ARTICLE TEMPLATE

On Higher Order Moment and Cumulant Estimation: Supplementary Material

ARTICLE HISTORY
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In this Supplementary Material, we have two sections. The first part gives the detailed derivations of the UMVUEs for the central moments, rawo moments and the cumulants when data are generated from the (negative) Binomial distribution, the Poisson distribution, the discrete/continuous uniform distribution and the exponential distribution. The second part provides extensive simulation studies to compare the performances of our proposed estimators with the maximum likelihood estimators (MLEs), the sample average estimators and the h/k-statistics.

1. Derivation Details

1.1. Binomial Distribution

Consider a sample of independent and identically distributed (i.i.d.) random variables

$$X_i \overset{i.i.d.}{\sim} \text{Binomial}(m, p)\,, i = 1, \ldots, n\,.$$

The sufficient statistics is $\boldsymbol{T} = \bar{X}$. The conditional density function of X_1 given the sufficient statistic $\bar{X}$ is

$$\begin{aligned} f_{X_1|\sum_{i=1}^n X_i = t}(x) &= \frac{f_{X_1, \sum_{i=1}^n X_i}(x, t)}{f_{\sum_{i=1}^n X_i}(t)} \\ &= \frac{f_{X_1}(x) f_{\sum_{i=2}^n X_i}(t-x)}{f_{\sum_{i=1}^n X_i}(t)} \\ &= \frac{\binom{m}{x} p^x (1-p)^{m-x} \binom{(n-1)m}{t-x} p^{t-x} (1-p)^{(n-1)m-(t-x)}}{\binom{nm}{t} p^t (1-p)^{nm-t}} \\ &= \frac{\binom{t}{x}\binom{nm-t}{m-x}}{\binom{nm}{m}}\,, \end{aligned}$$

implying that $X_1|\bar{X} \sim \text{Hypergeometric}(nm, n\bar{X}, m)$. It is well known that for the hypergeometric distribution, we have

$$\mathrm{E}[X_1(X_1-1)\cdots(X_1-k)|\bar{X}] = \bar{X}\,\frac{(m-1)(m-2)\cdots(m-k)(n\bar{X}-1)(n\bar{X}-2)\cdots(n\bar{X}-k)}{(nm-1)(nm-2)\cdots(nm-k)}\,. \tag{1}$$

Therefore, the UMVUEs of the raw moments $\hat{\mu}_1', \hat{\mu}_2', \hat{\mu}_3'$ and $\hat{\mu}_4'$ can be obtained by (1).

For the central moments and cumulants $\hat{\mu}_2 = \hat{\kappa}_2$ and $\hat{\mu}_3 = \hat{\kappa}_3$, we establish the following lemma.

Lemma 1.1. Consider a sample of i.i.d. random variables $X_i, i = 1, 2, \ldots, n$. If $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is a complete, sufficient statistic of the parameter of the distribution of X, then the UMVUEs of the second and third central moment, $\mu_2 = \mathrm{E}\big[(X - \mathrm{E}(X))^2\big]$ and $\mu_3 = \mathrm{E}\big[(X - \mathrm{E}(X))^3\big]$, are

$$\hat{\mu}_k = \hat{\eta}(\mu_k) = \int_{-\infty}^{\infty} C_k (x - \bar{X})^k f_{X_1|\bar{X}}(x)\, dx\,, \quad k = 2, 3\,,$$

respectively, where $C_2 = \frac{n}{n-1}$ and $C_3 = \frac{n^2}{(n-1)(n-2)}$.

Proof. As X_i's are identically distributed, straightforward calculations give

$$\mathrm{E}\Big[C_k (X_1 - \bar{X})^k\Big] = \mu_k, \quad k = 2, 3,$$

that is, $C_k(X_1 - \bar{X})^k$ is an unbiased estimator of μ_k. By the Lehmann-Scheffé theorem, $\mathrm{E}\big[C_k(X_1 - \bar{X})^k | \bar{X}\big]$ is an unbiased estimator that achieves the minimum variance among all unbiased estimators. Therefore,

$$\mathrm{E}\big[C_k (X_1 - \bar{X})^k | \bar{X}\big] = \int_{-\infty}^{\infty} C_k (x - \bar{X})^k f_{X_1|\bar{X}}(x)\, dx, \quad k = 2, 3\,,$$

is the UMVUE. □

By Lemma 1.1, $\hat{\mu}_2 = \hat{\kappa}_2$ and $\hat{\mu}_3 = \hat{\kappa}_3$ are the central moments of $X_1|\bar{X} \sim \mathrm{Hypergeometric}(nm, n\bar{X}, m)$, multiplied by the factor C_k.

For $\hat{\mu}_4$ and $\hat{\kappa}_4$, Theorems 2.3 and 2.4 state that their UMVUEs are the linear combinations of product moments. Thus, we need the joint densities of multiple X_is given $\boldsymbol{T} = \bar{X}$, for example,

$$\begin{aligned} f_{X_1, X_2 | \sum_{i=1}^n X_i = t}(x_1, x_2) &= \frac{f_{X_1, X_2, \sum_{i=1}^n X_i}(x_1, x_2, t)}{f_{\sum_{i=1}^n X_i}(t)} \\ &= \frac{f_{X_1}(x_1) f_{X_2}(x_2) f_{\sum_{i=3}^n X_i}(t - x_1 - x_2)}{f_{\sum_{i=1}^n X_i}(t)} \\ &= \frac{\binom{m}{x_1}\binom{m}{x_2}\binom{(n-2)m}{t - x_1 - x_2}}{\binom{nm}{t}}\,. \end{aligned}$$

Similarly, we have

$$f_{X_1,X_2,X_3|\sum_{i=1}^n X_i=t}(x_1,x_2,x_3) = \frac{\binom{m}{x_1}\binom{m}{x_2}\binom{m}{x_3}\binom{(n-3)m}{t-x_1-x_2-x_3}}{\binom{nm}{t}},$$

$$f_{X_1,X_2,X_3,X_4|\sum_{i=1}^n X_i=t}(x_1,x_2,x_3,x_4) = \frac{\binom{m}{x_1}\binom{m}{x_2}\binom{m}{x_3}\binom{m}{x_4}\binom{(n-4)m}{t-x_1-x_2-x_3-x_4}}{\binom{nm}{t}}.$$

Combining these joint conditional distributions and Lemma 2.2, we obtain

$$\begin{aligned}
\hat{\eta}(\mu_3'\mu_1') &= \mathrm{E}(X_1^3X_2|\boldsymbol{T}) = \frac{n\bar{X}\hat{\mu}_3' - \hat{\mu}_4'}{n-1}, \\
\hat{\eta}(\mu_2'^2) &= \mathrm{E}(X_1^2X_2^2|\boldsymbol{T}) = \frac{[(m-1)n\bar{X}+(n-2)m]n\bar{X}\hat{\mu}_2' - [2(m-1)n\bar{X}+(n-2)m]\hat{\mu}_3' + (m-1)\hat{\mu}_4'}{(n-1)[(n-1)m-1]}, \\
\hat{\eta}(\mu_2'\mu_1'^2) &= \mathrm{E}(X_1^2X_2X_3|\boldsymbol{T}) = \frac{m[n\bar{X}(n\bar{X}-1)\hat{\mu}_2' + (1-2n\bar{X})\hat{\mu}_3' + \hat{\mu}_4']}{(n-1)[(n-1)m-1]}, \\
\hat{\eta}(\mu_1'^4) &= \mathrm{E}(X_1X_2X_3X_4|\boldsymbol{T}) = \frac{m^2[n\bar{X}^2(n\bar{X}-1)(n\bar{X}-2) - (3n^2\bar{X}^2-6n\bar{X}+2)\hat{\mu}_2' + 3(n\bar{X}-1)\hat{\mu}_3' - \hat{\mu}_4']}{(n-1)[(n-1)m-1][(n-1)m-2]}.
\end{aligned}$$

Hence, we can obtain $\hat{\kappa}_4$ and $\hat{\mu}_4$ by (5) and (6), respectively.

1.2. Poisson Distribution

Consider a sample of i.i.d. random variables

$$X_i \overset{i.i.d.}{\sim} \text{Poisson}(\lambda)\,, i = 1,\ldots,n\,.$$

The sufficient statistic is found to be $\boldsymbol{T} = \bar{X}$. From Theorem 2.1, the conditional density function of X_1 given the sufficient statistic $\bar{X}$ is

$$\begin{aligned}
f_{X_1|\sum_{i=1}^n X_i=t}(x) &= \frac{f_{X_1,\sum_{i=1}^n X_i}(x,t)}{f_{\sum_{i=1}^n X_i}(t)} \\
&= \frac{f_{X_1}(x) f_{\sum_{i=2}^n X_i}(t-x)}{f_{\sum_{i=1}^n X_i}(t)} \\
&= \frac{\frac{\lambda^x \exp(-\lambda)}{x!}\,\frac{[(n-1)\lambda]^{t-x}\exp[-(n-1)\lambda]}{(t-x)!}}{\frac{(n\lambda)^t\exp(-n\lambda)}{t!}} \\
&= \binom{t}{x}\left(\frac{1}{n}\right)^x\left(\frac{n-1}{n}\right)^{t-x},
\end{aligned}$$

implying that $X_1|\bar{X} = \bar{X} \sim \text{Binomial}(n\bar{X}, \frac{1}{n})$. Therefore, the moments of the binomial distribution gives the UMVUEs for the raw moments $\hat{\mu}_1', \hat{\mu}_2', \hat{\mu}_3'$ and $\hat{\mu}_4'$.

