**Supplementary Material of Okpoti and Jeong, "A decentralized coordination algorithm for multi-objective linear programming with block angular structure",** ***Engineering Optimization*****, 2019.**

This file contains a table where the centralized and decentralized methods are compared based on WIP, Delayed Demand and Throughput. Also included, are the proofs to the Lemma discussed in the main article body.

## TABLES

**Table S1.** Comparison of centralized and decentralized methods

| WIP rate | Demand | | CFH | | | SFH | | | DEC | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | *WIP* | *DD* | *TH* | *WIP* | *DD* | *TH* | *WIP* | *DD* | *TH* |
| 1.0 | Type 1 | Avg. | 2419 | **2667*** | 173 | 2074 | 2876 | **187*** | **1852*** | 2684 | 171 |
| | | Min. | 1637 | 1804 | 136 | 1313 | 1849 | 152 | 1278 | 1822 | 134 |
| | | Max. | 3393 | 3586 | 208 | 3057 | 4250 | 221 | 3090 | 3618 | 207 |
| | Type 2 | Avg. | 2426 | 2759 | 171 | 2014 | 2983 | **184*** | **1900*** | **2682*** | 168 |
| | | Min. | 1684 | 2002 | 136 | 1283 | 2140 | 151 | 1210 | 1722 | 134 |
| | | Max. | 3446 | 3592 | 208 | 3168 | 4189 | 223 | 3601 | 3694 | 204 |
| | Type 3 | Avg. | 2352 | **2807*** | 170 | 2066 | 2996 | **183*** | **1815*** | 2903 | 164 |
| | | Min. | 1660 | 1948 | 140 | 1315 | 1969 | 156 | 1268 | 1134 | 139 |
| | | Max. | 3772 | 3742 | 203 | 3455 | 3792 | 213 | 3585 | 5517 | 185 |
| | Type 4 | Avg. | 1530 | 2685 | 175 | 1717 | 2906 | **190*** | **1236*** | **2656*** | 173 |
| | | Min. | 1258 | 1994 | 156 | 1373 | 2087 | 167 | 1065 | 1727 | 155 |
| | | Max. | 2409 | 4307 | 203 | 2791 | 4453 | 218 | 1345 | 4202 | 202 |
| 1.5 | Type 1 | Avg. | 2684 | **1966*** | 185 | 2356 | 2211 | **200*** | **2031*** | 1986 | 183 |
| | | Min. | 1709 | 996 | 148 | 1361 | 1137 | 165 | 1295 | 1001 | 145 |
| | | Max. | 4093 | 2957 | 224 | 3551 | 3623 | 235 | 3568 | 3000 | 223 |
| | Type 2 | Avg. | 2716 | 2089 | 182 | 2254 | 2340 | **196*** | **2072*** | **2013*** | 180 |
| | | Min. | 1865 | 1220 | 147 | 1306 | 1360 | 163 | 1225 | 926 | 145 |
| | | Max. | 4269 | 2964 | 222 | 3913 | 3487 | 238 | 4106 | 3147 | 218 |
| | Type 3 | Avg. | 2779 | **2149*** | 182 | 2343 | 2340 | **195*** | **2065** | 2291 | 175 |
| | | Min. | 1796 | 1230 | 152 | 1354 | 1250 | 166 | 1334 | 596 | 148 |
