Electronic Companion for "Machine Tools with Hidden Defects: Optimal Usage for Maximum Lifetime Value"

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EC.1. Solving for the Optimal Tool Usage Policy

Step 1 in Algorithm 1 considers the states where the tool is last observed as defective (i.e., $u=1$). It is optimal to retire the tool at any cumulative counter if the run counter is n_H-1 (i.e., line 3). This is because the tool fails with probability 1 if the process action is taken at run counter n_H-1 and $C_r \geq 0$. Then comparing the value of retiring the tool and continuing the production at the order specified in lines 4-8 allows us to calculate $V(v,\tau,w,1)$, $\forall(v,\tau,w,1)\in\mathcal{S}_1$. In Step 2, we consider the states where the tool is last observed as normal (i.e., $u=0$). For these states, the optimal action is to retire the tool at any run counter given that the cumulative counter is n_X+n_H-1 (i.e., line 11). This is again because the failure happens with probability 1 and $C_r\geq 0$. Comparing the values of the possible actions at the order specified in lines 12-23 allows us to calculate $V(v,\tau,0)$, $\forall(v,\tau,0)\in\mathcal{S}_0$. Algorithm 1 terminates with the calculation of $V(0,0,0)$, i.e., the maximum lifetime value of a new tool. The optimal policy is also generated as a by-product.

EC.2. Proofs

We first show a side result in Lemma EC.2.1 that will be used in the proof of Lemma 1.

Lemma EC.2.1. If the random variable X has a nondecreasing hazard rate and its pmf satisfies

$$\frac{f_X(2)}{f_X(1)} \geq \frac{f_X(3)}{f_X(2)} \geq \cdots \geq \frac{f_X(n_X)}{f_X(n_X-1)},$$

then it follows that

$$\frac{f_X(x+1)}{\sum_{t=x+2}^{n_X} f_X(t)} \geq \frac{f_X(x)}{\sum_{t=x+1}^{n_X} f_X(t)}, \quad \forall x\in\{1,\ldots,n_X-2\}.$$

Proof of Lemma EC.2.1. Since X has a nondecreasing failure rate, it follows that

$$\frac{f_X(x+2)}{\sum_{t=x+2}^{n_X} f_X(t)} \geq \frac{f_X(x+1)}{\sum_{t=x+1}^{n_X} f_X(t)}, \quad \forall x\in\{0,\ldots,n_X-2\},$$

which can be equivalently written as

$$\frac{f(x+2)}{f(x+1)} \geq \frac{\sum_{t=x+2}^{n_X} f_X(t)}{\sum_{t=x+1}^{n_X} f_X(t)}, \quad \forall x\in\{0,\ldots,n_X-2\}.$$

The result follows because $f(x+1)/f(x)\geq f(x+2)/f(x+1)$ for all $x\in\{1,\ldots,n_X-2\}$.

Algorithm 1 Calculation of the maximum lifetime value

Step 1: Calculate $V(v,\tau,w,1)$, $\forall(v,\tau,w,1)\in\mathcal{S}_1$:
  for $w=1$ **to** n_X-1 **do**
    $V(v,n_H-1,w,1)=C_r$ for $v\in\{w,\dots,n_X+n_H-2\}$.
    for $\tau=n_H-2$ **to** 0 **do**
      for $v=n_X-1+\tau$ **to** $\tau+w$ **do**
        $V(v,\tau,w,1)=\max\{C_r,(1-\pi_{f,1}(v,\tau,w)(m-C_d+V(v+1,\tau+1,w,1))\}$
      end for
    end for
  end for
Step 2: Calculate $V(v,\tau,0)$, $\forall(v,\tau,0)\in\mathcal{S}_0$:
  $V(n_X+n_H-1,\tau,0)=C_r$ for $\tau\in\{n_H,\dots,n_X+n_H-1\}$.
  for $v=n_X+n_H-2$ **to** n_X **do**
    for $\tau=v$ **to** $v-n_X+1$ **do**
      $V(v,\tau,0)=\max\{C_r,(1-\pi_{f,0}(v,\tau)(m-C_d+V(v+1,\tau+1,0))\}$
    end for
  end for
  for $v=n_X-1$ **to** 0 **do**
    $V(v,0,0)=\max\{C_r,(1-\pi_{f,0}(v,0)(m-C_d+V(v+1,1,0))\}$
    for $\tau=1$ **to** v **do**
      $$V(v,\tau,0)=\max\begin{cases}C_r\\(1-\pi_{f,0}(v,\tau)(m-\pi_d(v+1,\tau+1)C_d+V(v+1,\tau+1,0))\\-C_i+\pi_d(v,\tau)V(v,0,v-\tau+1,1)+(1-\pi_d(v,\tau))V(v,0,0)\end{cases}$$
    end for
  end for

Proof of Lemma 1. (i) By using the Bayes rule, we rewrite $\pi_d(v,\tau)$ as

$$\begin{aligned}\pi_d(v,\tau) &= \mathbb{P}(X\le v|X>v-\tau,X+H>v)\\ &= \frac{\mathbb{P}(v-\tau<X\le v,X+H>v)}{\mathbb{P}(X>v-\tau,X+H>v)}\end{aligned}\tag{EC.1}$$

for all $(v,\tau,0)\in\mathcal{S}_0$. Conditioning on X, we expand $\mathbb{P}(v-\tau<X\le v,X+H>v)$ as

$$f_X(v-\tau+1)\bar{F}_H(\tau)+f_X(v-\tau+2)\bar{F}_H(\tau-1)+\ldots+f_X(v)\bar{F}_H(1)\tag{EC.2}$$

Similarly, we condition on the value of X and rewrite $\mathbb{P}(X>v-\tau,X+H>v)$ as

$$f_X(v-\tau+1)\bar{F}_H(\tau)+f_X(v-\tau+2)\bar{F}_H(\tau-1)+\ldots+f_X(v)\bar{F}_H(1)+\bar{F}_X(v+1)\tag{EC.3}$$

The result follows from plugging the expressions in (EC.2) and (EC.3) into Equation (EC.1). Notice that (EC.2) is equal to (EC.3) for $v\ge n_X$, and therefore, $\pi_d(v,\tau)$ is then equal to 1.

(ii) We note that

$$\pi_d(v,\tau) = \frac{\sum_{x=v-\tau+1}^{v} f_X(x)\bar{F}_H(v-x+1)}{\sum_{x=v-\tau+1}^{v} f_X(x)\bar{F}_H(v-x+1)+\bar{F}_X(v+1)}$$

$$\leq \frac{\sum_{x=v-\tau+1}^{v} f_X(x)\bar{F}_H(v-x+1) + f_X(v-\tau)\bar{F}_H(\tau+1)}{\sum_{x=v-\tau+1}^{v} f_X(x)\bar{F}_H(v-x+1) + \bar{F}_X(v+1) + f_X(v-\tau)\bar{F}_H(\tau+1)}$$
$$= \pi_d(v, \tau+1)$$

for all $\tau \in \{0, 1, \ldots, v-1\}$ at any given value of $v \in \{1, \ldots, n_X - 1\}$, hence, the result follows.

(iii) We need to show that $\pi_d(v, \tau) \leq \pi_d(v+1, \tau)$, i.e.,

$$\frac{\mathbb{P}(v-\tau < X \leq v, X+H > v)}{\mathbb{P}(X > v-\tau, X+H > v)} \leq \frac{\mathbb{P}(v+1-\tau < X \leq v+1, X+H > v+1)}{\mathbb{P}(X > v+1-\tau, X+H > v+1)}, \tag{EC.4}$$

holds for all $v \in \{\tau, \ldots, n_X - 2\}$ at any given value of $\tau \in \{1, \ldots, n_X - 1\}$. We rewrite (EC.4) as

$$\frac{A}{A + \bar{F}_X(v+1)} \leq \frac{B}{B + \bar{F}_X(v+2)}, \tag{EC.5}$$

where

$$A \triangleq f_X(v-\tau+1)\bar{F}_H(\tau) + f_X(v-\tau+2)\bar{F}_H(\tau-1) + \ldots + f_X(v)\bar{F}_H(1),$$

$$B \triangleq f_X(v-\tau+2)\bar{F}_H(\tau) + f_X(v-\tau+3)\bar{F}_H(\tau-1) + \ldots + f_X(v+1)\bar{F}_H(1).$$

Notice that inequality (EC.5) holds if and only if

$$A\bar{F}_X(v+2) \leq B\bar{F}_X(v+1). \tag{EC.6}$$

Plugging the equations

$$\bar{F}_H(x) = \sum_{y=x}^{\tau-1} f_H(y) + \bar{F}_H(\tau),\ \ x = 1, \ldots, \tau$$

into A and B, and then expanding the right and left hand sides of (EC.6) reveals that (EC.6) holds when the inequalities

$$\frac{\sum_{x=0}^{t} f_X(v-x)}{\bar{F}_X(v+1)} \leq \frac{\sum_{x=0}^{t} f_X(v-x+1)}{\bar{F}_X(v+2)},\ \ t = 0, 1, \ldots, \tau-1 \tag{EC.7}$$

hold. The inequalities in (EC.7) are satisfied if and only if

$$\frac{f_X(v-t)}{\bar{F}_X(v+1-t)} \leq \frac{f_X(v+1)}{\bar{F}_X(v+2)},\ \ t = 0, 1, \ldots, \tau-1,$$

which always hold by Lemma EC.2.1. $\square$

We next show another side result that will be used in the proofs of Lemma 2 and Lemma 3.