Similarly,

$$f_{X_1,X_2|\sum_{i=1}^n X_i=t}(x_1,x_2) = \frac{t!}{x_1!x_2!(t-x_1-x_2)!}\left(\frac{1}{n}\right)^{x_1+x_2}\left(\frac{n-2}{n}\right)^{t-x_1-x_2},$$

implying that $X_1, X_2|\bar{X}$ given $\bar{X}$ follows multinomial distribution. Moreover, the conditional probability of X_1 given X_2 and $\bar{X}$ is

$$\begin{aligned}
f_{X_1|X_2,\sum_{i=1}^n X_i=t}(x_1) &= \frac{f_{X_1,X_2|\sum_{i=1}^n X_i=t}(x_1,x_2)}{f_{X_2|\sum_{i=1}^n X_i=t}(x_2)} \\
&= \frac{\frac{t!}{x_1!x_2!(t-x_1-x_2)!}\left(\frac{1}{n}\right)^{x_1+x_2}\left(\frac{n-2}{n}\right)^{t-x_1-x_2}}{\binom{t}{x_2}\left(\frac{1}{n}\right)^{x}_{2}\left(\frac{n-1}{n}\right)^{t-x_2}} \\
&= \binom{t-x_2}{x_1}\left(\frac{1}{n-1}\right)^{x_1}\left(\frac{n-2}{n-1}\right)^{t-x_1-x_2} .
\end{aligned}$$

Thus, $X_1|X_2,\bar{X} \sim \text{Binomial}(\bar{X}-x_2, \frac{1}{n-1})$. Taking conditional expectation, we have

$$\begin{aligned}
\hat{\eta}(\mu_2'^2) &= \mathrm{E}(X_1^2X_2^2|\bar{X}) = \mathrm{E}(X_2^2\mathrm{E}(X_1^2|X_2,\bar{X})|\bar{X}) \\
&= \mathrm{E}\left(X_2^2\left(\frac{n\bar{X}-X_2}{n-1}\left(\frac{n-2}{n-1}+\frac{n\bar{X}-X_2}{n-1}\right)\right)|\bar{X}\right) \\
&= \mathrm{E}\left(X_2^4\left(\frac{1}{(n-2)^2}\right)+X_2^3\left(-\frac{n-2+2n\bar{X}}{(n-1)^2}\right)+X_2^2\left(\frac{n\bar{X}(n-2)+n^2\bar{X}^2}{(n-1)^2}\right)|\bar{X}\right) \\
&= \bar{X}^4+\frac{2(n-3)}{n}\bar{X}^3+\frac{n^2-6n+11}{n^2}\bar{X}^2-\frac{n^2-4n+6}{n^3}\bar{X} , \\
\hat{\eta}(\mu_1'\mu_2') &= \mathrm{E}(X_1^2X_2|\bar{X}) = \mathrm{E}(X_2^2\mathrm{E}(X_1|X_2,\bar{X})|\bar{X}) \\
&= \mathrm{E}\left(X_2^2\left(\frac{n\bar{X}-X_2}{n-1}\right)|\bar{X}\right) \\
&= \mathrm{E}\left(X_2^2\left(\frac{n\bar{X}}{n-1}\right)-X_2^3\left(\frac{1}{n-1}\right)|\bar{X}\right) \\
&= \bar{X}^3+\frac{n-3}{n}\bar{X}^2-\frac{n-2}{n^2}\bar{X} , \\
\hat{\eta}(\mu_1'^2) &= \mathrm{E}(X_1X_2|\bar{X}) = \mathrm{E}(X_2\mathrm{E}(X_1|X_2,\bar{X})|\bar{X}) \\
&= \mathrm{E}\left(X_2\left(\frac{n\bar{X}-X_2}{n-1}\right)|\bar{X}\right) \\
&= \bar{X}^2-\frac{\bar{X}}{n} , \\
\hat{\eta}(\mu_1'\mu_3') &= \mathrm{E}(X_1X_2^3|\bar{X}) = \mathrm{E}(X_2^3\mathrm{E}(X_1|X_2,\bar{X})|\bar{X}) \\
&= \mathrm{E}\left(X_2^3\left(\frac{n\bar{X}-X_2}{n-1}\right)|\bar{X}\right) \\
&= \mathrm{E}\left(X_2^3\left(\frac{n\bar{X}}{n-1}\right)-X_2^4\left(\frac{1}{n-1}\right)|\bar{X}\right) \\
&= \bar{X}^4+\frac{3(n-2)}{n}\bar{X}^3+\frac{n^2-9n+11}{n^2}\bar{X}^2-\frac{n^2-6n+6}{n^3}\bar{X} .
\end{aligned}$$

Similarly, given $\bar{X}$, X_1, X_2, X_3 follows the multinomial distribution, that is,

$$\begin{aligned} f_{X_1,X_2,X_3|\sum_{i=1}^n X_i=t}(x_1,x_2,x_3) &= \frac{t!}{x_1!x_2!x_3!(t-x_1-x_2-x_3)!} \\ &\quad \times \left(\frac{1}{n}\right)^{x_1+x_2+x_3} \left(\frac{n-3}{n}\right)^{t-x_1-x_2-x_3} . \end{aligned}$$

Also, we have the conditional distribution

$$X_3|X_1,X_2,\bar{X} \sim \text{binomial}(\bar{X}-X_1-X_2, \frac{1}{n-2}) .$$

Thus, we have

$$\begin{aligned} \hat{\eta}(\mu_1'^2\mu_2') &= \mathrm{E}(X_1^2X_2X_3|\bar{X}) \\ &= \mathrm{E}(X_1^2X_2\mathrm{E}(X_3|X_1,X_2,\bar{X})|\bar{X}) \\ &= \mathrm{E}\left(X_1^2X_2\left(\frac{n\bar{X}-X_1-X_2}{n-2}\right)|\bar{X}\right) \\ &= \frac{n\bar{X}}{n-2}\mathrm{E}(X_1^2X_2|\bar{X}) - \frac{1}{n-2}\mathrm{E}(X_1^3X_2|\bar{X}) - \frac{1}{n-1}\mathrm{E}(X_1^2X_2^2|\bar{X}) \\ &= \bar{X}^4 + \frac{n-6}{n}\bar{X}^3 - \frac{3n-11}{n^2}\bar{X}^2 + \frac{2(n-3)}{n^3}\bar{X} , \\ \hat{\eta}(\mu_1'^3) &= \mathrm{E}(X_1X_2X_3|\bar{X}) \\ &= \mathrm{E}(X_1X_2\mathrm{E}(X_3|X_1,X_2,\bar{X})|\bar{X}) \\ &= \mathrm{E}\left(X_1X_2\left(\frac{n\bar{X}-X_1-X_2}{n-2}\right)|\bar{X}\right) \\ &= \frac{n\bar{X}}{n-2}\mathrm{E}(X_1X_2|\bar{X}) - \frac{2}{n-2}\mathrm{E}(X_1^2X_2|\bar{X}) \\ &= \bar{X}^3 - \frac{3}{n}\bar{X}^2 + \frac{2}{n^2}\bar{X} . \end{aligned}$$

Similarly, the conditional distribution of X_1, X_2, X_3, X_4 given $\bar{X}$ follows the multinomial distribution

$$\begin{aligned} & f_{X_1,X_2,X_3,X_4|\sum_{i=1}^n X_i=t}(x_1,x_2,x_3,x_4) \\ = & \frac{t!}{x_1!x_2!x_3!x_4!(t-x_1-x_2-x_3-x_4)!}\left(\frac{1}{n}\right)^{x_1+x_2+x_3+x_4}\left(\frac{n-4}{n}\right)^{t-x_1-x_2-x_4} . \end{aligned}$$

Also, we have

$$X_4|X_1,X_2,X_3,\bar{X} \sim \text{binomial}\left(\bar{X}-X_1-X_2-X_3, \frac{1}{n-3}\right) .$$

Thus, the UMVUE is given by

$$\begin{aligned}
\hat{\eta}(\mu_1'^4) &= \mathrm{E}(X_1X_2X_3X_4|\bar{X}) \\
&= \mathrm{E}(X_1X_2X_3\mathrm{E}(X_4|X_1,X_2,X_3\bar{X})|\bar{X}) \\
&= \mathrm{E}\left(X_1X_2X_3\left(\frac{n\bar{X}-X_1-X_2-X_3}{n-3}\right)|\bar{X}\right) \\
&= \frac{n\bar{X}}{n-3}\mathrm{E}(X_1X_2X_3|\bar{X}) - \frac{3}{n-3}\mathrm{E}(X_1^2X_2X_3|\bar{X}) \\
&= \bar{X}^4 - \frac{6}{n}\bar{X}^3 + \frac{11}{n^2}\bar{X}^2 - \frac{6}{n^3}\bar{X}\,.
\end{aligned}$$

Combining with $\hat{\mu}_4 = \bar{X} + 3\hat{\eta}(\mu_2'^2) - 6\hat{\eta}(\mu_2'\mu_1'^2) + 3\hat{\eta}(\mu_1'^4)$, we can obtain $\hat{\mu}_4$.

1.3. Negative Binomial Distribution

Consider a sample of i.i.d. random variables

$$X_i \overset{i.i.d.}{\sim} NB(r,p)\,,\ i=1,\ldots,n\,.$$

The sufficient statistic is found to be $\boldsymbol{T} = \bar{X}$. From Theorem 2.1, the conditional density function of X_1 given $\bar{X}$ is

$$\begin{aligned}
f_{X_1|\sum_{i=1}^n X_i=t}(x) &= \frac{f_{X_1,\sum_{i=1}^n X_i}(x,t)}{f_{\sum_{i=1}^n X_i}(t)} \\
&= \frac{f_{X_1}(x)f_{\sum_{i=2}^n X_i}(t-x)}{f_{\sum_{i=1}^n X_i}(t)} \\
&= \frac{\binom{x+r-1}{x}(1-p)^r p^x \binom{t-x+(n-1)r-1}{t-x}(1-p)^{(n-1)r}p^{t-x}}{\binom{t+nr-1}{t}(1-p)^{nr}p^t} \\
&= \frac{\binom{x+r-1}{x}\binom{t+(n-1)r-1-x}{(n-1)r-1}}{\binom{t+nr-1}{nr-1}}\,.
\end{aligned}$$

Therefore, $X_1|\bar{X}=\bar{X} \sim$ Negative Hypergeometric$(n(\bar{X}+r)-1, nr-1, r)$. Similar to the hypergeometric distribution, the negative hypergeometric distribution satisfies

$$\mathrm{E}[X_1(X_1-1)\cdots(X_1-k)|\bar{X}=\bar{X}] = \bar{X}\,\frac{(r+1)(r+2)\cdots(r+k)(n\bar{X}-1)(n\bar{X}-2)\cdots(n\bar{X}-k)}{(nr+1)(nr+2)\cdots(nr+k)}.$$

The calculations of raw moments $\hat{\mu}_1'\hat{\mu}_2', \hat{\mu}_3'$ and $\hat{\mu}_4'$ are almost the same as that of the binomial distributions. Next, $\hat{\mu}_2 = \hat{\kappa}_2$ and $\hat{\mu}_3 = \hat{\kappa}_3$ can be obtained from the population central moments of $X_1|\bar{X}=\bar{X} \sim$ Negative Hypergeometric$(n(\bar{X}+r)-1, nr-1, r)$ by Lemma 1.1. For $\hat{\mu}_4$ and $\hat{\kappa}_4$, similar to the case of binomial distribution, we need the joint densities of multiple X_is given $\boldsymbol{T} = \bar{X}$,

$$f_{X_1,X_2|\sum_{i=1}^n X_i=t}(x_1,x_2) = \frac{\binom{x_1+r-1}{x_1}\binom{x_2+r-1}{x_2}\binom{t-x_1-x_2+(n-2)r-1}{t-x_1-x_2}}{\binom{t+nr-1}{t}}\,.$$