| | | Max. | 4362 | 3163 | 215 | 4363 | 3273 | 227 | 4115 | 4957 | 198 |
| | Type 4 | Avg. | 1639 | 2004 | 188 | 1838 | 2232 | **203*** | **1320*** | **1976*** | 185 |
| | | Min. | 1275 | 1304 | 167 | 1432 | 1397 | 178 | 1139 | 1043 | 165 |
| | | Max. | 3173 | 3555 | 217 | 3095 | 3746 | 231 | 1582 | 3446 | 215 |
| 2.0 | | Avg. | 3160 | **1461*** | 195 | 2789 | 1711 | **210*** | **2442*** | 1486 | 193 |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | Type 1 | Min. | 1826 | 346 | 158 | 1450 | 688 | 176 | 1375 | 352 | 155 |
| | | Max. | 4634 | 2449 | 239 | 4240 | 3024 | 249 | 4207 | 2494 | 236 |
| | Type 2 | Avg. | 3179 | 1577 | 193 | 2617 | 1831 | **207*** | **2402*** | **1535*** | 189 |
| | | Min. | 1989 | 685 | 158 | 1339 | 786 | 175 | 1269 | 500 | 156 |
| | | Max. | 5133 | 2389 | 233 | 4709 | 2973 | 247 | 4747 | 2780 | 227 |
| | Type 3 | Avg. | 3337 | **1673*** | 192 | 2825 | 1895 | **205*** | **2468*** | 1850 | 183 |
| | | Min. | 1953 | 805 | 159 | 1428 | 825 | 174 | 1476 | 145 | 156 |
| | | Max. | 5695 | 2702 | 226 | 5048 | 3038 | 235 | 4772 | 4571 | 209 |
| | Type 4 | Avg. | 1915 | 1517 | 199 | 2070 | 1751 | **213*** | 1535 | **1491*** | 195 |
| | | Min. | 1431 | 788 | 175 | 1582 | 886 | 188 | 1259 | 601 | 172 |
| | | Max. | 3731 | 2891 | 229 | 3201 | 3181 | 242 | 2177 | 2767 | 225 |
| 3.0 | Type 1 | Avg. | 3033 | **845*** | 213 | 3383 | 1136 | **229*** | **2311*** | 854 | 206 |
| | | Min. | 1905 | 21 | 174 | 2304 | 72 | 195 | 1778 | 24 | 172 |
| | | Max. | 4479 | 1651 | 257 | 4932 | 2695 | 269 | 2937 | 1670 | 244 |
| | Type 2 | Avg. | 2857 | 920 | 210 | 3119 | 1142 | **226*** | **2098*** | **893*** | 204 |
| | | Min. | 1528 | 263 | 177 | 1706 | 299 | 197 | 1440 | 227 | 175 |
| | | Max. | 5075 | 1524 | 253 | 4710 | 2597 | 263 | 2938 | 1478 | 239 |
| | Type 3 | Avg. | 3105 | **1023*** | 207 | 3529 | 1339 | **223*** | **2292*** | 1033 | 201 |
| | | Min. | 1850 | 312 | 172 | 1931 | 483 | 189 | 1641 | 349 | 171 |
| | | Max. | 4783 | 2017 | 247 | 5344 | 2467 | 261 | 3095 | 2005 | 235 |
| | Type 4 | Avg. | 2733 | 872 | 217 | 3091 | 1109 | **232*** | **2223*** | **846*** | 209 |
| | | Min. | 1774 | 244 | 187 | 1907 | 367 | 205 | 1513 | 214 | 185 |
| | | Max. | 3938 | 1873 | 250 | 4742 | 2056 | 263 | 3223 | 1695 | 236 |

