Appendix

This section includes the proofs of the results presented in the preceding sections.

Proof of Proposition 1. v_1^e and v_2^e are functions of (v_1, v_2) that are uniquely determined by

$$\begin{cases} p_1 = (\alpha - 1)v_1 + v_2 - W_1(\mu_1, \lambda_1(v_1)) \\ p_2 = v_2 - W_2(\mu_2, \lambda_2(v_1, v_2)) \end{cases}, \tag{8}$$

of which $\lambda_1(v_1) = \Lambda(1 - v_1^+)^+$ and $\lambda_2(v_1, v_2) = \Lambda(\min\{1, v_1\} - v_2^+)^+$ with $x^+ = \max\{0, x\}$. It can be easily checked that v_1 is the valuation threshold at which customers is indifferent between services 1 and 2, and v_2 is the valuation threshold at which customer is indifferent between purchasing service 2 and not purchasing any service. Due to the various relationships between 0, 1, v_1 and v_2, there are four cases. Note that v_1^e and v_2^e are thresholds that determine the firms' effective arrival rate. Literally speaking, (v_1^e, v_2^e) does not necessarily equal (v_1, v_2): a customer who prefers service 1 over service 2 does not necessarily purchase service 1 eventually, because she may prefer balking over purchasing service 1. Aa a result, these equilibrium arrival rates comprise four different value-based market segmentations as shown in the following.

The specified functional relation between (v_1^e, v_2^e) and (v_1, v_2) takes the following forms.

1) If $0 \leq v_2 \leq v_1 \leq 1$, then $v_1^e = v_1$ and $v_2^e = v_2$.

2) If $0 \leq v_1 \leq v_2 \leq 1$, then $v_2^e = v_1^e$ and v_1^e is given by

$$\alpha v_1^e - W_1(\mu_1, \lambda_1^e) = \alpha v_1 - W_1(\mu_1, \Lambda(1 - v_1)) + v_2 - v_1. \tag{9}$$

3) If $v_1 < 0 \leq v_2 \leq 1$, then $v_2^e = v_1^e$ and v_1^e is given by

$$\alpha v_1^e - W_1(\mu_1, \lambda_1^e) = \alpha v_1 - W_1(\mu_1, \Lambda) + v_2 - v_1. \tag{10}$$

4) If $0 \leq v_2 \leq 1 < v_1$, then $v_1^e = 1$ and v_2^e is given by

$$v_2^e - W_2(\mu_2, \lambda_2^e) = p_2. \tag{11}$$

The proof of this Proposition involves lengthy analysis. We put it as a permanent working paper Huang *et al.* (2017). For brevity, we omit it in this paper, and refer to interested readers to Huang *et al.* (2017) for details. □

Proof of Lemma 1. 1. Note that if firm 1 chooses any strategy greater than 1, then $\lambda_1 = 0$. Considering this, firm 1's optimal strategy must be no greater than 1, and this includes two cases, $v_1 \geq v_2$ and $v_1 \leq v_2$.

1-1) Suppose that firm 1 chooses a strategy from $v_1 \geq v_2$. Then we have $\lambda_1(v_1) = \Lambda(1 - v_1)$, and firm 1's local best response (denoted by $v_1^1(v_2)$) is

$$v_1^1(v_2) = \underset{v_2 \leq v_1 \leq 1}{\operatorname{argmax}} \, \pi_1(v_1),$$

where $\pi_1(v_1) = \Lambda(1 - v_1)[(\alpha - 1)v_1 + v_2 - W_1(\mu_1, \lambda_1(v_1))]$. Let $\Lambda(1 - v_m) = \mu_1$. It is clear that if $v_1 \leq v_m$, then $W_1(\mu_1, \lambda_1(v_1)) = \infty$, and so $\pi_1(v_1) = -\infty$. Thus, the optimal strategy must satisfy $v_1 > v_m$, in which $\pi_1(v_1)$ is continuous in v_1. Given $v_m < v_1 \leq 1$, it can be easily calculated that

$$\pi_1'(v_1) = \Lambda(1 - v_1)\left[\alpha - 1 + \Lambda \frac{\partial W_1(\mu_1, \lambda_1(v_1))}{\partial \lambda_1}\right] - \Lambda[(\alpha - 1)v_1 + v_2 - W_1(\mu_1, \lambda_1(v_1))],$$
$$\pi_1''(v_1) = -\Lambda^3(1 - v_1)\frac{\partial^2 W_1(\mu_1, \lambda_1(v_1))}{\partial {\lambda_1}^2} - 2\Lambda\left[\alpha - 1 + \Lambda \frac{\partial W_1(\mu_1, \lambda_1(v_1))}{\partial \lambda_1}\right] < 0.$$

Thus, $\pi_1(v_1)$ is a strictly concave function with respect to v_1 given $v_m < v_1 \leq 1$. Let $v_1^0(v_2)$ be the stationary point of $\pi_1(v_1)$ with respect to v_1 given $v_m < v_1 \leq 1$, which is determined by the FOC $\pi_1'(v_1^0(v_2)) = 0$, i.e.,

$$\Lambda(1 - v_1^0(v_2))\left[\alpha - 1 + \Lambda \frac{\partial W_1(\mu_1, \lambda_1(v_1^0(v_2)))}{\partial \lambda_1}\right] - \Lambda[(\alpha - 1)v_1^0(v_2) + v_2 - W_1(\mu_1, \lambda_1(v_1^0(v_2)))] = 0. \tag{12}$$

Note that $\pi_1''(v_1) < 0$ and $\lim_{v_1 \to v_m} \pi_1'(v_1) = +\infty > 0$. It follows that $v_1^0(v_2)$ is well-defined only when $\pi_1'(1) \leq 0$; i.e. $v_2 \geq 1 - \alpha + W_1(\mu_1, 0)$. Thus, given $v_1 \in (v_m, 1]$, if $v_2 < 1 - \alpha + W_1(\mu_1, 0)$, then $\pi_1(v_1)$ is increasing in v_1; if $v_2 \geq 1 - \alpha + W_1(\mu_1, 0)$, then $\pi_1(v_1)$ peaks at $v_1^0(v_2)$ and $v_m < v_1^0(v_2) \leq 1$. Incorporating the constraint $v_2 \leq v_1$, $v_1^1(v_2)$ is given by

$$v_1^1(v_2) = \begin{cases} 1, & v_2 < 1 - \alpha + W_1(\mu_1, 0) \\ \max\{v_1^0(v_2), v_2\}, & v_2 \geq 1 - \alpha + W_1(\mu_1, 0) \end{cases}. \tag{13}$$

In what follows, we show how to simplify $v_1^1(v_2)$ with the condition that $v_2 \geq 1 - \alpha + W_1(\mu_1, 0)$. Differentiating (**??**) with respect to v_2, we have

$$\left[\Lambda^2(1 - v_1^0(v_2))\frac{\partial^2 W_1(\mu_1, \lambda_1(v_1^0(v_2)))}{\partial {\lambda_1}^2} + 2\left(\alpha - 1 + \Lambda \frac{\partial W_1(\mu_1, \lambda_1(v_1^0(v_2)))}{\partial \lambda_1}\right)\right]\frac{\partial v_1^0(v_2)}{\partial v_2} = -1,$$

and so

$$\frac{\partial v_1^0(v_2)}{\partial v_2} < 0, \ \frac{\partial v_1^0(v_2) - v_2}{\partial v_2} < 0.$$

Thus, there exists at most one solution with respect to v_2 for $v_1^0(v_2) = v_2$. Note that when $v_2 = 1 - \alpha + W_1(\mu_1, 0)$, $v_1^0(v_2) = 1$ and $v_1^0(v_2) - v_2 = \alpha - W_1(\mu_1, 0) > 0$. Note also that when