Similarly, we have

$$
\begin{aligned}
& f_{X_1,X_2,X_3|\sum_{i=1}^n X_i=t}(x_1,x_2,x_3) \\
= & \frac{\binom{x_1+r-1}{x_1}\binom{x_2+r-1}{x_2}\binom{x_3+r-1}{x_3}\binom{t-x_1-x_2-x_3+(n-3)r-1}{t-x_1-x_2-x_3}}{\binom{t+nr-1}{t}} \\
& f_{X_1,X_2,X_3,X_4|\sum_{i=1}^n X_i=t}(x_1,x_2,x_3,x_4) \\
= & \frac{\binom{x_1+r-1}{x_1}\binom{x_2+r-1}{x_2}\binom{x_3+r-1}{x_3}\binom{x_4+r-1}{x_4}\binom{t-x_1-x_2-x_3-x_4+(n-4)r-1}{t-x_1-x_2-x_3-x_4}}{\binom{t+nr-1}{t}} .
\end{aligned}
$$

Combining these joint conditional distributions and Lemma 2.2, we obtain

$$
\begin{aligned}
\hat{\eta}(\mu_3'\mu_1') &= \mathrm{E}(X_1^3X_2|\boldsymbol{T}) = \tfrac{n\bar{X}\hat{\mu}_3'-\hat{\mu}_4'}{n-1} \\
\hat{\eta}(\mu_2'^2) &= \mathrm{E}(X_1^2X_2^2|\boldsymbol{T}) = \tfrac{[(r+1)n\bar{X}+(n-2)r]n\bar{X}\hat{\mu}_2'-[2(r+1)n\bar{X}+(n-2)r]\hat{\mu}_3'+(r+1)\hat{\mu}_4'}{(n-1)[(n-1)r+1]} \\
\hat{\eta}(\mu_2'\mu_1'^2) &= \mathrm{E}(X_1^2X_2X_3|\boldsymbol{T}) = \tfrac{r[n\bar{X}(n\bar{X}-1)\hat{\mu}_2'+(1-2n\bar{X})\hat{\mu}_3'+\hat{\mu}_4']}{(n-1)[(n-1)r+1]} \\
\hat{\eta}(\mu_1'^4) &= \mathrm{E}(X_1X_2X_3X_4|\boldsymbol{T}) = \tfrac{r^2[n\bar{X}^2(n\bar{X}-1)(n\bar{X}-2)-(3n^2\bar{X}^2-6n\bar{X}+2)\hat{\mu}_2'+3(n\bar{X}-1)\hat{\mu}_3'-\hat{\mu}_4']}{(n-1)[(n-1)r+1][(n-1)r+2]} .
\end{aligned}
$$

Hence, $\hat{\kappa}_4$ and $\hat{\mu}_4$ are obtained by

$$
\begin{aligned}
\hat{\kappa}_4 &= \hat{\eta}(\mu_4') - 4\hat{\eta}(\mu_3'\mu_1') - 3\hat{\eta}(\mu_2'^2) + 12\hat{\eta}(\mu_2'\mu_1'^2) - 6\hat{\eta}(\mu_1'^4) , \\
\hat{\mu}_4 &= \hat{\kappa}_4 + 3\hat{\eta}(\mu_2'^2) - 6\hat{\eta}(\mu_2'\mu_1'^2) + 3\hat{\eta}(\mu_1'^4) ,
\end{aligned}
$$

respectively.

1.4. Discrete Uniform Distribution

Consider a sample of i.i.d. random variables

$$
X_i \overset{i.i.d.}{\sim} \mathrm{Uniform}(N), i = 1, \ldots, n .
$$

The sufficient statistics is found to be $\boldsymbol{T} = \max_{1\le j\le n} X_j \equiv X_{(n)}$. From Theorem 2.1, the UMVUE for the k-th order raw moment is

$$
\begin{aligned}
& \mathrm{E}(X_1^k|\max_j\{X_j\} = x_{(n)}) \\
= & \sum_{x=1}^{X_{(n)}} x^k \frac{P(X_1 = x, \max_j\{X_j\} = x_{(n)})}{P(\max_j\{X_j\} = x_{(n)})} \\
= & x_{(n)}^k \frac{P(X_1 = x_{(n)}, \max_{j\neq 1}\{X_j\} \le x_{(n)})}{P(\max_j\{X_j\} = x_{(n)})} + \sum_{x=1}^{x_{(n)}-1} x^k \frac{P(X_1 = x, \max_{j\neq 1}\{X_j\} = x_{(n)})}{P(\max_j\{X_j\} = x_{(n)})} \\
= & \frac{x_{(n)}^k \frac{1}{N}\frac{x_{(n)}^{n-1}}{N^{n-1}}}{\frac{1}{N^n}(x_{(n)}^n - (x_{(n)}-1)^n)} + \frac{(\sum_{x=1}^{x_{(n)}-1} x^k)\frac{1}{N}\frac{1}{N^{n-1}}(x_{(n)}^{n-1} - (x_{(n)}-1)^{n-1})}{\frac{1}{N^n}(x_{(n)}^n - (x_{(n)}-1)^n)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{x_{(n)}^{k+n-1} + (x_{(n)}^{n-1} - (x_{(n)}-1)^{n-1}) \sum_{x=1}^{x_{(n)}-1} x^k}{x_{(n)}^n - (x_{(n)}-1)^n} \\
&= \frac{x_{(n)}^{n-1} \sum_{x=1}^{x_{(n)}} x^k - (x_{(n)}-1)^{n-1} \sum_{x=1}^{x_{(n)}-1} x^k}{x_{(n)}^n - (x_{(n)}-1)^n} ,
\end{aligned}
$$

which leads to the desired results.

Note that Lemma 1.1 is not applicable in this case since $\boldsymbol{T} \neq \bar{X}$. Nevertheless, for all central moments and cumulants, we can apply Theorems 2.3 and 2.4 to obtain the UMVUEs. To find the product moments, we first derive the conditional joint densities of multiple X_is given $\boldsymbol{T}$:

$$
f_{X_1,X_2|\max_j\{X_j\}=x_{(n)}}(x_1,x_2) = \begin{cases} \frac{x_{(n)}^{n-2}-(x_{(n)}-1)^{n-2}}{x_{(n)}^n-(x_{(n)}-1)^n} , & x_1 < x_{(n)} \text{ and } x_2 < x_{(n)} , \\ \frac{x_{(n)}^{n-2}}{x_{(n)}^n-(x_{(n)}-1)^n} , & \text{otherwise} . \end{cases}
$$

Similarly, we have

$$
f_{X_1,X_2,X_3|\max_j\{X_j\}=x_{(n)}}(x_1,x_2,x_3) = \begin{cases} \frac{x_{(n)}^{n-3}-(x_{(n)}-1)^{n-3}}{x_{(n)}^n-(x_{(n)}-1)^n} , & x_1,x_2,x_3 < x_{(n)} , \\ \frac{x_{(n)}^{n-3}}{x_{(n)}^n-(x_{(n)}-1)^n} , & \text{otherwise} , \end{cases}
$$

$$
f_{X_1,X_2,X_3,X_4|\max_j\{X_j\}=x_{(n)}}(x_1,x_2,x_3,x_4) = \begin{cases} \frac{x_{(n)}^{n-4}-(x_{(n)}-1)^{n-4}}{x_{(n)}^n-(x_{(n)}-1)^n} , & x_1,x_2,x_3,x_4 < x_{(n)} , \\ \frac{x_{(n)}^{n-4}}{x_{(n)}^n-(x_{(n)}-1)^n} , & \text{otherwise} . \end{cases}
$$

By Theorems 2.3 and 2.4, we have

$$
\hat{\kappa}_2 = \hat{\mu}_2 = \hat{\mu}_2' - \mathrm{E}(X_1X_2|\boldsymbol{T}) = \frac{X_{(n)}^n(X_{(n)}+1)(7X_{(n)}+5)-(X_{(n)}-1)^n X_{(n)}(7X_{(n)}-2)}{12[X_{(n)}^n-(X_{(n)}-1)^n]} .
$$

For $\hat{\kappa}_4$ and $\hat{\mu}_4$, we use formula

$$
\begin{aligned}
\hat{\kappa}_4 &= \hat{\eta}(\mu_4') - 4\hat{\eta}(\mu_3'\mu_1') - 3\hat{\eta}(\mu_2'^2) + 12\hat{\eta}(\mu_2'\mu_1'^2) - 6\hat{\eta}(\mu_1'^4) , \\
\hat{\mu}_4 &= \hat{\kappa}_4 + 3\hat{\eta}(\mu_2'^2) - 6\hat{\eta}(\mu_2'\mu_1'^2) + 3\hat{\eta}(\mu_1'^4) ,
\end{aligned}
$$

respectively. In particular, by Lemma 2.2, we have

$$
\begin{aligned}
\hat{\eta}(\mu_3'\mu_1') &= \mathrm{E}(X_1^3X_2|\boldsymbol{T}) = \frac{X_{(n)}^{n+1}(X_{(n)}+1)^3-(X_{(n)}-1)^{n+1}X_{(n)}^3}{8[X_{(n)}^n-(X_{(n)}-1)^n]} , \\
\hat{\eta}(\mu_2'^2) &= \mathrm{E}(X_1^2X_2^2|\boldsymbol{T}) = \frac{X_{(n)}^n(2X_{(n)}+1)^2(X_{(n)}+1)^2-(X_{(n)}-1)^n(2X_{(n)}-1)^2X_{(n)}^2}{36[X_{(n)}^n-(X_{(n)}-1)^n]} , \\
\hat{\eta}(\mu_2'\mu_1'^2) &= \mathrm{E}(X_1^2X_2X_3|\boldsymbol{T}) = \frac{X_{(n)}^n(2X_{(n)}+1)(X_{(n)}+1)^3-(X_{(n)}-1)^n(2X_{(n)}-1)X_{(n)}^3}{24[X_{(n)}^n-(X_{(n)}-1)^n]} , \\
\hat{\eta}(\mu_1'^4) &= \mathrm{E}(X_1X_2X_3X_4|\boldsymbol{T}) = \frac{X_{(n)}^n(X_{(n)}+1)^4-(X_{(n)}-1)^nX_{(n)}^4}{16[X_{(n)}^n-(X_{(n)}-1)^n]} .
\end{aligned}
$$