*CFH: Customer First Heuristic*
*SFH: System First Heuristic*
*DEC: Proposed decentralized coordination mechanism*
**Bold asterisked** (*****) values indicate the best among the three methods

## APPENDIX

### A1. Proof of Lemma 4.1

Assuming that $\boldsymbol{x}_i^k - \boldsymbol{x}_c^k = \boldsymbol{0}$ and $\boldsymbol{x}_i^k = \boldsymbol{x}_i^{k+1}$, then it implies that $\boldsymbol{x}_i^{k+1} = \boldsymbol{x}_c^k$. Thus with the definition of $\boldsymbol{\pi}_i^{k+1}$ means $\boldsymbol{\pi}_i^{k+1} = \boldsymbol{\pi}_i^k$. Using the first-order optimality condition on the $\Theta_i(\boldsymbol{x}_i^{k+1})$ related problem in (12),

$$\frac{\partial \Theta_i}{\partial \boldsymbol{x}_i^{k+1}} = \boldsymbol{\zeta}_i^{k+1} + \boldsymbol{\pi}_i^k + \rho\left(2\boldsymbol{x}_i^{k+1} - \boldsymbol{x}_c^k - \boldsymbol{x}_i^{\boldsymbol{0}*}\right)$$

where $\boldsymbol{\zeta}_i^{k+1} \in \partial f_i(\boldsymbol{x}_i^{k+1})$ using the fact that the subdifferential of the sum of a differentiable function and a subdifferentiable function with domain $\mathbb{R}^m$ is the summation of the gradient and the subdifferential. If $\boldsymbol{x}_i^{k+1}$ is a solution to $min \left\{\Theta_i(\boldsymbol{x}_i^{k+1}) \middle| \boldsymbol{x}_i^{k+1} \in \boldsymbol{\mathcal{X}}\right\}$ then

$$\langle \boldsymbol{x}_i - \boldsymbol{x}_i^{k+1}, \boldsymbol{\zeta}_i^{k+1} + \boldsymbol{\pi}_i^k + \rho\left(2\boldsymbol{x}_i^{k+1} - \boldsymbol{x}_c^k - \boldsymbol{x}_i^{\boldsymbol{0}*}\right)\rangle \geq 0 \quad \forall \boldsymbol{x}_i \in \boldsymbol{\mathcal{X}} \tag{35}$$

Then it follows from $\boldsymbol{x}_i^{k+1} = \boldsymbol{x}_c^k$ and (13) that

$$\langle \boldsymbol{x}_i' - \boldsymbol{x}_i^{k+1}, \boldsymbol{\zeta}_i^{k+1} + \boldsymbol{\pi}_i^{k+1} + \rho\left(\boldsymbol{x}_c^k - \boldsymbol{x}_i^{\boldsymbol{0}*}\right)\rangle \geq 0 \quad \forall \boldsymbol{x}_i' \in \boldsymbol{\mathcal{X}}, \ \ i = 1, \ldots, m$$

and therefore $\left(x_{i1}^{k+1}, \cdots, x_{in}^{k+1}, \pi_{i1}^{k+1}, \cdots, \pi_{in}^{k+1}\right)$ is a solution to $MVI(\boldsymbol{Q}_i, \mathcal{U})$. **Q.E.D**

Lemma 4.1 shows that when inequality (15) holds, the iterate $\left(x_{i1}^{k+1}, \cdots, x_{in}^{k+1}, \pi_{i1}^{k+1}, \cdots, \pi_{in}^{k+1}\right)$ is a solution to $MVI(\boldsymbol{Q}, \mathcal{U})$.

### A2. Proof of Lemma 4.2

By substituting $\boldsymbol{x}_i$ in (35) with $\boldsymbol{x}_c^*$,

$$\langle \boldsymbol{x}_c^* - \boldsymbol{x}_i^{k+1}, \boldsymbol{\zeta}_i^{k+1} + \boldsymbol{\pi}_i^k + \rho\left(2\boldsymbol{x}_i^{k+1} - \boldsymbol{x}_c^k - \boldsymbol{x}_i^{\boldsymbol{0}*}\right)\rangle \geq 0 \tag{36}$$

Again, setting $\boldsymbol{x}_c^* = \boldsymbol{x}_i^{k+1}$ in (13) – (14) in the main text,

$$\langle \boldsymbol{x}_i^{k+1} - \boldsymbol{x}_c^*, \boldsymbol{\zeta}_i^* + \boldsymbol{\pi}_i^* + \rho\left(\boldsymbol{x}_i^{k+1} - \boldsymbol{x}_i^{\boldsymbol{0}*}\right)\rangle \geq 0 \tag{37}$$