Lemma EC.2.2. Suppose that $a_i \geq 0, b_i > 0$ for $i = 1, \ldots, k$ and it holds that

$$0 \leq \frac{a_1}{b_1} \leq \ldots \leq \frac{a_k}{b_k}. \tag{EC.8}$$

Then the following relations hold:

(i)

$$\frac{\sum_{i=1}^{k-1} a_i}{\sum_{i=1}^{k-1} b_i} \leq \frac{\sum_{i=1}^{k} a_i}{\sum_{i=1}^{k} b_i} \leq \frac{\sum_{i=2}^{k} a_i}{\sum_{i=2}^{k} b_i} \tag{EC.9}$$

(ii) For $0 \leq w_1 \leq w_2 \leq \ldots \leq w_k$,

$$\frac{a_1 + \ldots + a_k}{b_1 + \ldots + b_k} \leq \frac{w_j a_j + \ldots + w_k a_k}{w_j b_j + \ldots + w_k b_k}, \; j \in \{1, \ldots, k\}. \tag{EC.10}$$

Proof of Lemma EC.2.2. (i) The left inequality in (EC.9) holds if and only if

$$\frac{a_1 + \ldots + a_{k-1}}{b_1 + \ldots + b_{k-1}} \leq \frac{a_k}{b_k},$$

which further holds because $a_i b_k \leq a_k b_i$, $\forall i \in \{1, \ldots, k-1\}$ by (EC.8). Furthermore, it is easily verified that the right inequality in (EC.9) holds if and only if

$$\frac{a_1}{b_1} \leq \frac{a_2 + \ldots + a_k}{b_2 + \ldots + b_k},$$

which further holds because $a_1 b_i \leq b_1 a_i$, $\forall i \in \{2, \ldots, k\}$ by (EC.8).

(ii) We first show that

$$(a_j + \ldots + a_k)(w_j b_j + \ldots w_k b_k) \leq (b_j + \ldots + b_k)(w_j a_j + \ldots w_k a_k), \quad j \in \{1, \ldots, k\}. \tag{EC.11}$$

This holds because

$$(w_v - w_u) a_u b_v \leq (w_v - w_u) a_v b_u, \; u \in \{j, j+1, \ldots, k-1\}, v \in \{u+1, u+2, \ldots, k\}$$

holds by (EC.8) for all $j \in \{1, \ldots, k-1\}$ given the property $0 \leq w_1 \leq \ldots \leq w_k$. Consequently, (EC.10) holds for $j = 1$ by (EC.11). To show (EC.10) for $j \in \{2, \ldots, k\}$, we note that

$$(a_1 + \ldots + a_{j-1})(w_j b_j + \ldots w_k b_k) \leq (b_1 + \ldots + b_{j-1})(w_j a_j + \ldots w_k a_k), \quad j \in \{2, \ldots, k\} \tag{EC.12}$$

by (EC.8). Combining (EC.12) and (EC.11) for $j \in \{2, \ldots, k\}$ shows that (EC.10) holds. □

Proof of Lemma 2. (i) By using the Bayes rule, we rewrite $\pi_{f,0}(v, \tau)$ as

$$\begin{aligned} \pi_{f,0}(v, \tau) &= \mathbb{P}(X + H = v + 1 | X > v - \tau, X + H > v) \\ &= \frac{\mathbb{P}(X + H = v + 1, X > v - \tau)}{\mathbb{P}(X > v - \tau, X + H > v)}. \end{aligned} \tag{EC.13}$$

for all $(v, \tau, 0) \in \mathcal{S}_0$. We condition on the value of X and expand $\mathbb{P}(X + H = v + 1, X > v - \tau)$ as

$$f_X(v+1) f_H(0) + f_X(v) f_H(1) + \ldots + f_X(v - \tau + 2) f_H(\tau - 1) + f_X(v - \tau + 1) f_H(\tau). \tag{EC.14}$$

By conditioning on the value of H, we expand $\mathbb{P}(X > v-\tau, X+H > v)$ as

$$\bar{F}_X(v+1)f_H(0)+\bar{F}_X(v)f_H(1)+\ldots+\bar{F}_X(v-\tau+1)f_H(\tau)+\bar{F}_X(v+1-\tau)\bar{F}_H(\tau+1). \tag{EC.15}$$

The result follows from plugging the expressions in (EC.14) and (EC.15) into (EC.13).

(ii) We rewrite $\mathbb{P}(X > v-\tau, X+H > v)$ in the denominator of (EC.13) by conditioning on X as

$$f_X(v-\tau+1)\bar{F}_H(\tau)+f_X(v-\tau+2)\bar{F}_H(\tau-1)+\ldots+f_X(v)\bar{F}_H(1)+\bar{F}_X(v+1)\bar{F}_H(0). \tag{EC.16}$$

Notice that if the value of τ is increased by 1, then the numerator in (EC.14) increases by $f_X(v-\tau)f_H(\tau+1)$ and the denominator in (EC.16) increases by $f_X(v-\tau)\bar{F}_H(\tau+1)$. Taking $a_k = f_X(v-\tau)f_H(\tau)$ and $b_k = f_X(v-\tau)\bar{F}_H(\tau+1)$ in Lemma EC.2.2(i), the result follows from the first inequality in Lemma EC.2.2(i) because the random variable H has a nondecreasing hazard rate.

(iii) We need to show that $\pi_{f,0}(v,\tau) \le \pi_{f,0}(v+1,\tau)$, i.e.,

$$\begin{aligned}&\frac{f_X(v+1)f_H(0)+f_X(v)f_H(1)+\ldots+f_X(v-\tau+1)f_H(\tau)}{\bar{F}_X(v+1)f_H(0)+\bar{F}_X(v)f_H(1)+\ldots+\bar{F}_X(v-\tau+1)f_H(\tau)+\bar{F}_X(v+1-\tau)\bar{F}_H(\tau+1)}\\ \le\;&\frac{f_X(v+2)f_H(0)+f_X(v+1)f_H(1)+\ldots+f_X(v-\tau+2)f_H(\tau)}{\bar{F}_X(v+2)f_H(0)+\bar{F}_X(v+1)f_H(1)+\ldots+\bar{F}_X(v-\tau+2)f_H(\tau)+\bar{F}_X(v+2-\tau)\bar{F}_H(\tau+1)}\end{aligned} \tag{EC.17}$$

holds for all $v \in \{\tau, \tau+1, \ldots, \min\{n_X+n_H-2, n_X+\tau-2\}\}$ at any given value of $\tau \in \{0,\ldots,n_X+n_H-2\}$. For convenience in presentation, we let

$$C(v) \triangleq f_X(v+1)f_H(0)+f_X(v)f_H(1)+\ldots+f_X(v-\tau+1)f_H(\tau),$$

$$D(v) \triangleq \bar{F}_X(v+1)f_H(0)+\bar{F}_X(v)f_H(1)+\ldots+\bar{F}_X(v-\tau+1)f_H(\tau),$$

and rewrite (EC.17) as

$$\frac{C(v)}{D(v)+\bar{F}_X(v+1-\tau)\bar{F}_H(\tau+1)} \le \frac{C(v+1)}{D(v+1)+\bar{F}_X(v+2-\tau)\bar{F}_H(\tau+1)}. \tag{EC.18}$$

We prove (EC.18) in two steps: In the first step, we show that the inequality

$$\frac{C(v)}{D(v)} \le \frac{C(v+1)}{D(v+1)} \tag{EC.19}$$

holds. To this end, we first note that

$$\begin{aligned}\frac{C(v)}{D(v)} &= \frac{a_1+a_2+\ldots+a_\tau+a_{\tau+1}}{b_1+b_2+\ldots+b_\tau+b_{\tau+1}}\\ &\le \frac{a_2c_2+\ldots+a_\tau c_\tau+a_{\tau+1}c_{\tau+1}}{b_2c_2+\ldots+b_\tau c_\tau+b_{\tau+1}c_{\tau+1}},\end{aligned} \tag{EC.20}$$

where the equality follows from taking $a_i = f_X(v-\tau+i)f_H(\tau+1-i)$ and $b_i = \bar{F}_X(v-\tau+i)f_H(\tau+1-i)$ for $i \in \{1,\ldots,\tau+1\}$, and the inequality follows from Lemma EC.2.2(ii) by taking $c_i =$

$f_H(\tau+2-i)/f_H(\tau+1-i)$ for $i\in\{2,\ldots,\tau+1\}$ and using the property $c_2\le\ldots\le c_{\tau+1}$. In addition, it can easily be verified that

$$\begin{aligned}\frac{a_2c_2+\ldots+a_\tau c_\tau+a_{\tau+1}c_{\tau+1}}{b_2c_2+\ldots+b_\tau c_\tau+b_{\tau+1}c_{\tau+1}} &\le \frac{a_2c_2+\ldots+a_{\tau+1}c_{\tau+1}+f_X(v+2)f_H(0)}{b_2c_2+\ldots+b_{\tau+1}c_{\tau+1}+\bar F_X(v+2)f_H(0)}\\ &=\frac{C(v+1)}{D(v+1)}\end{aligned}\tag{EC.21}$$

since $f_X(v+2)b_i\ge \bar F_X(v+2)a_i$, $\forall i\in\{2,\ldots,\tau+1\}$. Combining (EC.20) and (EC.21) shows that (EC.19) holds.