1.5. Exponential Distribution

Consider a sample of i.i.d. random variables

$$X_i \overset{i.i.d.}{\sim} \text{Exponential}(\lambda)\,,\ i = 1, \ldots, n\,.$$

The sufficient statistic is $\boldsymbol{T} = \bar{X}$. From Theorem 2.1, the conditional density function of X_1 given $\bar{X}$ is

$$\begin{aligned} f_{X_1|\sum_{i=1}^n X_i = t}(x) &= \frac{f_{X_1, \sum_{i=1}^n X_i}(x, t)}{f_{\sum_{i=1}^n X_i}(t)} \\ &= \frac{t}{B(1,(n-1))}\Big(\frac{x}{t}\Big)^{1-1}\Big(1-\frac{x}{t}\Big)^{(n-1)-1}\,. \end{aligned}$$

Therefore, given $\bar{X}$, $\frac{X_1}{n\bar{X}} \sim \text{Beta}(1,(n-1))$. The beta distribution gives a general form for the uniform minimum-variance unbiased estimator of the kth moment,

$$\hat{\mu}'_k = (n\bar{X})^k \frac{\Gamma(n)\Gamma(k)}{\Gamma(n+k)} = \frac{n^{k-1}\bar{X}^k(k-1)!}{(n+1)(n+2)\cdots(n+k-1)}\,.$$

For the central moments $\hat{\mu}_2$ and $\hat{\mu}_3$, Lemma 1.1 states that the uniform minimum-variance unbiased estimator is the respective population central moment of the conditional distribution $X_1|\bar{X} \sim n\bar{X}\text{Beta}(1,(n-1))$, multiplied by the factor C_k. For $\hat{\kappa}_4$ and $\hat{\mu}_4$, we first derive the joint conditional density of multiple X_is to obtain the product moments,

$$\begin{aligned} f_{X_1,X_2|\sum_{i=1}^n X_i = t}(x_1, x_2) &= \frac{f_{X_1,X_2,\sum_{i=1}^n X_i}(x_1, x_2, t)}{f_{\sum_{i=1}^n X_i}(t)} \\ &= \frac{f_{X_1}(x_1) f_{X_2}(x_2) f_{\sum_{i=3}^n X_i}(t - x_1 - x_2)}{f_{\sum_{i=1}^n X_i}(t)} \\ &= \frac{t^2\Gamma(n)}{\Gamma(n-2)}\Big(\frac{x_1}{t}\Big)^{1-1}\Big(\frac{x_2}{t}\Big)^{1-1}\Big(\frac{t-x_1-x_2}{t}\Big)^{(n-2)-1}\,. \end{aligned}$$

Similarly, we have

$$\begin{aligned} & f_{X_1,X_2,X_3|\sum_{i=1}^n X_i = t}(x_1, x_2, x_3) \\ = \ & \frac{t^3\Gamma(n)}{\Gamma(n-3)}\Big(\frac{x_1}{t}\Big)^{1-1}\Big(\frac{x_2}{t}\Big)^{1-1}\Big(\frac{x_3}{t}\Big)^{1-1}\Big(\frac{t-x_1-x_2-x_3}{t}\Big)^{(n-3)-1} \\ & f_{X_1,X_2,X_3,X_4|\sum_{i=1}^n X_i = t}(x_1, x_2, x_3, x_4) \\ = \ & \frac{t^4\Gamma(n)}{\Gamma(n-4)}\Big(\frac{x_1}{t}\Big)^{1-1}\Big(\frac{x_2}{t}\Big)^{1-1}\Big(\frac{x_3}{t}\Big)^{1-1}\Big(\frac{x_4}{t}\Big)^{1-1}\Big(\frac{t-x_1-x_2-x_3-x_4}{t}\Big)^{(n-4)-1}\,. \end{aligned}$$

Combining these joint conditional distributions and Lemma 2.2, we obtain

$$\begin{aligned}
\hat{\eta}(\mu_3'\mu_1') &= \mathrm{E}(X_1^3 X_2|\boldsymbol{T}) = \tfrac{n\bar{X}\hat{\mu}_3'-\hat{\mu}_4'}{n-1}\,, \\
\hat{\eta}(\mu_2'^2) &= \mathrm{E}(X_1^2 X_2^2|\boldsymbol{T}) = \tfrac{(\alpha+1)(n^2\bar{X}^2\hat{\mu}_2'-2n\bar{X}\hat{\mu}_3'+\hat{\mu}_4')}{(n-1)[(n-1)\alpha+1]}\,, \\
\hat{\eta}(\mu_2'\mu_1'^2) &= \mathrm{E}(X_1^2 X_2 X_3|\boldsymbol{T}) = \tfrac{\alpha(n^2\bar{X}^2\hat{\mu}_2'-2n\bar{X}\hat{\mu}_3'+\hat{\mu}_4')}{(n-1)[(n-1)\alpha+1]}\,, \\
\hat{\eta}(\mu_1'^4) &= \mathrm{E}(X_1 X_2 X_3 X_4|\boldsymbol{T}) = \tfrac{\alpha^2[n^3\bar{X}^4-3n^2\bar{X}^2\hat{\mu}_2'+3n\bar{X}\hat{\mu}_3'-\hat{\mu}_4']}{(n-1)[(n-1)\alpha+1][(n-1)\alpha+2]}\,.
\end{aligned}$$

Hence, $\hat{\kappa}_4$ and $\hat{\mu}_4$ are obtained by

$$\begin{aligned}
\hat{\kappa}_4 &= \hat{\eta}(\mu_4') - 4\hat{\eta}(\mu_3'\mu_1') - 3\hat{\eta}(\mu_2'^2) + 12\hat{\eta}(\mu_2'\mu_1'^2) - 6\hat{\eta}(\mu_1'^4)\,, \\
\hat{\mu}_4 &= \hat{\kappa}_4 + 3\hat{\eta}(\mu_2'^2) - 6\hat{\eta}(\mu_2'\mu_1'^2) + 3\hat{\eta}(\mu_1'^4)\,,
\end{aligned}$$

respectively.

1.6. Continuous Uniform Distribution

Consider a sample of i.i.d. random variables

$$X_i \overset{i.i.d.}{\sim} \mathrm{Uniform}(a,b)\,, \quad i = 1,\ldots,n\,.$$

The sufficient statistic is found to be

$$\boldsymbol{T} = \left(\min_{1\le j\le n}\{X_j\}, \max_{1\le j\le n}\{X_j\}\right) = \left(X_{(1)}, X_{(n)}\right)\,.$$

From Theorem 1, the UMVUE of the k-th order moment is

$$\begin{aligned}
\hat{\mu}_k' &= \mathrm{E}(X_1^k|\min_j\{X_j\} = x_{(1)}, \max_j\{X_j\} = x_{(n)}) \\
&= \mathrm{E}(X_1^k|X_1 = x_{(1)})P(X_1 = x_{(1)}) \\
&\quad + \mathrm{E}(X_1^k|x_{(1)} < X_1 < x_{(n)})P(x_{(1)} < X_1 < x_{(n)}) \\
&\quad + \mathrm{E}(X_1^k|X_1 = x_{(n)})P(X_1 = x_{(n)}) \\
&= \frac{x_{(1)}^k}{n} + \frac{n-2}{n}\int_{x_{(1)}}^{x_{(n)}} x^k f_{X_1|x_{(1)}<X_1<x_{(n)}}(x)\,dx + \frac{x_{(n)}^k}{n} \\
&= \frac{x_{(n)}^k}{n} + \frac{x_{(1)}^k}{n} + \frac{n-2}{n}\int_{x_{(1)}}^{x_{(n)}} \frac{x^k}{x_{(n)}-x_{(1)}}\,dx \\
&= \frac{1}{n}\left(x_{(n)}^k + x_{(1)}^k + \frac{n-2}{k+1}\sum_{j=0}^{k} x_{(n)}^j x_{(1)}^{k-j}\right)\,.
\end{aligned}$$

Note that $\kappa_3 = \mu_3 = 0$. Similar to the discrete uniform distribution, Lemma 1.1 is not applicable to obtain $\hat{\mu}_2$. Again, we use Theorems 2.3 and 2.4 instead. Given the sufficient statistic $\boldsymbol{T} = \left(\min_j\{X_j\}, \max_j\{X_j\}\right)$, the observations $\{X_1, X_2, ..., X_n\}$ are no longer independent, and a joint conditional distribution of the random variables is required to obtain the uniform minimum-variance unbiased estimator of the products

of moments. Note that the conditional distribution of X_1 given $\boldsymbol{T}$ is partly discrete in the sense that

$$\begin{aligned} P(X_i = x_{(n)} | \min_j\{X_j\} = x_{(1)}, \max_j\{X_j\} = x_{(n)}) &= \frac{1}{n}, \\ P(X_i = x_{(1)} | \min_j\{X_j\} = x_{(1)}, \max_j\{X_j\} = x_{(n)}) &= \frac{1}{n}, \\ f_{X_i|x_{(1)}<X_i<x_{(n)}}(x)\,dx &= \frac{1}{x_{(n)} - x_{(1)}}, \end{aligned}$$

for $x \in (x_{(1)}, x_{(n)})$. By Theorems 2.3 and 2.4, we have

$$\hat{\kappa}_2 = \hat{\mu}_2 = \hat{\mu}_2' - \mathrm{E}(X_1X_2|\boldsymbol{T}) = \frac{(n+1)(n+2)(X_{(n)}^2 + X_{(1)}^2)}{12n(n-1)} - \frac{X_{(n)}X_{(1)}}{3(n-1)}.$$

Finally, $\hat{\kappa}_4$ and $\hat{\mu}_4$ are given by (10) and (11), respectively, where we have from Lemma 2.2 that

$$\begin{aligned} \hat{\eta}(\mu_3'\mu_1') &= \mathrm{E}(X_1^3X_2|\boldsymbol{T}) \\ &= \frac{(n-2)(n+3)(X_{(n)}^4 + X_{(1)}^4)}{8n(n-1)} + \frac{(n^2-n+2)(X_{(n)}^2 + X_{(1)}^2)X_{(n)}X_{(1)}}{4n(n-1)} \\ &+ \frac{(n-2)X_{(n)}^2X_{(1)}^2}{4n}, \\ \hat{\eta}(\mu_2'^2) &= \mathrm{E}(X_1^2X_2^2|\boldsymbol{T}) \\ &= \frac{(n-2)(n+3)(X_{(n)}^4 + X_{(1)}^4)}{9n(n-1)} + \frac{2(n-2)(X_{(n)}^2 + X_{(1)}^2)X_{(n)}X_{(1)}}{9(n-1)} \\ &+ \frac{(n^2-n+4)X_{(n)}^2X_{(1)}^2}{3n(n-1)}, \\ \hat{\eta}(\mu_2'\mu_1'^2) &= \mathrm{E}(X_1^2X_2X_3|\boldsymbol{T}) \\ &= \frac{(n-3)(n+3)(X_{(n)}^4 + X_{(1)}^4)}{12n(n-1)} + \frac{(3n^2-3n+2)(X_{(n)}^2 + X_{(1)}^2)X_{(n)}X_{(1)}}{12n(n-1)} \\ &+ \frac{(2n^2-3n+7)X_{(n)}^2X_{(1)}^2}{6n(n-1)}, \\ \hat{\eta}(\mu_1'^4) &= \mathrm{E}(X_1X_2X_3X_4|\boldsymbol{T}) \\ &= \frac{(n-4)(n+3)(X_{(n)} + X_{(1)})^4}{16n(n-1)} + \frac{3(X_{(n)} + X_{(1)})^2X_{(n)}X_{(1)}}{n(n-1)}. \end{aligned}$$

2. Simulation Studies

In this section, we conduct extensive simulations to investigate the performances of our proposed estimators and other estimators for the central moments, raw moments and cumulants when data are generated from different distributions.