Next, sum (36) and (37) to obtain

$$\langle \boldsymbol{x}_i^{k+1} - \boldsymbol{x}_c^*, \left(\boldsymbol{\zeta}_i^* - \boldsymbol{\zeta}_i^{k+1}\right) + \left(\boldsymbol{\pi}_i^* - \boldsymbol{\pi}_i^k\right) - \rho\left(\boldsymbol{x}_i^{k+1} - \boldsymbol{x}_c^k\right)\rangle \geq 0 \tag{38}$$

Rearrange (38) as follows:

$$\langle \boldsymbol{x}_i^{k+1} - \boldsymbol{x}_c^*, \left(\boldsymbol{\pi}_i^* - \boldsymbol{\pi}_i^k\right)\rangle \geq \langle \boldsymbol{x}_i^{k+1} - \boldsymbol{x}_c^*, \left(\boldsymbol{\zeta}_i^{k+1} - \boldsymbol{\zeta}_i^*\right) + \rho\left(\boldsymbol{x}_i^{k+1} - \boldsymbol{x}_c^k\right)\rangle$$

$$\langle \boldsymbol{x}_i^{k+1} - \boldsymbol{x}_c^*, \left(\boldsymbol{\pi}_i^* - \boldsymbol{\pi}_i^k\right)\rangle \geq \langle \boldsymbol{x}_i^{k+1} - \boldsymbol{x}_c^*, \left(\boldsymbol{\zeta}_i^{k+1} - \boldsymbol{\zeta}_i^*\right)\rangle + \langle \boldsymbol{x}_i^{k+1} - \boldsymbol{x}_c^*, \rho\left(\boldsymbol{x}_i^{k+1} - \boldsymbol{x}_c^k\right)\rangle$$

$$\langle \boldsymbol{x}_i^{k+1} - \boldsymbol{x}_c^*, \boldsymbol{\pi}_i^k - \boldsymbol{\pi}_i^*\rangle \leq -\sigma_i \left\|\boldsymbol{x}_i^{k+1} - \boldsymbol{x}_c^*\right\|^2 - \langle \boldsymbol{x}_i^{k+1} - \boldsymbol{x}_c^*, \rho\left(\boldsymbol{x}_i^{k+1} - \boldsymbol{x}_c^k\right)\rangle$$

where the first term on the right-hand side of the last inequality is from the strong monotonicity of the subdifferential mapping $\partial \Theta_i$. **Q.E.D**

**A3. Proof of Lemma 4.3**

*Proof*: From the last equality in (12) and (16) of the main text,

$$\rho^{-1}\left\|\boldsymbol{\pi}_i^{k+1}-\boldsymbol{\pi}_i^*\right\|^2 = \rho^{-1}\left\|\boldsymbol{\pi}_i^{k}-\boldsymbol{\pi}_i^* + \boldsymbol{\rho}\left(\boldsymbol{x}_i^{k+1}-\boldsymbol{x}_c^{k+1}\right)\right\|^2$$

$$\rho^{-1}\left\|\boldsymbol{\pi}_i^{k+1}-\boldsymbol{\pi}_i^*\right\|^2 = \rho^{-1}\left\|\boldsymbol{\pi}_i^{k}-\boldsymbol{\pi}_i^*\right\|^2 + \rho\left\|\left(\boldsymbol{x}_i^{k+1}-\boldsymbol{x}_c^{k+1}\right)\right\|^2 + 2\langle \boldsymbol{\pi}_i^{k}-\boldsymbol{\pi}_i^*, \boldsymbol{x}_i^{k+1}-\boldsymbol{x}_c^{k+1}\rangle \tag{39}$$