In the second step, we first note that (EC.18) holds when

$$\frac{\bar F_X(v-\tau+2)}{\bar F_X(v-\tau+1)}\le\frac{C(v+1)}{C(v)}\tag{EC.22}$$

because (EC.19) holds. We will verify that (EC.22) holds by considering the two cases $v\le n_X-2$ and $v>n_X-2$ separately. First, suppose $v\le n_X-2$. In this case, we note that if the inequality

$$\begin{aligned}&\frac{f_X(v-\tau+2)+f_X(v-\tau+3)+\ldots+f_X(n_X)}{f_X(v-\tau+1)+f_X(v-\tau+2)+\ldots+f_X(n_X-1)}\\ &\qquad\le\frac{f_X(v+2)f_H(0)+f_X(v+1)f_H(1)+\ldots+f_X(v-\tau+2)f_H(\tau)}{f_X(v+1)f_H(0)+f_X(v)f_H(1)+\ldots+f_X(v-\tau+1)f_H(\tau)}\end{aligned}\tag{EC.23}$$

holds, then (EC.22) holds. Since we know $f_H(\tau)\ge f_H(\tau-1)\ge\ldots,\ge f_H(0)$, taking $a_i=f_X(n_X-i+1)$ and $b_i=f_X(n_X-i)$ for $i\in\{1,2,\ldots,n_X-v+\tau-1\}$, $j=n_X-v-1$, and $w_i=f_H(i-n_X+v+1)$ for $i\in\{j,j+1,\ldots,n_X-v+\tau-1\}$ in Lemma EC.2.2(ii) implies that (EC.23) holds, and hence, the result follows. In the second case, we suppose that $v>n_X-2$ and note that if the inequality

$$\begin{aligned}&\frac{f_X(v-\tau+2)+f_X(v-\tau+3)+\ldots+f_X(n_X)}{f_X(v-\tau+1)+f_X(v-\tau+2)+\ldots+f_X(n_X-1)}\\ &\le\frac{f_X(n_X)f_H(v+2-n_X)+f_X(n_X-1)f_H(v+3-n_X)+\ldots+f_X(v-\tau+2)f_H(\tau)}{f_X(n_X-1)f_H(v+2-n_X)+f_X(n_X-2)f_H(v+3-n_X)+\ldots+f_X(v-\tau+1)f_H(\tau)}\end{aligned}\tag{EC.24}$$

holds, then (EC.22) holds. Taking $a_i=f_X(n_X-i+1)$, $b_i=f_X(n_X-i)$ and $w_i=f_H(i-n_X+v+1)$ for $i\in\{1,2,\ldots,n_X-v+\tau-1\}$ in Lemma EC.2.2(ii) implies that (EC.24) holds, and hence, the result follows. $\square$

Proof of Lemma 3. (i) By using the Bayes rule, we rewrite $\pi_{f,1}(v,\tau,w)$ as

$$\begin{aligned}\pi_{f,1}(v,\tau,w)&=\mathbb{P}(X+H=v+1|w\le X\le v-\tau,X+H>v)\\ &=\frac{\mathbb{P}(X+H=v+1,w\le X\le v-\tau)}{\mathbb{P}(w\le X\le v-\tau,X+H>v)}\end{aligned}\tag{EC.25}$$

for all $(v,\tau,w,1)\in\mathcal{S}_1$. We condition on X and expand $\mathbb{P}(X+H=v+1,w\le X\le v-\tau)$ as

$$f_X(w)f_H(v-w+1)+f_X(w+1)f_H(v-w)+\ldots+f_X(v-\tau)f_H(\tau+1).\tag{EC.26}$$

Similarly, we condition on X and rewrite $\mathbb{P}(X \le v-\tau, X+H > v)$ as

$$f_X(w)\bar{F}_H(v-w+1) + f_X(w+1)\bar{F}_H(v-w) + \ldots + f_X(v-\tau)\bar{F}_H(\tau+1). \tag{EC.27}$$

The result follows from plugging the expressions in (EC.26) and (EC.27) into Equation (EC.25).

(ii) We first show that $\pi_{f,1}(v,\tau,w) \le \pi_{f,1}(v,\tau+1,w)$, i.e.,

$$\frac{\mathbb{P}(X+H=v+1, w \le X \le v-\tau)}{\mathbb{P}(w \le X \le v-\tau, X+H>v)} \le \frac{\mathbb{P}(X+H=v+1, w \le X \le v-\tau-1)}{\mathbb{P}(w \le X \le v-\tau-1, X+H>v)} \tag{EC.28}$$

for all $v \in \{w+1,\ldots,n_X+n_H-3\}$ and $\tau \in \{0,\ldots,\min(n_H, v-w)-1\}$ at a given value of $w \in \{1,\ldots,n_X-1\}$. After writing (EC.28) as

$$\begin{aligned}&\frac{f_X(w)f_H(v-w+1)+\ldots+f_X(v-\tau-1)f_H(\tau+2)+f_X(v-\tau)f_H(\tau+1)}{f_X(w)\bar{F}_H(v-w+1)+\ldots+f_X(v-\tau-1)\bar{F}_H(\tau+2)+f_X(v-\tau)\bar{F}_H(\tau+1)}\\ &\qquad\qquad\le \frac{f_X(w)f_H(v-w+1)+\ldots+f_X(v-\tau-1)f_H(\tau+2)}{f_X(w)\bar{F}_H(v-w+1)+\ldots+f_X(v-\tau-1)\bar{F}_H(\tau+2)},\end{aligned}$$

it is easy to verify that this inequality holds follows from Lemma EC.2.2(i) with $k=v-\tau$, $a_i = f_X(v+1-\tau+i)f_H(\tau+1)$ and $b_i = f_X(v+1-\tau+i)\bar{F}(\tau+1)$ for $i \in \{1,\ldots,k\}$ because the random variable H has a nondecreasing hazard rate.

We next show that $\pi_{f,1}(v,\tau,w) \ge \pi_{f,1}(v,\tau,w+1)$, i.e.,

$$\begin{aligned}&\frac{f_X(w)f_H(v-w+1)+f_X(w+1)f_H(v-w)+\ldots+f_X(v-\tau)f_H(\tau+1)}{f_X(w)\bar{F}_H(v-w+1)+f_X(w+1)\bar{F}_H(v-w)+\ldots+f_X(v-\tau)\bar{F}_H(\tau+1)}\\ &\qquad\qquad\ge \frac{f_X(w+1)f_H(v-w)+\ldots+f_X(v-\tau)f_H(\tau+2)}{f_X(w+1)\bar{F}_H(v-w)+\ldots+f_X(v-\tau)\bar{F}_H(\tau+2)}\end{aligned} \tag{EC.29}$$

for all $(v,\tau,w,1) \in \mathcal{S}_1$ with $w \in \{1,\ldots,n_X-2\}$. Notice that this immediately follows from the first inequality in Lemma EC.2.2(i) because the random variable H has a nondecreasing hazard rate.

Finally, we show that $\pi_{f,0}(v,\tau) \le \pi_{f,1}(v,\tau,w)$ for all $(v,\tau,w,1) \in \mathcal{S}_1$. To do this, we first note that, after getting a common denominator and comparing the pairs of terms from the left and right hand sides, the inequality

$$\begin{aligned}&\frac{f_X(v-\tau+1)f_H(\tau)+\ldots+f_X(v+1)f_H(0)}{f_X(v-\tau+1)\bar{F}_H(\tau)+\ldots+f_X(v+1)\bar{F}_H(0)}\\ &\qquad\le \frac{f_X(w)f_H(v-w+1)+\ldots+f_X(v-\tau)f_H(\tau+1)}{f_X(w)\bar{F}_H(v-w+1)+\ldots+f_X(v-\tau)\bar{F}_H(\tau+1)} = \pi_{f,1}(v,\tau,w)\end{aligned} \tag{EC.30}$$

holds for all $(v,\tau,w,1) \in \mathcal{S}_1$ if the set of inequalities

$$\bar{F}_H(i)f_H(j) \le \bar{F}_H(j)f_H(i), \quad i \in \{\tau+1,\ldots,v-w+1\}, j \in \{0,1,\ldots,\tau\} \tag{EC.31}$$

holds. The set of inequalities in (EC.31) holds because the random variable H has a nondecreasing hazard rate. The result then follows by noting that $\pi_{f,0}(v,\tau)$ cannot be greater than the left hand side in (EC.30). This can be seen by noting that the numerator of the left hand side in (EC.30) is the numerator of $\pi_{f,0}(v,\tau)$ given by (EC.14), while the denominator of the left hand side in (EC.30) is obtained by expanding $\mathbb{P}(X > v-\tau, X+H > v)$ by conditioning on H and then subtracting the nonnegative term $\bar{F}_X(v+2)\bar{F}_H(0)$.