2.1. Poisson Distribution

For $X_i \overset{i.i.d.}{\sim}$ Poisson(λ), $i = 1, \ldots, n$, the cumulants $\kappa_k = \lambda$ for all k. Therefore, we only study the performances of the estimators for the first four central moments for poisson distribution. By simple derivation, we have $\mu_2 = \mu_3 = \lambda$, $\mu_4 = \lambda(1+3\lambda)$. Since MLE for λ is $\bar{X}$, MLEs for the central moments are $\tilde{\mu}_2 = \tilde{\mu}_3 = \bar{X}$ and $\tilde{\mu}_4 = \bar{X}(1+3\bar{X})$. In simulation, λ is randomly drawn from $(0, 20)$.

The estimation results are reported in Table 1.

$\lambda = 0.57$	Bias				MSE			
	$\hat{\mu}_2$	$\tilde{\mu}_2$	$\bar{\mu}_2$	h_2	$\hat{\mu}_2$	$\tilde{\mu}_2$	$\bar{\mu}_2$	h_2
$n = 10$	0.0014	0.0014	-0.0533	0.0041	0.0572	0.0572	0.1088	0.1308
$n = 100$	-0.0015	-0.0015	-0.0063	-0.0006	0.0056	0.0056	0.0123	0.0125
$n = 1000$	0.0002	0.0002	-0.00006	0.0005	0.0006	0.0006	0.0012	0.0012
	$\hat{\mu}_3$	$\tilde{\mu}_3$	$\bar{\mu}_3$	h_3	$\hat{\mu}_3$	$\tilde{\mu}_3$	$\bar{\mu}_3$	h_3
$n = 10$	0.0014	0.0014	-0.1507	0.0124	0.0572	0.0572	0.4896	0.9009
$n = 100$	-0.0015	-0.0015	-0.0155	0.0015	0.0056	0.0056	0.0739	0.0782
$n = 1000$	0.0002	0.0002	-0.0007	0.001	0.0006	0.0006	0.0076	0.0077
	$\hat{\mu}_4$	$\tilde{\mu}_4$	$\bar{\mu}_4$	h_4	$\hat{\mu}_4$	$\tilde{\mu}_4$	$\bar{\mu}_4$	h_4
$n = 10$	0.0062	0.1776	-0.3526	0.0411	1.2014	1.3911	6.6962	13.8045
$n = 100$	-0.0067	0.0103	-0.0363	0.0058	0.1096	0.1112	1.1885	1.2734
$n = 1000$	0.0008	0.0025	-0.0014	0.0028	0.0111	0.0111	0.1251	0.1260
$\lambda = 5.45$	Bias				MSE			
	$\hat{\mu}_2$	$\tilde{\mu}_2$	$\bar{\mu}_2$	h_2	$\hat{\mu}_2$	$\tilde{\mu}_2$	$\bar{\mu}_2$	h_2
$n = 10$	-0.0070	-0.0070	-0.5188	0.0291	0.5494	0.5494	6.0743	7.1677
$n = 100$	0.0015	0.0015	-0.0395	0.0151	0.0540	0.0540	0.6378	0.6494
$n = 1000$	-0.0007	-0.0007	-0.0068	-0.0013	0.0055	0.0055	0.0648	0.0649
	$\hat{\mu}_3$	$\tilde{\mu}_3$	$\bar{\mu}_3$	h_3	$\hat{\mu}_3$	$\tilde{\mu}_3$	$\bar{\mu}_3$	h_3
$n = 10$	-0.0070	-0.0070	-1.6542	-0.1781	0.5494	0.5494	99.53	186.75
$n = 100$	0.0015	0.0015	-0.1217	0.0419	0.0540	0.0540	14.715	15.618
$n = 1000$	-0.0007	-0.0007	-0.0210	-0.0047	0.0055	0.0055	1.5155	1.5242
	$\hat{\mu}_4$	$\tilde{\mu}_4$	$\bar{\mu}_4$	h_4	$\hat{\mu}_4$	$\tilde{\mu}_4$	$\bar{\mu}_4$	h_4
$n = 10$	-0.2215	1.4114	-18.82	-0.3101	627.49	640.58	9327.5	16426
$n = 100$	0.0479	0.2115	-1.2921	0.7151	61.48	61.64	1512.4	1603.7
$n = 1000$	-0.0224	-0.0060	-0.2237	-0.0238	6.2007	6.2014	157.5	158.4
$\lambda = 10$	Bias				MSE			
	$\hat{\mu}_2$	$\tilde{\mu}_2$	$\bar{\mu}_2$	h_2	$\hat{\mu}_2$	$\tilde{\mu}_2$	$\bar{\mu}_2$	h_2
$n = 10$	0.0123	0.0123	-0.9795	0.0228	1.0088	1.0088	19.78	23.24
$n = 100$	0.0024	0.0024	-0.3636	-0.0676	0.1008	0.1008	2.1045	2.1389
$n = 1000$	0.0004	0.0004	-0.0109	-0.0009	0.0099	0.0099	0.2092	0.2095
	$\hat{\mu}_3$	$\tilde{\mu}_3$	$\bar{\mu}_3$	h_3	$\hat{\mu}_3$	$\tilde{\mu}_3$	$\bar{\mu}_3$	h_3
$n = 10$	0.0123	0.0123	-2.7668	0.0461	1.0088	1.0088	537.68	1022.4
$n = 100$	0.0024	0.0024	-0.3636	-0.0676	0.1008	0.1008	73.52	77.97
$n = 1000$	0.0004	0.0004	-0.0304	-0.0005	0.0099	0.0099	7.7212	7.7668
	$\hat{\mu}_4$	$\tilde{\mu}_4$	$\bar{\mu}_4$	h_4	$\hat{\mu}_4$	$\tilde{\mu}_4$	$\bar{\mu}_4$	h_4
$n = 10$	0.7731	3.7768	-59.45	1.2223	3778.6	3829.3	90839	156900
$n = 100$	0.1520	0.4521	-6.2182	0.1391	375.4	375.9	12980	13681
$n = 1000$	0.0218	0.0518	-0.7936	-0.1544	37.01	37.02	1378.2	1385.3

Table 1. The bias and MSE of the UMVUEs, the MLEs, the sample average estimators and the h-statistics for the central moments μ_2, μ_3 and μ_4 when data are generated from the poisson distribution with rate λ.

2.2. Negative Binomial Distribution

Assume $X_i \overset{i.i.d.}{\sim}$ Negative Binomial(r,p), $i=1,\ldots,n$ with r known, p unknown. In simulation, the true parameter value of r and p are randomly drawn from the set of positive integers $\mathbb{Z}^+$ and the interval $(0,1)$, respectively. After using simple derivations, the central moments are calculated as $\mu_2 = \frac{rp}{(1-p)^2}$, $\mu_3 = \frac{rp(p+1)}{(1-p)^3}$ and $\mu_4 = \frac{rp(1+4p+p^2+3rp)}{(1-p)^4}$. Since the MLE of p is $\frac{\sum_{i=1}^n X_i}{nr+\sum_{i=1}^n X_i}$, the MLEs of central moments can be obtained as

$$\begin{aligned}
\tilde{\mu}_2 &= \frac{\sum_{i=1}^n X_i(nr+\sum_{i=1}^n X_i)}{n^2 r}, \\
\tilde{\mu}_3 &= \frac{\sum_{i=1}^n X_i(nr+\sum_{i=1}^n X_i)(nr+2\sum_{i=1}^n X_i)}{n^3 r^2}, \\
\tilde{\mu}_4 &= \frac{\sum_{i=1}^n X_i(nr+\sum_{i=1}^n X_i)^3+(4+3r)(\sum_{i=1}^n X_i)^2(nr+\sum_{i=1}^n X_i)^2+(\sum_{i=1}^n X_i)^3(nr+\sum_{i=1}^n X_i)}{n^4 r^3}.
\end{aligned}$$

Besides, the UMVUEs and h-statistics are given in Sections 3.3 and 4 respectively. The Monte Carlo bias and mean square error (MSE) are given in Table 2.