Replace

$$\boldsymbol{x}_i^{k+1}-\boldsymbol{x}_c^{k+1} = \left(\boldsymbol{x}_i^{k+1}-\boldsymbol{x}_c^{*}\right) + \left(\boldsymbol{x}_c^{*}-\boldsymbol{x}_c^{k+1}\right)$$

in the last term of (39) to obtain

$$\begin{aligned} 2\langle \boldsymbol{\pi}_i^{k}-\boldsymbol{\pi}_i^*, \boldsymbol{x}_i^{k+1}-\boldsymbol{x}_c^{k+1}\rangle &= 2\langle \boldsymbol{\pi}_i^{k}-\boldsymbol{\pi}_i^*, \left(\boldsymbol{x}_i^{k+1}-\boldsymbol{x}_c^{*}\right) + \left(\boldsymbol{x}_c^{*}-\boldsymbol{x}_c^{k+1}\right)\rangle \\ &= 2\langle \boldsymbol{\pi}_i^{k}-\boldsymbol{\pi}_i^*, \left(\boldsymbol{x}_i^{k+1}-\boldsymbol{x}_c^{*}\right)\rangle + 2\langle \boldsymbol{\pi}_i^{k}-\boldsymbol{\pi}_i^*, \left(\boldsymbol{x}_c^{*}-\boldsymbol{x}_c^{k+1}\right)\rangle \\ &\leq -2\sigma_i\left\|\boldsymbol{x}_i^{k+1}-\boldsymbol{x}_c^{*}\right\|^2 - 2\rho\langle \boldsymbol{x}_i^{k+1}-\boldsymbol{x}_c^{*}, \boldsymbol{x}_i^{k+1}-\boldsymbol{x}_c^{k}\rangle - 2\langle \boldsymbol{\pi}_i^{k}-\boldsymbol{\pi}_i^*, \boldsymbol{x}_c^{k+1}-\boldsymbol{x}_c^{*}\rangle \end{aligned}$$

Re-write (39) as follows:

$$\begin{aligned} \rho^{-1}\left\|\boldsymbol{\pi}_i^{k+1}-\boldsymbol{\pi}_i^*\right\|^2 \leq{}& \rho^{-1}\left\|\boldsymbol{\pi}_i^{k}-\boldsymbol{\pi}_i^*\right\|^2 + \rho\left\|\boldsymbol{x}_i^{k+1}-\boldsymbol{x}_c^{k+1}\right\|^2 - 2\sigma_i\left\|\boldsymbol{x}_i^{k+1}-\boldsymbol{x}_c^{*}\right\|^2 \\ &-2\rho\langle \boldsymbol{x}_i^{k+1}-\boldsymbol{x}_c^{*}, \boldsymbol{x}_i^{k+1}-\boldsymbol{x}_c^{k}\rangle - 2\langle \boldsymbol{\pi}_i^{k}-\boldsymbol{\pi}_i^*, \boldsymbol{x}_c^{k+1}-\boldsymbol{x}_c^{*}\rangle \end{aligned} \tag{40}$$

The second term in the right hand side of (40) can be represented as:

$$\begin{aligned} \rho\left\|\boldsymbol{x}_i^{k+1}-\boldsymbol{x}_c^{k+1}\right\|^2 &= \rho\left\|\left(\boldsymbol{x}_i^{k+1}-\boldsymbol{x}_i^{k}\right) + \left(\boldsymbol{x}_i^{k}-\boldsymbol{x}_c^{k+1}\right)\right\|^2 \\ &= \rho\left\|\left(\boldsymbol{x}_i^{k+1}-\boldsymbol{x}_i^{k}\right)\right\|^2 + \rho\left\|\left(\boldsymbol{x}_i^{k}-\boldsymbol{x}_c^{k+1}\right)\right\|^2 + 2\langle \boldsymbol{x}_i^{k+1}-\boldsymbol{x}_i^{k}, \boldsymbol{x}_i^{k}-\boldsymbol{x}_c^{k+1}\rangle \end{aligned}$$