$v_2 = 1$, Eq. (**??**) becomes

$$\Lambda(1 - v_1^0(1)) \left[\alpha - 1 + \Lambda \frac{\partial W_1(\mu_1, \lambda_1(v_1^0(1)))}{\partial \lambda_1}\right] - \Lambda[(\alpha - 1)v_1^0(1) + 1 - W_1(\mu_1, \lambda_1(v_1^0(1)))] = 0,$$

whose LHS is obviously decreasing in $v_1^0(1)$, and it is negative when $v_1^0(1) = 1$; so $v_1^0(1) < 1$. Thus, when $v_2 = 1$, $v_1^0(v_2) < 1 = v_2$. This means there exists a unique root with respect to v_2 of $v_1^0(v_2) = v_2$ between $1 - \alpha + W_1(\mu_1, 0)$ and 1. Denote such a root as $\underline{v}_2$. By Eq. (**??**), it satisfies the following equation:

$$\Lambda(1 - \underline{v}_2) \left[\alpha - 1 + \Lambda \frac{\partial W_1(\mu_1, \lambda_1(\underline{v}_2))}{\partial \lambda_1}\right] - \Lambda[\alpha \underline{v}_2 - W_1(\mu_1, \lambda_1(\underline{v}_2))] = 0. \tag{14}$$

By the monotonicity of $v_1^0(v_2) - v_2$, it is clear that $v_1^0(v_2) > v_2$ for $v_2 < \underline{v}_2$. Then, we can specify $v_1^1(v_2)$ (Eq. (7)) as

$$v_1^1(v_2) = \begin{cases} 1, & 0 \leq v_2 < 1 - \alpha + W_1(\mu_1, 0) \\ v_1^0(v_2), & 1 - \alpha + W_1(\mu_1, 0) \leq v_2 \leq \underline{v}_2 \\ v_2, & v_2 > \underline{v}_2 \end{cases} . \tag{15}$$

1-2) Suppose that firm 1 chooses a strategy from $v_1 \leq v_2$. Then we have $\lambda_1 = \Lambda(1 - v_1)$, and firm 1's local best response (denoted by $v_1^2(v_2)$) is

$$v_1^2(v_2) = \underset{0 \leq v_1 \leq v_2}{\operatorname{argmax}}\, \pi_1(v_1),$$

where $\pi_1(v_1) = \Lambda(1 - v_1)[\alpha v_1 - W_1(\mu_1, \lambda_1(v_1))]$. Given $v_1 \leq 1$, it can be easily calculated that

$$\begin{aligned} \pi_1'(v_1) &= \Lambda(1 - v_1) \left[\alpha + \Lambda \frac{\partial W_1(\mu_1, \lambda_1(v_1))}{\partial \lambda_1}\right] - \Lambda[\alpha v_1 - W_1(\mu_1, \lambda_1(v_1))], \\ \pi_1''(v_1) &= -\Lambda^3(1 - v_1) \frac{\partial W_1(\mu_1, \lambda_1(v_1))}{\partial \lambda_1} - 2\Lambda \left[\alpha + \Lambda \frac{\partial W_1(\mu_1, \lambda_1(v_1))}{\partial \lambda_1}\right] < 0. \end{aligned}$$

This means that $\pi_1(\cdot)$ is strictly concave. Let v_1^M be the stationary point of $\pi_1(\cdot)$, which is determined with the following FOC:

$$\Lambda(1 - v_1^M) \left[\alpha + \Lambda \frac{\partial W_1(\mu_1, \lambda_1(v_1^M))}{\partial \lambda_1}\right] - \Lambda[\alpha v_1^M - W_1(\mu_1, \lambda_1(v_1^M))] = 0. \tag{16}$$

Furthermore, as $\pi_1''(v_1) < 0$, $\pi_1'(0) > 0$ and $\pi_1'(1) < 0$, it is clear that v_1^M is well-defined, $0 < v_1^M < 1$, and

$$v_1^2(v_2) = \begin{cases} v_2, & 0 \leq v_2 < v_1^M \\ v_1^M, & v_2 \geq v_1^M \end{cases} . \tag{17}$$

In the following, we compare the two local best responses and pick the one resulting in larger profit as the global best response. Before this, we first show that $0 < \underline{v}_2 < v_1^M < 1$. Denote

$$f_3(x) := (2\alpha - 1)(1 - x)\Lambda - \alpha\Lambda + \Lambda \left[W_1(\mu_1, \lambda_1(x)) + \Lambda(1 - x)\frac{\partial W_1(\mu_1, \lambda_1(x))}{\partial \lambda_1} \right], \tag{18}$$

$$f_4(x) := 2\alpha(1 - x)\Lambda - \alpha\Lambda + \Lambda \left[W_1(\mu_1, \lambda_1(x)) + \Lambda(1 - x)\frac{\partial W_1(\mu_1, \lambda_1(x))}{\partial \lambda_1} \right]. \tag{19}$$

It is easily seen that given $x \in [0, 1]$, $f_3(x)$ and $f_4(x)$ are decreasing in x, and $f_3(x) \leq f_4(x)$ (the equality holds only when $x = 1$). And, by Eqs. (8) and (**??**), it is clear that

$$f_3(\underline{v}_2) = 0, \;\; f_4(v_1^M) = 0.$$

Since

$$f_3(0) = (\alpha - 1)\Lambda + \Lambda \left[W_1(\mu_1, \lambda_1(0)) + \Lambda \frac{\partial W_1(\mu_1, \lambda_1(0))}{\partial \lambda_1} \right] > 0,$$

$$f_4(1) = -\alpha + W_1(\mu_1, 0) < 0.$$

it follows that $0 < \underline{v}_2$, $v_1^M < 1$ and $f_3(v_1^M) < f_4(v_1^M) = 0$. Thus,

$$0 < \underline{v}_2 < v_1^M < 1.$$

Then, we capture firm 1's global best response ($v_1^*(v_2)$) based on the preceding results, the local best response $v_1^1(v_2)$ and $v_1^2(v_2)$ (see Eqs. (9) and (10)).

a. When $0 \leq v_2 < 1 - \alpha + W_1(\mu_1, 0)$, $v_1^2(v_2) = v_2$ is a feasible strategy contained by $v_1 \geq v_2$. Note that $v_1^1(v_2)$ is the local best response for all $v_1 \geq v_2$, so $v_1^2(v_2)$ is dominated by $v_1^1(v_2)$, thus indicating that $v_1^*(v_2) = v_1^1(v_2) = 1$.

b. When $1 - \alpha + W_1(\mu_1, 0) \leq v_2 \leq \underline{v}_2$, we have $v_1^2(v_2) = v_2$. Similar to Case a, it follows that $v_1^*(v_2) = v_1^1(v_2) = v_1^0(v_2)$.

c. When $\underline{v}_2 < v_2 < v_1^M$, $v_1^1(v_2) = v_2 = v_1^2(v_2)$, and so $v_1^*(v_2) = v_2$.

d. When $v_2 \geq v_1^M$, $v_1^1(v_2) = v_2$ is a feasible strategy contained by $v_1 \leq v_2$. Note that $v_1^2(v_2)$ is the local best response for all $v_1 \leq v_2$, so $v_1^1(v_2)$ is dominated by $v_1^2(v_2)$, thus indicating that $v_1^*(v_2) = v_1^2(v_2) = v_1^M$.