$r=17, p=0.44$	Bias				MSE			
	$\hat{\mu}_2$	$\tilde{\mu}_2$	$\bar{\mu}_2$	h_2	$\hat{\mu}_2$	$\tilde{\mu}_2$	$\bar{\mu}_2$	h_2
$n=10$	0.0271	0.1675	-2.4917	-0.1184	15.64	15.86	123.6	144.9
$n=100$	-0.0238	-0.0098	-0.2991	-0.0612	1.5871	1.5885	13.39	13.58
$n=1000$	0.0031	0.0045	-0.0259	-0.0020	0.1572	0.1572	1.3476	1.3496
	$\hat{\mu}_3$	$\tilde{\mu}_3$	$\bar{\mu}_3$	h_3	$\hat{\mu}_3$	$\tilde{\mu}_3$	$\bar{\mu}_3$	h_3
$n=10$	0.0920	1.1802	-17.857	-0.9495	212.2	221.2	10275	19206
$n=100$	-0.0868	0.0213	-1.9053	-0.0799	21.29	21.36	1666.2	1766.3
$n=1000$	0.0113	0.0221	-0.3211	-0.1376	2.1098	2.1109	167.5	168.5
	$\hat{\mu}_4$	$\tilde{\mu}_4$	$\bar{\mu}_4$	h_4	$\hat{\mu}_4$	$\tilde{\mu}_4$	$\bar{\mu}_4$	h_4
$n=10$	3.7147	80.1416	-429.7	-38.94	417951	456342	4590249	8408579
$n=100$	-3.7937	3.7466	-49.17	-6.4257	41161	41468	814916	865108
$n=1000$	0.4946	1.2495	-6.5849	-2.2844	4076.2	4080.6	85345	85841
$r=9, p=0.6$	Bias				MSE			
	$\hat{\mu}_2$	$\tilde{\mu}_2$	$\bar{\mu}_2$	h_2	$\hat{\mu}_2$	$\tilde{\mu}_2$	$\bar{\mu}_2$	h_2
$n=10$	-0.0606	0.3137	-3.5974	-0.2472	54.28	55.59	284.6	335.4
$n=100$	-0.0150	0.0224	-0.2952	0.0427	5.3998	5.4121	34.15	31.69
$n=1000$	-0.0009	0.0028	-0.0466	-0.0129	0.5413	0.5414	3.0554	3.0595
	$\hat{\mu}_3$	$\tilde{\mu}_3$	$\bar{\mu}_3$	h_3	$\hat{\mu}_3$	$\tilde{\mu}_3$	$\bar{\mu}_3$	h_3
$n=10$	-0.3485	4.1731	-39.51	-2.3768	1902.0	2049.2	44179	82216
$n=100$	-0.0882	0.3618	-3.6506	0.3839	186.5	187.9	6794.3	7204.1
$n=1000$	-0.0055	0.0395	-0.6947	-0.2909	18.69	18.70	699.3	703.1
	$\hat{\mu}_4$	$\tilde{\mu}_4$	$\bar{\mu}_4$	h_4	$\hat{\mu}_4$	$\tilde{\mu}_4$	$\bar{\mu}_4$	h_4
$n=10$	-14.38	281.8	-963.1	-57.69	3.5×10^6	4.1×10^6	3.2×10^7	6.2×10^7
$n=100$	-3.7292	25.424	-89.364	10.078	3.3×10^5	3.4×10^5	5.5×10^6	5.9×10^6
$n=1000$	-0.2284	2.6839	-19.01	-9.0417	33440	33493	571450	574927
$r=5, p=0.3$	Bias				MSE			
	$\hat{\mu}_2$	$\tilde{\mu}_2$	$\bar{\mu}_2$	h_2	$\hat{\mu}_2$	$\tilde{\mu}_2$	$\bar{\mu}_2$	h_2
$n=10$	0.0993	0.8791	-3.7748	0.1268	125.6	131.4	459.9	550.2
$n=100$	0.0703	0.1482	-0.2390	0.1514	12.50	12.57	49.31	50.28
$n=1000$	-0.0019	0.0058	-0.0073	0.0316	1.2206	1.2211	4.8841	4.8948
	$\hat{\mu}_3$	$\tilde{\mu}_3$	$\bar{\mu}_3$	h_3	$\hat{\mu}_3$	$\tilde{\mu}_3$	$\bar{\mu}_3$	h_3
$n=10$	0.7876	14.23	-58.85	3.9561	9128.9	1.0×10^4	1.2×10^5	2.2×10^5
$n=100$	0.5894	1.9169	-4.4219	2.2110	888.6	902.6	1.6×10^4	1.7×10^4
$n=1000$	-0.0183	0.1139	-0.0694	0.5930	86.38	86.49	1745.2	1756.1
	$\hat{\mu}_4$	$\tilde{\mu}_4$	$\bar{\mu}_4$	h_4	$\hat{\mu}_4$	$\tilde{\mu}_4$	$\bar{\mu}_4$	h_4
$n=10$	28.27	840.1	-1346	218.1	1.4×10^7	1.9×10^7	1.2×10^8	2.6×10^8
$n=100$	22.87	101.4	-101.9	63.21	1.3×10^6	1.4×10^6	1.7×10^7	1.9×10^7
$n=1000$	-0.7732	7.0221	3.1942	19.73	1.3×10^5	1.3×10^5	1.9×10^6	1.9×10^6

Table 2. The bias and MSE of the UMVUEs, the MLEs, the sample average estimators and the h-statistics for the central moments μ_2, μ_3 and μ_4 when data are generated from the negative binomial distribution with parameters r and p.

Next, we study the performances of our proposed estimators for the cumulants $\hat{\kappa}_k$ for the negative binomial distribution. By simple derivations, we have $\kappa_2 = \frac{rp}{(1-p)^2}$, $\kappa_3 = \frac{rp(p+1)}{(1-p)^3}$ and $\kappa_4 = \frac{rp(p^2+4p+1)}{(1-p)^4}$, then the MLEs of the cumulants are

$$\begin{aligned}
\tilde{\kappa}_2 &= \frac{\sum_{i=1}^n X_i(nr+\sum_{i=1}^n X_i)}{n^2 r}, \\
\tilde{\kappa}_3 &= \frac{\sum_{i=1}^n X_i(nr+\sum_{i=1}^n X_i)(nr+2\sum_{i=1}^n X_i)}{n^3 r^2}, \\
\tilde{\kappa}_4 &= \frac{\sum_{i=1}^n X_i(nr+\sum_{i=1}^n X_i)^3+4(\sum_{i=1}^n X_i)^2(nr+\sum_{i=1}^n X_i)^2+(\sum_{i=1}^n X_i)^3(nr+\sum_{i=1}^n X_i)}{n^4 r^3}.
\end{aligned}$$

The UMVUEs and k-statistics are given in Section 3.3 and Section 4, respectively. The results of bias and MSE are given in Table 3.

$r=17, p=0.44$	Bias				MSE			
	$\hat{\kappa}_2$	$\tilde{\kappa}_2$	$\bar{\kappa}_2$	k_2	$\hat{\kappa}_2$	$\tilde{\kappa}_2$	$\bar{\kappa}_2$	k_2
$n=10$	0.0072	0.1476	-2.5519	-0.1853	15.73	15.95	122.5	143.3
$n=100$	0.0021	0.0162	-0.2632	-0.0249	1.6042	1.6063	13.68	13.89
$n=1000$	-0.0079	-0.0065	-0.0393	-0.0155	0.1552	0.1552	1.3611	1.3625
	$\hat{\kappa}_3$	$\tilde{\kappa}_3$	$\bar{\kappa}_3$	k_3	$\hat{\kappa}_3$	$\tilde{\kappa}_3$	$\bar{\kappa}_3$	k_3
$n=10$	0.0237	1.1107	-19.15	-2.75	212.9	221.8	10011	18611
$n=100$	0.0087	0.1170	-1.8075	0.0208	21.55	21.64	1663	1763
$n=1000$	-0.0289	-0.0181	-0.3369	-0.1534	2.0820	2.0822	166.6	167.6
	$\hat{\kappa}_4$	$\tilde{\kappa}_4$	$\bar{\kappa}_4$	k_4	$\hat{\kappa}_4$	$\tilde{\kappa}_4$	$\bar{\kappa}_4$	k_4
$n=10$	0.1011	8.2604	-452.6	-43.95	5123.9	5567.7	1.1×10^6	4.5×10^6
$n=100$	0.0472	0.8551	-48.26	0.9984	512.3	516.6	323065	373998
$n=1000$	-0.1414	-0.0607	-6.6571	-1.6934	49.37	49.39	35258	35754
$r=9, p=0.6$	Bias				MSE			
	$\hat{\kappa}_2$	$\tilde{\kappa}_2$	$\bar{\kappa}_2$	k_2	$\hat{\kappa}_2$	$\tilde{\kappa}_2$	$\bar{\kappa}_2$	k_2
$n=10$	0.0333	0.4087	-3.3275	0.0527	54.04	55.41	273.9	324.6
$n=100$	-0.0018	0.0357	-0.3228	0.0148	5.3907	5.4039	30.51	31.02
$n=1000$	-0.0049	-0.0012	-0.0345	-0.0008	0.5337	0.5338	3.0732	3.0781
	$\hat{\kappa}_3$	$\tilde{\kappa}_3$	$\bar{\kappa}_3$	k_3	$\hat{\kappa}_3$	$\tilde{\kappa}_3$	$\bar{\kappa}_3$	k_3
$n=10$	0.1793	4.7186	-39.53	-2.3971	1904.8	2057.1	40287	74706
$n=100$	-0.0112	0.4391	-3.8222	0.2069	186.5	187.9	6742	7147
$n=1000$	-0.0292	0.0158	-0.1453	0.2602	18.43	18.44	706.4	710.7
	$\hat{\kappa}_4$	$\tilde{\kappa}_4$	$\bar{\kappa}_4$	k_4	$\hat{\kappa}_4$	$\tilde{\kappa}_4$	$\bar{\kappa}_4$	k_4
$n=10$	1.3167	55.76	-1108	-65.21	1.2×10^5	1.4×10^5	6.2×10^6	2.8×10^7
$n=100$	-0.0911	5.2441	-110.9	12.23	1.1×10^4	1.2×10^4	2.5×10^6	2.9×10^6
$n=1000$	-0.2304	0.3020	-5.4377	6.9806	1129.9	1131.5	2.9×10^5	3.0×10^5
$r=5, p=0.3$	Bias				MSE			
	$\hat{\kappa}_2$	$\tilde{\kappa}_2$	$\bar{\kappa}_2$	k_2	$\hat{\kappa}_2$	$\tilde{\kappa}_2$	$\bar{\kappa}_2$	k_2
$n=10$	-0.0157	0.0452	-0.3228	-0.0186	1.0349	1.0785	2.8408	3.3789
$n=100$	-0.0026	0.0035	-0.0271	0.0035	0.1062	0.1067	0.3239	0.3297
$n=1000$	0.0019	0.0025	0.0009	0.0039	0.0106	0.0106	0.0333	0.0334
	$\hat{\kappa}_3$	$\tilde{\kappa}_3$	$\bar{\kappa}_3$	k_3	$\hat{\kappa}_3$	$\tilde{\kappa}_3$	$\bar{\kappa}_3$	k_3
$n=10$	-0.0428	0.3003	-1.6738	-0.1138	6.6965	7.6236	60.27	110.9
$n=100$	-0.0064	0.0277	-0.1575	0.0123	0.6752	0.6841	9.9716	10.56
$n=1000$	0.0047	0.0081	-0.0128	0.0043	0.0671	0.0672	1.0466	1.0527
	$\hat{\kappa}_4$	$\tilde{\kappa}_4$	$\bar{\kappa}_4$	k_4	$\hat{\kappa}_4$	$\tilde{\kappa}_4$	$\bar{\kappa}_4$	k_4
$n=10$	-0.1522	1.6714	-13.58	-0.5443	76.83	100.3	1195	6315
$n=100$	-0.0206	0.1575	-1.5204	0.0233	7.4759	7.6850	543.5	629.6
$n=1000$	0.0156	0.0334	-0.1992	-0.0432	0.7388	0.7415	58.37	59.21

Table 3. The bias and MSE of the UMVUEs, the MLEs, the sample average estimators and the k-statistics for the cumulants κ_2, κ_3 and κ_4 when data are generated from the negative binomial distribution with parameters r and p.