Using the fact that for any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$

$$2\left|\langle \boldsymbol{a}, \boldsymbol{b}\rangle\right| \leq \|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2 \tag{41}$$

$$\rho\left\|\boldsymbol{x}_i^{k+1}-\boldsymbol{x}_c^{k+1}\right\|^2 \leq 2\rho\left\|\left(\boldsymbol{x}_i^{k+1}-\boldsymbol{x}_i^{k}\right)\right\|^2 + 2\rho\left\|\left(\boldsymbol{x}_i^{k}-\boldsymbol{x}_c^{k+1}\right)\right\|^2 \tag{42}$$

Also the first term in the right hand side of (42) can be expressed as:

$$\rho\left\|\left(\boldsymbol{x}_i^{k+1}-\boldsymbol{x}_i^{k}\right)\right\|^2 = \rho\left\|\left(\boldsymbol{x}_i^{k+1}-\boldsymbol{x}_c^{*}\right) - \left(\boldsymbol{x}_i^{k}-\boldsymbol{x}_c^{*}\right)\right\|^2$$

$$\rho\left\|\left(\boldsymbol{x}_i^{k+1}-\boldsymbol{x}_i^{k}\right)\right\|^2 \leq 2\rho\left\|\left(\boldsymbol{x}_i^{k+1}-\boldsymbol{x}_c^{*}\right)\right\|^2 + 2\rho\left\|\left(\boldsymbol{x}_i^{k}-\boldsymbol{x}_c^{*}\right)\right\|^2$$

That means (42) can be represented as shown below:

$$\rho\left\|\boldsymbol{x}_i^{k+1}-\boldsymbol{x}_c^{k+1}\right\|^2 \leq 2\rho\left\|\left(\boldsymbol{x}_i^{k+1}-\boldsymbol{x}_c^{*}\right)\right\|^2 + 2\rho\left\|\left(\boldsymbol{x}_i^{k}-\boldsymbol{x}_c^{*}\right)\right\|^2 + 2\rho\left\|\left(\boldsymbol{x}_i^{k}-\boldsymbol{x}_c^{k+1}\right)\right\|^2$$

And (40) becomes

$$\rho^{-1}\left\|\boldsymbol{\pi}_{\boldsymbol{i}}^{\boldsymbol{k+1}}-\boldsymbol{\pi}_{\boldsymbol{i}}^{*}\right\|^{2} \leq \rho^{-1}\left\|\boldsymbol{\pi}_{\boldsymbol{i}}^{\boldsymbol{k}}-\boldsymbol{\pi}_{\boldsymbol{i}}^{*}\right\|^{2} + 2\rho\left\|\left(\boldsymbol{x}_{\boldsymbol{i}}^{\boldsymbol{k+1}}-\boldsymbol{x}_{\boldsymbol{c}}^{*}\right)\right\|^{2} + 2\rho\left\|\left(\boldsymbol{x}_{\boldsymbol{i}}^{\boldsymbol{k}}-\boldsymbol{x}_{\boldsymbol{c}}^{*}\right)\right\|^{2} + 2\rho\left\|\left(\boldsymbol{x}_{\boldsymbol{i}}^{\boldsymbol{k}}-\boldsymbol{x}_{\boldsymbol{c}}^{\boldsymbol{k+1}}\right)\right\|^{2}$$
$$-2\sigma_i\left\|\boldsymbol{x}_{\boldsymbol{i}}^{\boldsymbol{k+1}}-\boldsymbol{x}_{\boldsymbol{c}}^{*}\right\|^{2} - 2\rho\langle \boldsymbol{x}_{\boldsymbol{i}}^{\boldsymbol{k+1}}-\boldsymbol{x}_{\boldsymbol{c}}^{*}, \boldsymbol{x}_{\boldsymbol{i}}^{\boldsymbol{k+1}}-\boldsymbol{x}_{\boldsymbol{c}}^{\boldsymbol{k}}\rangle$$
$$-2\langle \boldsymbol{\pi}_{\boldsymbol{i}}^{\boldsymbol{k}}-\boldsymbol{\pi}_{\boldsymbol{i}}^{*}, \boldsymbol{x}_{\boldsymbol{c}}^{\boldsymbol{k+1}}-\boldsymbol{x}_{\boldsymbol{c}}^{*}\rangle$$