(iii) We need to show that $\pi_{f,1}(v,\tau,w) \le \pi_{f,1}(v+1,\tau,w)$, i.e.,

$$\begin{aligned}&\frac{f_X(w)f_H(v-w+1)+\ldots+f_X(v-\tau)f_H(\tau+1)}{f_X(w)\bar{F}_H(v-w+1)+\ldots+f_X(v-\tau)\bar{F}_H(\tau+1)}\\ &\quad\le \frac{f_X(w)f_H(v-w+2)+f_X(w+1)f_H(v-w+1)+\ldots+f_X(v+1-\tau)f_H(\tau+1)}{f_X(w)\bar{F}_H(v-w+2)+f_X(w+1)\bar{F}_H(v-w+1)+\ldots+f_X(v+1-\tau)\bar{F}_H(\tau+1)}. \end{aligned} \tag{EC.32}$$

holds for all $v \in \{w+\tau, w+\tau+1, \ldots, \tau+n_X-2\}$ at any given value of $\tau \in \{0,\ldots,n_H-1\}$.

We first consider the inequality

$$\begin{aligned}&\frac{f_X(w)f_H(v-w+1)+\ldots+f_X(v-\tau)f_H(\tau+1)}{f_X(w)\bar{F}_H(v-w+1)+\ldots+f_X(v-\tau)\bar{F}_H(\tau+1)}\\ &\quad\le \frac{f_X(w+1)f_H(v-w+1)+\ldots+f_X(v+1-\tau)f_H(\tau+1)}{f_X(w+1)\bar{F}_H(v-w+1)+\ldots+f_X(v+1-\tau)\bar{F}_H(\tau+1)}. \end{aligned} \tag{EC.33}$$

After obtaining a common denominator in Inequality (EC.33) and comparing the pairs of terms from the left and right hand sides of the numerator, we observe that it is sufficient that the following set of inequalities is satisfied for (EC.33) to hold:

$$\begin{aligned}&f_X(v+2-j)f_X(v+1-i)\left\{\bar{F}_H(j)f_H(i)-f_H(j)\bar{F}_H(i)\right\}\\ &\quad\le f_X(v+1-j)f_X(v+2-i)\left\{\bar{F}_H(j)f_H(i)-f_H(j)\bar{F}_H(i)\right\} \qquad i,j\in\{\tau+1,\ldots,v-w+1\} \end{aligned} \tag{EC.34}$$

Notice that (EC.34) holds as equality for $i=j$. For $j<i$, we know that $\bar{F}_H(j)f_H(i)-f_H(j)\bar{F}_H(i)\ge 0$ because the random variable H has a nondecreasing failure rate, and therefore (EC.34) holds if

$$\frac{f_X(v+2-j)}{f_X(v+1-j)} \le \frac{f_X(v+2-i)}{f_X(v+1-i)}. \tag{EC.35}$$

On the other hand, for $j>i$, the nondecreasing failure rate of the random variable H implies that $\bar{F}_H(j)f_H(i)-f_H(j)\bar{F}_H(i)\le 0$. Consequently, (EC.34) holds if

$$\frac{f_X(v+2-j)}{f_X(v+1-j)} \ge \frac{f_X(v+2-i)}{f_X(v+1-i)} \tag{EC.36}$$

for $j>i$. We notice that the conditions in (EC.35) and (EC.36) hold under the property in (4). Therefore, we conclude that Inequality (EC.33) holds.

We next apply Lemma EC.2.2(i) by taking $k=v-w+2-\tau$, $a_i=f_X(v+2-\tau-i)f_H(\tau+i)$ and $b_i=f_X(v+2-\tau-i)\bar{F}_H(\tau+i)$ for $i\in\{1,\ldots,k\}$ and find that

$$\begin{aligned}&\frac{f_X(2)f_H(v)+\ldots+f_X(v+1-\tau)f_H(\tau+1)}{f_X(2)\bar{F}_H(v)+\ldots+f_X(v+1-\tau)\bar{F}_H(\tau+1)}\\ &\quad\le \frac{f_X(1)f_H(v+1)+f_X(2)f_H(v)+\ldots+f_X(v+1-\tau)f_H(\tau+1)}{f_X(1)\bar{F}_H(v+1)+f_X(2)\bar{F}_H(v)+\ldots+f_X(v+1-\tau)\bar{F}_H(\tau+1)}. \end{aligned} \tag{EC.37}$$

It follows from (EC.37) and (EC.33) that (EC.32) holds, and hence, the result follows. $\square$

Proof of Proposition 1. We use induction on the steps of the value iteration algorithm as the proof technique. Let $V^k(v,\tau,w,1)$ denote the total expected reward (i.e., also referred to as value function) at the kth iteration of the value iteration algorithm. Since the absorbing end-of-life state of the MDP is reached in a finite number of steps, $V^k(v,\tau,w,1)$ converges to the optimal infinite-horizon expected reward as k increases (Proposition 7.2.1, Bertsekas 2005).

(i) We first show that $V(v,\tau,w,1)$ is a nonincreasing function of v. To start with, we let $V^0(v,\tau,w,1) = C_r$ for all $(v,\tau,w,1) \in \mathcal{S}_1$, which is trivially a nonincreasing function. As the induction hypothesis, we assume $V^k(v,\tau,w,1) \geq V^k(v+1,\tau,w,1)$ for $k > 0$, and want to show $V^{k+1}(v,\tau,w,1) \geq V^{k+1}(v+1,\tau,w,1)$ for all $(v,\tau,w,1) \in \mathcal{S}_1$ with $v < n_X + \tau - 1$. To this end, we first show that

$$\begin{aligned}
V_P^k(v,\tau,w,1) &= (1-\pi_{f,1}(v,\tau,w))(m - C_d + V^k(v+1,\tau+1,w,1)) \\
&\geq (1-\pi_{f,1}(v,\tau,w))(m - C_d + V^k(v+2,\tau+1,w,1)) \\
&\geq (1-\pi_{f,1}(v+1,\tau,w))(m - C_d + V^k(v+2,\tau+1,w,1)) \\
&= V_P^k(v+1,\tau,w,1)
\end{aligned}$$

for all $(v,\tau,w,1) \in \mathcal{S}_1$ with $v < n_X + \tau - 1$, where the first inequality follows from the induction hypothesis, and the second inequality follows from Assumption 1(i) and $m - C_d + V^k(v+2,\tau+1,w,1)$ being positive. Therefore, it follows that

$$V^{k+1}(v,\tau,w,1) = \max\{0, V_P^k(v,\tau,w,1)\} \geq \max\{0, V_P^k(v+1,\tau,w,1)\} = V^{k+1}(v+1,\tau,w,1).$$

To show $V(v,\tau,w,1)$ is nonincreasing in τ, the proof follows the similar steps with the main difference of using Assumption 1(ii) instead of Assumption 1(i). Therefore, we omit the details.

(ii) Let $V^0(v,\tau,w,1) = 0$ for all $(v,\tau,w,1) \in \mathcal{S}_1$, which is trivially a nondecreasing function. As the induction hypothesis, we assume $V^k(v,\tau,w,1) \leq V^k(v,\tau,w+1,1)$ for $k > 0$ and want to show $V^{k+1}(v,\tau,w,1) \leq V^{k+1}(v,\tau,w+1,1)$ for all $(v,\tau,w,1) \in \mathcal{S}_1$ with $w < v - \tau$. Note that

$$\begin{aligned}
V_P^k(v,\tau,w,1) &= (1-\pi_{f,1}(v,\tau,w))(m - C_d + V^k(v+1,\tau+1,w,1)) \\
&\leq (1-\pi_{f,1}(v,\tau,w))(m - C_d + V^k(v+1,\tau+1,w+1,1)) \\
&\leq (1-\pi_{f,1}(v,\tau,w+1))(m - C_d + V^k(v+1,\tau+1,w+1,1)) \\
&= V_P^k(v,\tau,w+1,1)
\end{aligned}$$

for all $(v,\tau,w,1) \in \mathcal{S}_1$ with $w < v - \tau$, where the first inequality follows from the induction hypothesis, and the second inequality follows from Assumption 1(iii) and $m - C_d + V^k(v+1,\tau+1,w+1,1)$ being positive. Thus, it follows that

$$V^{k+1}(v,\tau,w,1) = \max\{0, V_P^k(v,\tau,w,1)\} \leq \max\{0, V_P^k(v,\tau,w+1,1)\} = V^{k+1}(v,\tau,w+1,1). \quad \square$$

Proof of Theorem 1. (i) We first show the existence of a critical threshold $i^*(t,w)$ for a given value of t and w. It is known from Proposition 1 that $V(t+i,i,w,1)$, and therefore,

$$V_P(t+i,i,w,1) = (1-\pi_{f,1}(t+i,i,w))\,(m - C_d + V(t+i+1,i+1,w,1))$$

are nonincreasing functions of i for all $t \in \{w,\ldots,n_X-1\}$ and $w \in \{1,\ldots,n_X-1\}$. Recall that $V(t+i,i,w,1) = \max\{V_P(t+i,i,w,1), C_r\}$ for all $i \in \{0,\ldots,n_H-1\}$, $t \in \{w,\ldots,n_X-1\}$ and $w \in \{1,\ldots,n_X-1\}$. Since $V_P(t+i,i,w,1)$ is nonincreasing in i and C_r is constant, a critical threshold $i^*(t,w) \in \{0,\ldots,n_H-1\}$ exists such that the 'process' action is optimal for $i < i^*(t,w)$ and the 'retire-the-tool' action is optimal for $i \geq i^*(t,w)$ at states $(t+i,i,w,1) \in \mathcal{S}_1$.