2. Given $v_1 \leq 1$, firm 2's revenue (denoted by $\pi_2(v_2)$) can be divided into two cases; that is

$$\pi_2(v_2) = \begin{cases} \Lambda(v_1 - v_2)[v_2 - W_2(\mu_2, \lambda_2(v_2))], & v_2 \leq v_1 \\ 0, & v_2 > v_1 \end{cases}.$$

Obviously, any strategy greater than v_1 is dominated by those no greater than v_1. Thus, firm 2's best response is

$$v_2^*(v_1) = \underset{0 \leq v_2 \leq v_1}{\operatorname{argmax}}\ \pi_2(v_2) = \Lambda(v_1 - v_2)[v_2 - W_2(\mu_2, \lambda_2(v_2))].$$

Given $0 \leq v_2 \leq v_1$, it can be easily calculated that

$$\begin{aligned}
\pi_2'(v_2) &= \Lambda(v_1 - v_2)\left[1 + \Lambda \frac{\partial W_2(\mu_2, \lambda_2(v_2))}{\partial \lambda_2}\right] - \Lambda[v_2 - W_2(\mu_2, \lambda(v_2))], \\
\pi_2''(v_2) &= -\Lambda^3(v_1 - v_2)\frac{\partial^2 W_2(\mu_2, \lambda_2(v_2))}{\partial {\lambda_2}^2} - 2\Lambda\left[1 + \Lambda \frac{\partial W_2(\mu_2, \lambda_2(v_2))}{\partial \lambda_2}\right] < 0.
\end{aligned}$$

That is, $\pi_2(\cdot)$ is a strictly concave function. Let $v_2^0(v_1)$ be the stationary point of $\pi_2(\cdot)$ with respect to v_2 with the constraint that $0 \leq v_2 \leq v_1$, which is determined by

$$\Lambda(v_1 - v_2^0(v_1))\left[1 + \Lambda \frac{\partial W_2(\mu_2, \lambda_2(v_2^0(v_1)))}{\partial \lambda_2}\right] - \Lambda[v_2^0(v_1) - W_2(\mu_2, \lambda(v_2^0(v_1)))] = 0. \tag{20}$$

Note that $\pi_2''(v_2) < 0$ and $\pi_2'(0) > 0$. $v_2^0(v_1)$ is well-defined only when $\pi_2'(v_1) \geq 0$; i.e., $v_1 \geq W_2(\mu_2, 0)$. Thus, given $0 \leq v_2 \leq v_1$, if $v_1 < W_2(\mu_2, 0)$, then $\pi_2(v_2)$ is increasing in v_2; if $v_1 \geq W_2(\mu_2, 0)$, then $\pi_2(v_2)$ peaks at $v_2 = v_2^0(v_1)$. Thus, $v_2^*(v_1)$ is given by

$$v_2^*(v_1) = \begin{cases} v_1, & 0 \leq v_1 < W_2(\mu_2, 0) \\ v_2^0(v_1), & W_2(\mu_2, 0) \leq v_1 \leq 1 \end{cases}. \tag{21}$$

In particular, if $v_1 = W_2(\mu_2, 0)$, then $v_2^0(v_1) = W_2(\mu_2, 0)$.

Furthermore, differentiating Eq. (**??**) with respect to v_1, we have

$$\begin{aligned}
&\Lambda\left[2 + 2\Lambda \frac{\partial W_2(\mu_2, \lambda_2(v_2^0(v_1)))}{\partial \lambda_2} + \Lambda^2(v_1 - v_2^0(v_1))\frac{\partial^2 W_2(\mu_2, \lambda_2(v_2^0(v_1)))}{\partial {\lambda_2}^2}\right]\frac{\partial v_2^0(v_1)}{\partial v_1} \\
=&\Lambda\left[1 + 2\Lambda \frac{\partial W_2(\mu_2, \lambda_2(v_2^0(v_1)))}{\partial \lambda_2} + \Lambda^2(v_1 - v_2^0(v_1))\frac{\partial^2 W_2(\mu_2, \lambda_2(v_2^0(v_1)))}{\partial {\lambda_2}^2}\right],
\end{aligned}$$

and so

$$0 < \frac{\partial v_2^0(v_1)}{\partial v_1} < 1.$$

This proves the monotonicity of $v_2^0(v_1)$ with respective to v_1. $\square$

Proof of Proposition 2. In particular, $\mu_2^T(\mu_1)$ is the unique solution to $W_2(\mu_2^T(\mu_1), 0) = 1-\lambda_1^T/\Lambda$. We first show that $\mu_2 < \mu_2^T(\mu_1)$ is equivalent to $W_2(\mu_2, 0) > v_1^M$. By the definition of $\mu_2^T(\mu_1)$, it is easily seen that $\mu_2 < \mu_2^T(\mu_1)$ indicates

$$4\alpha t - \alpha\Lambda + \Lambda\left[W_1(\mu_1, 2t) + 2t\frac{\partial W_1(\mu_1, 2t)}{\partial \lambda_1}\right] < 0,$$

where $t = \Lambda[1 - W_2(\mu_2, 0)]/2$. By the definition of $f_4(x)$ (see Eq. (**??**)) and v_1^M, this further indicates that $f_4(W_2(\mu_2, 0)) < 0 = f_4(v_1^M)$. Finally, by the monotonicity of $f_4(\cdot)$, it follows that $W_2(\mu_2, 0) > v_1^M$. According to Lemma 1, we have Figure 9.

Figure 9: The Nash equilibrium

As shown in Figure 9, there exists continuum equilibria, from $(\underline{v}_2, \underline{v}_2)$ to (v_1^M, v_1^M). Among these equilibria, $\lambda_2^T = 0$ and so $\pi_2 = 0$ holds. As for firm 1, given $v_2 = v_1$, it solves

$$\max_{\underline{v}_2 \le v_1 \le v_1^M} \pi_1(v_1) = \Lambda(1 - v_1)[\alpha v_1 - W_1(\mu_1, \lambda_1(v_1))].$$

By the definition of v_1^M, it is clear that $\pi_1(v_1)$ peaks at v_1^M. Thus, the equilibrium, (v_1^M, v_1^M), is Pareto dominating. Let $(v_1^*, v_2^*) = (v_1^M, v_1^M)$, it is clear that $\lambda_1^T = \Lambda(1 - v_1^M)$, so we have

$$2\alpha\lambda_1^T - \alpha\Lambda + \Lambda\left[W_1(\mu_1, \lambda_1^T) + \lambda_1^T\frac{\partial W_1(\mu_1, \lambda_1^T)}{\partial \lambda_1}\right] = 0.$$

□

Proof of Proposition 3. In particular, $\mu_2^{T'}(\mu_1)$ is the unique solution to

$$\alpha - (2\alpha - 1)\frac{\lambda_1^T}{\Lambda} - \left[W_1(\mu_1, \lambda_1^T) + \lambda_1^T\frac{\partial W_1(\mu_1, \lambda_1^T)}{\partial \lambda_1}\right] = 0. \quad (22)$$

We first show that $\mu_2^T(\mu_1) \le \mu_2 < \mu_2^{T'}(\mu_1)$ is equivalent to $\underline{v}_2 < W_2(\mu_2, 0) \le v_1^M$. In the proof of Proposition 2, we showed that $\mu_2^T(\mu_1) \le \mu_2$ is equivalent to $W_2(\mu_2, 0) \le v_1^M$, and so we just need to verify that $\mu_2 < \mu_2^{T'}(\mu_1)$ is equivalent to $\underline{v}_2 < W_2(\mu_2, 0)$.

By the definition of $\mu_2^{T'}(\mu_1)$, it is easily seen that $\mu_2 < \mu_2^{T'}(\mu_1)$ indicates

$$2(2\alpha - 1)t - \alpha\Lambda + \Lambda\left[W_1(\mu_1, 2t) + 2t\frac{\partial W_1(\mu_1, 2t)}{\partial \lambda_1}\right] < 0.$$

By the definition of $f_3(x)$ (see Eq. (**??**)) and $\underline{v}_2$ (see Eq. (8)), this further indicates that $f_3(W_2(\mu_2, 0)) < 0 = f_3(\underline{v}_2)$. Finally, by the monotonicity of $f_3(\cdot)$, it follows that $W_2(\mu_2, 0) > \underline{v}_2$, and so $\underline{v}_2 < W_2(\mu_2, 0) \le v_1^M$ holds. According to Lemma 1, we have Figure 10.