Re-arranging the last inequality,

$$\rho^{-1}\left\|\boldsymbol{\pi}_{\boldsymbol{i}}^{\boldsymbol{k+1}}-\boldsymbol{\pi}_{\boldsymbol{i}}^{*}\right\|^{2} + 2\rho\left\|\left(\boldsymbol{x}_{\boldsymbol{i}}^{\boldsymbol{k+1}}-\boldsymbol{x}_{\boldsymbol{c}}^{*}\right)\right\|^{2} \leq \rho^{-1}\left\|\boldsymbol{\pi}_{\boldsymbol{i}}^{\boldsymbol{k}}-\boldsymbol{\pi}_{\boldsymbol{i}}^{*}\right\|^{2} + 2\rho\left\|\left(\boldsymbol{x}_{\boldsymbol{i}}^{\boldsymbol{k}}-\boldsymbol{x}_{\boldsymbol{c}}^{*}\right)\right\|^{2} + 2\rho\left\|\left(\boldsymbol{x}_{\boldsymbol{i}}^{\boldsymbol{k}}-\boldsymbol{x}_{\boldsymbol{c}}^{\boldsymbol{k+1}}\right)\right\|^{2}$$
$$-2\sigma_i\left\|\boldsymbol{x}_{\boldsymbol{i}}^{\boldsymbol{k+1}}-\boldsymbol{x}_{\boldsymbol{c}}^{*}\right\|^{2} - 2\rho\langle \boldsymbol{x}_{\boldsymbol{i}}^{\boldsymbol{k+1}}-\boldsymbol{x}_{\boldsymbol{c}}^{*}, \boldsymbol{x}_{\boldsymbol{i}}^{\boldsymbol{k+1}}-\boldsymbol{x}_{\boldsymbol{c}}^{\boldsymbol{k}}\rangle$$
$$-2\langle \boldsymbol{\pi}_{\boldsymbol{i}}^{\boldsymbol{k}}-\boldsymbol{\pi}_{\boldsymbol{i}}^{*}, \boldsymbol{x}_{\boldsymbol{c}}^{\boldsymbol{k+1}}-\boldsymbol{x}_{\boldsymbol{c}}^{*}\rangle + 4\rho\left\|\left(\boldsymbol{x}_{\boldsymbol{i}}^{\boldsymbol{k+1}}-\boldsymbol{x}_{\boldsymbol{c}}^{*}\right)\right\|^{2}$$

The fifth and sixth terms can be expressed as follows using (41)

$$-2\rho\langle \boldsymbol{x}_{\boldsymbol{i}}^{\boldsymbol{k+1}}-\boldsymbol{x}_{\boldsymbol{c}}^{*}, \boldsymbol{x}_{\boldsymbol{i}}^{\boldsymbol{k+1}}-\boldsymbol{x}_{\boldsymbol{c}}^{\boldsymbol{k}}\rangle \leq -2\rho\left\|\left(\boldsymbol{x}_{\boldsymbol{i}}^{\boldsymbol{k+1}}-\boldsymbol{x}_{\boldsymbol{c}}^{*}\right)\right\|^{2} - 2\rho\left\|\left(\boldsymbol{x}_{\boldsymbol{i}}^{\boldsymbol{k+1}}-\boldsymbol{x}_{\boldsymbol{c}}^{\boldsymbol{k}}\right)\right\|^{2}$$

$$-2\langle \boldsymbol{\pi}_{\boldsymbol{i}}^{\boldsymbol{k}}-\boldsymbol{\pi}_{\boldsymbol{i}}^{*}, \boldsymbol{x}_{\boldsymbol{c}}^{\boldsymbol{k+1}}-\boldsymbol{x}_{\boldsymbol{c}}^{*}\rangle \leq -2\left\|\left(\boldsymbol{\pi}_{\boldsymbol{i}}^{\boldsymbol{k}}-\boldsymbol{\pi}_{\boldsymbol{i}}^{*}\right)\right\|^{2} - 2\left\|\left(\boldsymbol{x}_{\boldsymbol{c}}^{\boldsymbol{k+1}}-\boldsymbol{x}_{\boldsymbol{c}}^{*}\right)\right\|^{2}$$