Due to the existence of a critical threshold, one of the following two cases must hold for any t and w value:

(A) it is always optimal to retire, i.e., $i^*(t,w) = 0$. Then, $(1-\pi_{f,1}(t,0,w))(m - C_d + C_r) \leq C_r$.

(B) it is optimal to process at state $(t+i^*(t,w)-1, i^*(t,w)-1, w, 1)$ and to retire at state $(t+i^*(t,w), i^*(t,w), w, 1)$, implying that

$$(1-\pi_{f,1}(t+i^*(t,w)-1, i^*(t,w)-1, w))(m - C_d + C_r) > C_r \qquad \text{(EC.38)}$$

and

$$(1-\pi_{f,1}(t+i^*(t,w), i^*(t,w), w))(m - C_d + C_r) \leq C_r, \qquad \text{(EC.39)}$$

respectively.

In case A, the inequality can be rewritten as $\pi_{f,1}(t,0,w) \geq (m-C_d)/(m-C_d+C_r)$, and hence, the result follows. In case B, the inequality (EC.38) can be rewritten as $\pi_{f,1}(t+i^*(t,w)-1, i^*(t,w)-1, w) < (m-C_d)/(m-C_d+C_r)$. It then follows that $\pi_{f,1}(t+i,i,w) < (m-C_d)/(m-C_d+C_r)$ at all the states $(t+i,i,w,1)$ with $i \leq i^*(t,w)-1$ because $\pi_{f,1}(t+i,i,w)$ cannot increase as i decreases. Furthermore, the inequality (EC.39) can be rewritten as $\pi_{f,1}(t+i^*(t,w), i^*(t,w), w) \geq (m-C_d)/(m-C_d+C_r)$. It then follows that $\pi_{f,1}(t+i,i,w) \geq (m-C_d)/(m-C_d+C_r)$ at all the states $(t+i,i,w,1)$ with $i \geq i^*(t,w)$ because $\pi_{f,1}(t+i,i,w)$ is nondecreasing in i.

(ii) The nonincreasing behavior of $i^*(t,w)$ in t follows from the characterization of $i^*(t,w)$ and the nondecreasing behavior of $\pi_{f,1}(v,\tau,w)$ in v (Assumption 1.i). Similarly, the nondecreasing behavior of $i^*(t,w)$ in w follows because $\pi_{f,1}(v,\tau,w)$ is nonincreasing in w (Assumption 1.iii). $\square$

Proof of Proposition 2. Theorem 2(i) shows that $i^*(v,w)$ is the optimal number of products to process with a tool at state $(v,0,w,1)$. There are two cases to consider: (i) if $i^*(v,w)$ is zero, then it is optimal to take the retire-the-tool action right after the inspection and no production reward can be gained; i.e., $g(v,w) = C_r$. (ii) If $i^*(v,w) \in \{1,\ldots,n_H-1\}$, the expected production reward from the processing of the ith product is given by $(m-C_d)\prod_{j=1}^{i}(1-\pi_{f,1}(v+j-1,j-1,w))$ for $i=1,\ldots,i^*(v,w)$. On the other hand, the expected reward of retiring at the $i^*(v,w)$th product is given by $C_r\prod_{j=1}^{i^*(v,w)}(1-\pi_{f,1}(v+j-1,j-1,w))$. The result follows from adding these terms. $\square$

Proof of Proposition 3. We prove parts (i)-(iii) together. Let $V^k(v,\tau,0)$ denote the value function at the kth iteration of the value iteration algorithm. As stated in the proof of Proposition 1, the value function $V^k(v,\tau,0)$ converges to the optimal infinite-horizon expected reward as k increases. As the induction hypothesis, at a given $k>0$, we assume that

(H1) $V^k(v,\tau,0)\geq V^k(v+1,\tau,0)$, $\forall v\in\{0,\ldots,n_X+n_H-2\}$ and $\tau\in\{\max(v-n_X+1,0),\ldots,v\}$,

(H2) $V^k(v,\tau+1,0)\geq V^k(v,\tau,0)$, $\forall v\in\{1,\ldots,n_X+n_H-1\}$ and $\tau\in\{\max(v-n_X+1,0),\ldots,v-1\}$,

(H3) $V^k(v,\tau,0)\geq V^k(v,\tau,v-\tau,1)$, $\forall(v,\tau,v-\tau,1)\in\mathcal{S}_1$ (i.e., it is sufficient to consider $w=v-\tau$ as it is already known from Proposition 1(ii) that $V(v,\tau,w,1)$ cannot increase as w decreases),

(H4) $V^k(v,0,w,1)\geq V^k(v+1,0,w+1,1)$ for $w\in\{1,\ldots,n_X-2\}$ and $v\in\{w,\ldots,n_X-2\}$.

For $k=0$, we let $V^0(\mathbf{s})=C_r$ for all $\mathbf{s}\in\mathcal{S}_0\cup\mathcal{S}_1\cup\{\nabla\}$, which is trivially a nonincreasing function. We do the proof in two parts. In the first part, we will show that H1, H3 and H4 hold for $k+1$ under the Assumption 2(i), 2(iii) and 2(iv). In the second part, we will show that H2, H3 and H4 hold for $k+1$ under the Assumption 2(ii), 2(iii) and 2(iv). We start with noting that

$$\begin{aligned}
V_P^k(v,\tau,0) &= (1-\pi_{f,0}(v,\tau))\left(m-\pi_d(v+1,\tau+1)C_d+V^k(v+1,\tau+1,0)\right)\\
&\geq (1-\pi_{f,0}(v,\tau))\left(m-\pi_d(v+1,\tau+1)C_d+V^k(v+2,\tau+1,0)\right)\\
&\geq (1-\pi_{f,0}(v+1,\tau))(m-\pi_d(v+2,\tau+1)C_d+V^k(v+2,\tau+1,0))\\
&= V_P^k(v+1,\tau,0)
\end{aligned}$$

for all $v\in\{0,\ldots,n_X+n_H-3\}$ and $\tau\in\{\max(v-n_X+1,0),\ldots,v\}$. The first inequality above follows from the induction hypothesis H1. The second inequality follows from $\pi_{f,0}(v+1,\tau)\geq\pi_{f,0}(v,\tau)$ and $\pi_d(v+2,\tau+1)\geq\pi_d(v+1,\tau+1)$ (Assumption 2.i), the assumption $m-C_d+C_r>0$, and $V^k(v+2,\tau+1,v-\tau+2,0)\geq C_r$ (i.e., this is because at any $k>0$ we already know $V^k(v,\tau,0)\geq C_r$ for all $(v,\tau,0)\in\mathcal{S}_0$ as the 'retire-the-tool' is always a feasible action). We next show that

$$\begin{aligned}
V_I^k(v,\tau) &= -C_i+\pi_d(v,\tau)V^k(v,0,v-\tau+1,1)+(1-\pi_d(v,\tau))V^k(v,0,0)\\
&\geq -C_i+\pi_d(v,\tau)V^k(v+1,0,v-\tau+2,1)+(1-\pi_d(v,\tau))V^k(v+1,0,0)\\
&\geq -C_i+\pi_d(v+1,\tau)V^k(v+1,0,v-\tau+2,1)+(1-\pi_d(v+1,\tau))V^k(v+1,0,0)\\
&= V_I^k(v+1,\tau)
\end{aligned}$$

for all $v \in \{0, \ldots, n_X - 2\}$ and $\tau \in \{1, \ldots, v\}$. The first inequality above follows from the induction hypotheses H1 and H4, and the second inequality follows from $\pi_d(v+1, \tau) \geq \pi_d(v, \tau)$ (Assumption 2.i) and the induction hypothesis H3. Consequently, it holds that