2.3. Discrete Uniform Distribution

Now, we study the performance of the estimators for the continuous uniform distribution. For $X_i \overset{i.i.d.}{\sim} \text{Uniform}(N),\ i = 1, \ldots, n$, we have $\mu_3 = \kappa_3 = 0$. Thus, we only need to estimate μ_2 and μ_4 for the central moment. By simple calculations, we obtain $\mu_2 = \frac{1}{12}(N-1)(N+1)$ and $\mu_4 = \frac{1}{240}(N-1)(N+1)(3N^2-7)$. Since the MLE for N is $X_{(n)}$, the MLEs for the central moments can be obtained as $\tilde{\mu}_2 = \frac{1}{12}(X_{(n)}-1)(X_{(n)}+1)$ and $\tilde{\mu}_4 = \frac{1}{240}(X_{(n)}-1)(X_{(n)}+1)(3X_{(n)}^2-7)$. The average estimators for μ_2 and μ_4 are $\bar{\mu}_2$ and $\bar{\mu}_4$ as before. The corresponding UMVUEs and h-statistics are given in Sections 3.4 and 4, respectively. In the simulation, we randomly draw N from $\{1, \ldots, 20\}$. The estimation results are reported in Table 4. Note that when sample size $n = 1000$, there exists computation overflow in the simulation, therefore, we only show the simulation results for n =10 and 100.

$N = 5$	Bias				MSE			
	$\hat{\mu}_2$	$\tilde{\mu}_2$	$\bar{\mu}_2$	h_2	$\hat{\mu}_2$	$\tilde{\mu}_2$	$\bar{\mu}_2$	h_2
$n = 10$	0.0033	-0.0810	-0.1932	0.0076	0.0665	0.0644	0.3353	0.3679
$n = 100$	0.0153	0	-0.0183	0.0017	2.3×10^{-20}	0	0.0279	0.0282
	$\hat{\mu}_4$	$\tilde{\mu}_4$	$\bar{\mu}_4$	h_4	$\hat{\mu}_4$	$\tilde{\mu}_4$	$\bar{\mu}_4$	h_4
$n = 10$	0.0165	-0.4513	-0.4488	0.0544	2.1048	1.9671	8.1505	8.5642
$n = 100$	8.6×10^{-10}	0	-0.0262	0.0076	7.4×10^{-19}	0	0.5784	0.5648
$N = 9$	Bias				MSE			
	$\hat{\mu}_2$	$\tilde{\mu}_2$	$\bar{\mu}_2$	h_2	$\hat{\mu}_2$	$\tilde{\mu}_2$	$\bar{\mu}_2$	h_2
$n = 10$	0.0079	-0.5521	-0.6448	0.0243	1.0785	1.1564	3.9678	4.3859
$n = 100$	1.1×10^{-5}	0	-0.0641	0.0025	1.2×10^{-10}	0	0.3515	0.3544
	$\hat{\mu}_4$	$\tilde{\mu}_4$	$\bar{\mu}_4$	h_4	$\hat{\mu}_4$	$\tilde{\mu}_4$	$\bar{\mu}_4$	h_4
$n = 10$	0.1566	-11.06	-6.1047	0.5411	447.4	435.7	1287	1434
$n = 100$	2.3×10^{-4}	0	-0.4954	0.0018	5.3×10^{-8}	0	103.7	102.6
$N = 14$	Bias				MSE			
	$\hat{\mu}_2$	$\tilde{\mu}_2$	$\bar{\mu}_2$	h_2	$\hat{\mu}_2$	$\tilde{\mu}_2$	$\bar{\mu}_2$	h_2
$n = 10$	-0.0345	-1.8063	-1.6048	0.0224	7.0510	8.8510	23.88	26.31
$n = 100$	-0.0002	-0.0016	-0.1576	-0.0050	0.0035	0.0035	2.1182	2.1359
	$\hat{\mu}_4$	$\tilde{\mu}_4$	$\bar{\mu}_4$	h_4	$\hat{\mu}_4$	$\tilde{\mu}_4$	$\bar{\mu}_4$	h_4
$n = 10$	-2.0330	-89.86	-40.60	-0.1163	18224	19621	48777	55297
$n = 100$	-0.0116	-0.0854	-2.8858	0.2658	10.43	10.43	3880.3	3847.7

Table 4. The UMVUEs, the MLEs, the sample average estimators and the h-statistics for the central moments μ_2 and μ_4 when data are generated from the discrete uniform distribution with parameter N. Note that $\mu_3 = 0$.

Similarly, since $\kappa_3 = 0$, we only need to estimate κ_2 and κ_4. By similar derivations as the central moment, the MLEs of the cumulants are $\tilde{\kappa}_2 = \frac{1}{12}(X_{(n)} - 1)(X_{(n)} + 1)$ and $\tilde{\kappa}_4 = -\frac{1}{120}(X_{(n)}-1)(X_{(n)}+1)(X_{(n)}^2+1)$. The UMVUEs and k-statistics are given in Sections 3.4 and 4, respectively. The estimation results are given in Table 5.

2.4. Exponential Distribution

We investigate the performances of the estimators for the first four central moments for the exponential distribution. Consider $X_i \overset{i.i.d.}{\sim} \text{Exponential}(\lambda),\ i = 1, \ldots, n$ with sample size $n \in \{10, 100, 1000\}$. The true parameter value of λ is randomly drawn from $(0, 10)$. By simple derivations, we have $\mu_2 = 1/\lambda^2$, $\mu_3 = 2/\lambda^3$ and $\mu_4 = 9/\lambda^4$. Since the MLE for λ is $1/\bar{X}$, the MLEs for the central moments can be obtained as $\tilde{\mu}_2 = \bar{X}^2$, $\tilde{\mu}_3 = 2\bar{X}^3$ and $\tilde{\mu}_4 = 9\bar{X}^4$. In addition, our proposed UMVUEs and h-statistics are given in Sections 3.5 and 4, respectively. The estimation results are reported in Table

$N=5$	Bias				MSE			
	$\hat{\kappa}_2$	$\tilde{\kappa}_2$	$\bar{\kappa}_2$	h_2	$\hat{\kappa}_2$	$\tilde{\kappa}_2$	$\bar{\kappa}_2$	k_2
$n=10$	-0.0057	-0.0894	-0.2089	-0.0099	0.0728	0.0713	0.3431	0.3698
$n=100$	1.5×10^{-10}	0	-0.0216	-0.0016	2.3×10^{-20}	0	0.0284	0.0285
	$\hat{\kappa}_4$	$\tilde{\kappa}_4$	$\bar{\kappa}_4$	k_4	$\hat{\kappa}_4$	$\tilde{\kappa}_4$	$\bar{\kappa}_4$	k_4
$n=10$	0.0243	0.3612	0.9469	0.0925	1.2098	1.1468	14.78	34.86
$n=100$	-6.3×10^{-10}	0	0.1357	0.0162	3.9×10^{-19}	0	1.5969	1.7074
$N=9$	Bias				MSE			
	$\hat{\kappa}_2$	$\tilde{\kappa}_2$	$\bar{\kappa}_2$	k_2	$\hat{\kappa}_2$	$\tilde{\kappa}_2$	$\bar{\kappa}_2$	k_2
$n=10$	0.0119	-0.5486	-0.6501	0.0184	1.0790	1.1526	3.9914	4.4062
$n=100$	1.1×10^{-5}	0	-0.0697	-0.0031	1.2×10^{-10}	0	0.3467	0.3488
	$\hat{\kappa}_4$	$\tilde{\kappa}_4$	$\bar{\kappa}_4$	k_4	$\hat{\kappa}_4$	$\tilde{\kappa}_4$	$\bar{\kappa}_4$	k_4
$n=10$	-0.1701	7.5039	7.9453	0.0376	209.1	203.2	1849	4429
$n=100$	-1.6×10^{-4}	0	1.1901	0.0710	2.5×10^{-8}	0	193.8	207.4
$N=14$	Bias				MSE			
	$\hat{\kappa}_2$	$\tilde{\kappa}_2$	$\bar{\kappa}_2$	k_2	$\hat{\kappa}_2$	$\tilde{\kappa}_2$	$\bar{\kappa}_2$	k_2
$n=10$	-0.0002	-1.7767	-1.6618	-0.0409	7.1257	8.5330	23.9990	26.2209
$n=100$	0.0007	-0.0007	-0.1601	0.0024	0.0015	0.0015	2.1044	2.1210
	$\hat{\kappa}_4$	$\tilde{\kappa}_4$	$\bar{\kappa}_4$	k_4	$\hat{\kappa}_4$	$\tilde{\kappa}_4$	$\bar{\kappa}_4$	k_4
$n=10$	-0.0260	59.41	47.29	3.2875	8315	8809	62679	148204
$n=100$	-0.0250	0.0246	6.5304	0.2072	2.0247	2.0233	6891	7383

Table 5. The UMVUEs, the MLEs, the sample average estimators and the k-statistics for the cumulants κ_2 and κ_4 when data are generated from the discrete uniform distribution with parameter N. Note that $\kappa_3 = 0$.

6.

Next, we investigate the performances of our proposed estimators for the cumulants $\hat{\kappa}_k$ for the exponential distribution. After using simple derivations, we have $\kappa_2 = 1/\lambda^2$, $\kappa_3 = 2/\lambda^3$ and $\kappa_4 = 8/\lambda^4$. Then the MLEs are $\tilde{\kappa}_2 = \bar{X}^2$, $\tilde{\kappa}_3 = 2\bar{X}^3$ and $\tilde{\kappa}_4 = 6\bar{X}^4$. The UMVUEs and k-statistics are given in Sections 3.5 and 4, respectively. The results of bias and MSE are reported in Table 7.