Figure 10: The Nash equilibrium

As shown in Figure 10, there exist continuum equilibria, from $(\underline{v}_2, \underline{v}_2)$ to $(W_2(\mu_2, 0), W_2(\mu_2, 0))$. Among these equilibria, $\lambda_2^T = 0$, and so $\pi_2 = 0$ holds. As for firm 1, given $v_2 = v_1$, it solves

$$\max_{\underline{v}_2 \leq v_1 \leq W_2(\mu_2, 0)} \pi_1(v_1) = \Lambda(1 - v_1)[\alpha v_1 - W_1(\mu_1, \lambda_1(v_1))].$$

By the definition of v_1^M (see Eq. (**??**)), it is clear that given $W_2(\mu_2, 0) \leq v_1^M$, $\pi_1(v_1)$ is increasing in the feasible domain. Thus, the equilibrium, $(W_2(\mu_2, 0), W_2(\mu_2, 0))$, is Pareto dominating. Let $(v_1^*, v_2^*) = (W_2(\mu_2, 0), W_2(\mu_2, 0))$, it is clear that

$$\lambda_1^T = \Lambda(1 - v_1^*) = \Lambda[1 - W_2(\mu_2, 0)], \ \lambda_2^T = 0.$$

□

Proof of Proposition 4. In particular, $\mu_1^T(\mu_2)$ is the unique solution to $W_1(\mu_1^T(\mu_2), 0) = \alpha - \lambda_2^T/\Lambda$. We first show that $\mu_1 < \mu_1^T(\mu_2)$ is equivalent to $1 - \alpha + W_1(\mu_1, 0) > v_2^0(1)$. By the definition of $\mu_1^T(\mu_2)$, it is easily seen that $\mu_1 < \mu_1^T(\mu_2)$ indicates

$$4s - \Lambda + \Lambda \left[W_2(\mu_2, 2s) + 2s \frac{\partial W_2(\mu_2, 2s)}{\partial \lambda_2} \right] < 0,$$

where $s = \Lambda[\alpha - W_1(\mu_1, 0)]/2$. Let $f_5(x) := 2(1-x)\Lambda - \Lambda + \Lambda \left[W_2(\mu_2, \Lambda(1-x)) + \Lambda(1-x) \frac{\partial W_2(\mu_2, \Lambda(1-x))}{\partial \lambda_2} \right]$. It is clear that $f_5(x)$ is decreasing in x, and $f_5(1 - \alpha + W_1(\mu_1, 0)) = 4s - \Lambda + \Lambda \left[W_2(\mu_2, 2s) + 2s \frac{\partial W_2(\mu_2, 2s)}{\partial \lambda_2} \right]$. According to Eq. (**??**), it is clear that $f_5(v_2^0(1)) = 0$. Thus, we have $f_5(1 - \alpha + W_1(\mu_1, 0)) < 0 = f_5(v_2^0(1))$. Finally, by the monotonicity of $f_5(\cdot)$, it follows that $1 - \alpha + W_1(\mu_1, 0) > v_2^0(1)$. According to Lemma 1, we have Figure 11.

Figure 11: The Nash equilibrium

As shown in Figure 11, the equilibrium is $(v_1^*, v_2^*) = (1, v_2^0(1))$, and so $\lambda_1^T = 0$ and $\lambda_2^T = \Lambda[1 - v_2^0(1)]$. Let $v_2^0(1) = 1 - \frac{\lambda_2^T}{\Lambda}$ and substitute it into $f_5(v_2^0(1)) = 0$. It follows that

$$2\lambda_2^T - \Lambda + \Lambda \left[W_2(\mu_2, \lambda_2^T) + \lambda_2^T \frac{\partial W_2(\mu_2, \lambda_2^T)}{\partial \lambda_2} \right] = 0.$$

□

Proof of Proposition 5. By the definition of $\mu_1^T(\mu_2)$ and $\mu_2^{T'}(\mu_1)$, when $\mu_1 \geq \mu_1^T(\mu_2)$ and $\mu_2 \geq \mu_2^{T'}(\mu_1)$, we have the following figure.

Figure 12: The Nash equilibrium

As shown in Figure 12, the equilibrium, (v_1^*, v_2^*), satisfies

$$\begin{cases} \Lambda(1-v_1^*)\left[\alpha-1+\Lambda\frac{\partial W_1(\mu_1,\Lambda(1-v_1^*))}{\partial\lambda_1}\right]-(\alpha-1)\Lambda v_1^*+\Lambda W_1(\mu_1,\Lambda(1-v_1^*))-\Lambda v_2^*=0, \\ \Lambda(v_1^*-v_2^*)\left[1+\Lambda\frac{\partial W_2(\mu_2,\Lambda(v_1^*-v_2^*))}{\partial\lambda_2}\right]-\Lambda v_2^*+\Lambda W_2(\mu_2,\Lambda(v_1^*-v_2^*))=0. \end{cases} \quad (23)$$

On this occasion, $\lambda_1^T = \Lambda(1-v_1^*)$ and $\lambda_2^T = \Lambda(v_1^* - v_2^*)$. Hence, we have

$$\begin{cases} (2\alpha-1)\lambda_1^T+\lambda_2^T-\alpha\Lambda+\Lambda\left[W_1(\mu_1,\lambda_1^T)+\lambda_1^T\frac{\partial W_1(\mu_1,\lambda_1^T)}{\partial\lambda_1}\right]=0, \\ \lambda_1^T+2\lambda_2^T-\Lambda+\Lambda\left[W_2(\mu_2,\lambda_2^T)+\lambda_2^T\frac{\partial W_2(\mu_2,\lambda_2^T)}{\partial\lambda_2}\right]=0. \end{cases}$$

□

Proof of Proposition 6. 1. According to Eq. (**??**), it is clear that when $\mu_1 = \underline{\mu_1}$, $\lambda_1^T = 0$, and thus, $\mu_2^T(\mu_1) = \underline{\mu_2}$ ($\mu_2^{T'}(\mu_1) = \underline{\mu_2}$). According to Eq. (**??**), we have $\mu_2 = \underline{\mu_2}$, $\lambda_2^T = 0$. Thus, $\mu_1^T(\mu_2) = \underline{\mu_1}$.

2. Let

$$f_1(x) = \alpha\left(1 - 2\frac{x}{\Lambda}\right) - \left[W_1(\mu_1, x) + x\frac{\partial W_1(\mu_1, x)}{\partial\lambda_1}\right].$$

It is easy to see that $f_1'(x) < 0$ and $\frac{\partial f_1(x)}{\partial\mu_1} > 0$. According to Eq. (**??**), it is clear that λ_1^T is increasing in μ_1. From $W_2(\mu_2^T(\mu_1), 0) = 1 - \frac{\lambda_1^T}{\Lambda}$, it is clear that $\mu_2^T(\mu_1)$ is increasing in μ_1.

Let

$$f_2(x) = \alpha - (2\alpha - 1)\frac{x}{\Lambda} - \left[W_1(\mu_1, x) + x\frac{\partial W_1(\mu_1, x)}{\partial\lambda_1}\right].$$

It is easy to see that $f_2'(x) < 0$ and $\frac{\partial f_2(x)}{\partial\mu_1} > 0$. According to Eq. (**??**), it is clear that λ_1^T is increasing in μ_1. From $W_2(\mu_2^{T'}(\mu_1), 0) = 1 - \frac{\lambda_1^{T'}}{\Lambda}$, it is clear that $\mu_2^{T'}(\mu_1)$ is increasing in μ_1.

Let

$$f_3(x) = 1 - 2\frac{x}{\Lambda} - \left[W_2(\mu_2, x) + x\frac{\partial W_2(\mu_2, x)}{\partial\lambda_2}\right].$$

It is easy to see that $f_3'(x) < 0$ and $\frac{\partial f_3(x)}{\partial\mu_2} > 0$. According to Eq. (**??**), it is clear that λ_2^T is increasing in μ_2. From $W_1(\mu_1^T(\mu_2), 0) = \alpha - \frac{\lambda_2^T}{\Lambda}$, it is clear that $\mu_2^T(\mu_1)$ is increasing in μ_1.