$$V^{k+1}(v,\tau,0) = \begin{cases} \max\left\{V_P^k(v,\tau,0), V_I^k(v,\tau), C_r\right\} & \forall (v,\tau) \in \{(v,\tau,0) \in \mathcal{S}_0 : v < n_X, 0 < \tau\} \\ \max\left\{V_P^k(v,\tau,0), C_r\right\} & \forall (v,\tau) \in \{(v,\tau,0) \in \mathcal{S}_0 : v \geq n_X \text{ or } \tau = 0\} \end{cases}$$

is a nonincreasing function of v because the maximum of nonincreasing functions is also a nonincreasing function. Furthermore, $V^{k+1}(v+i, 0, v-\tau+i, 1)$ is nondecreasing in i because

$$\begin{aligned} V^{k+1}(v,0,v-\tau,1) &= \max\{(1-\pi_{f,1}(v,0,v-\tau+1))\left(m - C_d + V^k(v,0,v-\tau+1,1)\right), C_r\} \\ &\geq \max\{(1-\pi_{f,1}(v+1,0,v-\tau+2))\left(m - C_d + V^k(v+1,0,v-\tau+2,1)\right), C_r\} \\ &= V^{k+1}(v+1,0,v-\tau+1,1), \end{aligned}$$

where the inequality follows from the induction hypothesis H4 and Assumption 2(iv). Thus, H1 and H4 hold for $k+1$. We next show H3 for $k+1$. To this end, we first show

$$\begin{aligned} V_P^k(v,\tau,0) &= (1-\pi_{f,0}(v,\tau))(m - \pi_d(v+1,\tau+1)C_d + V^k(v+1,\tau+1,0)) \\ &\geq (1-\pi_{f,0}(v,\tau))(m - C_d + V^k(v+1,\tau+1,v-\tau,1)) \\ &\geq (1-\pi_{f,1}(v,\tau,v-\tau))(m - C_d + V^k(v+1,\tau+1,v-\tau,1)) = V_P^k(v,\tau,v-\tau,1) \end{aligned}$$

for all $(v,\tau) \in \{(v,\tau) : (v,\tau,w,1) \in \mathcal{S}_1, w = v - \tau\}$, where the first inequality follows from $C_d \geq 0$, $\pi_d(v+1,\tau+1) \leq 1$ and the induction hypothesis H3, and the second inequality follows from Assumption 2(iii) and $m - C_d + V^k(v+1,\tau+1,v-\tau,1) > 0$. Consequently, for all $(v,\tau) \in \{(v,\tau,0) \in \mathcal{S}_0 : v < n_X, 0 < \tau\}$, where inspection is a feasible action, it follows that

$$\begin{aligned} V^{k+1}(v,\tau,0) &= \max\{V_P^k(v,\tau,0), V_I^k(v,\tau), C_r\} \\ &\geq \max\{V_P^k(v,\tau,0), C_r\} \\ &\geq \max\{V_P^k(v,\tau,v-\tau,1), C_r\} = V^{k+1}(v,\tau,v-\tau,1), \end{aligned}$$

and, for all $(v,\tau) \in \{(v,\tau,0) \in \mathcal{S}_0 : v \geq n_X \text{ or } \tau = 0\}$, where inspection is not a feasible action, it similarly follows that

$$\begin{aligned} V^{k+1}(v,\tau,v-\tau+1,0) &= \max\{V_P^k(v,\tau,v-\tau+1,0), 0\} \\ &\geq \max\{V_P^k(v,\tau,v-\tau,1), 0\} = V^{k+1}(v,\tau,v-\tau,1). \end{aligned}$$

Thus, H3 holds $k+1$.

We omit the details of the second part of the proof as it is very similar to the first part except using the induction hypothesis H2 and Assumption 2(ii) instead of H1 and Assumption 2(i). □

Proof of Theorem 2. (i) The existence of $u^*(t)$, $t=0,\dots,n_X-1$, will be proved by showing that if retire-the-tool action is optimal at state $(v,\tau,0)$, then the retire-the-tool action is also optimal at state $(v+1,\tau+1,0)$ for all possible v and τ values in $\mathcal{S}_0$.

Suppose that the retire-the-tool action is optimal at state $(v,\tau,0)$, but the retire-the-tool action is not optimal at state $(v+1,\tau+1,0)$. This would imply $V_P(v,\tau,0)<C_r$ and $V_I(v,\tau)<C_r$ and either $V_P(v+1,\tau+1,0)>C_r$ or $V_I(v+1,\tau+1)>C_r$. We first note that $V_P(v+1,\tau+1,0)>C_r$ is not possible because

$$\begin{aligned}V_P(v,\tau,0) &= (1-\pi_{f,0}(v,\tau))\,(m-\pi_d(v+1,\tau+1)C_d+V(v+1,\tau+1,0))\\ &\geq (1-\pi_{f,0}(v+1,\tau+1))\,(m-\pi_d(v+2,\tau+2)C_d+V(v+2,\tau+2,0))=V_P(v+1,\tau+1,0)\end{aligned}$$

but $V_P(v,\tau,0)<C_r$, leading to a contradiction. Notice that the inequality above follows from Assumption 2(i)-(ii), monotonicity of the value function established in Proposition 3(i), $C_d\geq 0$, the assumption $m-C_d+C_r\geq 0$, and $V(v+2,\tau+2,0)\geq C_r$.

We also note that $V_I(v+1,\tau+1,0)>C_r$ is not possible because

$$\begin{aligned}V_I(v,\tau) &= -C_i+\pi_d(v,\tau)V(v,0,v-\tau+1,1)+(1-\pi_d(v,\tau))V(v,0,0)\\ &\geq -C_i+\pi_d(v,\tau)V(v+1,0,v-\tau+1,1)+(1-\pi_d(v,\tau))V(v+1,0,0)\\ &\geq -C_i+\pi_d(v+1,\tau+1)V(v+1,0,v-\tau+1,1)+(1-\pi_d(v+1,\tau+1))V(v+1,0,0)\\ &= V_I(v+1,\tau+1)\end{aligned}$$

but $V_I(v,\tau)<C_r$, leading to a contradiction. Notice that the first inequality above follows from the monotonicity of the value function established in Proposition 1(i) and 3(i). The second inequality follows from Assumption 2(i)-(ii) and Proposition 3(ii).

Consequently, we show by contradiction that the retire-the-tool action must be optimal at state $(v+1,\tau+1,0)$ if the retire-the-tool action is already optimal at state $(v,\tau,0)$. This implies, at a fixed value of $t=v-\tau$, there exists a threshold $u^*(t)$ such that it is optimal to take the retire-the-tool action at states $(t+i,i,0)\in\mathcal{S}_0$ for $i\in\{u^*(t),u^*_w+1,\dots,n_X+n_H-1-t\}$ for any $t\in\{1,\dots,n_X-1\}$.

(ii) The threshold $\ell^*(t)$ exists when the following holds: if 'process' action is optimal at $(v,\tau,0)$, then the 'process' action is also optimal at $(v-1,\tau-1,0)$ for all possible v and τ values in $\mathcal{S}_0$ with a fixed $t=v-\tau$. The states on the 45-degree line with $t\in\{0,\dots,n_X-1\}$ can be partitioned into three groups of states: (a) $(n_X+i,n_X-t+i,0)$ for $i=1,\dots,n_H-1$, (b) $(n_X,n_X-t,0)$, and (c) $(t+i,i,0)$ for $i=0,\dots,n_X-1-t$.

For any t, suppose that the state $(v,\tau,0)$ is in group (a) or in group (c) with $i=1$. It is easy to verify that if 'process' is optimal at state $(v,\tau,0)$, then 'process' is also optimal at state $(v-1,\tau-1,0)$. Specifically, if it is known $V(v,\tau,0)>C_r$ (since the 'process' action is optimal), it is not possible to have $V(v-1,\tau-1,0)=C_r$ (from Proposition 3.i), and hence, the 'process' must also be optimal at state $(v-1,\tau-1,0)$ (since 'inspection' is not feasible at $(v-1,\tau-1,0)$). So, in the remainder of the proof, we consider the remaining states.

