Online Supplementary Material

Title of the manuscript: Parameter estimation of Cambanis type Bivariate Uniform distribution with Ranked Set Sampling

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The supporting details of Section 6 of the manuscript are as follows.

- **Details for Section 6.1**

i) Simulated data1

The sample of size 10 under RSS, ERSS_1, LRSS and URSS schemes for simulated data1 from $CTBU(-0.1, 0.9, 1,1)$ distribution is given in the following.

Sample under RSS scheme:

r	1	2	3	4	5	6	7	8	9	10
$X_{(r)r}$	0.1522	0.3255	0.0690	0.5385	0.5125	0.6810	0.5456	0.7444	0.9616	0.9893
$Y_{[r]r}$	0.4363	0.0058	0.1210	0.4709	0.6124	0.9619	0.4506	0.5753	0.5827	0.9469

Sample under ERSS_1 scheme:

r	1	2	3	4	5	6	7	8	9	10
$X_{(1)2r-1}$	0.1522	--	0.0206	--	0.0570	--	0.0803	--	0.0133	--
$Y_{[1]r}$	0.4363	--	0.4636	--	0.3186	--	0.4345	--	0.4828	--
$X_{(n)2r}$	--	0.9784	--	0.8015	--	0.9439	--	0.9019	--	0.9893
$Y_{[n]2r}$	--	0.4818	--	0.8180	--	0.5770	--	0.3020	--	0.9469

Sample under LRSS scheme:

r	1	2	3	4	5	6	7	8	9	10
$X_{(1)r}$	0.1522	0.1120	0.0206	0.0676	0.0570	0.1394	0.0803	0.0034	0.0133	0.0705
$Y_{[1]r}$	0.4363	0.1171	0.4636	0.2044	0.3186	0.5088	0.4345	0.4510	0.4828	0.6853

Sample under URSS scheme:

r	1	2	3	4	5	6	7	8	9	10
$X_{(n)r}$	0.9061	0.9784	0.9863	0.8015	0.9697	0.9439	0.9707	0.9019	0.9808	0.9893
$Y_{[n]r}$	0.2823	0.4818	0.3121	0.8180	0.9793	0.5770	0.8031	0.3020	0.3738	0.9469

ii) Simulated data2

Another sample of size 10 under RSS, ERSS_1, LRSS and URSS schemes for simulated data from $CTBU(-0.1, -0.9, 1,1)$ distribution is given in the following.

Sample under RSS scheme:

r	1	2	3	4	5	6	7	8	9	10
$X_{(r)r}$	0.0095	0.2020	0.2051	0.3982	0.3587	0.6753	0.7570	0.7292	0.8087	0.7728
$Y_{[r]r}$	0.7072	0.8917	0.8287	0.6629	0.8283	0.3550	0.7908	0.5064	0.5211	0.1794

Sample under $ERSS_1$ scheme:

r	1	2	3	4	5	6	7	8	9	10
$X_{(1)2r-1}$	0.0095	--	0.0337	--	0.0272	--	0.0160	--	0.0141	--
$Y_{[1]r}$	0.7072	--	0.7756	--	0.8460	--	0.8196	--	0.7631	--
$X_{(n)2r}$	--	0.9726	--	0.9150	--	0.9182	--	0.9025	--	0.7728
$Y_{[n]2r}$	--	0.5415	--	0.0871	--	0.0803	--	0.0446	--	0.1794

Sample under LRSS scheme:

r	1	2	3	4	5	6	7	8	9	10
$X_{(1)r}$	0.0095	0.0140	0.0337	0.2529	0.0272	0.0850	0.0160	0.0382	0.0141	0.0368
$Y_{[1]r}$	0.7072	0.2840	0.7756	0.5700	0.8460	0.9282	0.8196	0.5985	0.7631	0.7705

Sample under URSS scheme:

r	1	2	3	4	5	6	7	8	9	10
$X_{(n)r}$	0.8502	0.9726	0.8277	0.9150	0.9076	0.9182	0.9060	0.9025	0.8875	0.7728
$Y_{[n]r}$	0.8422	0.5415	0.3848	0.0871	0.3900	0.0803	0.0358	0.0446	0.4186	0.1794

iii) Simulated data3

The sample of size 10 under RSS, $ERSS_1$, LRSS and URSS schemes for simulated data from $CTBU(-0.1, 0.9, 1, 2)$ distribution is given in the following.