3. According to Eq. (**??**), it is clear that λ_1^T is increasing in α. From $W_2(\mu_2^T(\mu_1), 0) = 1 - \frac{\lambda_1^T}{\Lambda}$, it is clear that $\mu_2^T(\mu_1)$ is increasing in α. According to Eq. (**??**), it is clear that λ_2^T is constant in α. From $W_1(\mu_1^T(\mu_2), 0) = \alpha - \frac{\lambda_2^T}{\Lambda}$, it is clear that $\mu_1^T(\mu_2)$ is decreasing in α.

4. We verify this via a contradiction. Assume that there is at least one intersecting point between $\mu_2 = \mu_2^T(\mu_1)$ and $\mu_1 = \mu_1^T(\mu_2)$. Then there exists a pair of (μ_1, μ_2) such that

$$\underline{\mu_1} < \mu_1 \leq \mu_1^T(\mu_2), \ \underline{\mu_2} < \mu_2 \leq \mu_2^T(\mu_1). \tag{24}$$

According to Theorems 2 and 4, this indicates that

$$\lambda_1^T = 0, \ \lambda_2^T = 0. \tag{25}$$

However, this is true only when

$$\mu_1 < \underline{\mu_1} \ \& \ \mu_2 < \underline{\mu_2}.$$

Otherwise, the firm can post a small positive price to obtain some customers, and thus, Eq. (14) cannot hold. According to Eq. (13), there is a contradiction. Similarly, we can show that there is no intersecting point between $\mu_2 = \mu_2^{T'}(\mu_1)$ and $\mu_1 = \mu_1^T(\mu_2)$.

To show that there is no intersecting point between $\mu_2 = \mu_2^{T'}(\mu_1)$ and $\mu_2 = \mu_2^T(\mu_1)$, it suffices to show that $\mu_2^{T'}(\mu_1) > \mu_2^S(\mu_1)$ for any $\mu_1 > \underline{\mu_1}$. Note that $\mu_2^S(\mu_1)$ is determined by $W_2(\mu_2^S(\mu_1), 0) = 1 - x$, where x satisfies $\alpha\left(1 - 2\frac{x}{\Lambda}\right) - \left[W_1(\mu_1, x) + x\frac{\partial W_1(\mu_1,x)}{\partial \lambda_1}\right]$; $\mu_2^{T'}(\mu_1)$ is determined by $W_2(\mu_2^{T'}(\mu_1), 0) = 1 - y$, where x satisfies $\alpha\left(1 - 2\frac{y}{\Lambda}\right) + \frac{y}{\Lambda} - \left[W_1(\mu_1, y) + y\frac{\partial W_1(\mu_1,y)}{\partial \lambda_1}\right]$. As a result, we need to show that $y > x$. Recall that $f_1(x) = 0 = f_2(y)$; $f_2'(\cdot)$ is decreasing; $f_2(x) - f_1(x) = x/\Lambda > 0$. It follows that $f_2(y) = 0 = f_1(x) < f_2(x)$, and so $y > x$.

5. From Eq. (**??**), it is easy to see that $\lambda_1^T < \Lambda/2$, and thus, $W_2(\mu_2^T(\mu_1), 0) = 1 - \frac{\lambda_1^T}{\Lambda} > \frac{1}{2}$. From Eq. (**??**), it is easy to see that $\lambda_1^T < \frac{\alpha}{2\alpha-1}\Lambda$, and thus, $W_2(\mu_2^{T'}(\mu_1), 0) = 1 - \frac{\lambda_1^T}{\Lambda} > \frac{\alpha-1}{2\alpha-1}$. From Eq. (**??**), it is easy to see that $\lambda_1^T < \Lambda/2$, and so $W_2(\mu_2^T(\mu_1), 0) = 1 - \frac{\lambda_1^T}{\Lambda} > \frac{1}{2}$. □

Proof of Proposition 7. The proofs of the first three cases are straightforward; thus, we give the last two cases.

For Case 4, differentiating Eq. (1) w.r.t. μ_1, we have

$$\frac{1}{\Lambda}\frac{\partial \lambda_1^T}{\partial \mu_1} = -\left[\frac{2}{\Lambda} + 2\frac{\partial W_2(\mu_2, \lambda_2^T)}{\partial \lambda_2} + \lambda_2^T \frac{\partial^2 W_2(\mu_2, \lambda_2^T)}{\partial {\lambda_2}^2}\right] \frac{\partial \lambda_2^T}{\partial \mu_1} \tag{26}$$

and

$$\frac{1}{\Lambda}\frac{\partial \lambda_2^T}{\partial \mu_1} = -\left[\frac{2\alpha - 1}{\Lambda} + 2\frac{\partial W_1(\mu_1, \lambda_1^T)}{\partial \lambda_1} + \lambda_1^T \frac{\partial^2 W_1(\mu_1, \lambda_1^T)}{\partial {\lambda_1}^2}\right] \frac{\partial \lambda_1^T}{\partial \mu_1} - \left[\frac{\partial W_1(\mu_1, \lambda_1^T)}{\partial \mu_1} + \lambda_1^T \frac{\partial^2 W_1(\mu_1, \lambda_1^T)}{\partial \lambda_1 \partial \mu_1}\right]. \tag{27}$$

This gives $\frac{\partial\lambda_1^T}{\partial\mu_1}\times\frac{\partial\lambda_2^T}{\partial\mu_1}<0$. Furthermore, substituting $\frac{\partial\lambda_2^T}{\partial\mu_1}$ given in Eq. (**??**) into Eq. (**??**), we have

$$\left[2\frac{\partial W_1(\mu_1,\lambda_1^T)}{\partial\lambda_1}+\lambda_1^T\frac{\partial^2 W_1(\mu_1,\lambda_1^T)}{\partial{\lambda_1}^2}+\frac{2\alpha-1}{\Lambda}-\frac{1}{\Lambda}\frac{1/\Lambda}{2\frac{\partial W_2(\mu_2,\lambda_2^T)}{\partial\lambda_2}+\lambda_2^T\frac{\partial^2 W_2(\mu_2,\lambda_2^T)}{\partial{\lambda_2}^2}+\frac{2}{\Lambda}}\right]\frac{\partial\lambda_1^T}{\partial\mu_1}$$
$$=-\left[\frac{\partial W_1(\mu_1,\lambda_1^T)}{\partial\mu_1}+\lambda_1^T\frac{\partial^2 W_1(\mu_1,\lambda_1^T)}{\partial\lambda_1\partial\mu_1}\right].$$

Note that

$$\frac{2\alpha-1}{\Lambda}-\frac{1}{\Lambda}\frac{1/\Lambda}{2\frac{\partial W_2(\mu_2,\lambda_2^T)}{\partial\lambda_2}+\lambda_2^T\frac{\partial^2 W_2(\mu_2,\lambda_2^T)}{\partial{\lambda_2}^2}+\frac{2}{\Lambda}}>\frac{2\alpha-1}{\Lambda}-\frac{1}{\Lambda}\frac{1/\Lambda}{\frac{2}{\Lambda}}>0.$$

It is clear that $\frac{\partial\lambda_1^T}{\partial\mu_1}>0$ and $\frac{\partial\lambda_2^T}{\partial\mu_1}<0$. Now, $\frac{\partial\lambda_1^T}{\partial\mu_1}+\frac{\partial\lambda_2^T}{\partial\mu_1}=\left[\frac{\frac{1}{\Lambda}+2\frac{\partial W_2(\mu_2,\lambda_2^T)}{\partial\lambda_2}+\lambda_2^T\frac{\partial^2 W_2(\mu_2,\lambda_2^T)}{\partial{\lambda_2}^2}}{\frac{2}{\Lambda}+2\frac{\partial W_2(\mu_2,\lambda_2^T)}{\partial\lambda_2}+\lambda_2^T\frac{\partial^2 W_2(\mu_2,\lambda_2^T)}{\partial{\lambda_2}^2}}\right]\frac{\partial\lambda_1^T}{\partial\mu_1}>0$. Similarly, we can show that $\frac{\partial\lambda_1^T}{\partial\mu_2}<0$, $\frac{\partial\lambda_2^T}{\partial\mu_2}>0$ and $\frac{\partial\lambda_1^T}{\partial\mu_2}+\frac{\partial\lambda_2^T}{\partial\mu_2}>0$.