For $t\in\{0,\ldots,n_X-1\}$, suppose that the state $(v,\tau,0)$ is in group (b). We want to show that if 'process' is optimal at state $(v,\tau,0)$, then 'process' is also optimal at state $(v-1,\tau-1,0)$. For $t=n_X-1$, this reduces to showing that if 'process' is optimal at state $(n_X,1,0)$, then 'process' is also optimal at state $(n_X-1,0,0)$. This trivially holds because 'inspection' is not feasible at $(n_X-1,0,0)$ and $V(n_X-1,0,0)>C_r$ (from Proposition 3.i). For $t\le n_X-2$, suppose that 'process' is optimal at state (n_X,n_X-t) and 'inspection' is optimal at state (n_X-1,n_X-t-1). This implies that $V_P(n_X,n_X-t,0)>C_r$ and $V_I(n_X-1,n_X-t-1)>V_P(n_X-1,n_X-t-1,0)$; i.e.,

$$(1-\pi_{f,0}(n_X,n_X-t))(m-C_d+V(n_X+1,n_X-t+1,0))>C_r \tag{EC.40}$$

and

$$\begin{aligned}-C_i+\pi_d(n_X-1,n_X-t-1)V(n_X-1,0,t+1,1)+(1-\pi_d(n_X-1,n_X-t-1))V(n_X-1,0,0)\\ >(1-\pi_{f,0}(n_X-1,n_X-t-1))(m-C_d+V(n_X,n_X-t,0)).\end{aligned} \tag{EC.41}$$

Replacing $V(n_X-1,0,t+1,1)$ with $g(n_X-1,t+1)$ and adding up (EC.40) and (EC.41), we obtain

$$\begin{aligned}&\pi_d(n_X-1,n_X-t-1)g(n_X-1,t+1)+(1-\pi_d(n_X-1,n_X-t-1))V(n_X-1,0,0)\\ &\quad+(1-\pi_{f,0}(n_X,\tau))V(n_X+1,n_X-t+1,0)\\ &>C_i+\Delta_f(n_X,n_X-t;1,1)(m-C_d)+(1-\pi_{f,0}(n_X-1,n_X-t-1))V(n_X,n_X-t,0)+C_r.\end{aligned} \tag{EC.42}$$

Since $V(n_X,n_X-t,0)\ge V(n_X+1,n_X-t+1,0)$ (Proposition 3.i), it can be shown that if (EC.42) holds then the inequality

$$\begin{aligned}\pi_d(n_X-1,n_X-t-1)g(n_X-1,t+1)+(1-\pi_d(n_X-1,n_X-t-1))V(n_X-1,0,0)\\ >C_i+\Delta_f(n_X,n_X-t;1,1)(m-C_d+V(n_X,n_X-t,0))+C_r\end{aligned} \tag{EC.43}$$

also holds. For $v\in\{0,\ldots,n_X-1\}$, let $r(v)$ denote $m(\mathbb{E}(X|X>v)-v-1)+(m-C_d)(\mathbb{E}(H)+1)$, representing the production reward of a tool, which is just found normal in an inspection performed at cumulative counter v, if it is run until failure. Plugging the inequality $V(n_X-1,0,0)\le r(n_X-1)+C_r$ (i.e., the maximum lifetime value of a tool cannot be greater than a policy which runs the tool until failure and still collects the salvage reward) on the left-hand-side of the inequality (EC.43) and plugging $V(n_X,n_X-t,0)\ge C_r$ on the right hand side of the inequality (EC.43) give

$$\begin{aligned}\pi_d(n_X-1,n_X-t-1)g(n_X-1,t+1)+(1-\pi_d(n_X-1,n_X-t-1))(r(n_X-1)+C_r)\\ >C_i+\Delta_f(n_X,n_X-t;1,1)(m-C_d+C_r)+C_r.\end{aligned} \tag{EC.44}$$

However, this inequality cannot be true if the condition (8) holds. Thus, by contradiction, if 'process' is optimal at $(n_X, n_X - t, 0)$, then 'process' is also optimal at $(n_X - 1, n_X - t - 1, 0)$.

Finally, for $t \in \{0, \ldots, n_x - 3\}$, suppose that the state $(v, \tau, 0)$ is in group (c) with $i \geq 2$. We want to show that if 'process' is optimal at state $(v, \tau, 0)$, then 'process' is also optimal at state $(v - 1, \tau - 1, 0)$. Suppose that 'process' is optimal at $(v, \tau, 0)$ but 'inspection' is optimal at $(v - 1, \tau - 1, 0)$. This implies $V_P(v, \tau, 0) > V_I(v, \tau)$ and $V_I(v - 1, \tau - 1) > V_P(v - 1, \tau - 1, 0)$; i.e.,

$$\begin{aligned}(1 - \pi_{f,0}(v, \tau))(m - \pi_d(v + 1, \tau + 1)C_d + V(v + 1, \tau + 1, 0)) \\ > -C_i + \pi_d(v, \tau)V(v, 0, v - \tau + 1, 1) + (1 - \pi_d(v, \tau))V(v, 0, 0)\end{aligned} \tag{EC.45}$$

and

$$\begin{aligned}-C_i + \pi_d(v - 1, \tau - 1)V(v - 1, 0, v - \tau + 1, 1) + (1 - \pi_d(v - 1, \tau - 1))V(v - 1, 0, 0) \\ > (1 - \pi_{f,0}(v - 1, \tau - 1))(m - \pi_d(v, \tau)C_d + V(v, \tau, 0)).\end{aligned} \tag{EC.46}$$

The following inequalities can easily be verified by using the monotonicity of the value function:

$$V(v - 1, 0, v - \tau + 1, 1) \leq m - C_d + V(v, 0, v - \tau + 1, 1) \tag{EC.47}$$

$$V(v - 1, 0, 0) \leq m + V(v, 0, 0). \tag{EC.48}$$

Plugging (EC.47) and (EC.48) into (EC.46) leads to the inequality

$$\begin{aligned}-C_i + \pi_d(v - 1, \tau - 1)(m - C_d + V(v, 0, v - \tau + 1, 1)) + (1 - \pi_d(v - 1, \tau - 1))(m + V(v, 0, 0)) \\ > (1 - \pi_{f,0}(v - 1, \tau - 1))(m - \pi_d(v, \tau)C_d + V(v, \tau, 0)).\end{aligned} \tag{EC.49}$$

Since $V(v + 1, \tau + 1, v - \tau + 1, 0) \leq V(v, \tau, v - \tau + 1, 0)$, (EC.45) implies that

$$\begin{aligned}(1 - \pi_{f,0}(v, \tau))(m - \pi_d(v + 1, \tau + 1)C_d + V(v, \tau, v - \tau + 1, 0)) \\ > -C_i + \pi_d(v, \tau)V(v, 0, v - \tau + 1, 1) + (1 - \pi_d(v, \tau))V(v, 0, 0).\end{aligned} \tag{EC.50}$$

Noting that $V(v, 0, v - \tau + 1, 1) = g(v, v - \tau + 1)$, and adding (EC.49) and (EC.50) lead to

$$\begin{aligned}\Delta_d(v, \tau; 1, 1)\,(V(v, 0, 0) - g(v, v - \tau + 1)) + \big(\pi_{f,0}(v, \tau)\pi_d(v + 1, \tau + 1) - \pi_{f,0}(v - 1, \tau - 1)\pi_d(v, \tau)\big)C_d \\ > \Delta_f(v, \tau; 1, 1)(V(v, \tau, 0) + m) - m + C_d\big(\Delta_d(v + 1, \tau + 1; 1, 1) + \pi_d(v - 1, \tau - 1)\big),\end{aligned}$$

which further implies

$$\begin{aligned}\Delta_d(v, \tau; 1, 1)\,(r(v) + C_r - g(v, v - \tau + 1)) + \big(\pi_{f,0}(v, \tau)\pi_d(v + 1, \tau + 1) - \pi_{f,0}(v - 1, \tau - 1)\pi_d(v, \tau)\big)C_d \\ > \Delta_f(v, \tau; 1, 1)(C_r + m) - m + C_d\big(\Delta_d(v + 1, \tau + 1; 1, 1) + \pi_d(v - 1, \tau - 1)\big)\end{aligned}$$

because $V(v, 0, 0) \leq r(v) + C_r$ and $V(v, \tau, 0) \geq C_r$. However, this inequality cannot be true if the condition (9) holds. Thus, it follows that if 'process' is optimal at state $(v, \tau, 0)$ then 'process' is optimal at state $(v - 1, \tau - 1, 0)$. $\square$

Proof of Theorem 3. (i) Recall that the existence of $u^*(t)$, $t=0,\ldots,n_X-1$, is proved in Theorem 2(i) by showing that if 'retire-the-tool' is optimal at state $(v,\tau,0)$, then 'retire-the-tool' is also optimal at state $(v+1,\tau+1,0)$ for all possible v and τ values in $\mathcal{S}_0$. Similarly, the following two results together imply that $u^*(t)$ is nonincreasing in t: (a) If 'retire-the-tool' is optimal at state $(v,\tau,0)$, then 'retire-the-tool' is also optimal at state $(v+1,\tau,0)$. (b) If 'retire-the-tool' is optimal at state $(v,\tau,0)$, then 'retire-the-tool' is also optimal at state $(v,\tau+1,0)$. Both results (a) and (b) can be shown by following the same steps as in part Theorem 2(i), so we omit a detailed proof. To show result (a), Assumption 2(i), Proposition 1(ii) and Proposition 3(ii)-(iii) are used. To show result (b), Assumption 2(ii), Proposition 1(i)-(ii) and Proposition 3(ii) are used.