Sample under RSS scheme:

r	1	2	3	4	5	6	7	8	9	10
$X_{(r)r}$	0.0294	0.2765	0.2553	0.4781	0.3313	0.4428	0.5067	0.6799	0.5202	0.7363
$Y_{[r]r}$	0.3932	0.5083	1.7179	0.4252	0.6323	0.6788	1.1077	1.3031	0.2002	1.989

Sample under $ERSS_1$ scheme:

r	1	2	3	4	5	6	7	8	9	10
$X_{(1)2r-1}$	0.0294	--	0.1652	--	0.0242	--	0.0901	--	0.0304	--
$Y_{[1]r}$	0.3932	--	0.1323	--	0.0658	--	0.3401	--	0.9087	--
$X_{(n)2r}$	--	0.9089	--	0.9549	--	0.9717	--	0.8957	--	0.7363
$Y_{[n]2r}$	--	1.2707	--	1.0702	--	1.7159	--	1.6295	--	1.989

Sample under LRSS scheme:

r	1	2	3	4	5	6	7	8	9	10
$X_{(1)r}$	0.0294	0.1918	0.1652	0.0435	0.0242	0.0650	0.0901	0.0569	0.0304	0.0478
$Y_{[1]r}$	0.3932	1.5825	0.1323	1.1416	0.0658	1.9955	0.3401	0.4303	0.9087	0.0120

Sample under URSS scheme:

r	1	2	3	4	5	6	7	8	9	10
$X_{(n)r}$	0.7700	0.9089	0.9506	0.9549	0.8726	0.9717	0.8062	0.8957	0.8348	0.7363
$Y_{[n]r}$	1.3003	1.2707	0.8966	1.0702	1.0058	1.7159	0.7179	1.6295	1.5753	1.9890

- **Details for Section 6.2**

Consider bivariate samples of size 8 under RSS, $ERSS_1$, LRSS and URSS schemes on purslane plants given by Tahmasebi and Jafari [28] where Y represents the shoot diameter (in cm) and X represents the shoot height (in cm) of the plant.

Sample under RSS scheme:

r	1	2	3	4	5	6	7	8
$X_{(r)r}$	10.37	7.25	8.33	9.00	8.87	11.37	10.50	8.50
$Y_{[r]r}$	1.37	1.27	1.10	1.15	1.72	1.75	1.57	1.70

Sample under $ERSS_1$ scheme:

r	1	2	3	4	5	6	7	8
$X_{(1)2r-1}$	10.37	--	4.00	--	7.12	--	4.25	--
$Y_{[1]r}$	1.37	--	1.40	--	1.32	--	1.12	--
$X_{(n)2r}$	--	10.00	--	11.25	--	12.50	--	8.50
$Y_{[n]2r}$	--	1.57	--	1.57	--	1.80	--	1.70

Sample under LRSS scheme:

r	1	2	3	4	5	6	7	8
$X_{(1)r}$	10.37	6.70	4.00	6.75	7.12	7.16	4.25	5.12
$Y_{[1]r}$	1.37	1.72	1.40	1.22	1.32	1.70	1.12	0.97

Sample under URSS scheme:

r	1	2	3	4	5	6	7	8
$X_{(n)r}$	15.00	10.00	15.37	11.25	9.50	12.50	11.83	8.50
$Y_{[n]r}$	1.60	1.57	1.75	1.57	1.45	1.80	1.53	1.70

We assume that $(X,Y)\sim CTBU(\alpha_2,\alpha_3,\theta_1,\theta_2)$ distribution and wish to obtain an estimator θ_2 based on the corresponding RSS schemes when α_2 and α_3 are known. But while dealing with the above sample data, to obtain an estimator θ_2, first we need to estimate α_2 and α_3 based on ranked set sample. As estimator of θ_2 is the function of parameters α_2 and α_3. Hence, we obtain the MLE of α_2 and α_3 based on the ranked set sample of size $n=8$.

Estimation of α_2 and α_3:

To obtain MLE of α_2 and α_3, we use the following R software options.

- *optim* function
- *mle2* function

We use these procedures to cross verify the results. To obtain MLE of α_2 and α_3 we use the joint density of (X,Y) is given by,

$$h(x,y)=\frac{1}{\theta_1\theta_2}\left[1+\alpha_2\left(1-\frac{2y}{\theta_2}\right)+\alpha_3\left(1-\frac{2x}{\theta_1}\right)\left(1-\frac{2y}{\theta_2}\right)\right];0<x<\theta_1,0<y<\theta_2,$$

where $|\alpha_2+\alpha_3|\le 1,|\alpha_2-\alpha_3|\le 1,\theta_1,\theta_2>0.$ (1)

The *optim* function in R software minimizes the negative of the log-likelihood. The log-likelihood function is given by,

$$logL(\alpha_2,\alpha_3,\theta_1,\theta_2|x_1,x_2,\dots,x_n,y_1,y_2,\dots,y_n)$$
$$=-nlog(\theta_1)-nlog(\theta_2)+\sum_{i=1}^{n}log\left[1+\ \alpha_2\left(1-\frac{2y_i}{\theta_2}\right)+\alpha_3\left(1-\frac{2x_i}{\theta_1}\right)\left(1-\frac{2y_i}{\theta_2}\right)\right]$$

The R code to obtain MLE is given in Appendix 1. In the following table we summarize the MLE's obtained under different ranges of α_2 and α_3 using *optim* function and justify the feasible ranges for the same.