For Case 5, differentiating Eq. (1) w.r.t. α, we have

$$-\frac{1}{\Lambda}\frac{\partial\lambda_1^T}{\partial\alpha}=\left[2\frac{\partial W_2(\mu_2,\lambda_2^T)}{\partial\lambda_2}+\lambda_2^T\frac{\partial^2 W_2(\mu_2,\lambda_2^T)}{\partial{\lambda_2}^2}+\frac{2}{\Lambda}\right]\frac{\partial\lambda_2^T}{\partial\alpha}$$

and

$$1-2\frac{\lambda_1^T}{\Lambda}=\left[2\frac{\partial W_1(\mu_1,\lambda_1^T)}{\partial\lambda_1}+\lambda_1^T\frac{\partial^2 W_1(\mu_1,\lambda_1^T)}{\partial{\lambda_1}^2}+\frac{2\alpha-1}{\Lambda}\right]\frac{\partial\lambda_1^T}{\partial\alpha}+\frac{1}{\Lambda}\frac{\partial\lambda_2^T}{\partial\alpha}.$$

This gives $\frac{\partial\lambda_1^T}{\partial\alpha}\times\frac{\partial\lambda_2^T}{\partial\alpha}<0$, $\frac{\partial(\lambda_1^T+\lambda_2^T)}{\partial\alpha}=-\left[2\Lambda\frac{\partial W_2(\mu_2,\lambda_2^T)}{\partial\lambda_2}+\Lambda\lambda_2^T\frac{\partial^2 W_2(\mu_2,\lambda_2^T)}{\partial{\lambda_2}^2}+1\right]\frac{\partial\lambda_2^T}{\partial\alpha}$ and

$$\left[2\frac{\partial W_1(\mu_1,\lambda_1^T)}{\partial\lambda_1}+\lambda_1^T\frac{\partial^2 W_1(\mu_1,\lambda_1^T)}{\partial{\lambda_1}^2}+\frac{2\alpha-1}{\Lambda}-\frac{1}{\Lambda}\frac{1/\Lambda}{2\frac{\partial W_2(\mu_2,\lambda_2^T)}{\partial\lambda_2}+\lambda_2^T\frac{\partial^2 W_2(\mu_2,\lambda_2^T)}{\partial{\lambda_2}^2}+\frac{2}{\Lambda}}\right]\frac{\partial\lambda_1^T}{\partial\alpha}=1-2\frac{\lambda_1^T}{\Lambda}.$$

Thus, if $1-2\frac{\lambda_1^T}{\Lambda}\geq 0$, then $\frac{\partial\lambda_1^T}{\partial\alpha}\geq 0$, $\frac{\partial\lambda_2^T}{\partial\alpha}\leq 0$, and $\frac{\partial(\lambda_1^T+\lambda_2^T)}{\partial\alpha}\geq 0$; if $1-2\frac{\lambda_1^T}{\Lambda}<0$, then $\frac{\partial\lambda_1^T}{\partial\alpha}<0$, $\frac{\partial\lambda_2^T}{\partial\alpha}>0$ and $\frac{\partial(\lambda_1^T+\lambda_2^T)}{\partial\alpha}<0$. According to Proposition 7, it is clear that as μ_2 increases, λ_1^T first remains unchanged, then increases, and then decreases. In particular, λ_1^T peaks at $\mu_2=\mu_2^{T'}(\mu_1)$ with value $\Lambda[1-W_2(\mu_2^{T'}(\mu_1),0)]$. Let $1-2\frac{\Lambda[1-W_2(\mu_2^{T'}(\mu_1),0)]}{\Lambda}>0$. We have $W_2(\mu_2^{T'}(\mu_1),0)>1/2$. Recall that $\mu_2^{T'}(\mu_1)$ is increasing in μ_1. Thus, when μ_1 is small, $W_2(\mu_2^{T'}(\mu_1),0)>1/2$ holds, and so $\frac{\partial\lambda_1^T}{\partial\alpha}\geq 0$, $\frac{\partial\lambda_2^T}{\partial\alpha}\leq 0$ and $\frac{\partial(\lambda_1^T+\lambda_2^T)}{\partial\alpha}\geq 0$. In contrast, when μ_1 is large, $W_2(\mu_2^{T'}(\mu_1),0)\leq 1/2$ holds. Note also that λ_1^T is decreasing in μ_2 for $\mu_2\geq\mu_2^{T'}(\mu_1)$. It follows that (1) when μ_2 is small, $1-2\frac{\lambda_1^T}{\Lambda}\leq 0$, and so $\frac{\partial\lambda_1^T}{\partial\alpha}\leq 0$, $\frac{\partial\lambda_2^T}{\partial\alpha}\geq 0$ and $\frac{\partial(\lambda_1^T+\lambda_2^T)}{\partial\alpha}\leq 0$; (2) when μ_2 is large, $1-2\frac{\lambda_1^T}{\Lambda}>0$, and so $\frac{\partial\lambda_1^T}{\partial\alpha}>0$, $\frac{\partial\lambda_2^T}{\partial\alpha}<0$ and $\frac{\partial(\lambda_1^T+\lambda_2^T)}{\partial\alpha}>0$. □

Proof of Lemma 2. Differentiating (2) with respect to p_1, we have

$$1 = \left[-\alpha\frac{1}{\Lambda} - \frac{\partial W_1(\mu_1,\lambda_1)}{\partial \lambda_1}\right]\frac{\partial \lambda_1}{\partial p_1} - \frac{1}{\Lambda}\frac{\partial \lambda_2}{\partial p_1},$$
$$0 = -\frac{1}{\Lambda}\frac{\partial \lambda_1}{\partial p_1} + \left[-\frac{1}{\Lambda} - \frac{\partial W_2(\mu_2,\lambda_2)}{\partial \lambda_2}\right]\frac{\partial \lambda_2}{\partial p_1}.$$

From the second equation, we have $sign[\frac{\partial \lambda_1}{\partial p_1}] = -sign[\frac{\partial \lambda_2}{\partial p_1}]$. Combining these equations and eliminating $\frac{\partial \lambda_2}{\partial p_1}$, we have

$$1 = \left[-(\alpha-1)\frac{1}{\Lambda} - \frac{\partial W_1(\mu_1,\lambda_1)}{\partial \lambda_1} - \frac{1}{\Lambda} + \frac{\left(\frac{-1}{\Lambda}\right)^2}{\frac{1}{\Lambda} + \frac{\partial W_2(\mu_2,\lambda_2)}{\partial \lambda_2}}\right]\frac{\partial \lambda_1}{\partial p_1}$$

This gives $\frac{\partial \lambda_1}{\partial p_1} < 0$ and $\frac{\partial \lambda_2}{\partial p_1} > 0$. Moreover, we have

$$\frac{\partial \lambda_1}{\partial p_1} + \frac{\partial \lambda_2}{\partial p_1} = \left[\frac{\frac{\partial W_2(\mu_2,\lambda_2)}{\partial \lambda_2}}{\frac{1}{\Lambda} + \frac{\partial W_2(\mu_2,\lambda_2)}{\partial \lambda_2}}\right]\frac{\partial \lambda_1}{\partial p_1} < 0.$$

In the same way, it can be shown that $\frac{\partial \lambda_1}{\partial p_2} > 0$, $\frac{\partial \lambda_2}{\partial p_2} < 0$ and $\frac{\partial \lambda_1}{\partial p_2} + \frac{\partial \lambda_2}{\partial p_2} < 0$. □