After telescoping re-arrangement and using (41)

$$\rho^{-1}\left\|\boldsymbol{\pi}_{\boldsymbol{i}}^{\boldsymbol{k+1}}-\boldsymbol{\pi}_{\boldsymbol{i}}^{*}\right\|^{2} + 2\rho\left\|\left(\boldsymbol{x}_{\boldsymbol{i}}^{\boldsymbol{k+1}}-\boldsymbol{x}_{\boldsymbol{c}}^{*}\right)\right\|^{2} \leq \rho^{-1}\left\|\boldsymbol{\pi}_{\boldsymbol{i}}^{\boldsymbol{k}}-\boldsymbol{\pi}_{\boldsymbol{i}}^{*}\right\|^{2} + 2\rho\left\|\left(\boldsymbol{x}_{\boldsymbol{i}}^{\boldsymbol{k}}-\boldsymbol{x}_{\boldsymbol{c}}^{*}\right)\right\|^{2}$$
$$-2\sigma_i\left\|\boldsymbol{x}_{\boldsymbol{i}}^{\boldsymbol{k+1}}-\boldsymbol{x}_{\boldsymbol{c}}^{*}\right\|^{2} - 2\rho\left\|\left(\boldsymbol{x}_{\boldsymbol{c}}^{*}-\boldsymbol{x}_{\boldsymbol{c}}^{\boldsymbol{k}}\right)\right\|^{2}$$
$$-2\rho\left\|\left(\boldsymbol{x}_{\boldsymbol{c}}^{\boldsymbol{k+1}}-\boldsymbol{x}_{\boldsymbol{i}}^{\boldsymbol{k}}\right)\right\|^{2} - 2\left\|\boldsymbol{\pi}_{\boldsymbol{i}}^{\boldsymbol{k}}-\boldsymbol{\pi}_{\boldsymbol{i}}^{*}\right\|^{2} - 2\left\|\boldsymbol{x}_{\boldsymbol{c}}^{\boldsymbol{k+1}}-\boldsymbol{x}_{\boldsymbol{c}}^{*}\right\|^{2}$$

Letting

$$\left\|\boldsymbol{u}_{\boldsymbol{i}}^{\boldsymbol{k+1}}-\boldsymbol{u}_{\boldsymbol{i}}^{*}\right\|^{2} = \rho^{-1}\left\|\boldsymbol{\pi}_{\boldsymbol{i}}^{\boldsymbol{k+1}}-\boldsymbol{\pi}_{\boldsymbol{i}}^{*}\right\|^{2} + 2\rho\left\|\left(\boldsymbol{x}_{\boldsymbol{i}}^{\boldsymbol{k+1}}-\boldsymbol{x}_{\boldsymbol{c}}^{*}\right)\right\|^{2}$$

and

$$\left\|\boldsymbol{u}_{\boldsymbol{i}}^{\boldsymbol{k}}-\boldsymbol{u}_{\boldsymbol{i}}^{*}\right\|^{2} = \rho^{-1}\left\|\boldsymbol{\pi}_{\boldsymbol{i}}^{\boldsymbol{k}}-\boldsymbol{\pi}_{\boldsymbol{i}}^{*}\right\|^{2} + 2\rho\left\|\left(\boldsymbol{x}_{\boldsymbol{i}}^{\boldsymbol{k}}-\boldsymbol{x}_{\boldsymbol{c}}^{*}\right)\right\|^{2}$$

completes the proof. **Q.E.D**