Restrictions on $\boldsymbol{\alpha_2}$ and $\boldsymbol{\alpha_3}$	**Sampling Scheme**	**MLE: $(\hat{\alpha}_2,\hat{\alpha}_3)$**	**Remark**
$\alpha_2,\alpha_3\in[-1,1]$	RSS $ERSS_1$ LRSS URSS	$(-1,1)$	Here $\hat{\alpha}_2$ and $\hat{\alpha}_3$ do not satisfy the conditions specified in (1). Hence are infeasible.
$\alpha_2\in[-1,1]$, $\alpha_3\in[-0.5,0.5]$	RSS $ERSS_1$ LRSS URSS	$(-1,0.5)$	
$\alpha_2\in[-0.5,0.5]$, $\alpha_3\in[-1,1]$	RSS $ERSS_1$ LRSS URSS	$(-0.5,1)$	
$\alpha_2,\alpha_3\in[-0.5,0.5]$	RSS $ERSS_1$ LRSS URSS	$(-0.5,0.5)$	These estimates satisfy the conditions and hence are feasible

Observe that feasible MLE's of α_2 and α_3 under RSS, LRSS, URSS and $ERSS_1$ scheme obtained are the boundary points of parameter space.

Appendix 1. The R code to obtain MLE of α_2 and α_3 for samples in Table 5

```
#------------------------------------------------------------------------------------------------------------------------------------
# R code to find MLE's of α2 and α3 using optim function for samples under RSS, ERSS1, LRSS and URSS schemes
#------------------------------------------------------------------------------------------------------------------------------------
rm(list=ls(all=TRUE))
#------------------------------------------------------------------------------------------------------------------------------------
# Sample under RSS scheme
x=c(10.37,7.25,8.33,9,8.87,11.37,10.50,8.50)
y=c(1.37,1.27,1.10,1.15,1.72,1.75,1.57,1.70)
#------------------------------------------------------------------------------------------------------------------------------------
# Sample under ERSS1 scheme
x=c(10.37,10,4,11.25,7.12,12.5,4.25,8.5)
y=c(1.37,1.57,1.40,1.57,1.32,1.80,1.12,1.70)
#------------------------------------------------------------------------------------------------------------------------------------
# Sample under LRSS scheme
x=c(10.37,6.70,4.00,6.75,7.12,7.16,4.25,5.12)
y=c(1.37,1.72,1.40,1.22,1.32,1.70,1.12,0.97)
#------------------------------------------------------------------------------------------------------------------------------------
# Sample under URSS scheme
x=c(15.00,10.00,15.37,11.25,9.50,12.50,11.83,8.50)
y=c(1.60,1.57,1.75,1.57,1.45,1.80,1.53,1.70)
#------------------------------------------------------------------------------------------------------------------------------------
n=length(y)
#------------------------------------------------------------------------------------------------------------------------------------
# The negative of the log-likelihood function of h(x,y) when (X,Y)~CTBU(a2, a3, theta1, theta2)
logL=function(a,x,y)
{
  a2=a[1];a3=a[2];theta1=a[3];theta2=a[4];
  return(n*log(theta1)+n*log(theta2)-sum(log(1+a2*(1-2*y/theta2)+a3*(1-2*x/theta1)*
(1-2*y/theta2))))
}
#------------------------------------------------------------------------------------------------------------------------------------
fit=optim(c(-0.1,0.1,max(x),max(y)),logL,y=y,x=x,lower = c(-0.5,-0.5,max(x),max(y)), upper = c(0.5,0.5,16,2),
hessian = T,method = "L-BFGS-B")
fit
MLE=fit$par                          # MLE of a2, a3, theta1, theta2
m1=fit$hess                          # Hessian matrix
vhat=solve(m1)                       # Inverse of hessian matrix
std.errors=sqrt(diag(vhat))          # standard erros of MLE's
cbind(MLE,std.errors)                # To combine MLE and standard error
```

```
#------------------------------------------------------------------------------------------------------------------------------------
# R code to find MLE's of α2 and α3 using mle2 function for samples under RSS, ERSS1, LRSS and URSS schemes
# Library required: bbmle
#------------------------------------------------------------------------------------------------------------------------------------
library(bbmle)
#------------------------------------------------------------------------------------------------------------------------------------
n=length(y)
d=data.frame(x,y)
#------------------------------------------------------------------------------------------------------------------------------------
# The negative of the log-likelihood function of h(x,y) when (X,Y)~CTBU(a2, a3, theta1, theta2)
logL=function(a2,a3,theta1,theta2,x,y)
{
  n*log(theta1)+n*log(theta2)-sum(log(1+a2*(1-2*y/theta2)+a3*(1-2*x/theta1)*(1-2*y/theta2)))
}
#------------------------------------------------------------------------------------------------------------------------------------
fit=mle2(logL,data=d,start =list(a2=-0.1,a3=0.1,theta1=max(x),theta2=max(y)), lower = c(-0.5,-
0.5,max(x),max(y)), upper = c(0.5,0.5,16,2),method = "L-BFGS-B")             # MLE of a2, a3, theta1, theta2
fit
summary(fit)
```