Proof of Lemma 3. According to

$$\pi_1(\lambda_1) = \lambda_1\left[\alpha - \alpha\frac{\lambda_1}{\Lambda} - \frac{\lambda_2}{\Lambda} - W_1(\mu_1,\lambda_1)\right],$$

we have

$$\pi_1'(\lambda_1) = \alpha - 2\alpha\frac{\lambda_1}{\Lambda} - \frac{\lambda_2}{\Lambda} - W_1(\mu_1,\lambda_1) - \lambda_1\frac{\partial W_1(\mu_1,\lambda_1)}{\partial \lambda_1},$$
$$\pi_1''(\lambda_1) = -2\frac{\alpha}{\Lambda} - 2\frac{\partial W_1(\mu_1,\lambda_1)}{\partial \lambda_1} - \lambda_1\frac{\partial^2 W_1(\mu_1,\lambda_1)}{\partial {\lambda_1}^2} < 0.$$

This proves the concavity of $\pi_1(\lambda_1)$ in λ_1. As for $\pi_2(\lambda_2)$, we have

$$\pi_2'(\lambda_2) = 1 - 2\frac{\lambda_2}{\Lambda} - \frac{\lambda_1}{\Lambda} - W_2(\mu_2,\lambda_2) - \lambda_2\frac{\partial W_2(\mu_2,\lambda_2)}{\partial \lambda_2},$$
$$\pi_2''(\lambda_2) = -\frac{2}{\Lambda} - 2\frac{\partial W_2(\mu_2,\lambda_2)}{\partial \lambda_2} - \lambda_2\frac{\partial^2 W_2(\mu_2,\lambda_2)}{\partial {\lambda_2}^2} < 0.$$

This proves the concavity of $\pi_2(\lambda_2)$ in λ_2. □

Proof of Lemma 4. Note that each firm's best response is characterized by the FOC. Firm 1's best response, say $\lambda_1^*(\lambda_2)$, is given by

$$\alpha - 2\alpha\frac{\lambda_1^*(\lambda_2)}{\Lambda} - \frac{\lambda_2}{\Lambda} - W_1(\mu_1,\lambda_1^*(\lambda_2)) - \lambda_1^*(\lambda_2)\frac{\partial W_1(\mu_1,\lambda_1^*(\lambda_2))}{\partial \lambda_1} = 0,$$

and firm 2's best response, say $\lambda_2^*(\lambda_1)$, is given by

$$1 - 2\frac{\lambda_2^*(\lambda_1)}{\Lambda} - \frac{\lambda_1}{\Lambda} - W_2(\mu_2, \lambda_2^*(\lambda_1)) - \lambda_2^*(\lambda_1)\frac{\partial W_2(\mu_2, \lambda_2^*(\lambda_1))}{\partial \lambda_2} = 0.$$

This gives

$$\frac{\partial \lambda_1^*(\lambda_2)}{\partial \lambda_2} = -\frac{1}{2\alpha + 2\Lambda\frac{\partial W_1(\mu_1,\lambda_1^*(\lambda_2))}{\partial \lambda_1} + \lambda_1^*(\lambda_2)\frac{\partial^2 W_1(\mu_1,\lambda_1^*(\lambda_2))}{\partial \lambda_1^2}} < 0,\ |\frac{\partial \lambda_1^*(\lambda_2)}{\partial \lambda_2}| < 1,$$

and

$$\frac{\partial \lambda_2^*(\lambda_1)}{\partial \lambda_1} = -\frac{1}{2 + 2\Lambda\frac{\partial W_2(\mu_2,\lambda_2^*(\lambda_1))}{\partial \lambda_2} + \lambda_2^*(\lambda_1)\frac{\partial^2 W_2(\mu_2,\lambda_2^*(\lambda_1))}{\partial \lambda_2^2}} < 0,\ |\frac{\partial \lambda_2^*(\lambda_1)}{\partial \lambda_1}| < 1.$$

□

Proof of Proposition 8.

- For Region (I), firm 1's effective arrival rate λ_1^S is given by

$$\alpha\left(1 - 2\frac{\lambda_1^S}{\Lambda}\right) - \left[W_1(\mu_1, \lambda_1^S) + \lambda_1^S\frac{\partial W_1(\mu_1, \lambda_1^S)}{\partial \lambda_1}\right] = 0. \tag{28}$$

- For Region (II), firm 2's effective arrival rate λ_2^S is given by

$$\left(1 - 2\frac{\lambda_2^S}{\Lambda}\right) - \left[W_2(\mu_2, \lambda_2^S) + \lambda_2^S\frac{\partial W_2(\mu_2, \lambda_2^S)}{\partial \lambda_2}\right] = 0. \tag{29}$$

- For Region (III), λ_1^S and λ_2^S are given by

$$\begin{cases} \frac{\lambda_2^S}{\Lambda} = \alpha\left(1 - 2\frac{\lambda_1^S}{\Lambda}\right) - \left[W_1(\mu_1, \lambda_1^S) + \lambda_1^S\frac{\partial W_1(\mu_1,\lambda_1^S)}{\partial \lambda_1}\right], \\ \frac{\lambda_1^S}{\Lambda} = \left(1 - 2\frac{\lambda_2^S}{\Lambda}\right) - \left[W_2(\mu_2, \lambda_2^S) + \lambda_2^S\frac{\partial W_2(\mu_2,\lambda_2^S)}{\partial \lambda_2}\right]. \end{cases} \tag{30}$$

- $\mu_2^S(\mu_1)$ and $\mu_1^S(\mu_2)$ are the solutions to $W_2(\mu_2^S(\mu_1), 0) = 1 - \lambda_1^S/\Lambda$ and $W_1(\mu_1^S(\mu_2), 0) = \alpha - \lambda_2^S/\Lambda$, respectively, where λ_1^S is determined by (15) and λ_2^S is determined by (16).

According to Lemmas 3 and 4, the equilibrium of the effective arrival rates $(\lambda_1^S, \lambda_2^S)$ is derived from $\pi_1'(\lambda_1^S) = 0$ and $\pi_2'(\lambda_2^S) = 0$; i.e.,

$$\frac{\lambda_2^S}{\Lambda} = \alpha\left(1 - 2\frac{\lambda_1^S}{\Lambda}\right) - \left[W_1(\mu_1, \lambda_1^S) + \lambda_1^S\frac{\partial W_1(\mu_1, \lambda_1^S)}{\partial \lambda_1}\right], \tag{31}$$

$$\frac{\lambda_1^S}{\Lambda} = \left(1 - 2\frac{\lambda_2^S}{\Lambda}\right) - \left[W_2(\mu_2, \lambda_2^S) + \lambda_2^S\frac{\partial W_2(\mu_2, \lambda_2^S)}{\partial \lambda_2}\right]. \tag{32}$$

Next, we derive the condition for $\lambda_i^S \geq 0$ for $i = 1, 2$. We begin by showing that $\frac{\partial \lambda_1}{\partial \mu_1} > 0$ and $\frac{\partial \lambda_2}{\partial \mu_2} > 0$. Differentiating Eq. (**??**) w.r.t. μ_1, we have

$$\frac{1}{\Lambda}\frac{\partial \lambda_1}{\partial \mu_1} = -\left[\frac{2}{\Lambda} + 2\frac{\partial W_2(\mu_2, \lambda_2^S)}{\partial \lambda_2} + \lambda_2^S \frac{\partial^2 W_2(\mu_2, \lambda_2^S)}{\partial {\lambda_2}^2}\right]\frac{\partial \lambda_2}{\partial \mu_1}. \tag{33}$$