(ii) The existence of threshold $\ell^*(t)$, $t=0,\ldots,n_X-1$, has been proved in Theorem 2(ii). It is sufficient to show the following additional result to establish $\ell^*(t)\geq\ell^*(t+1)$ for any $t\{0,\ldots,n_X-2\}$: if 'process' is optimal at a state on the 45-degree line with $t+1$ as the difference between cumulative and run counters, then 'process' is also optimal at the state with one less run counter (i.e., the left neighbor state on the 45-degree line with t as the difference between cumulative and run counters). For a specific $t\in\{0,\ldots,n_X-2\}$, there are two sets of states that we need to consider separately: (a) $(n_X,n_X-t-1,0)$, and (b) $(n_X-i,n_X-t-i,0)$, $i=1,\ldots,n_X-2-t$ (i.e., for $t=n_X-2$, this set is empty).

For case (a), suppose that 'process' is optimal at state $(n_X,n_X-t-1,0)$ but 'inspection' is optimal at state (n_X-1,n_X-t-1). This implies that $V_P(n_X,n_X-t-1,0)>C_r$ and $V_I(n_X-1,n_X-t-1)>V_P(n_X-1,n_X-t-1)$; i.e.,

$$(1-\pi_{f,0}(n_X,n_X-t-1))(m-C_d+V(n_X+1,n_X-t,0))>C_r \tag{EC.51}$$

and

$$\begin{aligned}&-C_i+\pi_d(n_X-1,n_X-t-1)V(n_X-1,0,t+1,1)+(1-\pi_d(n_X-1,n_X-t-1))V(n_X-1,0,0)\\&\quad>(1-\pi_{f,0}(n_X-1,n_X-t-1))(m-C_d+V(n_X,n_X-t,0)).\end{aligned} \tag{EC.52}$$

Replacing $V(n_X-1,0,t+1,1)$ with $g(n_X-1,t+1)$ and adding up inequalities (EC.51) and (EC.52), we obtain that

$$\begin{aligned}&\pi_d(n_X-1,n_X-t-1)g(n_X-1,t+1)+(1-\pi_d(n_X-1,n_X-t-1))V(n_X-1,0,0)\\&\qquad+(1-\pi_{f,0}(n_X,n_X-t-1))V(n_X+1,n_X-t,0)\\&>C_i+\Delta_f(n_X,n_X-t-1;1,0)(m-C_d)+(1-\pi_{f,0}(n_X-1,n_X-t-1))V(n_X,n_X-t,0)+C_r.\end{aligned} \tag{EC.53}$$

Since $V(n_X,n_X-t,0)\geq V(n_X+1,n_X-t,0)$ (Proposition 3.i), it can be shown that if (EC.53) holds then the inequality

$$\begin{aligned}&\pi_d(n_X-1,n_X-t-1)g(n_X-1,t+1)+(1-\pi_d(n_X-1,n_X-t-1))V(n_X-1,0,0)\\&\qquad>C_i+\Delta_f(n_X,n_X-t-1;1,0)(m-C_d+V(n_X,n_X-t,0))+C_r.\end{aligned}$$

also holds, which further implies

$$\pi_d(n_X-1,n_X-t-1)g(n_X-1,t+1)+(1-\pi_d(n_X-1,n_X-t-1))(r(n_X-1)+C_r)$$
$$>C_i+\Delta_f(n_X,n_X-t-1;1,0)(C_r+m-C_d)+C_r$$

because $r(n_X-1)+C_r \geq V(n_X-1,0,0)$ and $V(n_X,n_X-t,0)\geq C_r$. However, this inequality cannot be true if the condition (10) holds. Thus, by contradiction, if 'process' is optimal at $(n_X,n_X-t,0)$ then 'process' is also optimal at $(n_X,n_X-t-1,0)$.

For case (b), consider the states $(v,\tau,0)$ with $v=n_X-i$ and $\tau=n_X-t-i$, for $i=1,\ldots,n_X-2-t$. Suppose that 'process' is optimal at $(v,\tau,0)$ and 'inspection' is optimal at $(v-1,\tau,0)$. This implies that $V_P(v,\tau,0)>V_I(v,\tau)$ and $V_I(v-1,\tau)>V_P(v-1,\tau,0)$; i.e.,

$$(1-\pi_{f,0}(v,\tau))(m-\pi_d(v+1,\tau+1)C_d+V(v+1,\tau+1,0))$$
$$>-C_i+\pi_d(v,\tau)V(v,0,v-\tau+1,1)+(1-\pi_d(v,\tau))V(v,0,0) \tag{EC.54}$$

and

$$-C_i+\pi_d(v-1,\tau)V(v-1,0,v-\tau,1)+(1-\pi_d(v-1,\tau))V(v-1,0,0)$$
$$>(1-\pi_{f,0}(v-1,\tau))(m-\pi_d(v,\tau+1)C_d+V(v,\tau+1,0)). \tag{EC.55}$$

Notice that Proposition 1(ii) and (EC.47) imply $V(v-1,0,v-\tau,1)\leq m-C_d+V(v,0,v-\tau+1,1)$. Plugging this inequality and the inequality (EC.48) on the left-hand side of (EC.55), plugging $V(v,\tau+1,0)\geq V(v+1,\tau+1,0)$ (Proposition 3.i) on the left-hand side of (EC.54), and finally, adding (EC.54) and (EC.55) lead to

$$\Delta_d(v,\tau;1,0)\left(V(v,0,0)-g(v,v-\tau+1)\right)+\big(\pi_{f,0}(v,\tau)\pi_d(v+1,\tau+1)-\pi_{f,0}(v-1,\tau)\pi_d(v,\tau+1)\big)C_d$$
$$>\Delta_f(v,\tau;1,0)(V(v,\tau+1,0)+m)-m+C_d\big(\Delta_d(v+1,\tau+1;1,0)+\pi_d(v-1,\tau)\big),$$

which further implies

$$\Delta_d(v,\tau;1,0)\left(r(v)+C_r-g(v,v-\tau+1)\right)+\big(\pi_{f,0}(v,\tau)\pi_d(v+1,\tau+1)-\pi_{f,0}(v-1,\tau)\pi_d(v,\tau+1)\big)C_d$$
$$>\Delta_f(v,\tau;1,0)(C_r+m)-m+C_d\big(\Delta_d(v+1,\tau+1;1,0)+\pi_d(v-1,\tau)\big),$$

because $V(v,0,0)\leq r(v)+C_r$ and $V(v,\tau+1,0)\geq C_r$. However, this inequality cannot be true if the condition (11) holds. Thus, it follows that if 'process' is optimal at state $(v,\tau,0)$, then 'process' is also optimal at state $(v-1,\tau,0)$. $\square$

EC.3. Estimation of the Degradation-Process Parameters

We let the probability distributions of X and H have a known parametric form with unknown parameters $\boldsymbol{\theta}_X$ and $\boldsymbol{\theta}_H$. In this section, we present how the parameters $\boldsymbol{\theta}_X$ and $\boldsymbol{\theta}_H$ can be estimated from historical data. Suppose that the last inspection of tool i had been performed when the cumulative counter was equal to y_i. Let z_i denote the current value of the cumulative counter for tool i, $i=1,\ldots,n$. There are four scenarios to consider:

(a) The tool is found in the normal phase in the last inspection, and it has not failed yet by the time it is retired. This event occurs with probability $\mathbb{P}(X+H>z_i, X>y_i)$. Let $\mathcal{T}_a$ denote the set of indices of these tools.

(b) The tool is found normal in the last inspection and it has failed afterwards. This event occurs with probability $\mathbb{P}(X+H=z_i, X>y_i)$. Let $\mathcal{T}_b$ be the set of indices of these tools.

(c) The tool is found defective in the last inspection, and it has not failed yet by the time it is retired. This event occurs with probability $\mathbb{P}(X+H>z_i, X\le y_i)$. Let $\mathcal{T}_c$ denote the set of indices of these tools.

(d) The tool is found defective in the last inspection and it has failed afterwards. This event occurs with probability $\mathbb{P}(X+H=z_i, X\le y_i)$. Let $\mathcal{T}_d$ denote the set of indices of these tools.

We maximize the log-likelihood function, which can be characterized as

$$\sum_{i\in\mathcal{T}_a}\log\sum_{x=y_i+1}^{n_X} f_X(x;\boldsymbol{\theta}_X)(1-F_H(z_i-x;\boldsymbol{\theta}_H))+\sum_{i\in\mathcal{T}_b}\log\sum_{x=y_i+1}^{z_i} f_X(x;\boldsymbol{\theta}_X)f_H(z_i-x;\boldsymbol{\theta}_H)$$
$$+\sum_{i\in\mathcal{T}_c}\log\sum_{x=0}^{y_i} f_X(x;\boldsymbol{\theta}_X)(1-F_H(z_i-x;\boldsymbol{\theta}_H))+\sum_{i\in\mathcal{T}_d}\log\sum_{x=0}^{y_i} f_X(x;\boldsymbol{\theta}_X)f_H(z_i-x;\boldsymbol{\theta}_H),$$

in terms of the unknown parameters $\boldsymbol{\theta}_X$ and $\boldsymbol{\theta}_H$. Although a closed-form solution is not necessarily available for the resulting maximum-likelihood estimates, they can easily be identified numerically by using standard optimization packages. In Section 6, we apply this estimation technique by using the real-world tool maintenance logs from the production lines at our industrial partner.