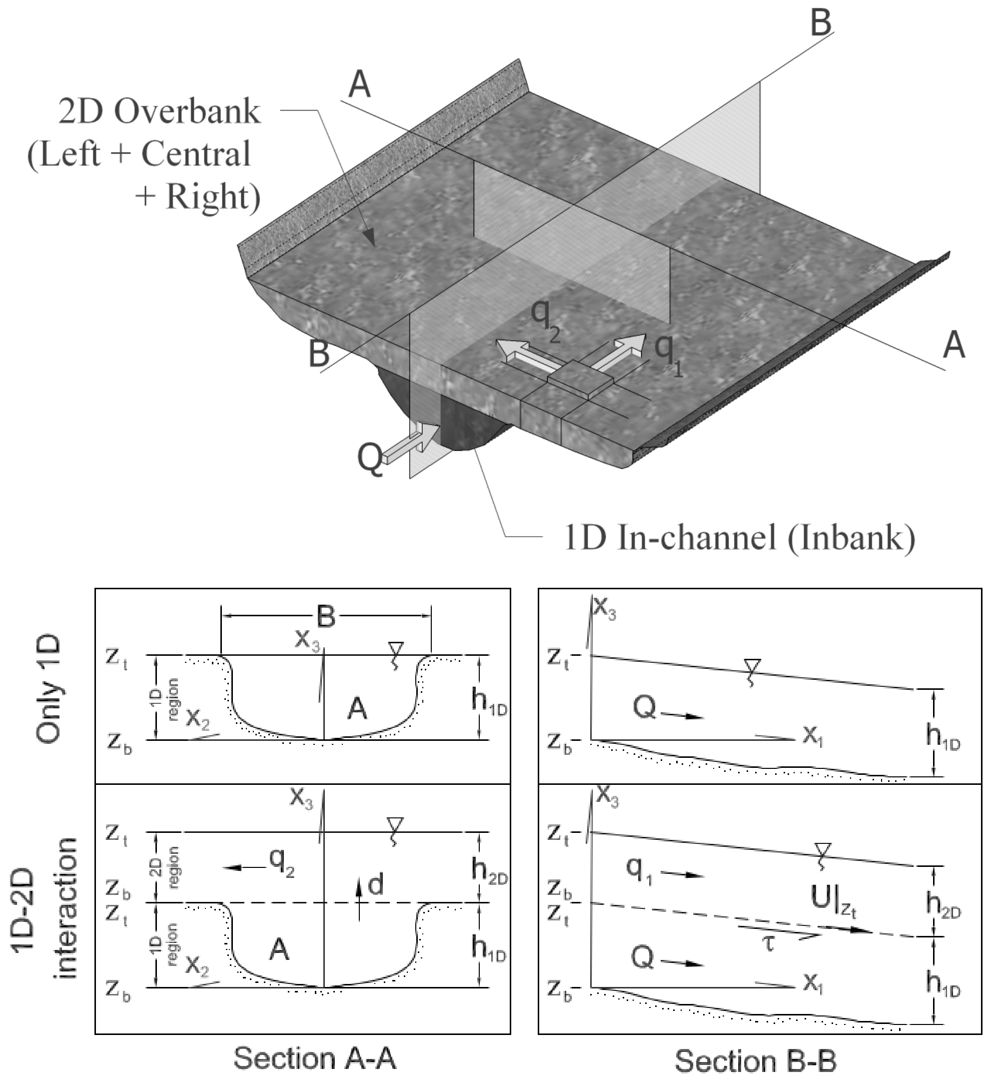

Figure 2: Vertical coupling method. Top figure: three-dimensional view of the coupling of 1D (inbank) and 2D (overbank) flows. Bottom figures: A-A: transversal cross section, B-B: longitudinal profile along the main channel. Notice that the flow above the in-channel region is considered as the overbank (left+central+right) region.