This gives $\frac{\partial \lambda_1}{\partial \mu_1} \times \frac{\partial \lambda_2}{\partial \mu_1} < 0$. Furthermore, differentiating Eq. (**??**) w.r.t. μ_1 and substituting $\frac{\partial \lambda_2}{\partial \mu_1}$ given in Eq. (**??**), we have

$$\begin{aligned}&\left[2\frac{\partial W_1(\mu_1, \lambda_1^S)}{\partial \lambda_1} + \lambda_1^S \frac{\partial^2 W_1(\mu_1, \lambda_1^S)}{\partial {\lambda_1}^2} + \frac{2\alpha}{\Lambda} - \frac{1}{\Lambda}\frac{1/\Lambda}{2\frac{\partial W_2(\mu_2, \lambda_2^S)}{\partial \lambda_2} + \lambda_2^S \frac{\partial^2 W_2(\mu_2, \lambda_2^S)}{\partial {\lambda_2}^2} + \frac{2}{\Lambda}}\right]\frac{\partial \lambda_1}{\partial \mu_1} \\ &= -\left[\frac{\partial W_1(\mu_1, \lambda_1^S)}{\partial \mu_1} + \lambda_1^S \frac{\partial^2 W_1(\mu_1, \lambda_1^S)}{\partial \lambda_1 \partial \mu_1}\right].\end{aligned}$$

Note that

$$2\frac{\partial W_1(\mu_1, \lambda_1^S)}{\partial \lambda_1} + \lambda_1^S \frac{\partial^2 W_1(\mu_1, \lambda_1^S)}{\partial {\lambda_1}^2} + \frac{2\alpha}{\Lambda} - \frac{1}{\Lambda}\frac{1/\Lambda}{2\frac{\partial W_2(\mu_2, \lambda_2^S)}{\partial \lambda_2} + \lambda_2^S \frac{\partial^2 W_2(\mu_2, \lambda_2^S)}{\partial {\lambda_2}^2} + \frac{2}{\Lambda}} > \frac{2\alpha}{\Lambda} - \frac{1}{\Lambda}\frac{1/\Lambda}{\frac{2}{\Lambda}} > 0.$$

It is clear that $\frac{\partial \lambda_1}{\partial \mu_1} > 0$ and $\frac{\partial \lambda_2}{\partial \mu_1} < 0$. Similarly, we can show that $\frac{\partial \lambda_1}{\partial \mu_2} < 0$ and $\frac{\partial \lambda_2}{\partial \mu_2} > 0$.

Let $\lambda_2^S = 0$. According to Eqs. (**??**) and (**??**), we have

$$1 - \frac{\lambda_1^S}{\Lambda} = W_2(\mu_2, 0), \tag{34}$$

$$\alpha\left(1 - 2\frac{\lambda_1^S}{\Lambda}\right) - \left[W_1(\mu_1, \lambda_1^S) + \lambda_1^S \frac{\partial W_1(\mu_1, \lambda_1^S)}{\partial \lambda_1}\right] = 0. \tag{35}$$

Thus, $\lambda_2^S \geq 0$ if and only if $\mu_2 \geq \mu_2^S(\mu_1)$, where $\mu_2^S(\mu_1)$ satisfies $W_2(\mu_2^S(\mu_1), 0) = 1 - \frac{\lambda_1^S}{\Lambda}$ with λ_1^S determined by Eq. (19). Let $\lambda_1^S = 0$. We have

$$\alpha - \frac{\lambda_2^S}{\Lambda} = W_1(\mu_1, 0), \tag{36}$$

$$\left(1 - 2\frac{\lambda_2^S}{\Lambda}\right) - \left[W_2(\mu_2, \lambda_2^S) + \lambda_2^S \frac{\partial W_2(\mu_2, \lambda_2^S)}{\partial \lambda_2}\right] = 0. \tag{37}$$

Then, $\lambda_1^S \geq 0$ if and only if $\mu_1 \geq \mu_1^S(\mu_2)$, where $\mu_1^S(\mu_2)$ satisfies $W_1(\mu_1^S(\mu_2), 0) = \alpha - \frac{\lambda_2^S}{\Lambda}$ with λ_2^S determined by Eq. (21). Thus, $\lambda_i^D \geq 0$ if and only if both $\mu_2 \geq \mu_2^S(\mu_1)$ and $\mu_1 \geq \mu_1^S(\mu_2)$ hold. Moreover, if one of them does not hold, the market structure becomes a monopoly. In particular, it is easy to show that if $\mu_2 < \mu_2^S(\mu_1)$, λ_1^S is exactly characterized by Eq. (19); if $\mu_1 < \mu_1^S(\mu_2)$, λ_2^S is exactly characterized by Eq. (21). □

Proof of Proposition 9. We omit this proof because it is similar to but less complicated than that of Proposition 6. □

Proof of Proposition 10. We omit this proof because it is similar to but less complicated than that of Proposition 7. $\square$

Proof of Proposition 11. It follows immediately from Propositions 2 and 4 and Parts (1) and (2) of Proposition 8. $\square$

Proof of Proposition 12. The proof is divided into two cases: (1) $\mu_2^T(\mu_1) < \mu_2 < \mu_2^{T'}(\mu_1)$; (2) $\mu_1 > \mu_1^T(\mu_2)$ and $\mu_2 \geq \mu_2^{T'}(\mu_1)$. (1) When $\mu_2^T(\mu_1) < \mu_2 < \mu_2^{T'}(\mu_1)$, we have $\lambda_2^T = 0 < \lambda_2^S$. Note that $\lambda_1^T = \lambda_1^S$ when $\mu_2 = \mu_2^T(\mu_1)$. Note also that when $\mu_2^T(\mu_1) < \mu_2 < \mu_2^{T'}(\mu_1)$, $\partial\lambda_1^T/\partial\mu_2 > 0$ and $\partial\lambda_1^S/\partial\mu_2 < 0$. It is immediate that $\lambda_1^T > \lambda_1^S$. (2) When $\mu_1 > \mu_1^T(\mu_2)$ and $\mu_2 \geq \mu_2^{T'}(\mu_1)$, the equilibrium market structure is a duopoly in the value-based and size-based competitions and $\lambda_i^j > 0$, $i = 1, 2$ and $j = T, S$. We prove by contradiction that $\lambda_1^T > \lambda_1^S$. Suppose otherwise that $\lambda_1^T \leq \lambda_1^S$. In the previous case, we have shown that $\lambda_1^T > \lambda_1^S$ when $\mu_2 = \mu_2^{T'}(\mu_1)$. By the continuity of λ_1^T and λ_1^S. There exists at least one threshold with respect to μ_2 such that $\lambda_1^T = \lambda_1^S$. Then from the second line of Eqs. (1) and (17), we have $\lambda_2^T = \lambda_2^S$, which, according to the first line of Eqs. (1) and (17), gives us $\lambda_1^T = \lambda_1^S = 0$, thus contradicting with $\lambda_1^T > 0$ and $\lambda_1^S > 0$. Therefore, we have $\lambda_1^T > \lambda_1^S$, and from the second line of Eqs. (1) and (17), we have $\lambda_2^T < \lambda_2^S$. $\square$

Proof of Proposition 13. Note from Proposition 12 that $\lambda_2^S > \lambda_2^T$, and thus, we can immediately conclude that $p_2^T < p_2^S$ since $p_2^j = \frac{\lambda_2^j}{\Lambda} + \lambda_2^j \frac{\partial W_2(\mu_2, \lambda_2^j)}{\partial \lambda_2}$ is increasing in λ_2^j for $j = S, T$. Next, we prove $p_1^T < p_1^S$ by contradiction. Note that the equilibrium effective arrival rates and prices in a duopoly satisfy (2). Suppose now that $p_1^T \geq p_1^S$. Recall that $p_2^T < p_2^S$. From Lemma 2, we have λ_1 increases in p_2 and decreases in p_1. Thus, $p_2^T < p_2^S$ and $p_1^T \geq p_1^S$ result in $\lambda_1^T < \lambda_1^S$, which contradicts $\lambda_1^T > \lambda_1^S$. Hence, $p_1^T < p_1^S$ holds. Now, we have $p_1^T < p_1^S$ and $p_2^T < p_2^S$, from Lemma 2, it is clear that $\lambda_1^T + \lambda_2^T > \lambda_1^S + \lambda_2^S$. $\square$