$\lambda = 0.7$	Bias				MSE			
	$\hat{\mu}_2$	$\tilde{\mu}_2$	$\bar{\mu}_2$	h_2	$\hat{\mu}_2$	$\tilde{\mu}_2$	$\bar{\mu}_2$	h_2
$n = 10$	0.0078	0.2094	-0.1169	0.0073	0.6138	0.7604	0.9857	1.1981
$n = 100$	-0.0085	0.0097	-0.0248	-0.0126	0.0639	0.0652	0.1117	0.1135
$n = 1000$	7.5×10^{-4}	8.42×10^{-4}	3.45×10^{-3}	2.21×10^{-3}	0.0059	0.0093	0.0116	0.0114
	$\hat{\mu}_3$	$\tilde{\mu}_3$	$\bar{\mu}_3$	h_3	$\hat{\mu}_3$	$\tilde{\mu}_3$	$\bar{\mu}_3$	h_3
$n = 10$	0.0181	0.9018	-0.7974	-0.0406	7.1631	13.294	19.464	36.321
$n = 100$	-0.0258	0.0562	-0.1722	-0.0933	0.7147	0.761	2.9997	3.164
$n = 1000$	-0.0026	0.0056	-0.0134	-0.0052	0.0663	0.0767	0.3913	0.3935
	$\hat{\mu}_4$	$\tilde{\mu}_4$	$\bar{\mu}_4$	h_4	$\hat{\mu}_4$	$\tilde{\mu}_4$	$\bar{\mu}_4$	h_4
$n = 10$	-4.0133	9.8027	-4.122	-3.0815	185	1093	944	2024
$n = 100$	-0.6037	0.671	-1.3658	-0.951	30.5	36.7	184.8	198.1
$n = 1000$	-0.0638	0.0642	0.0543	0.0504	2.95	3.01	34.89	35.16
$\lambda = 0.9$	Bias				MSE			
	$\hat{\mu}_2$	$\tilde{\mu}_2$	$\bar{\mu}_2$	h_2	$\hat{\mu}_2$	$\tilde{\mu}_2$	$\bar{\mu}_2$	h_2
$n = 10$	0.088	0.1332	-0.1235	-0.047	0.6138	0.7604	0.9857	1.1981
$n = 100$	-0.0038	0.0086	-0.0224	-0.0101	0.0617	0.0629	0.1184	0.1204
$n = 1000$	-6.34×10^{-4}	5.99×10^{-4}	-0.0029	-0.0017	0.0059	0.0059	0.0117	0.0116
	$\hat{\mu}_3$	$\tilde{\mu}_3$	$\bar{\mu}_3$	h_3	$\hat{\mu}_3$	$\tilde{\mu}_3$	$\bar{\mu}_3$	h_3
$n = 10$	0.0181	0.9018	-0.7974	-0.0406	7.1631	13.294	19.464	36.321
$n = 100$	-0.0258	0.0562	-0.1722	-0.0933	0.7147	0.761	2.9997	3.164
$n = 1000$	-0.0026	0.0056	-0.0134	-0.0052	0.0663	0.0767	0.3913	0.3935
	$\hat{\mu}_4$	$\tilde{\mu}_4$	$\bar{\mu}_4$	h_4	$\hat{\mu}_4$	$\tilde{\mu}_4$	$\bar{\mu}_4$	h_4
$n = 10$	-3.214	11.74	-2.904	2.554	106	876	792	1749
$n = 100$	-0.5254	0.5697	-0.7105	-0.2683	30.01	32.3	149.5	148.8
$n = 1000$	-0.0061	0.0064	-0.0447	0.01	2.94	2.01	35.17	35.44
$\lambda = 1.5$	Bias				MSE			
	$\hat{\mu}_2$	$\tilde{\mu}_2$	$\bar{\mu}_2$	h_2	$\hat{\mu}_2$	$\tilde{\mu}_2$	$\bar{\mu}_2$	h_2
$n = 10$	0.0041	0.0049	-0.0464	-0.0052	0.0833	0.1031	0.1378	0.1675
$n = 100$	-0.003	0.0034	-0.0087	-0.0043	0.0085	0.0086	0.0145	0.0148
$n = 1000$	-2.9×10^{-4}	3.4×10^{-4}	-0.013	-8.7×10^{-4}	7.6×10^{-4}	8.6×10^{-4}	0.0015	0.0015
	$\hat{\mu}_3$	$\tilde{\mu}_3$	$\bar{\mu}_3$	h_3	$\hat{\mu}_3$	$\tilde{\mu}_3$	$\bar{\mu}_3$	h_3
$n = 10$	0.0086	0.201	-0.1655	0.0062	0.3589	0.6656	1.2051	2.2719
$n = 100$	-0.0054	0.0124	-0.0366	-0.0195	0.0345	0.0367	0.1491	0.1573
$n = 1000$	-0.0006	0.0011	-0.0033	-0.0015	0.0031	0.0031	0.0184	0.0185
	$\hat{\mu}_4$	$\tilde{\mu}_4$	$\bar{\mu}_4$	h_4	$\hat{\mu}_4$	$\tilde{\mu}_4$	$\bar{\mu}_4$	h_4
$n = 10$	-0.4998	1.3196	-0.5902	0.0549	3.48	20.74	30.64	68.81
$n = 100$	-0.0766	0.0888	-0.1668	-0.1125	0.5379	0.6484	3.96	4.253
$n = 1000$	-0.0025	0.008	-0.0029	0.0031	0.0495	0.0504	0.5838	0.5882

Table 6. The bias and MSE of the UMVUEs, the MLEs, the sample average estimators and the h-statistics for the central moments μ_2, μ_3 and μ_4 when data are generated from the exponential distribution with rate λ.

$\lambda = 0.7$	Bias				MSE			
	$\hat{\kappa}_2$	$\tilde{\kappa}_2$	$\bar{\kappa}_2$	k_2	$\hat{\kappa}_2$	$\tilde{\kappa}_2$	$\bar{\kappa}_2$	k_2
$n = 10$	-0.0232	0.1786	-0.1978	0.0169	1.99	2.44	3.25	3.96
$n = 100$	0.0062	-0.014	-0.0393	-0.0191	0.1685	0.1717	0.3004	0.3053
$n = 1000$	-0.0014	0.0007	-0.0056	-0.0035	0.0159	0.0159	0.0327	0.0328
	$\hat{\kappa}_3$	$\tilde{\kappa}_3$	$\bar{\kappa}_3$	k_3	$\hat{\kappa}_3$	$\tilde{\kappa}_3$	$\bar{\kappa}_3$	k_3
$n = 10$	0.0037	1.8708	-1.319	0.4353	44.75	81.48	117	223
$n = 100$	-0.0583	0.1161	-0.3532	-0.1849	3.099	3.298	13.44	14.18
$n = 1000$	-0.0062	0.0113	-0.0279	-0.0105	0.2923	0.2942	1.815	1.826
	$\hat{\kappa}_4$	$\tilde{\kappa}_4$	$\bar{\kappa}_4$	k_4	$\hat{\kappa}_4$	$\tilde{\kappa}_4$	$\bar{\kappa}_4$	k_4
$n = 10$	-6.819	9.5	-25.44	-8.95	1314	3399	2500	9172
$n = 100$	3.984	-8.218	-12.36	-10.55	221	166	1130	1245
$n = 1000$	2.461	-7.147	-8.347	-8.146	37.3	77.2	282	282
$\lambda = 0.9$	Bias				MSE			
	$\hat{\kappa}_2$	$\tilde{\kappa}_2$	$\bar{\kappa}_2$	k_2	$\hat{\kappa}_2$	$\tilde{\kappa}_2$	$\bar{\kappa}_2$	k_2
$n = 10$	-0.0077	0.115	-0.1451	-0.0241	0.6296	0.775	0.9204	1.111
$n = 100$	-0.0087	0.0036	-0.0213	-0.0095	0.0574	0.0585	0.1201	0.1222
$n = 1000$	-0.0005	0.0007	-0.0025	-0.0013	0.0062	0.0062	0.0116	0.0117
	$\hat{\kappa}_3$	$\tilde{\kappa}_3$	$\bar{\kappa}_3$	k_3	$\hat{\kappa}_3$	$\tilde{\kappa}_3$	$\bar{\kappa}_3$	k_3
$n = 10$	-0.0272	0.8419	-0.8693	-0.1405	7.935	14.53	14.22	25.99
$n = 100$	-0.031	0.0509	-0.1125	-0.0317	0.6416	0.648	3.892	4.122
$n = 1000$	-0.0017	0.0065	-0.0141	-0.0059	0.0689	0.0694	0.4098	0.4121
	$\hat{\kappa}_4$	$\tilde{\kappa}_4$	$\bar{\kappa}_4$	k_4	$\hat{\kappa}_4$	$\tilde{\kappa}_4$	$\bar{\kappa}_4$	k_4
$n = 10$	-2.601	3.306	-9.82	-4.529	216.9	876	792	1749
$n = 100$	1.467	-3.001	-3.935	-3.229	27.83	21.49	239.7	270.1
$n = 1000$	0.8604	-2.645	-3.104	-3.03	5.199	10.38	42.54	42.57
$\lambda = 1.5$	Bias				MSE			
	$\hat{\kappa}_2$	$\tilde{\kappa}_2$	$\bar{\kappa}_2$	k_2	$\hat{\kappa}_2$	$\tilde{\kappa}_2$	$\bar{\kappa}_2$	k_2
$n = 10$	-0.0033	0.0408	-0.0512	-0.0075	0.0817	0.1005	0.1486	0.1803
$n = 100$	0.0009	0.0054	-0.0065	-0.0021	0.0082	0.0084	0.0146	0.0149
$n = 1000$	3.9×10^{-5}	0.0005	-0.0005	5.1×10^{-5}	0.0007	0.0007	0.0015	0.0015
	$\hat{\kappa}_3$	$\tilde{\kappa}_3$	$\bar{\kappa}_3$	k_3	$\hat{\kappa}_3$	$\tilde{\kappa}_3$	$\bar{\kappa}_3$	k_3
$n = 10$	-0.0084	0.1786	-0.1669	-0.014	0.4125	0.7505	1.365	2.579
$n = 100$	0.002	0.0199	-0.0224	-0.0049	0.0334	0.0358	0.1742	0.1846
$n = 1000$	1.7×10^{-5}	0.0018	-0.0039	-0.0021	0.0029	0.003	0.0181	0.0182
	$\hat{\kappa}_4$	$\tilde{\kappa}_4$	$\bar{\kappa}_4$	k_4	$\hat{\kappa}_4$	$\tilde{\kappa}_4$	$\bar{\kappa}_4$	k_4
$n = 10$	-0.3387	0.4259	1.158	-0.3009	4.299	12.41	12.14	66.67
$n = 100$	0.1914	-0.3881	-0.4808	-0.4165	0.5368	0.3757	3.96	4.253
$n = 1000$	0.1476	-0.3164	-0.4258	-0.3872	0.0836	0.1719	0.6267	0.6252

Table 7. The UMVUEs, the MLEs, the sample average estimators and the k-statistics for the cumulants κ_2, κ_3 and κ_4 when data are generated from the exponential distribution with rate λ.